q-Chebyshev polynomials

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Abstract

We give a short elementary introduction to $q$–Chebyshev polynomials.

1. The classical Chebyshev polynomials

In [4] I considered some polynomials related to the Al Salam and Ismail – polynomials introduced in [1] which can be interpreted as $q$–analogues of the Chebyshev polynomials. In this note I want to give a short direct approach to these polynomials.

The (classical) Chebyshev polynomials of the first kind $T_n(x)$ are characterized by the recurrence

$$T_n(x) = 2x T_{n-1}(x) - T_{n-2}(x)$$

(1.1)

with initial values $T_0(x) = 1$ and $T_1(x) = x$.

They have the determinant representation

$$T_n(x) = \det \begin{pmatrix} x & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2x & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2x & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2x \end{pmatrix}$$

(1.2)

The (classical) Chebyshev polynomials of the second kind $U_n(x)$ satisfy the same recurrence

$$U_n(x) = 2x U_{n-1}(x) - U_{n-2}(x)$$

(1.3)

but with initial values $U_{-1}(x) = 0$ and $U_0(x) = 1$, which gives $U_1(x) = 2x$.

Their determinant representation is
These polynomials are related by the identity

\[
\left( x + \sqrt{x^2 - 1} \right)^n = T_n(x) + U_{n-1}(x)\sqrt{x^2 - 1},
\]

which in turn implies

\[
T_n(x)^2 - (x^2 - 1)U_{n-1}(x)^2 = 1.
\]

**Remark 1.1**

For \( x = \cos \theta \) identity (1.5) becomes

\[
\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n = T_n(\cos \theta) + iU_{n-1}(\cos \theta) \sin \theta
\]

or equivalently

\[
T_n(\cos \theta) = \cos n\theta
\]

\[
U_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}.
\]

Identity (1.6) reduces to

\[
\cos^2 n\theta + \sin^2 n\theta = 1.
\]

It would be nice to find a \( q \) – analogue of this trigonometric definition.

The Chebyshev polynomials are orthogonal polynomials. A sequence \( (p_n)_{n \geq 0} \) of polynomials with \( p_0 = 1 \) and \( \deg p_n = n \) is called orthogonal with respect to a linear functional \( \Lambda \) on the vector space of polynomials if \( \Lambda(p_m p_n) = 0 \) for \( m \neq n \). The linear functional is uniquely determined by \( \Lambda(p_0) = 1 \) and \( \Lambda(p_n) = 0 \) for \( n > 0 \) which as usual we abbreviate with \( \Lambda(p_n) = [n = 0] \). The values \( \Lambda(x^n) \) are called moments of \( \Lambda \).

For the polynomials \( T_n(x) \) the corresponding linear functional \( L \) is given by

\[
L(p(x)) = \frac{1}{\pi} \int_{-1}^{1} \frac{p(x)}{\sqrt{1-x^2}} dx.
\]

For

\[
\frac{1}{\pi} \int_{-1}^{1} T_n(x) dx = \frac{1}{\pi} \int_{0}^{\pi} \cos(n\theta) d\theta = [n = 0].
\]
The corresponding moments are

\[ L(x^{2n}) = \frac{1}{\pi} \int_{-1}^{1} \frac{x^{2n}}{\sqrt{1-x^2}} = \frac{1}{2^n} \left( \begin{array}{c} 2n \\ n \end{array} \right) \] (1.9)

and \( L(x^{2n+1}) = 0 \).

For the polynomials \( U_n(x) \) we get the functional \( M \) with \( M(p_n) = \frac{2}{\pi} \int_{-1}^{1} p(x) \sqrt{1-x^2} \, dx \) since

\[ \frac{2}{\pi} \int_{-1}^{1} U_n(x) \sqrt{1-x^2} \, dx = \frac{2}{\pi} \int_{0}^{\pi} \sin((n+1)\theta) \sin \theta \, d\theta = [n = 0]. \]

The corresponding moments are

\[ M(x^{2n}) = \frac{2}{\pi} \int_{-1}^{1} x^{2n} \sqrt{1-x^2} \, dx = \frac{1}{2^n} \frac{1}{n+1} \left( \begin{array}{c} 2n \\ n \end{array} \right) \] (1.10)

and \( M(x^{2n+1}) = 0 \).

Instead of the classical Chebyshev polynomials of the first kind we will consider the \textit{bivariate Chebyshev polynomials} \( T_n(x,s) \) of the first kind which satisfy the recurrence

\[ T_n(x,s) = 2x T_{n-1}(x,s) + s T_{n-2}(x,s) \] (1.11)

with initial values \( T_0(x,s) = 1 \) and \( T_1(x,s) = x \).

Then of course \( T_n(x) = T_n(x,-1) \).

They have the determinant representation

\[ T_n(x,s) = \det \begin{pmatrix} x & s & 0 & \cdots & 0 & 0 \\ -1 & 2x & s & \cdots & 0 & 0 \\ 0 & -1 & 2x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2x & s \\ 0 & 0 & 0 & \cdots & -1 & 2x \end{pmatrix}, \] (1.12)

In the same way we consider the \textit{bivariate Chebyshev polynomials of the second kind} \( U_n(x,s) \) which satisfy the same recurrence

\[ U_n(x,s) = 2x U_{n-1}(x,s) + s U_{n-2}(x,s) \] (1.13)
but with initial values $U_0(x,s) = 1$ and $U_1(x,s) = 2x$.

Their determinant representation is

$$ U_n(x,s) = \det \begin{bmatrix} 2x & s & 0 & \cdots & 0 & 0 \\ -1 & 2x & s & \cdots & 0 & 0 \\ 0 & -1 & 2x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2x & s \\ 0 & 0 & 0 & \cdots & -1 & 2x \end{bmatrix}. \quad (1.14) $$

Both polynomials are connected via

$$ \left( x + \sqrt{x^2 + s} \right)^n = T_n(x,s) + U_{n-1}(x,s)\sqrt{x^2 + s}. \quad (1.15) $$

This also implies

$$ T_n(x,s)^2 - (x^2 + s)U_{n-1}(x,s)^2 = (-s)^n. \quad (1.16) $$

2. q-analogues

We assume that $q \neq 1$ is a real number. All $q$–identities in this paper reduce to known identities when $q$ tends to 1. We assume that the reader is familiar with the most elementary notions of $q$–analysis (cf. e.g. [3]). The $q$–binomial coefficients

$$ \binom{n}{k} = \frac{[1][2] \cdots [n]}{[1][k] \cdots [n-k]} \quad \text{with} \quad [n] = \frac{1-q^n}{1-q} $$

satisfy the recurrences

$$ \binom{n}{k} = q^k \binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n-1}{k} + q^{n-k} \binom{n-1}{k-1}. \quad (2.1) $$

We also need the $q$–binomial theorem in the form

$$ p_n(x,y) = (x + y)(q x + y) \cdots (q^{n-1} x + y) = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \quad (2.2) $$
Definition 2.1

The polynomials

\[
T_n(x,s,q) = \det \begin{pmatrix}
x & qs & 0 & \cdots & 0 & 0 \\
-1 & (1 + q)x & q^2s & \cdots & 0 & 0 \\
0 & -1 & (1 + q^2)x & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & (1 + q^{n-2})x & q^{n-1}s \\
0 & 0 & 0 & \cdots & -1 & (1 + q^{n-1})x \\
\end{pmatrix}
\]

will be called \(q\)-Chebyshev polynomials of the first kind.

The first terms are \(1, x, [2] x^2 + qs, [4] x^3 + q[3] sx, \cdots\).

By expanding this determinant with respect to the last column we get

Proposition 2.1

The \(q\)-Chebyshev polynomials of the first kind satisfy the recurrence

\[
T_n(x,s,q) = (1 + q^{n-1})xT_{n-1}(x,s,q) + q^{n-1}sT_{n-2}(x,s,q)
\] (2.3)

with initial values \(T_0(x,s,q) = 1\) and \(T_1(x,s,q) = x\).

Here \(T_0(x,s,q)\) is not defined by the determinant. But if we set \(T_0(x,s,q) = 1\) then (2.3) gives

\[
T_2(x,s,q) = (1 + q)xT_1(x,s,q) + qsT_0(x,s,q) = (1 + q)x^2 + qs = \det \begin{pmatrix}
x & qs \\
-1 & (1 + q)x \\
\end{pmatrix}
\]

Definition 2.2

The polynomials

\[
U_n(x,s,q) = \det \begin{pmatrix}
(1 + q)x & qs & 0 & \cdots & 0 & 0 \\
-1 & (1 + q^2)x & q^2s & \cdots & 0 & 0 \\
0 & -1 & (1 + q^3)x & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & (1 + q^{n-1})x & q^{n-1}s \\
0 & 0 & 0 & \cdots & -1 & (1 + q^n)x \\
\end{pmatrix}
\]

will be called \(q\)-Chebyshev polynomials of the second kind.

Proposition 2.2

The q–Chebyshev polynomials of the second kind satisfy the recurrence

\[ U_n(x,s,q) = (1 + q^n)xU_{n-1}(x,s,q) + q^{n-1}sU_{n-2}(x,s,q) \]  \hspace{1cm} (2.4)

with initial values \( U_0(x,s,q) = 1 \) and \( U_{-1}(x,s,q) = 0 \).

It remains to verify that \( U_1(x,s,q) = (1 + q)x \) and \( U_2(x,s,q) = (1 + q)(1 + q^2)x^2 + qs \), which follows immediately from (2.4).

In [2] and [5] a tiling interpretation of the classical Chebyshev polynomials has been given. This can easily be extended to the \( q \)–case. As in the classical case it is easier to begin with polynomials of the second kind.

We consider an \( n \times 1 \)–rectangle (called \( n \)–board) where the \( n \) cells of the board are numbered 1 to \( n \). As in [2] and [5] we consider tilings with two sorts of squares, say white and black squares, and dominoes (which cover two adjacent cells of the board).

Definition 2.3

To each tiling of a board we assign a weight \( w \) in the following way: Each white square has weight \( x \). A black square at position \( i \) has weight \( q^ix \) and a domino which covers positions \( i−1, i \) has weight \( q^{i−1}s \). The weight of a tiling is the product of its elements.

The weight of a set of tilings is the sum of their weights.

Each tiling can be represented by a word in the letters \( \{a, b, dd\} \). Here \( a \) denotes a white square, \( b \) a black square and \( dd \) a domino.

For example the word \( abbdaddaabd \) represents the tiling with white squares at positions 1,6,9,10, black squares at 2,3,11 and dominoes at \( \{4,5\} \) and \( \{7,8\} \). Its weight is \( x \cdot q^2x \cdot q^3x \cdot q^4s \cdot x \cdot q^7s \cdot x \cdot x \cdot q^{11}x = q^27s^2x^7 \).

Theorem 2.1

The weight \( w(V_n) \) of the set \( V_n \) of all tilings of an \( n \)–board is \( w(V_n) = U_n(x,s,q) \).

Proof

This holds for \( n = 1 \) and \( n = 2 \). Each \( n \)–tiling \( u_n \) has one of the following forms:

\( u_{n−1}a, u_{n−2}b, u_{n−2}dd \).

Therefore

\[ w(V_n) = \sum_{u_n} w(u_n) = \sum_{u_{n−1}} w(u_{n−1})x + \sum_{u_{n−1}} w(u_{n−1})q^n x + \sum_{u_{n−2}} w(u_{n−2})q^{n−1}s \]

\[ = w(V_{n−1})(1 + q^n)x + w(V_{n−2})q^{n−1}s \]

which implies Theorem 2.1.
The same reasoning as above gives

**Proposition 2.3**

Let \( u(n,k,s) \) be the weight of all tilings on \( \{1, \ldots, n\} \) with exactly \( k \) dominoes. Then

\[
    u(n,k,s) = u(n-1,k,s)(1 + q^n)x + u(n-2,k-1,s)q^{n-1}s
\]

with initial values

\[
    u(n,0,s) = (1 + q)(1 + q^2) \cdots (1 + q^n)x^n
\]

and

\[
    u(1,0,s) = (1 + q)x \quad \text{and} \quad u(1,k,s) = 0 \quad \text{for} \quad k > 0.
\]

It is now easy to verify

**Theorem 2.2**

The weight \( u(n,k,s) \) of the set of all tilings on \( \{1, \ldots, n\} \) with exactly \( k \) dominoes is

\[
    u(n,k,s) = q^k \left[ \frac{n-k}{k} \right] (1 + q^{k+1}) \cdots (1 + q^{n-k})s^kx^{n-2k}
\]

for \( 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \) and \( u(n,k,s) = 0 \) for \( k > \left\lfloor \frac{n}{2} \right\rfloor \).

**Proof**

The initial values coincide and by induction

\[
    u(n-1,k,s)(1 + q^n)x + u(n-2,k-1,s)q^{n-1}s = q^k \left[ \frac{n-k-1}{k} \right] (1 + q^{k+1}) \cdots (1 + q^{n-k-1})s^kx^{n-2k}
\]

\[
    + q^{(k-1)^2} \left[ \frac{n-k-1}{k-1} \right] (1 + q^k) \cdots (1 + q^{n-k-1})q^{n-1}s^kx^{n-2k}
\]

\[
    = q^k (1 + q^{k+1}) \cdots (1 + q^{n-k-1})s^kx^{n-2k} \left[ \frac{n-k-1}{k} \right] (1 + q^n) + q^{n-k}(n-k-1)q^{n-2k}(1 + q^k)
\]

\[
    = q^k (1 + q^{k+1}) \cdots (1 + q^{n-k-1})s^{k}x^{n-2k} \left[ \frac{n-k-1}{k} \right] (1 + q^{n-2k}) + q^{n-k}(q^k \left[ \frac{n-k-1}{k} \right] + \left[ \frac{n-k-1}{k-1} \right])
\]

\[
    = q^k (1 + q^{k+1}) \cdots (1 + q^{n-k-1})s^kx^{n-2k} \left[ \frac{n-k}{k} \right] (1 + q^{n-k}).
\]

Here we used the recurrence relations for the \( q \)–binomial coefficients (2.1).
**Remark 2.1**

Formula (2.6) is the product of 
\[ q^k \binom{n-k}{k} s^k x^{n-2k} \text{ and } (1 + q^{k+1}) \cdots (1 + q^{n-k}) \text{.} \]

The first term is the weight of all tilings without black squares. It would be nice to find a combinatorial model from which this product representation becomes obvious.

We have thus proved

**Theorem 2.3**

\[ U_n(x, s, q) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^k \binom{n-k}{k} (1 + q^{k+1}) \cdots (1 + q^{n-k}) s^k x^{n-2k} \text{.} \quad (2.7) \]

For the \( q \)–Chebyshev polynomials of the first kind the situation is somewhat more complicated. Here we get

**Theorem 2.4**

\( T_n(x, s, q) \) is the weight of the subset of all tilings of \{1, \cdots, n\} whose last block is either a white square or a domino.

Therefore for \( n > 0 \)

\[ T_n(x, s, q) = x U_{n-1}(x, s, q) + q^{n-1} s U_{n-2}(x, s, q) \text{.} \quad (2.8) \]

**Theorem 2.5**

The \( q \)–Chebyshev polynomials of the first kind are given by

\[ T_n(x, s, q) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^k (1 + q^{k+1}) \cdots (1 + q^{n-k-1}) \frac{n}{n-k} \binom{n-k}{k} s^k x^{n-2k} \text{.} \quad (2.9) \]

**Proof of Theorems 2.4 and 2.5**

Consider the subset of all tilings of an \( n \)–board whose last block is not a black square. Let \( t(n, k, s) \) be the weight of all these tilings with exactly \( k \) dominoes.

Then

\[ t(n, k, s) = u(n-1, k, s)x + u(n-2, k-1)q^{n-1}s \text{.} \quad (2.10) \]

We first show that
\[ t(n,k,s) = q^{k^2} \left( 1 + q^{k+1} \right) \cdots \left( 1 + q^{n-k-1} \right) \left[ \frac{n}{n-k} \right] \left[ \frac{n-k-1}{k} \right] s^k x^{n-2k}. \]  \hspace{1em} (2.11)

This is true for \( n = 1 \) and \( n = 2 \). By induction we get

\[
t(n,k,s) = u(n-1,k,s) + u(n-2,k-1) q^{n-1} s
\]

\[= q^{k^2} \left[ \frac{n-k-1}{k} \right] \left( 1 + q^{k+1} \right) \cdots \left( 1 + q^{n-k-1} \right) s^k x^{n-2k} + q^{n-1} q^{k^2-2k+1} \left[ \frac{n-1-k}{k-1} \right] \left( 1 + q^k \right) \cdots \left( 1 + q^{n-k} \right) s^k x^{n-2k} \]

\[= q^{k^2} \left( 1 + q^{k+1} \right) \cdots \left( 1 + q^{n-k-1} \right) \left[ \frac{n-k-1}{k} \right] + q^k \left[ n-1-k \right] \left[ k-1 \right] s^k x^{n-2k} \]

\[= q^{k^2} \left( 1 + q^{k+1} \right) \cdots \left( 1 + q^{n-k-1} \right) + q^k \left[ n-1-k \right] \left[ n-1-k \right] s^k x^{n-2k} \]

\[= q^{k^2} \left( 1 + q^{k+1} \right) \cdots \left( 1 + q^{n-k-1} \right) \left[ \frac{n}{n-k} \right] \left[ \frac{n-k-1}{k} \right] s^k x^{n-2k}.
\]

It remains to verify that the recurrence (2.3) holds.

This recurrence is equivalent with

\[ t(n,k,s) = (1 + q^{n-1}) xt(n-1,k,s) + q^{n-1} st(n-2,k-1,s) \]  \hspace{1em} (2.12)

for all \( k \).

For \( k = 0 \) this gives \( t(n,0,s) = x^1 \prod_{j=1}^{n-1} (1 + q^j) \), which coincides with (2.11).

\[ t(n,k,s) = 0 \text{ for } n < 2k \text{ and } t(2n,k,s) = q s \cdot q^3 s \cdots q^{2k-1} s = q^{k^2} s^k. \]

If (2.11) holds for \( k-1 \) and for \( k \) and \( n-1 \) then

\[ t(n,k,s) = (1 + q^{n-1}) q^{k^2} \left( 1 + q^{k+1} \right) \cdots \left( 1 + q^{n-k-2} \right) \left[ \frac{n-1}{n-k-1} \right] \left[ \frac{n-k-1}{k} \right] s^k x^{n-2k} + q^{n-1} q^{(k-1)^2} \left[ \frac{n-2}{n-k-1} \right] \left[ \frac{n-k-1}{k-1} \right] s^k x^{n-2k} \]

\[= q^{k^2} \left( 1 + q^{k+1} \right) \cdots \left( 1 + q^{n-k-2} \right) s^k x^{n-2k} \left[ \frac{n-1}{n-k-1} \right] \left[ \frac{n-k-1}{k} \right] \left( 1 + q^{n-1} \right) + q^{n-2k} \left( 1 + q^k \right) \left[ \frac{n-2}{n-k-1} \right] \left[ \frac{n-k-1}{k-1} \right] \left[ n-k-1 \right] \left[ k-1 \right]. \]
For $q = 1$ the polynomial $T_n(x,s)$ can also be interpreted as the weight of the set $T_n$ of all tilings which begin with a domino or with a white square since in this case the weights of the words $c_1 \cdots c_n$ and $c_n \cdots c_1$ coincide.

In the general case this is not true. For example for $n = 2$ the set $T_2 = \{aa, ab, dd\}$ has weight $w(T_2) = x^2 + q^2 x^2 + qs \neq T_2(x,s,q) = x^2 + qx^2 + qs$.

We have instead

**Theorem 2.6**

$$T_n(x,s,q) = xU_{n-1}(x,q^2s,q) + qsU_{n-2}(x,q^2s,q). \quad (2.13)$$

**Proof**

It suffices to show that the right-hand side satisfies recurrence (2.3).

\begin{align*}
 \left(1 + q^{n-1}\right)x \left(xU_{n-2}(x,q^2s,q) + qsU_{n-3}(x,q^2s,q)\right) &+ q^{n-2}s \left(xU_{n-2}(x,q^2s,q) + qsU_{n-3}(x,q^2s,q)\right) \\
&= x^2U_{n-2}(x,q^2s,q) + q^{n-2}s^2U_{n-4}(x,q^2s,q) \\
&+ q^{n-1}xU_{n-2}(x,q^2s,q) + q^{n-2}s^2U_{n-3}(x,q^2s,q) \\
&= x \left( \left(1 + q^{n-1}\right)xU_{n-2}(x,q^2s,q) + q^{n-2}s^2U_{n-3}(x,q^2s,q) \right) \\
&+ qs \left( \left(1 + q^{n-2}\right)xU_{n-3}(x,q^2s,q) + q^{n-3}s^2U_{n-4}(x,q^2s,q) \right) \\
&= xU_{n-1}(x,q^2s,q) + qsU_{n-2}(x,q^2s,q).
\end{align*}

In order to find a $q$ – analogue of (1.15) let us first consider this identity in more detail.

\begin{align*}
 \left( x + \sqrt{x^2 + s} \right)^n & = T_n(x,s) + U_{n-1}(x,s) \sqrt{x^2 + s} \\
\text{is equivalent with} & \\
T_{n+1}(x,s) + U_n(x,s) \sqrt{x^2 + s} & = \left( x + \sqrt{x^2 + s} \right)^{n+1} = \left( x + \sqrt{x^2 + s} \right) \left( x + \sqrt{x^2 + s} \right)^n \\
& = \left( x + \sqrt{x^2 + s} \right) \left( T_n(x,s) + U_{n-1}(x,s) \sqrt{x^2 + s} \right) \\
& = T_n(x,s) + \left( x^2 + s \right) U_{n-1}(x,s) + \left( T_n(x,s) + U_{n-1}(x,s) x \right) \sqrt{x^2 + s}.
\end{align*}
Therefore (1.15) is equivalent with both identities

\[ T_{n+1}(x,s) = T_n(x,s)x + \left(x^2 + s\right)U_{n-1}(x,s) \]  \hspace{1cm} (2.14)

and

\[ U_n(x,s) = T_n(x,s) + U_{n-1}(x,s)x. \]  \hspace{1cm} (2.15)

To prove identity (2.14) observe that for \( q = 1 \) a tiling of an \((n+1)\)-board which does not end with a black square either ends with two white squares \(aa\) or with a domino and a white square \(dda\). The weight \( w \) of these tilings is \( T_n(x,s)x \). Or it ends with \( ba \) or \( dd \). Their weight is \((x^2 + s)U_{n-1}(x,s)\).

Identity (2.15) simply means that an arbitrary tiling either ends with a black square which gives the weight \( U_{n-1}(x,s)x \) or does not end with a black square which gives \( T_n(x,s) \).

In the general case \( q \neq 1 \) this classification of the tilings implies the identities

\[ T_{n+1}(x,s,q) = xT_n(x,s,q) + q^n(x^2 + s)U_{n-1}(x,s,q) \]  \hspace{1cm} (2.16)

and

\[ U_n(x,s,q) = T_n(x,s,q) + q^n x U_{n-1}(x,s,q). \]  \hspace{1cm} (2.17)

But we need another \( q \)-analogue of (2.14):

\[ T_{n+1}(x,s,q) = q^n x T_n(x,s,q) + (x^2 + qs)U_{n-1}(x,q^2s,q). \]  \hspace{1cm} (2.18)

By (2.13) we have \( T_{n+1}(x,s,q) = xU_n(x,q^2s,q) + qsU_{n-1}(x,q^2s,q) \).

Eliminating \( T_{n+1}(x,s,q) \) we need only show that

\[ U_n(x,q^2s,q) = q^n T_n(x,s,q) + x U_{n-1}(x,q^2s,q). \]  \hspace{1cm} (2.19)

This means that for each \( k \)

\[ xq^{2k}u(n-1,k,s) + q^n t(n,k,s) = q^{2k}u(n,k,s). \]  \hspace{1cm} (2.20)

This can easily be verified:
As our \( q \)-analogue of (2.14) and (2.15) we now choose the identities (2.17) and (2.18) which we write in the form

\[
T_n(x,s,q) = q^n x T_n(x,s,q) + (x^2 + qs)\eta^2 U_{n-1}(x,s,q)
\]

\[
U_n(x,s,q) = T_n(x,s,q) + q^n x U_{n-1}(x,s,q).
\]

(2.21)

Here \( \eta \) denotes the linear operator on the polynomials in \( s \) defined by \( \eta p(s) = p(qs) \).

To stress the analogy with (1.15) we introduce a formal square root \( A = \sqrt{(x^2 + s)\eta^2} \) which commutes with \( x \) and real or complex numbers and satisfies \( A^2 = (x^2 + qs)\eta^2 \) and write (2.21) in the form

\[
T_{n+1}(x,s,q) + AU_n(x,s,q) = (q^n x + A)(T_n(x,s,q) + AU_{n-1}(x,s,q)).
\]

(2.22)

Since \( (q^j x + A)(q^j x + A) = (q^j x + A)(q^j x + A) \) using the \( q \)-binomial theorem (2.2) we get as analogue of (1.15)

\[
p_n(x,A) = (x + A)(q x + A) \cdots (q^{n-1} x + A) = T_n(x,s,q) + AU_{n-1}(x,s,q).
\]

(2.23)

This gives

**Theorem 2.7**

For the \( q \)-Chebyshev polynomials the following formulae hold:

\[
T_n(x,s,q) = \frac{p_n(x,A) + p_n(x,-A)}{2} = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} q^{\frac{n-2k}{2}} \left[ \begin{array}{c} n \\ 2k \\ \end{array} \right] x^{n-2k} \prod_{j=0}^{k-1} \left( x^2 + q^{2j+1} s \right)
\]

(2.24)

and

\[
U_n(x,s,q) = \frac{p_{n+1}(x,A) - p_{n+1}(x,-A)}{2A} = \sum_{k=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} q^{\frac{n-2k}{2}} \left[ \begin{array}{c} n+1 \\ 2k+1 \end{array} \right] x^{n-2k} \prod_{j=0}^{k} \left( x^2 + q^{2j+1} s \right).
\]

(2.25)
Proof
This follows from (2.2) and the observation that
\[ A^{2k} = \left( (x^2 + qs) \eta \right)^k = \prod_{j=0}^{k-1} (x^2 + q^{2j+1} s) \eta^{2k}. \]

Remark 2.2
For \( q = 1 \) we get from (1.15)
\[ T_n(x, s)^2 - (x^2 + s) U_{n-1}(x, s)^2 = (-s)^n. \]
Since \( A \) does not commute with polynomials in \( s \) we cannot deduce a \( q \) – analogue of this formula from (2.23).

But we can instead consider the matrices
\[ A_n = \begin{pmatrix} x & q^n(x^2 + s) \\ 1 & q^n x \end{pmatrix}. \]  

We then get

Theorem 2.8
\[ \begin{pmatrix} T_n(x, s, q) & (x^2 + s) U_{n-1}(x, qs, q) \\ U_{n-1}(x, s, q) & T_n(x, \frac{s}{q}, q) \end{pmatrix} = A_{n-1} A_{n-2} \cdots A_0. \]  

Proof
We must show that
\[ \begin{pmatrix} T_{n+1}(x, s, q) & (x^2 + s) U_n(x, qs, q) \\ U_n(x, s, q) & T_{n+1}(x, \frac{s}{q}, q) \end{pmatrix} = \begin{pmatrix} x & q^n(x^2 + s) \\ 1 & q^n x \end{pmatrix} \begin{pmatrix} T_n(x, s, q) & (x^2 + s) U_{n-1}(x, qs, q) \\ U_{n-1}(x, s, q) & T_n(x, \frac{s}{q}, q) \end{pmatrix} \]

or equivalently
\[ T_{n+1}(x, s, q) = x T_n(x, s, q) + q^n(x^2 + s) U_{n-1}(x, s, q), \]
\[ U_n(x, s, q) = T_n(x, s, q) + q^n x U_{n-1}(x, s, q), \]
\[ U_n(x, q^2 s, q) = q^n T_n(x, s, q) + x U_{n-1}(x, q^2 s, q), \]
\[ T_{n+1}(x, s, q) = q^n x T_n(x, s, q) + (x^2 + qs) U_{n-1}(x, q^2 s, q). \]

This follows from the recurrences (2.16), (2.17), (2.18) and (2.19).
If we take determinants in (2.27) we get the desired $q -$ analogue of
\[ T_n(x,s)^2 - (x^2 + s)U_{n-1}(x,s)^2 = (-s)^n. \]

**Theorem 2.9**

\[ T_n(x,s,q)T_n(x,qs,q) - (x^2 + qs)U_{n-1}(x,qs,q)U_{n-1}(x,q^2s,q) = q^{\binom{n+1}{2}}(-s)^n. \]  

(2.28)

In [4] many other identities occur. These follow in an easy manner from the identities obtained above.

Since the $q -$ Chebyshev polynomials satisfy a three-term recurrence they are orthogonal with respect to some linear functionals, i.e. $L(T_n(x,s,q)T_m(x,s,q)) = 0$ and
\[ M(U_n(x,s,q)U_m(x,s,q)) = 0 \text{ for } n \neq m. \]
These linear functionals are uniquely determined by
\[ L(T_n(x,s,q)) = [n = 0] \text{ and } M(U_n(x,s,q)) = [n = 0]. \]
Of special interest are the moments of these linear functionals, i.e. the values $L(x^n)$ and $M(x^n)$. To find these values it suffices to find the uniquely determined representation of $x^n$ as a linear combination of the $q -$ Chebyshev polynomials.

These have been calculated in [4] for the corresponding monic polynomials. Therefore I only state the results in the present notation:

For the $q -$ Chebyshev polynomials of the first kind we have
\[
 x^n = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{k} a(n-2k)(-qs)^k \frac{T_{n-2k}(x,s,q)}{(1+q) \cdots (1+q^k) \cdots (1+q^{n-k})} 
\]
(2.29)

with $a(0) = 1$ and $a(n) = 1 + q^n$ for $n > 0$.

This gives
\[
 L(x^{2n}) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{k} \frac{(-qs)^n}{n!} \prod_{j=1}^{n} (1+q^j)^2 
\]
(2.30)

and $L(x^{2n+1}) = 0$.

For the $q -$ Chebyshev polynomials of the second kind the corresponding formulae are
\[
 x^n = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{k} \binom{n}{k-1} (-s)^k \frac{1+q^{n-2k+1}}{\prod_{j=1}^{k} (1+q^j) \prod_{j=1}^{k-1} (1+q^j)} \prod_{j=1}^{n-2k+1} U_{n-2k}(x,s,q) 
\]
(2.31)

and therefore
\[ M(x^{2n}) = \frac{1}{[n+1]} \binom{2n}{n} \frac{1+q}{1+q^{n+1}} \prod_{j=1}^{n} \frac{(-s)^n}{(1+q^j)^2} \] (2.32)

and \( M(x^{2n+1}) = 0. \)

References


