Some remarks on the paper „On a curious q-hypergeometric identity”

by M. J. Cantero and A. Iserles

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Abstract

We give a short proof for an identity of M. J. Cantero and A. Iserles.

1. Statement of the result

Let $|q|<1$ and $(z;q)_n = (1-z)(1-qz)\cdots(1-q^{n-1}z).$

In their paper [3] M. J. Cantero and A. Iserles prove that the rational functions $a_n(z,q)$ defined by $a_0(z,q)=1$ and

$$\sum_{j=0}^{n} \frac{a_{n-j}(z,q)}{(q;q)_j (z;q)_j} = \frac{q^n}{(q;q)_n (z;q)_n} \tag{1}$$

for $n>0$ satisfy

$$\lim_{q\to 1} a_n(z,q) = (-1)^n C_{n-1} \frac{z^{n-1}}{(1-z)^{2n-1}} \tag{2}$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is a Catalan number.

2. The proof

Their proof uses orthogonal polynomials on the unit circle. In this note we give a simpler proof of their result.

The equations (1) are equivalent with the identity for formal power series in x

$$\sum_{k\ge 0} a_k(z,q)x^k \sum_{\ell\ge 0} \frac{1}{(z;q)_\ell (q;q)_\ell} x^\ell = \sum_{n\ge 0} \frac{1}{(z;q)_n (q;q)_n} (qx)^n. \tag{3}$$

Let

$$F(z,x,q)=\sum_{n\geq 0}\frac{1}{(z;q)_n(q;q)_n}x^n. \tag{4}$$

Then

$$\sum_{n\geq 0}a_n(z,q)x^n=\frac{F(z,qx,q)}{F(z,x,q)}. \tag{5}$$

If we set

$$R(z,x,q)=\frac{F(qz,x,q)}{F(z,x,q)} \tag{6}$$

then

$$\frac{F(z,qx,q)}{F(z,x,q)}=1-\frac{x}{1-z}R(z,x,q). \tag{7}$$

Moreover

$$F(z,x,q)-F(qz,x,q)=\sum_{n\geq 0}\frac{1}{(z;q)_{n+1}(q;q)_n}x^n\left(\left(1-q^nz\right)-\left(1-z\right)\right)=\frac{zx}{(1-z)(1-qz)}F\left(q^2z,x,q\right).$$

This implies

$$R(z,x,q)=1-\frac{xz}{(1-z)(1-qz)}R(z,x,q)R(qz,x,q). \tag{8}$$

Let $f(z,x)=\lim_{q\to 1}R(z,x,q)$ in the sense that all coefficients converge.

This gives

$$f(z,x)=1-\frac{xz}{(1-z)^2}f(z,x)^2. \tag{9}$$

As is well known (see e.g. [4]) this means that

$$f(z,x)=\sum_{n\geq 0}C_n\left(-\frac{xz}{(1-z)^2}\right)^n. \tag{10}$$

Therefore by (8) and (7) we get for $n>0$ the desired result (2).

3. Remarks

3.1. From (8) we get by comparing coefficients

$$R(z,x,q)=\sum_{n\geq 0} r(n,z,q)x^n \tag{11}$$

where $r(0,z,q)=1$ and

$$r(n,z,q)=\frac{-z}{(1-z)(1-qz)}\sum_{k=0}^{n-1} r(k,z,q)r(n-1-k,qz,q). \tag{12}$$

This implies that $r(n,z,q)$ is a rational function in z and satisfies

$$r\left(n,\frac{1}{z},\frac{1}{q}\right)=q^n r(n,z,q). \tag{13}$$

From (5) and (7) follows that

$$a_n(z,q)=\frac{-1}{1-z}r(n-1,z,q). \tag{14}$$

If we set

$$r(n,z,q)=\left(\frac{-z}{1-z}\right)^n \frac{c(n,z,q)}{(z;q)_n}, \tag{15}$$

we get

$$c(n,z,q)=\sum_{k=0}^{n-1} q^{n-1-k}\left(\frac{1-z}{1-qz}\right)^{n-k}\frac{(q^k z;q)_{n-k}}{(z;q)_{n-k}}c(k,z,q)c(n-1-k,qz,q). \tag{16}$$

Therefore $c(n,0,q)=C_n(q)$ are the (Carlitz-) $q-$ Catalan numbers (cf. e.g. [4]) defined by $C_0(q)=1$ and

$$C_n(q)=\sum_{k=0}^{n-1} q^k C_k(q)C_{n-k-1}(q). \tag{17}$$

The first terms are

$$\left(C_n(q)\right)_{n\geq 0}=\left(1,\ 1,\ 1+q,\ 1+2q+q^2+q^3,\ 1+3q+3q^2+3q^3+2q^4+q^5+q^6,\cdots\right).$$

The first terms of the sequence $\left(c(n,z,q)\right)_{n\geq 0}$ are

$$\left(c(n,z,q)\right)_{n\geq 0}=\left(C_0(q),\ \frac{1-z}{1-qz}C_1(q),\ \frac{1-z}{1-q^2z}C_2(q),\ \frac{1-z}{(1-qz)\left(1-q^3z\right)}\left(C_3(q)-q^4zC_3\left(\frac{1}{q}\right)\right),\right)$$

From (16) we see that

$$\lim_{q \to 1} c(n,z,q) = C_n. \tag{18}$$

Therefore numerator and denominator of $c(n,z,q)$ have the same degree.

3.2. The functions $r(n,z,q)$ can also be obtained in the following way:

Consider the polynomials $f_n(x,z)$ defined by

$$f_n(x,z) = x f_{n-1}(x,qz) + \frac{z}{(1-z)(1-qz)} f_{n-2}(x,q^2 z) \tag{19}$$

with initial values $f_0(x,z)=1$ and $f_1(x,z)=x.$

Then we get (cf. [5] or [6])

$$f_n(x,z) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \begin{bmatrix} n-k \\ k \end{bmatrix} q^{2\binom{k}{2}} \frac{z^k}{(z;q)_k \left(q^{n-k} z;q\right)_k} x^{n-2k}. \tag{20}$$

These polynomials also satisfy the recurrence

$$f_n(x,z) = x f_{n-1}(x,z) + \frac{q^{n-2} z}{(1-q^{n-2}z)(1-q^{n-1}z)} f_{n-2}(x,z) \tag{21}$$

and are therefore orthogonal with respect to the linear functional Λ_z on the polynomials in x defined by

$$\Lambda_z\left(f_n(x,z)\right) = [n=0]. \tag{22}$$

As shown in [5] their moments $\Lambda_z\left(x^n\right)$ are given by $\Lambda_z\left(x^{2n+1}\right)=0$ and

$$\Lambda_z\left(x^{2n}\right) = r(n,z,q). \tag{23}$$

It should be noted that ansatz (5) is useful to obtain other interesting sequences.

For $F_1(x,q) = \sum_{n\ge 0} \frac{q^{n^2-n}}{(q;q)_n} (-x)^n$ we get $\frac{F_1(qx,q)}{F_1(x,q)} = \sum_{n\ge 0} C_n(q) x^n,$

for $F_2(x,z,q) = \sum_k \frac{q^{k^2}}{(q;q)_k (qz;q)_k} x^k$ we get $\frac{F_2(qx,x,q)}{F_2(x,x,q)} = \sum_{n\ge 0} (-1)^n q^{\binom{n+1}{2}} x^n$

and further examples of this sort can be found in [4] and [7].

4. An interesting special case

For $z=-q$ and $|x|<1$ we get by the $q-$ binomial theorem in the form [2], Theorem 10.2.1,

$$F(-q,x,q)=\sum_{n\geq 0}\frac{x^n}{\left(q^2;q^2\right)_n}=\frac{1}{\left(x;q^2\right)_\infty}.$$

Therefore again by the $q-$ binomial theorem

$$\sum_{n\geq 0}a_n(-q,q)x^n=\frac{F(-q,qx,q)}{F(-q,x,q)}=\frac{\left(x;q^2\right)_\infty}{\left(qx;q^2\right)_\infty}=\sum_{n\geq 0}\frac{\left(q^{-1};q^2\right)_n}{\left(q^2;q^2\right)_n}(qx)^n$$

which gives

$$a_{n+1}(-q,q)=q^{n+1}\frac{\left(q^{-1};q^2\right)_{n+1}}{\left(q^2;q^2\right)_{n+1}}=-\frac{1}{[n+1]}\begin{bmatrix}2n\\n\end{bmatrix}\frac{q^n}{(-q;q)_n(-q;q)_{n+1}} \tag{24}$$

and therefore

$$r(n,-q,q)=\frac{q^n}{[n+1]}\begin{bmatrix}2n\\n\end{bmatrix}\frac{1}{(-q;q)_n\left(-q^2;q\right)_n}. \tag{25}$$

These numbers are related to the $q-$ Catalan numbers introduced by George Andrews [1].

Of course this result also follows from the above considerations.

For $z=-q$ we have shown in [6], Theorem 4.2, (with a slightly different notation)

$$x^n=\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\left(\begin{bmatrix}n\\k\end{bmatrix}-\begin{bmatrix}n\\k-1\end{bmatrix}\right)\frac{1}{(-q;q)_k\left(-q^{n+2-2k};q\right)_k}f_{n-2k}(x,-q,q). \tag{26}$$

This implies

$$\Lambda_{-q}\left(x^{2n}\right)=\left(\begin{bmatrix}2n\\n\end{bmatrix}-\begin{bmatrix}2n\\n-1\end{bmatrix}\right)\frac{1}{(-q;q)_n\left(-q^2;q\right)_n}=\frac{q^n}{[n+1]}\begin{bmatrix}2n\\n\end{bmatrix}\frac{1}{(-q;q)_n\left(-q^2;q\right)_n}$$

and thus again (25).

References

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