Abstract
This note gives a simple approach to $q-$ analogues of some results associated with Abel polynomials.

0. Introduction
In this note I want to give a simple approach to $q-$ analogues of the following well-known results about Abel polynomials:
Let $a_n(x,a) = x(x-na)^{n-1}$ be the Abel polynomials.
N.H. Abel [1] has found the beautiful formula

$$ (x + y)^n = \sum_{k=0}^{n} \binom{n}{k} a_k(x,a)(y+ka)^{n-k}. \quad (0.1) $$

We want to state it in a slightly more general form by changing $x \to x-b, y \to y+b$ and defining

$$ a_n(x,a,b) = a_n(x-b,a) = (x-b)(x-b-na)^{n-1}. \quad (0.2) $$

Then we get

$$ (x + y)^n = \sum_{k=0}^{n} \binom{n}{k} a_k(x,a,b)(y+ka+b)^{n-k}. \quad (0.3) $$

If we denote by $\partial$ the differentiation operator we see that

$$ a_n(x,a,b) = (1+a\partial)(x-b-na)^n. \quad (0.4) $$

By Taylor’s theorem which may be stated as $p(y+x) = e^{x\partial}p(y)$ for polynomials $p(y)$ formula (0.3) is equivalent with

$$ e^{x\partial}p(y) = \sum_{k} \frac{a_k(x,a,b)}{k!} e^{(b+ka)\partial^k} \left( \frac{\partial y}{y} \right)^k p(y) \quad (0.5) $$

for all polynomials $p(y)$. This gives the operator identity

$$ e^{x\partial} = \sum_{k} \frac{a_k(x,a,b)}{k!} e^{(b+ka)\partial^k} \left( \frac{\partial y}{y} \right)^k \quad (0.6) $$
which by the isomorphism \( \partial \leftrightarrow z \) is equivalent with the identity of formal power series

\[
e^{x^z} = \sum_k \frac{a_k(x,a,b)}{k!} z^k e^{(b+ka)z}.
\]  

(0.7)

The linear operator

\[
Q = \partial e^{a\partial}.
\]

(0.8)
called Abel operator, satisfies

\[
Q a_n(x,a,b) = na_{n-1}(x,a,b).
\]

(0.9)

Since \( a_n(0,a) = [n = 0] \) we get

\[
LQ^k a_n(x,a) = L\partial^k e^{ka} a_n(x,a) = n! [k = n],
\]

(0.10)

where \( L \) denotes the linear functional on the polynomials \( p(x) \) defined by \( Lp = p(0) \) and 

\[
[k = n]
\]
is Knuth’s symbol defined by \([k = n] = 1\) if \( k = n \) is true and \([k = n] = 0\) if \( k \neq n \).

This implies that the coefficients of the expansion

\[
f(z) = \sum_k \frac{c_k}{k!} z^k e^{ka}.
\]

(0.11)

are given by the Lagrange formula

\[
c_n = L\partial^n e^{-ax} f'(x).
\]

(0.12)

In the same way we get that the coefficients of the expansion

\[
\frac{f(z)}{1 + az} = \sum_k \frac{c_k}{k!} z^k e^{ka}.
\]

(0.13)

are given by the formula of Lagrange-Bürmann

\[
c_n = L\partial^n e^{-ax} f(x).
\]

(0.14)

A special case is

\[
e^{x^z} \frac{1}{1 - az} = \sum_{k \geq 0} (x + ak)^k \frac{z^k}{k!}
\]

(0.15)

This can also be written in the form

\[
\frac{1}{1 - az} = \sum_{k \geq 0} (x + ak)^k \frac{z^k}{k!} e^{-(ak + x)z}.
\]

(0.16)

Expanding \( e^{-(ak + x)z} \) into a power series and comparing coefficients of \( z^n \) shows that this is equivalent with

\[
n!a^n = \sum_{k = 0}^n (-1)^{n-k} \binom{n}{k} (n + ak)^n.
\]

(0.17)
Historical notes about some of these formulas can be found in [10]. The first \( q \)-analogues of some of these results have been given by F. H. Jackson ([5]) in 1910. In [3] I have given another proof depending on Rota’s Finite Operator Calculus ([8]). The case \( b \neq 0 \) has first been considered by J. Hofbauer in [4]. It also appeared in papers by W.P. Johnson ([6]), B. Bhatnagar and St.C. Milne ([2]), C. Krattenthaler and M. Schlosser ([7]) and M. Schlosser ([9]). In the following I want to give a self-contained exposition and some simplifications of these results. I want to thank Michael Schlosser for some useful comments.

1. \( q \)-Abel polynomials

Let \( D \) or \( D_q \) be the \( q \)– differentiation operator, defined by \( Df(x) = \frac{f(x) - f(qx)}{(1-q)x} \). Instead of \( Df(x) \) we also write \( f'(x) \).

In order to simplify notation we set \( (y + x)^n := \prod_{j=0}^{n-1} (y + q^j x) \) and \( (y - x)^n := \prod_{j=0}^{n-1} (y - q^j x) \).

The other notations from \( q \)– calculus are the usual ones. The \( q \)– binomial coefficients are denoted by \( \binom{n}{k} := \frac{[n]!}{k!(n-k)!} \) for \( 0 \leq k \leq n \). Here \( [n] = \frac{1-q^n}{1-q} \) and \( n! = \prod_{j=1}^{n} [j] \).

We need the analogues of the exponential series \( e(z) = \sum_{k \geq 0} \frac{z^k}{[k]!} \) and \( E(z) = \sum_{k \geq 0} q^{\frac{k}{2}} \frac{z^k}{[k]!} \) which are related by \( e(z)E(-z) = 1 \) and the well-known facts that \( \sum_{k \geq 0} \frac{(x+y)^k}{[k]!} z^k = \frac{e(xz)}{e(yz)} \) and \( E(aD)y^n = (y + a) \cdots (y + q^{n-1} a) \).

The following \( q \)– analogue of (0.3) holds:

\[
\prod_{j=0}^{n-1} (y + q^j x) = \sum_{k=0}^{n} \binom{n}{k} A_k(x,a,b) \prod_{j=0}^{n-k-1} (y + q^j [k]a + q^{k+1} b). \tag{1.1}
\]

Here

\[
A_k(x,a,b) = (x - b) \prod_{j=1}^{n-k} \left( q^j x - a [n] - q^k b \right) \tag{1.2}
\]

is a \( q \)– analogue of the general Abel polynomial \( a_k(x,a,b) \).

We will write (1.1) in the form

\[
(y + x)^n = \sum_{k=0}^{n} \binom{n}{k} A_k(x,a,b)(y + ([k]a + q^k b))^n-k \tag{1.3}
\]
or

\[
E(xD_y)y^n = \sum_{k} \frac{A_k(x,a,b)}{[k]!} E\left( ([k]a + q^k b) D_y \right) D_y^k y^n. \tag{1.4}
\]
By the isomorphism $D \leftrightarrow z$ it is also equivalent with the identity of formal power series

$$E(xz) = \sum_{k} \frac{A_k(x,a,b)}{[k]!} E(([k]a + q^k b)z)z^k. \quad (1.5)$$

For $b = 0$ this $q$–Abel theorem has been found by F. H. Jackson ([5]). The general case has been considered by J. Hofbauer ([4]), W. P. Johnson ([6]) and is also contained in more general results by B. Bhatnagar and St. C. Milne ([2]) and by C. Krattenthaler and M. Schlosser ([7]) and M. Schlosser ([9]).

In order to prove (1.2) we write (1.3) with unknown coefficients $A_k(x,a,b)$ and try to determine their values.

We consider this formula for $n \to n - 1$ and multiply with $c[n]$ for some constant $c$.

We thus get

$$c[n](y + x)^{n-1} = c \sum_{k=0}^{n-1} [n]A_k(x,a,b)(y + ([k]a + q^k b))^{n-k-1}$$

$$= \sum_{k=0}^{n-1} \left[ \frac{n}{n-k} \right] A_k(x,a,b)(y + ([k]a + q^k b))^{n-k} \frac{c[n-k]}{y + q^{n-k-1}([k]a + q^k b)}.$$

Comparing with (1.3) we see that the first $n - 1$ terms coincide if

$$\frac{c[n-k]}{y + q^{n-k-1}([k]a + q^k b)} = 1.$$

This gives

$$y = c[n-k] - q^{-1-k}a[k] - q^{-1}b = \frac{c(1-q^{n-k}) - q^{-1-k}(1-q^k)a}{1-q} - q^{-1}b.$$

This is independent on $k$ if $q^{-k}c + q^{-1-k}a = 0$, i.e. $c = -\frac{a}{q}$.

We then get

$$y = -\frac{a(1-q^{n-k}) + q^{-k}(1-q^k)}{1-q} - q^{-1}b = -\frac{a[n]}{q} - q^{-1}b.$$

Therefore

$$A_n(x,a,b) = q^{n-1}(x-b)\prod_{j=0}^{n-2} \left( q^j x - \frac{a}{q}[n] - q^{-1}b \right) = (x-b)\prod_{j=1}^{n-1} \left( q^j x - a[n] - q^b \right).$$

For $b = 0$ we get

$$A_n(x,a,0) = x^{n-1} \prod_{j=1}^{n-1} \left( q^j x - [n]a \right) \quad (1.6)$$

for $n > 0$ and $A_0(x,a,q) = 1$. 
This implies Jackson’s identity.

By letting \( a \rightarrow a + (1 - q)b \) we get

\[
G_n(x,a,b) = A_n(x,a + (1 - q)b,b) = (x - b) \prod_{j=0}^{n-1} (q^j x - [n]a - b). \tag{1.7}
\]

This is another form of the general \( q \)-Abel polynomials. Whereas \( A_n(x,a,b) \) and \( G_n(x,a,b) \) are equivalent some formulas become simpler by using \( G_n(x,a,b) \).

The polynomials \( G_n(x,a,b) \) have first been considered by J. Hofbauer ([4]). They are also the special case \( h = 0 \) of the polynomials \( a_n(x;b,h,w,q) \), which have been studied by W.P. Johnson in [6].

The corresponding identity is

\[
\prod_{j=0}^{n-1} (y + q^j x) = \sum_{k=0}^{n} \binom{n}{k} G_k(x,a,b) \prod_{j=0}^{n-k-1} (y + q^j [k]a + q^j b). \tag{1.8}
\]

This is equivalent with identity (8.4) in [7]. There it is stated in the form

\[
(c ; q)_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{-\frac{k(k+1)}{2}} \frac{1 - (a + b)}{1 - q^{-k} a + b} \left( q^{-k} a + b ; q \right)_k \left( c(q^k a + b) ; q \right)_{n-k},
\]

where as usual \((x ; q)_n = (1 - x)(1 - qx) \cdots (1 - q^{n-1} x)\). To obtain (1.8) make the substitutions

\[
a \rightarrow \frac{b(1 - q) - a}{(1 - q)x}, b \rightarrow \frac{a}{(1 - q)x}, c \rightarrow -\frac{x}{y}.
\]

The corresponding formula

\[
E(xz) = \sum_{k} \frac{G_k(x,a,b)}{[k]!} E((k)[a + b]z)z^k \tag{1.9}
\]

has been obtained in [4] and is also equivalent with formula [7], (7.4).

### 2. Abel expansions

If we apply the operator \( D \) to (1.9) we get

\[
zE(qxz) = DE(xz) = \sum_{n \geq 1} \frac{DG_n(x,a,b)}{[n]!} z^n E((b + [n]a)z)
\]

or

\[
E(qxz) = \sum_{n \geq 0} \frac{DG_{n+1}(x,a,b)}{[n+1]!} z^n E((b + [n + 1]a)z)).
\]

On the other hand (1.9) also gives
\[ E(qxz) = \sum_{n \geq 0} \frac{G_n(qx, qa, b + a)}{[n]!} z^n E((b + [n + 1]a)z). \]

Comparing these two formulas we get

\[ DG_{n+1}(x, a, b) = [n + 1]G_n(qx, qa, b + a). \]

By induction this implies

\[
D^k G_n(x, a, b) = q^{\binom{k}{2}} \frac{[n]!}{[n-k]!} G_{n-k}(q^k x, q^k a, b + [k]a). \tag{2.1}
\]

This means that

\[ D^k G_n(x, a, b) = q^{\binom{k}{2}} [n]! \]

and for \( k < n \)

\[
D^k G_n(x, a, b) = q^{\binom{k}{2}} \frac{[n]!}{[n-k]!} \left( q^k x - [k]a - b \right) \prod_{j=k+1}^{n-1} \left( q^j x - [n]a - b \right). \]

An important consequence is

\[
G_n^{(k)}(b + [k]a, a, b) = q^{\binom{k}{2}} [k]! [k = n]. \tag{2.2}
\]

Thus each polynomial \( f(x) \) has the following Abel expansion

\[
f(x) = \sum_k f^{(k)}(q^{-k}(b + [k]a)) q^{\binom{k}{2}} G_k(x, a, b). \tag{2.3}
\]

Choosing \( f(x) = \prod_{j=0}^{n-1} (y + q^j x) \) we get again (1.8).

By choosing \( f(x) = G_n(x, -a, -y - b) \) we get

\[
G_n(x, -a, -y - b) = \sum_{k=0}^{n} \left[ \frac{\binom{n}{k}}{[k]!} G_k(x, -a, b) y \prod_{j=1}^{n-k} \left( y + (1 - q^j)b + ([n] - q^j[k])a \right) \right]. \tag{2.4}
\]

This is the special case \( h = 0 \) of Theorem 4 by W.P. Johnson [6].

The expansion (2.3) also holds for formal power series. If we choose \( f(x) = E(xz) \) we get (1.9).

From (1.9) we get e.g.

\[
z^n = \sum_{k \geq 0} \left( \frac{(-1)^k}{[k]!} ([n]a + b)([n + k]a + b)^{k-1} z^{n+k} E(([n + k]a + b)z) \right)
\]

by applying \( q \) - differentiation \( k \) times and setting \( x = 0 \).
3. A q - analogue of the Abel operator

Let
\[ w_n(x, a, b) = \prod_{j=0}^{n-1} \left( q^j x - [n]a - b \right). \]  
(3.1)

This can be written in the form
\[
\begin{align*}
q_n(x, a, b) &= \sum_{k=0}^{n} (-1)^k \left( \begin{array}{c} n \\ k \end{array} \right) ^{n-k} \left( [n]a + b \right)^k x^{n-k} = q \left( \begin{array}{c} n \\ k \end{array} \right) ^{n-k} \left( \frac{[n]a + b}{q^{n-1}} \right)^k x^{n-k} \\
&= q^{\frac{n}{2}} E \left( -\frac{[n]a + b}{q^{n-1}} D \right) x^n.
\end{align*}
\]

In analogy to (0.4) we get
\[
G_n(x, a, b) = (1 + aD) \prod_{j=0}^{n-1} \left( q^j x - [n]a - b \right) = (1 + aD)w_n(x, a, b). \]  
(3.2)

Therefore we have
\[
G_n(x, a, b) = (1 + aD)w_n(x, a, b) = (1 + aD)q^{\frac{n}{2}} E \left( -\frac{[n]a + b}{q^{n-1}} D \right) x^n
\]
\[
= q^{\frac{n}{2}} E \left( -\frac{[n]a + b}{q^{n-1}} D \right) \left( x^n + [n]ax^{n-1} \right).
\]

If we write \( S_n(x, a) = q^{\frac{n}{2}} e \left( \frac{[n]a + b}{q^{n-1}} D \right) G_n(x, a, b) = x^n + [n]ax^{n-1}, \) then
\[ DS_n(x, a) = [n]S_{n-1}(x, a). \]

Let now \( Q_n \) be the operator
\[
Q_n = \frac{D}{q^{n-1}} e \left( \frac{[n]a + b}{q^{n-1}} D \right).
\]  
(3.3)

Then we get as a q – analogue of (0.9)
\[
Q_n G_n(x, a, b) = [n]G_{n-1}(x, a, b). \]  
(3.4)

Therefore the operators \( Q_n \) can be interpreted as q – analogues of the Abel operator \( Q \).

Unfortunately they are depending on \( n \).
(3.4) is a consequence of

\[
\frac{D}{q^{n-1}} e^{\left(\frac{[n]a + b}{q^{n-1} D}\right)} G_n(x, a, b) = \frac{1}{q^{n-1}} E\left(-\frac{[n-1]a + b}{q^{n-2} D}\right) q^{\left(\frac{n}{2}\right)} DS_n(x, a, b)
\]

\[
= q^{\left(\frac{n-1}{2}\right)} E\left(-\frac{[n-1]a + b}{q^{n-2} D}\right) [n] S_{n-1}(x, a, b) = [n] G_{n-1}(x, a, b).
\]

The operator $Q_n$ can also be written as

\[
Q_n = \sum_{k=0}^{\infty} \prod_{j=0}^{k-1} \frac{([j+1] - q^n[j])a + (1 - q^{j+1})b}{[k]!} \left(\frac{D}{q^{n-1}}\right)^{k+1}.
\] (3.5)

For

\[
[j+1] - q^n[j] = \frac{1 - q^{j+1} - q^n(1 - q^j)}{1 - q} = \frac{(1 - q^n) - q^{j+1}(1 - q^{n+1})}{1 - q} = [n] - q^{j+1}[n-1]
\]

and therefore

\[
([j+1] - q^n[j])a + (1 - q^{j+1})b = ([n] - q^{j+1}[n-1])a + (1 - q^{j+1})b = ([n]a + b) - q^{j+1}([n-1]a + b).
\]

This implies

\[
\prod_{j=0}^{k-1} \left(\frac{([j+1] - q^n[j])a + (1 - q^{j+1})b}{[k]!}\right) = \left(\left([n]a + b\right) - q\left([n-1]a + b\right)\right)^{\frac{k}{4}}.
\]

4. A q - Lagrange formula

For a formal power series $f(z)$ we want to find the coefficients in the expansion

\[
f(z) = \sum_{n} \frac{c_n}{[n]!} z^n E([n]az).
\] (4.1)

Consider first the expansion of $f(z) = e(xz)$. Let

\[
e(xz) = \sum_{k} \frac{B_k(x,a)}{[k]!} z^k E([k]az).
\] (4.2)
We know from (1.9) that $E(xz) = \sum_k A_k(x,a,0)E([k]az)z^k$.

If we let $V$ be the linear operator defined by $Vq^\frac{n}{2}x^n = x^n$, then it is clear that

$$B_n(x,a) = VA_n(x,a,0) = \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k ([n]a)^k \sum_{k=0}^{n-1} \binom{n-k}{k} x^{n-k} = \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k ([n]a)^k x^{n-k}.$$  

Therefore we get

$$B_n(x,a) = x \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} a^j x^{n-1-j} [n]^j = xe\left(-[n]aDx^{n-1}\right). \quad (4.3)$$

Here we have a direct analogue of the formula $a_n(x,a) = xe^{-ma^2}x^{n-1}$.

A possible disadvantage is that there is no simple factorization.

The property we are interested in is

$$LE([k]aD)D^kB_n(x,a) = [n][k = n], \quad (4.4)$$

which generalizes (0.10).

To prove it observe that $D^k e(xz) = z^k e(xz)$. Therefore

$$E([k]aD)D^k e(xz) = \sum_k E([k]aD)D^k B_n(x,a)z^k E([k]az)$$

and for $x = 0$

$$E([k]az)z^k = \sum_k LE([k]aD)D^k B_n(x,a)z^k E([k]az).$$

Comparing coefficients we get (4.4).

This implies the following

$q$ – Lagrange formula ([3]):

The coefficients $c_n$ in the expansion

$$f(x) = \sum_n \frac{c_n}{[n]!} x^n E([n]ax) \quad (4.5)$$

are given by

$$c_n = Lf'(D)e(-[n]aD)x^{n-1} = LD^{n-1}e(-[n]a)x f'(x). \quad (4.6)$$

For by (4.4) we have $c_n = Lf(D)B_n(x,a) = Lf(D)x e(-[n]aD)x^{n-1} = LD^{n-1}e(-[n]aD)f'(x)$. 

9
The last equation follows from \( L(D^k x^n) = [n]! [k = n] = L(D^n x^k) \) and the \( q \)–Pincherle derivative
\[
f(D)x - xf(qD) = f'(D). \tag{4.7}
\]
Here \( x \) denotes the operator multiplication by \( x \).
This well-known fact follows from \((Dx - qxD)x^n = Dx^{n+1} - qx Dx^n = ([n + 1] - q[n])x^n = x^n\),
which implies \( Dx - qxD = 1 \) by induction \( D^n x - q^n D^n = nD^{n-1} \).

More generally let \( B_n(x,a,b) = VA_n(x,a,b) \). Then the coefficients of the expansion
\[
f(z) = \sum_n C_n [n]^n z^n E \left( \left( [n] a + q^n b \right) z \right) \tag{4.8}
\]
are given by
\[
c_n = Lf(D)B_n(x,a,b) = Lf(D)xe\left( - ([n] a + q^n b) D \right) x^{n-1} - q^{n-1} b Lf(D) e\left( - \left( \frac{[n] a + q^n b}{q} \right) D \right) x^{n-1}
= LD^{n-1} e\left( - ([n] a + q^n b) x \right) f'(x) - q^{n-1} b LD^{n-1} e\left( - \left( \frac{[n] a + q^n b}{q} \right) x \right) f(x).
\]
For the proof observe that
\[
LE \left( (q^k b + [k] a) D \right) D^k B_n(x,a,b) = [n]! [k = n] \tag{4.9}
\]
by the same argument as above and that
\[
B_n(x,a,b) = VA_n(x,a,b) = V(x - b) \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \left( [n] a + q^n b \right)^k \left( \frac{q}{2} \right)^{n-1-k} x^{n-1-k}
= V \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \left( [n] a + q^n b \right)^k \left( \frac{q}{2} \right)^{n-1-k} x^{n-1-k} - b V \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \left( [n] a + q^n b \right)^k \left( \frac{q}{2} \right)^{n-1-k} x^{n-1-k}
= \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \left( [n] a + q^n b \right)^k x^{n-k} - q^{n-1} b \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \left( [n] a + q^n b \right)^k \left( \frac{q}{2} \right)^{n-1-k} x^{n-1-k}
= xe\left( - ([n] a + q^n b) D \right) x^{n-1} - q^{n-1} b e\left( - \left( \frac{[n] a + q^n b}{q} \right) D \right) x^{n-1}.
\]

\( B_n(x,a,b) \) can also be written in the form
\[
B_n(x,a,b) = \sum_{k=0}^{n} (-1)^k x^{n-k} \binom{n}{k} (q^n b + [n] a)^{k-1} (q^{n-k} b + [n - k] a).
\]
In order to obtain an analogue of the **Lagrange-Bürmann formula** we note that

\[
\left(1 + \frac{a}{q}D\right)e\left(-\frac{q^b + [n]a}{q}D\right)x^n = xe\left(-\left([n]a + q^b\right)D\right)x^{n-1} - q^{-1}be\left(-\left([n]a + q^b\right)D\right)x^{n-1}.
\] (4.10)

\[= B_n(x,a,b).
\]

This follows from the \(q\) – Pincherle derivative

\[
\left(e\left(-\frac{q^b + [n]a}{q}D\right)x - xe\left(-\left(q^b + [n]a\right)D\right)x^{n-1} = -\frac{q^a + [n]a}{q}e\left(-\frac{q^b + [n]a}{q}D\right)x^{n-1}.\]

Thus we get a

**q - Lagrange - Bürmann type formula:**

The coefficients of

\[
\frac{f(z)}{1 + \frac{az}{q}} = \sum_{k} \frac{c_k}{[k]!} z^{k} E\left((q^a + [k]a)z\right)
\] (4.11)

are given by

\[
c_n = LD^n e\left(-\frac{q^b + [n]a}{q}x\right)f(x).
\] (4.12)

This is an immediate consequence of

\[
c_n = Lf(D)\frac{1}{1 + \frac{ad}{q}} B_n(x,a,b) = Lf(D)e\left(-\frac{q^b + [n]a}{q}D\right)x^n = LD^n e\left(-\frac{q^b + [n]a}{q}x\right)f(x).
\]

If we choose \(f(z) = E(-yz)\) we get

\[
c_n = LD^n e\left(-\frac{q^b + [n]a}{q}x\right)E(-xy) = LD^n \frac{e\left(-\frac{q^b + [n]a}{q}x\right)}{e(xy)} = \left(-\frac{q^b + [n]a}{q}y\right)^n
\]

This is equivalent with

\[
\frac{E(xz)}{1 - az} = \sum_{k} \frac{(q^a + [k]a + x)^k}{[k]!} z^k E(-q(q^b + [k]a)z).
\] (4.13)

This \(q\) – analogue of (0.15) has been found by C. Krattenthaler and M. Schlosser ( cf. [7] and [9], (5.4)) in another context.
5. Other methods of proof

We give now another proof of formula (4.13) by using a $q$–analogue of the difference operator:

Let $U$ be the linear operator on the polynomials in $q^n$ defined by

$$U q^n = q^{(n-1)}$$

(5.1)

for all $i \in \mathbb{N}$. As a special case we get $U[n]^m = [n-1]^m$.

Define now

$$\Delta^k = (1-qU) \cdots (1-q^kU).$$

(5.2)

Then it is clear that

$$\Delta^k q^n = 0$$

(5.3)

for $k \geq i > 0$ and

$$\Delta^k 1 = (1-q)^k[k]!.$$  

(5.4)

Furthermore

$$\Delta^k [n]^m = [k]!(1-q)^{k-m}$$

(5.5)

for $k \geq m$ and

$$\Delta^k q^n [n]^m = 0$$

(5.6)

for $k \geq m+i$ if $i > 0$.

It suffices to show (5.5), which follows from

$$\Delta^k [n]^m = \frac{1}{(1-q)^m} \Delta^k (1-q^n)^m = \frac{1}{(1-q)^m} \sum (-1)^j \binom{m}{j} \Delta^j q^n = \frac{\Delta^k 1}{(1-q)^m} = (1-q)^k[k]!.$$ 

This again implies

$$\Delta^k (q^n x + [n]a) = 0$$

(5.7)

if $j > 0$ and

$$\Delta^k (q^n x + [n]a)^n = [n]! a^n.$$ 

(5.8)

We conclude that

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \left( q^{n-k} b + [n-k]a + x \right)^{n-k} \left( x + q (q^{n-k} b + [n-k]a) \right)^k = [n]! a^n.$$ 

(5.9)

For
\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^k \left( q^{n-k}b + a[n-k] + x \right)^{n-k} \left( x + q(q^{n-k}b + a[n-k]) \right)^k \\
= \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{\binom{k+1}{2}} \left( q^{n-k}b + a[n-k] + q^{-k}x \right)^n \\
= \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{\binom{k+1}{2}} \sum_{j=0}^{k} q^{-j} x^j q^{\binom{j}{2}} \left( q^{n-k}b + a[n-k] \right)^{n-j} \\
= \sum_{j=0}^{n} q^{-j} x^j q^{\binom{j}{2}} \Delta^n \left( q^n \left( q^n b + a[n] \right)^{n-j} \right) = \Delta^n \left( \left( q^n b + a[n] \right)^n \right) = [n]! a^n.
\]

By comparing coefficients we see that (5.9) is equivalent with
\[
\frac{1}{1-az} = \sum_{k} \binom{k}{a} x^k \sum_{j} \binom{j}{a} \left( x + q(a[k] + q^j b) \right)^j (-z)^j \\
= \sum_{k} \binom{k}{a} \frac{e(-xz)}{e(q(q^k b + [k]a)z)}.
\]

This is equivalent with (4.13).

By applying the \( q - \) differentiation operator \( k \) times and then setting \( x = 0 \) we get from (4.13)
\[
\frac{z^n}{1-az} = \sum_{k=0}^{n} \lambda^k \binom{n+k}{a} \left( q^n b + [k]a + x \right)^k \sum_{j} \binom{j}{a} \left( x + q(a[k] + q^j b) \right)^j (-z)^j.
\]

By considering the isomorphism \( z \rightarrow D_y \) and applying it to \( y^n \) (4.13) gives
\[
(y+x)^n = \sum_{k=0}^{n} \binom{n}{k} (q^k b + [k]a + x)^k \nu(n,k,a,b,y),
\]
with \( \nu(n,k,a,b,y) = \left( y + q(q^k b + [k]a) \right)^{n-k} \left( y - q^n b - [n]a \right) \) for \( k < n \) and \( \nu(n,n,a,b,y) = 1 \).

If we substitute \( a \rightarrow a + (1-q)b \) in (4.13) we get
\[
\frac{E(xz)}{1-(a+(1-q)b)z} = \sum_{k} \frac{(b+[k]a+x)^k}{[k]!} z^k E(-q(b+[k]a)z). \]
References


