Remarks on some sequences of binomial sums

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Abstract
We give simple proofs for the recurrence relations of some sequences of binomial sums which have previously been obtained by other more complicated methods.

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1. Introduction
Modifying an idea of E. Brietzke [2] we give simple proofs for the recurrence relations of sequences of binomial sums of the form
\[ a(n,m,k,z) = \sum_{j=\mathbb{Z}} z^j \left( \frac{n}{2} - mj + k \right), \]
which have been obtained by other methods in [3].

In order to motivate the method we consider first the well-known special case
\[ a(n,5,k,-1) = \sum_{j=\mathbb{Z}} (-1)^j \left( \frac{n}{2} - 5j + k \right) = (-1)^k \sum_{j=0}^n t(n,k-5j), \]
with \( t(n,k) = (-1)^k \left( \frac{n+k}{2} \right) \).

We use the fact that \( t(n,k) = -t(n-1,k-1) - t(n-1,k+1) \) with \( t(0,0) = 1, t(0,1) = -1 \) and \( t(0,k) = 0 \) for all other \( k \in \mathbb{Z} \).

Define the operator \( K \) by \( Kf(n,k) = f(n,k-1) \) and the operator \( N \) by \( Nf(n,k) = f(n+1,k) \). Then
\[ t(n) = Nt(n-1) = -(K+K^{-1})t(n-1) = (-1)^n(K+K^{-1})^nt(0). \]

Let \( s(n,k) \) be the function which satisfies the same recurrence with initial values \( s(0,k) = [k = 0] \). Then we have \( t(0) = (1-K)s(0) \). Since \( K \) is a linear operator we also have \( t(n) = (1-K)s(n) \).

Let \( \mathcal{F} \) be the vector space of all functions on \( \mathbb{N} \times \mathbb{Z} \) which are finite linear combinations of functions \( K^js, j \in \mathbb{Z} \). For \( f \in \mathcal{F} \) we have \( Nf = -(K+K^{-1})f \).

Let \( T \) be the linear operator on \( \mathcal{F} \) defined by
\[ Tf = N^2f - Nf - f = (K+K^{-1})^2f + (K+K^{-1})f - f = (K^{-2} + K^{-1} + 1 + K + K^2)f. \]
Then $\sum_{j \in \mathbb{Z}} K^{5j}TK^i s(0) = \sum_{j \in \mathbb{Z}} K^j s(0) = 1$ for all $i \in \mathbb{Z}$ since $KT = KT$.

Furthermore

$$\sum_{j \in \mathbb{Z}} K^{5j} Tl(n) = \sum_{j \in \mathbb{Z}} K^{5j} (K + K^{-1})^n (1 - K) s(0) = (-1)^n (K + K^{-1})^n (1 - K) \sum_{j \in \mathbb{Z}} K^{5j} T s(0) = 0.$$ 

Since $a(n,5,k,-1) = (-1)^i \sum_j t(n,k - j)$ is a finite sum for each $k$, the sequence $(a(n,5,k,-1))$ satisfies the recurrence $a(n + 2,5,k,-1) - a(n + 1,5,k,-1) - a(n,5,k,-1) = 0$ for $n \geq 0$.

Since the Fibonacci numbers $F_n$ satisfy the same recurrence with initial values $F_0 = 0$ and $F_1 = 1$, we get the following results (cf. G.E. Andrews [1]):

**Proposition 1**

For $k \equiv 0,1 (\text{mod} 10)$ the initial conditions are $a(0,5,k) = a(1,5,k) = 1$ and therefore $a(n,5,k) = F_{n+1}$.

For $k \equiv 2,9 (\text{mod} 10)$ we have $a(0,5,k) = a(1,5,k) = 1$ and therefore $a(n,5,k) = F_n$.

For $k \equiv 3,8 (\text{mod} 10)$ we get $a(0,5,k) = a(1,5,k) = 0$ and therefore $a(n,5,k) = 0$. Furthermore $a(n,5,k + 5) = -a(n,5,k)$.

It is interesting to observe that this result has first been proved by I. Schur [6] in a strengthened version: Let $\binom{n}{k} = \frac{(1-q^{n-k+1})\cdots (1-q^n)}{(1-q)\cdots (1-q^k)}$ be a $q$–binomial coefficient. Then the following polynomial version of the celebrated Rogers-Ramanujan identity

$$\sum_{k=0}^{n} q^k \binom{n-k}{k} = \sum_{k \in \mathbb{Z}} (-1)^k q^{\frac{k(k+1)}{2}} \binom{n}{n+5k}$$

holds, which for $q = 1$ reduces to

$$\sum_{k=0}^{n} \binom{n-k}{k} = F_{n+1} = \sum_{j \in \mathbb{Z}} (-1)^j \binom{n}{n + 5j}.$$ 

An elementary proof of this $q$–identity may be found in [5].

**2. A useful method**

After this example let us consider a more general case.

For $a,b \in \mathbb{R}$ let $s_{a,b}$ be the function on $\mathbb{N} \times \mathbb{Z}$ defined by $s_{a,b}(0,k) = [k = 0]$ and the recurrence relation

$$s_{a,b}(n,k) = as_{a,b}(n-1,k-1) + bs_{a,b}(n-1,k) + as_{a,b}(n-1,k+1).$$  \hspace{1cm} (1)
This can be written in the form
\[ s_{a,b}(n) = (aK^{-1} + b + aK)s_{a,b}(n-1) = (aK^{-1} + b + aK)^n s_{a,b}(0). \]

Let \( \mathcal{F} \) be the vector space of all functions on \( \mathbb{N} \) which are finite linear combinations of functions \( K^j s_{a,b}, j \in \mathbb{Z} \).

For any polynomial \( p(x) = \sum_{i=0}^{m} a_i x^i \) we denote by \( p(N) \) the linear operator on \( \mathcal{F} \) defined by
\[ p(N) f(n) = \sum_{i=0}^{m} a_i f(n+i). \]
Then we have \( p(N) = p(aK^{-1} + b + aK) \).

We are looking for an operator \( p(N) \) with analogous properties as \( T \) had in the above example.

To this end we define a sequence of polynomials \( p_n(x,a,b) = \sum_{k=0}^{m} p_{n,k}(a,b)x^k \) by the recurrence
\[ p_n(x,a,b) = (x-b)p_{n-1}(x,a,b) - a^2 p_{n-2}(x,a,b) \quad (2) \]
with initial values \( p_0(x,a,b) = 1 \) and \( p_1(x,a,b) = x + a - b \).

**Lemma 1**
For all \( k \in \mathbb{Z} \) the following identity holds
\[ p_m(N,a,b)s_{a,b}(0,k) = \sum_{i=0}^{m} p_{m,i}(a,b)s_{a,b}(i,k) = a^m \left[ k \leq m \right]. \quad (3) \]

**Proof**
It suffices to show that on \( \mathcal{F} \)
\[ p_m(N,a,b) = a^m \sum_{j=-m}^{m} K^j. \quad (4) \]

It is immediately verified that (4) is true for \( m = 0 \) and \( m = 1 \), since \( (N + a - b) = (aK + a + aK^{-1}) \).

If has already been shown for \( m-1 \) and \( m-2 \) we get
\[ p_m(N,a,b) = (N-b)p_{m-1}(N,a,b) - a^2 p_{m-2}(N,a,b) \]
\[ = a(K + K^{-1})a^{m-1} \sum_{j=-m+1}^{m-1} K^j - a^2 a^{m-2} \sum_{j=-m+2}^{m-2} K^j = a^m \sum_{j=-m}^{m} K^j. \]

From (3) we get
\[ \sum_{i=0}^{m} p_{m,i}(a,b) \sum_{j \in \mathbb{Z}} s_{a,b}(i,k - (2m+1)j) = a^m \quad (5) \]
for each \( k \in \mathbb{Z} \).
Application
As an application we consider for each \( m \in \mathbb{N} \) the sequence

\[
a(n, 2m + 1, k, -1) = \sum_{j \in \mathbb{Z}} (-1)^j \left\lfloor \frac{n - (2m + 1)j + k}{2} \right\rfloor = (-1)^j \sum_{j} t(n, k - (2m + 1)j).
\]

As shown above we have \( t = (1 - K)_s \). Therefore by (5) we get

\[
\sum_{i=0}^{m} p_{m,i}(-1,0)a(0, 2m + 1, k, -1) = 0.
\]

Formula (1) implies that \( t(n) \) is a finite linear combination of functions \( K^j t(0) \). Therefore we also get

\[
p_m(N,-1,0)a(n, 2m + 1, k, -1) = \sum_{i=0}^{m} p_{m,i}(-1,0)a(n, 2m + 1, k, -1) = 0.
\]

Now we look for an explicit expression for \( p_n(x, -1, 0) \).
We know that it satisfies the recurrence \( p_n(x, -1, 0) = x p_{n-1}(x, -1, 0) - p_{n-2}(x, -1, 0) \) with initial values \( p_0(x, -1, 0) = 1 \) and \( p_1(x, -1, 0) = x - 1 \).

Recall that the Fibonacci polynomials

\[
F_n(x,s) = \sum_{k=0}^{n-1} \binom{n-1-k}{k} s^k x^{n-2k-1} = \frac{1}{\sqrt{x^2 + 4s}} \left( \frac{x + \sqrt{x^2 + 4s}}{2} \right)^n - \left( \frac{x - \sqrt{x^2 + 4s}}{2} \right)^n
\]

are characterized by the recurrence

\[
F_n(x,s) = xF_{n-1}(x,s) + sF_{n-2}(x,s)
\]

with initial conditions \( F_0(x,s) = 0 \) and \( F_1(x,s) = 1 \). Therefore

\[
p_n(x, -1, 0) = F_{n+1}(x, -1) - F_n(x, -1).
\]

The first values of the polynomials \( p_n(x, -1, 0) \) are

\[1, x-1, x^2-x-1, x^3-x^2-2x+1, x^4-x^3-3x^2+2x+1, \ldots \]

This gives

**Theorem 1**

The sequence \( a(n, 2m + 1, k, -1) \) satisfies the recurrence relation of order \( m \)

\[
(F_{m+1}(N, -1) - F_m(N, -1)) a(n, 2m + 1, k, -1) = 0
\]

for each \( k \in \mathbb{Z} \).
Remark
This theorem has been proved in [3] with a more complicated method. The recurrence (8) is not for all $k$ the minimal recurrence, because e.g. $a(n,2m+1,m+1,-1) \equiv 0$. But it is so for $a(n,2m+1,0,-1)$, which has a simple combinatorial interpretation. It is the number of the set of all lattice paths in $\mathbb{R}^2$ which start at the origin, consist of $\left\lceil \frac{n}{2} \right\rceil$ northeast steps $(1,1)$ and $\left\lfloor \frac{n+1}{2} \right\rfloor$ southeast steps $(1,-1)$ and which are contained in the strip $-m-1 < y < m$. (cf. e.g. [4], [5]).

It is easy to see that the initial values of $a(n,2m+1,0,-1)$ are $a(j,2m+1,0,-1) = \binom{j}{\frac{j}{2}}$ for $0 \leq j < 2m$.

As a special case of Theorem 1 we mention that $a(n,3,0,-1) = 1$. This means

$$\sum_{j=0}^{n} (-1)^j \binom{n}{n - 3j} = 1 \text{ for all } n \in \mathbb{N}.$$ 

The generating function of the sequence $(a(n,2m+1,0,-1))_{n \geq 0}$ has the form

$$\sum_{n \geq 0} a(n,2m+1,0,-1)x^n = \frac{c_m(x)}{d_m(x)},$$

where

$$d_m(x) = p_m \left( \frac{1}{x}, -1, 0 \right) x^m = x^m \left( F_{m+1} \left( \frac{1}{x}, -1 \right) - F_{m+1} \left( \frac{1}{x}, 1 \right) \right) = F_{m+1} \left( 1, -x^2 \right) - x F_m \left( 1, -x^2 \right)$$

and $c_m(x)$ is a polynomial of degree less than $m$.

The first values of $(c_m(x))_{m \geq 1}$ are

$c_1(x) = 1, c_2(x) = 1, c_3(x) = 1 - x^2, c_4(x) = 1 - 2x^2, c_5(x) = 1 - 3x^2 + x^4, \cdots$.

This suggests that for $m \geq 2$

$$c_m(x) = \sum_{j=0}^{m-1} (-1)^j \binom{m-1-j}{j} x^{2j} = F_m(1, -x^2).$$

This can be proved in the following way: Both $d_m(x)$ and $d_m(1, -x^2)$ satisfy the same recurrence $h_m(x) = h_{m-1}(x) - x^2 h_{m-2}(x)$. This implies that for $a_{2m+1}(x) = \sum_{n \geq 0} a(n,2m+1,0,-1)x^n$ we have

$$d_m(x)a_{2m+1}(x) - d_{m-1}(x)a_{2m-1}(x) + x^2 d_{m-2}(x)a_{2m-3}(x) = (d_m(x) - d_{m-1}(x) - x^2 d_{m-2}(x))a_{2m+1}(x)$$

$$+ d_{m-1}(x)(a_{2m+1}(x) - a_{2m-1}(x)) + x^2 d_{m-2}(x)(a_{2m+1}(x) - a_{2m-3}(x)).$$

Since the coefficients of $x^j$ for $0 \leq j \leq 2m - 5$ of $a_{2m-3}(x)$ are the same as those of $a_{2m-1}(x)$ and $a_{2m+1}(x)$ we see that for $2m - 4 \geq m - 1$ the polynomial

$$d_m(x)a_{2m+1}(x) - d_{m-1}(x)a_{2m-1}(x) + x^2 d_{m-2}(x)a_{2m-3}(x)$$

which has degree $< m$ must identically vanish. This implies that

$$c_m(x) = d_m(x)a_{2m+1}(x) = F_m(1, -x^2).$$
Corollary 1

For $m \geq 2$ the generating function for $a(n, 2m+1, 0, -1)$ is given by

$$\sum_{n \geq 0} a(n, 2m+1, 0, -1)x^n = \frac{F_m(1, -x^2)}{F_{m+1}(1, -x^2) - xF_m(1, -x^2)}. \quad (9)$$

3. A modification of the above method

In order to obtain an analogous result for the sequences $a(n, 2m, k, -1)$ we define a sequence of polynomials $q_n(x, a, b) = \sum_{k=0}^{n} q_{n,k}(a, b)x^k$ by the same recurrence

$$q_n(x, a, b) = (x-b)q_{n-1}(x, a, b) - a^2 q_{n-2}(x, a, b), \quad (10)$$

but with initial values $q_0(x, a, b) = 2$ and $q_1(x, a, b) = x - b$.

Lemma 2

For all $k \in \mathbb{Z}$ the following identity holds

$$q_m(N, a, b)s_{a,b}(0, k) = \sum_{i=0}^{m} q_{m,i}(a, b)s_{a,b}(i, k) = a^m \binom{k}{m}. \quad (11)$$

Proof

It suffices to show that on $\mathcal{F}$

$$q_m(N, a, b) = a^m(K^m + K^{-m}). \quad (12)$$

(12) is true for $m = 0$ and $m = 1$ by inspection. If it is already shown for $m - 1$ and $m - 2$ we get

$$q_m(N, a, b) = a(K + K^{-1})a^{m-1}(K^{m-1} + K^{-(m-1)}) - a^2 a^{m-2}(K^{m-2} + K^{-(m-2)})$$

$$= a^m(K^m + K^{-m}).$$

Application

As an application let $u(n, k) = \binom{n}{n+k/2}$. Then $u(n, k) = u(n-1, k-1) + u(n-1, k+1)$ and $u(0, k) = \binom{k}{0}$. Therefore $u(n, k) = s_{i,0}(n, k) + s_{i,0}(n, k-1)$ or $u = (1+K)s_{i,0}$.
\[ a(n, 2m, k, -1) = \sum_{j \in \mathbb{Z}} (-1)^j \left( \frac{n - (2m) j + k}{2} \right) = \sum_{j \in \mathbb{Z}} \left( \left( \frac{n - (2m) j + k}{2} \right) - \left( \frac{n - (2m) (2j + 1) + k}{2} \right) \right) \]

\[ = \sum_{j \in \mathbb{Z}} (s_{i, 0}(n, k - 4mj) - s_{i, 0}(n, k - 2m - 4mj)) \]

\[ + \sum_{j \in \mathbb{Z}} (s_{i, 0}(n, k - 4mj) - s_{i, 0}(n, k - 1 - 2m - 4mj)) \]

Here we get \( q_m(N, 1, 0) \sum_{j \in \mathbb{Z}} (s_{i, 0}(0, i - 4mj) - s_{i, 0}(0, i - 2m - 4mj)) = 0 \) for each \( i \),
because for \( i - 4mj = m \) we get \( i - 4mj - 2m = -m \) and the sums cancel and for \( i - 4mj = -m \)
we get \( i - 4m(j - 1) - 2m = m \). For other values the sum vanishes.

In the same way as above we conclude that \( q_m(N, 1, 0) \sum_{j \in \mathbb{Z}} (s_{i, 0}(n, i - 4mj) - s_{i, 0}(n, i - 2m - 4mj)) = 0 \)
too.

In order to give a concrete representation of \( q_m(x, 1, 0) \) recall that the Lucas polynomials

\[ L_n(x, s) = \sum_{k=0}^{n-1} \left( \frac{n - k}{k} \right) \frac{n}{n - k} s^k x^{n-2k} = \left( \frac{x + \sqrt{x^2 + 4s}}{2} \right)^n + \left( \frac{x - \sqrt{x^2 + 4s}}{2} \right)^n \]  

are characterized by the recurrence

\[ L_n(x, s) = xL_{n-1}(x, s) + sL_{n-2}(x, s) \]  

with initial conditions \( L_0(x, s) = 2 \) and \( L_1(x, s) = x \).

Therefore \( q_n(x, 1, 0) = L_n(x, -1) \).

The first values of the sequence \( L_n(x, -1) \) are

\[ x, x^2 - 2, x^3 - 3x, x^4 - 4x^2 + 2, \cdots. \]

**Theorem 2**

For \( m \geq 1 \) the sequence \( a(n, 2m, k, -1) = \sum_{j \in \mathbb{Z}} (-1)^j \left( \frac{n - (2m) j + k}{2} \right) \) satisfies the recurrence relation

\[ L_m(N, -1)a(n, 2m, k, -1) = 0. \]  

**Remark**

It should be noted that \( a(n, 2m, 0, -1) \) has the following combinatorial interpretation. It is the number of the set of all lattice paths in \( \mathbb{R}^2 \) which start at the origin, consist of \( \left\lfloor \frac{n}{2} \right\rfloor \) northeast steps \((1,1)\) and \( \left\lfloor \frac{n+1}{2} \right\rfloor \) southeast steps \((1,-1)\) and which are contained in the strip \(-m < y < m. \) (cf. e.g. [5]).
The generating function of the sequence \((a(n, 2m, 0, -1))_{n \geq 0}\) is given by

\[
\sum_{n \geq 0} a(n, 2m, 0, -1)x^n = \frac{c_m(x)}{d_m(x)},
\]
where \(d_m(x) = q_m \left( \frac{1}{x}, 1, 0 \right) x^m = x^m L_m \left( \frac{1}{x}, -1 \right) = L_m(1, -x^2)\) and \(c_m(x)\) is a polynomial of degree less than \(m\).

The first values of \((c_m(x))_{m \geq 1}\) are
\[
c_1(x) = 1, c_2(x) = 1 + x, c_3(x) = 1 + x - x^2, c_4(x) = 1 + x - 2x^2 - x^3, c_5(x) = 1 + x - 3x^2 - 2x^3 + x^4, \cdots.
\]

This implies as above that \(c_m(x) = F_m(1, -x^2) + xF_{m-1}(1, -x^2)\).

**Corollary 2**

For \(m \geq 2\) the generating function for \(a(n, 2m, 0, -1)\) is given by

\[
\sum_{n \geq 0} a(n, 2m, 0, -1)x^n = \frac{F_m(1, -x^2) + xF_{m-1}(1, -x^2)}{L_m(1, -x^2)}.
\] (16)

**4. Further applications**

4.a) The same method can be applied to the general sum

\[
a(n, m, k, z) = \sum_{j \in \mathbb{Z}} z^j \left( \frac{n}{n - mj + k} \right) = \sum_{j \in \mathbb{Z}} z^{2j} \left( \frac{n}{n - 2mj + k} \right) + \sum_{j \in \mathbb{Z}} z^{2j-1} \left( \frac{n}{n - 2mj + k + m} \right).
\]

Here we get

\[
L_m(N, -1)a(0, m, k, z) = L_m(N, -1) \sum_{j \in \mathbb{Z}} z^{2j} u(0, k - 2mj) + L_m(N, -1) \sum_{j \in \mathbb{Z}} z^{2j-1} u(0, k + m - 2mj).
\]

In this case we have \(L_m(N, -1)u(0, k - 2mj) = \begin{cases} 1 & \text{if } k = 2mj - m \\ 1 & \text{if } k = 2mj + m \\ 0 & \text{else} \end{cases}\)

or \(L_m(N, -1)u(0, k - 2mj) = u(0, k - m - 2mj) + u(0, k + m - 2mj)\). This implies

\[
L_m(N, -1)a(0, m, k, z) = \sum_{j \in \mathbb{Z}} z^{2j} (u(0, k - m - 2mj) + u(0, k + m - 2mj))
+ \sum_{j \in \mathbb{Z}} z^{2j-1} (u(0, k + 2m - 2mj) + u(0, k - 2mj)) = \left( z + \frac{1}{z} \right) a(0, m, k, z).
\]
Thus we get
\[
\left( L_m(N, -1) - \left( z + \frac{1}{z} \right) \right) a(0, m, k, z) = 0.
\]

**Theorem 3**

The sequence \( a(n, m, k, z) = \sum_{j=2}^{\infty} \left( \frac{n}{n-mj+k} \right) \) satisfies the recurrence relation
\[
\left( L_m(N, -1) - \left( z + \frac{1}{z} \right) \right) a(n, m, k, z) = 0. \tag{17}
\]

**Remark**

It is easy to see that the initial values of \( a(n, m, 0, z) \) are
\[
a(n, m, 0, z) = \left\lfloor \frac{j}{2} \right\rfloor \quad \text{for} \quad 0 \leq j < m - 1, \quad a(m-1, m, 0, z) = \left\lfloor \frac{m-1}{2} \right\rfloor + \frac{1}{z}, \quad a(m, m, 0, z) = \left\lfloor \frac{m}{2} \right\rfloor + \frac{1}{z} + z.
\]

The generating function of the sequence \( a(n, m, 0, z) \) for \( m \geq 1 \) has the form
\[
\sum_{n \geq 0} a(n, m, 0, z)x^n = \frac{c_m(x, z)}{d_m(x, z)} \quad \text{with} \quad d_m(x, z) = x^n \left( L_m \left( \frac{1}{x}, -1 \right) - \left( z + \frac{1}{z} \right) \right) = d_m(x) - x^n \left( z + \frac{1}{z} \right)
\]
and \( c_m(x, z) = \frac{x^{m-1}}{z} + F_m(1, -x^2) + xF_{m-1}(1, -x^2) \).

Since \( d_m(x) = L_m(1, -x^2) \) and \( F_m(1, -x^2) + xF_{m-1}(1, -x^2) \) satisfy the same recurrence
\[
h_m(x) = h_{m-1}(x) - x^2h_{m-2}(x)
\]
we get
\[
\left( d_m(x) - x^n \left( z + \frac{1}{z} \right) \right) a_m(x) = \left( d_{m-1}(x) - x^{m-1} \left( z + \frac{1}{z} \right) \right) a_{m-1}(x) + x^2 \left( d_{m-2}(x) - x^{m-2} \left( z + \frac{1}{z} \right) \right) a_{m-2}(x)
\]
\[
= d_{m-1}(x)(a_m(x) - a_{m-1}(x)) + x^2d_{m-2}(x)(a_m(x) - a_{m-2}(x)) - x^n \left( z + \frac{1}{z} \right) a_m(x) + x^{m-1} \left( z + \frac{1}{z} \right) a_{m-1}(x) - x^m \left( z + \frac{1}{z} \right) a_{m-2}(x).
\]

Since \( d_m(0) = 1 \) it is easy to verify that for \( m \geq 3 \)
\[
d_{m-1}(x)(a_m(x) - a_{m-1}(x)) = -\frac{x^{m-2}}{z} - x^{m-1}z + x^n(\cdots) \quad \text{and}
\]
\[
x^2d_{m-2}(x)(a_m(x) - a_{m-2}(x)) = -\frac{x^{m-1}}{z} + x^n(\cdots).
\]
Therefore we get
\[ d_m(x, z)a_m(x) - d_{m-1}(x, z)a_{m-1}(x) + x^2 d_{m-2}(x, z)a_{m-2}(x) = -\frac{x^{m-2}}{z} + x^m \cdot \ldots. \]

Now the left hand side must be a polynomial of degree less than \( m \). Therefore we have in fact
\[ d_m(x, z)a_m(x) - d_{m-1}(x, z)a_{m-1}(x) + x^2 d_{m-2}(x, z)a_{m-2}(x) = -\frac{x^{m-2}}{z}. \]

Now \( c_m(x, z) \) satisfies the same recurrence. Since the initial values coincide, we get

**Corollary 3**

For \( m \geq 2 \) the generating function for \( a(n, m, 0, z) \) is given by

\[
\sum_{n \geq 0} a(n, m, 0, z)x^n = \frac{x^{m-1} + F_m(1, -x^2) + xF_{m-1}(1, -x^2)}{L_m(1, -x^2) - x^m\left(z + \frac{1}{z}\right)}. \quad (18)
\]

**Remark**

In the same way we get

\[
\sum_{n \geq 0} a(n, 2m + 1, m + 1, z)x^n = \frac{(1 + z)x^m(F_{m+1}(1, -x^2) + xF_m(1, -x^2))}{L_{2m+1}(1, -x^2) - x^{2m+1}\left(z + \frac{1}{z}\right)}.
\]

For \( z = -1 \) the right hand side vanishes and therefore we get again \( a(n, 2m + 1, m + 1, -1) = 0 \).

**4.b)** For the special case \( z = 1 \) also simpler recurrences can be found.

It is easy to verify that
\[ \left(x + \frac{1}{x} - 2\right)F_m(x + \frac{1}{x}, -1)(1 + x) = \frac{1}{x^m} - \frac{1}{x^{m-1}} - x^m + x^{m+1}. \]

This implies as above
\[ (N - 2)F_m(N, -1)u(0) = (K^m - K^{m-1} - K^{-m} + K^{-m-1})S_{1,0}(0). \]

Therefore we get
\[ (N - 2)F_m(N, -1)\sum_j K^{2jm}u(0) = \sum_j K^{2jm}(K^m - K^{m-1} - K^{-m} + K^{-m-1})S_{1,0}(0) = 0. \]
From this we conclude as above

**Theorem 4**

The sequence $a(n,2m,k,1) = \sum_{j \in \mathbb{Z}} \left( n - \frac{(2m)j + k}{2} \right)$ satisfies the recurrence relation

$$(N - 2)L_m(N,-1)a(n,2m,k,1) = 0.$$  \hfill (19)

**Corollary 4**

For $m \geq 1$ the generating function for $a(n,2m,0,1)$ is given by

$$\sum_{n \geq 0} a(n,2m,0,1)x^n = \frac{F_m(1,-x^2) - xf_{m-1}(1,-x^2)}{(1 - 2x)F_m(1,-x^2)}.$$  \hfill (20)

4.c) It is again easy to verify that

$$(L_m(x + \frac{1}{x},-1) - L_{m-1}(x + \frac{1}{x},-1))(1 + x) = \frac{1}{x^m} - \frac{1}{x^{m-2}} - x^{m-1} + x^{m+1}.$$  

Therefore we get

$$\left( L_m(K + K^{-1},-1) - L_{m-1}(K + K^{-1},-1) \right) \sum_j K^{(2m-1)j} u(0)

= \sum_j K^{(2m-1)j} (K^m - K^{m-2} - K^{-m+1} + K^{-m-1}) s_{1,0}(0) = 0.$$

This implies

**Theorem 5**

The sequence $a(n,2m-1,k,1) = \sum_{j \in \mathbb{Z}} \left( n - \frac{(2m-1)j + k}{2} \right)$ satisfies the recurrence relation

$$\left( L_m(N,-1) - L_{m-1}(N,-1) \right) a(n,2m-1,k,1) = 0.$$  \hfill (21)

**Corollary 5**

For $m \geq 2$ the generating function for $a(n,2m-1,0,1)$ is given by

$$\sum_{n \geq 0} a(n,2m-1,0,1)x^n = \frac{L_{m-1}(1,-x^2)}{L_m(1,-x^2) - xL_{m-1}(1,-x^2)}.$$  \hfill (22)
Remark

For the special cases \( z = \pm 1 \) numerator and denumerator of the generating function

\[
\frac{x^{m-1}}{z} + F_m(1,-x^2) + xF_{m-1}(1,-x^2)
\]

\[
L_m(1,-x^2) - x^m \left( z + \frac{1}{z} \right)
\]

have common divisors which can be cancelled.

This can be verified by using the following identities, which are easily deduced from the representations (6) and (13) (cf. e.g. [3]) :

\[
L_{2m}(x,-1) - 2 = (x^2 - 4) \left( F_m(x,-1) \right)^2,
\]

\[
L_{2m-1}(x,-1) - 2 = \left( \frac{L_m(x,-1) - L_{m-1}(x,-1)}{x - 2} \right)^2,
\]

\[
L_{2m}(x,-1) + 2 = \left( L_m(x,-1) \right)^2,
\]

\[
L_{2m-1}(x,-1) + 2 = (x + 2) \left( F_m(x,-1) - F_{m-1}(x,-1) \right)^2.
\]

References


