Problem
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Let $(2n+1)!!=1\cdot 3\cdot 5\cdots(2n+1)$ and $(-1)!!=1.$ The Hermite polynomials $H_n(x,s)$ are given by $H_n(x,s)=\sum_{k=0}^{n/2}\binom{n}{2k}(2k-1)!!(-s)^k x^{n-2k}$. The first terms are $1, x, x^2-s, x^3-3sx, x^4-6sx^2+3s^2,\cdots.$

Consider x as a parameter and s as a variable or indeterminate and define a linear functional L on the vector space of polynomials in s by

$$L\left(H_{2n}\right)=\delta_{n,0}. \tag{1}$$

Show that
1)

$$L\left(H_{2n+1}\right)=(-1)^n T_{2n+1}x^{2n+1} \tag{2}$$

where T_{2n+1} are the tangent numbers defined by

$$\tan z=\sum_{n\geq 0}T_{2n+1}\frac{z^{2n+1}}{(2n+1)!}=z+2\frac{z^3}{3!}+16\frac{z^5}{5!}+272\frac{z^7}{7!}+\cdots \tag{3}$$

and

2)

$$L(s^n)=\frac{E_{2n}x^{2n}}{(2n-1)!!}, \tag{4}$$

where E_{2n} are the Euler numbers defined by

$$\frac{1}{\cos z}=\sum_{n\geq 0}\frac{E_{2n}}{(2n)!}z^{2n}=1+\frac{z^2}{2!}+5\frac{z^4}{4!}+61\frac{z^6}{6!}+\cdots. \tag{5}$$

Examples:
$L\left(H_1\right)=L\left(x\right)=x,$ because x is a parameter.
$L(H_2)=0$ implies that $L(x^2-s)=0,$ thus $L(s)=x^2.$
$L(H_3)=L\left(x^3-3sx\right)=x^3-3xL(s)=x^3-3x^3=-2x^3=(-1)^1T_3x^3,$
$0=L(H_4)=L\left(x^4-6x^2s+3s^2\right)=x^4-6x^4+3L(s^2)$ implies
$L\left(s^2\right)=\frac{5x^4}{3}=\frac{E_4x^4}{3!!}.$

3) Show analogous results for the (discrete) $q-$Hermite polynomials

$$h_n(x,s,q)=\sum_{k=0}^{n/2}\begin{bmatrix} n \\ 2k \end{bmatrix}_q q^{k^2}[2k-1]_q!!(-s)^k x^{n-2k}.$$

Here $\begin{bmatrix} n \\ k \end{bmatrix}_q=\frac{[n]_q!}{[k]_q![n-k]_q!}$, $[n]_q!=[1]_q[2]_q\cdots[n]_q$, $[2n-1]_q!!=[1]_q[3]_q\cdots[2n-1]_q$ and $[n]_q=1+q+\cdots+q^{n-1}$.

Solution

1) It is well known or easily verified that the exponential generating function of the Hermite polynomials is

$$H(z,x,s)=\sum_{n\geq 0}H_n(x,s)\frac{z^n}{n!}=e^{xz-s\frac{z^2}{2}}. \tag{6}$$

This implies $H(z,x,s)=e^{2xz}H(-z,x,s)$ and therefore also

$$\sum_{k\geq 1}H_{2k-1}(x,s)\frac{z^{2k-1}}{(2k-1)!}=\frac{H(z,x,s)-H(-z,x,s)}{2}=\frac{e^{2xz}-1}{2}H(-z,x,s)$$

and

$$\sum_{k\geq 0}H_{2k}(x,s)\frac{z^{2k}}{(2k)!}=\frac{H(z,x,s)+H(-z,x,s)}{2}=\frac{e^{2xz}+1}{2}H(-z,x,s).$$

This gives

$$\sum_{k\geq 1}H_{2k-1}(x,s)\frac{z^{2k-1}}{(2k-1)!}=\frac{e^{2xz}-1}{e^{2xz}+1}\sum_{k\geq 0}H_{2k}(x,s)\frac{z^{2k}}{(2k)!}. \tag{7}$$

Now apply L and observe that $\frac{e^{2z}-1}{e^{2z}+1}=\sum_{n\geq 0}(-1)^n T_{2n+1}\frac{z^{2n+1}}{(2n+1)!}$.

Caution: L is a linear functional on the polynomials and cannot a priori be applied to an infinite series. But H_{2n+1} is the coefficient of $\frac{z^{2n+1}}{(2n+1)!}$ of the right-hand side. This coefficient is a finite linear combination of terms $H_{2k}(x,s)$. Therefore there are no problems to apply the linear functional.

Another consequence of (7) is the expansion

$$H_{2n+1}(x,s)=\sum_{k=0}^{n}(-1)^k T_{2k+1}\binom{2n+1}{2k+1}x^{2k+1}H_{2n-2k}(x,s). \tag{8}$$

2) From (6) and using (1) we see that

$$L\left(e^{-\frac{z^2}{2}s}\right)=L\left(e^{-xz}H(z,x,s)\right)=e^{-xz}L\left(\sum_{n\geq 0}\frac{H_n(x,s)z^n}{n!}\right)=e^{-xz}\left(1+\sum_{n\geq 0}\frac{L\left(H_{2n+1}(x,s)\right)z^{2n+1}}{(2n+1)!}\right).$$

Now we can apply (2) and get

$$1+\sum_{n\ge 0}\frac{L\left(H_{2n+1}(x,s)\right)z^{2n+1}}{(2n+1)!}=1+\sum_{n\ge 0}\frac{(-1)^n T_{2n+1}x^{2n+1}z^{2n+1}}{(2n+1)!}=1+\frac{e^{2xz}-1}{e^{2xz}+1}=\frac{2e^{2xz}}{e^{2xz}+1}.$$

Combining these two equations we conclude that

$$L\left(e^{-\frac{z^2}{2}s}\right)=\frac{2}{e^{xz}+e^{-xz}}=\sum_{n\ge 0}\frac{(-1)^n E_{2n}x^{2n}}{(2n)!}z^{2n},$$

which implies (4).

3) It is well-known and easily verified (cf. J. Cigler, Elementare $q-$Identitäten, Sem. Lotharingien Comb., B05a (1981), 29 pp.) that

$$h(z,x,s,q)=\sum_{n\ge 0}h_n(x,s,q)\frac{z^n}{[n]_q!}=\frac{e_q(xz)}{e_{q^2}\left(\frac{qsz^2}{[2]_q}\right)}$$

where $e_q(z)=\sum_{n\ge 0}\frac{z^n}{[n]_q!}$ denotes the $q-$exponential series.

This implies $h(z,x,s,q)=\frac{e_q(xz)}{e_q(-xz)}h(-z,x,s,q)$ and therefore also

$$\sum_{k\ge 1}h_{2k+1}(x,s,q)\frac{z^{2k+1}}{[2k+1]_q!}=\frac{e_q(xz)-e_q(-xz)}{e_q(xz)+e_q(-xz)}\sum_{k\ge 0}h_{2k}(x,s,q)\frac{z^{2k}}{[2k]_q!}.$$

Now apply L and observe that $\frac{e_q(xz)-e_q(-xz)}{e_q(xz)+e_q(-xz)}=:\sum_{n\ge 0}(-1)^n T_{2n+1}(q)\frac{x^{2n+1}z^{2n+1}}{[2n+1]_q!}.$

On the other hand we have

$$L\left(\frac{1}{e_{q^2}\left(\frac{qsz^2}{[2]}\right)}\right)=L\left(\frac{1}{e_q(xz)}h(z,x,s,q)\right)=\frac{1}{e_q(xz)}L\left(\sum_{n\ge 0}\frac{h_n(x,s,q)z^n}{[n]_q!}\right)$$

$$=\frac{1}{e_q(xz)}\left(1+\sum_{n\ge 0}\frac{L\left(h_{2n+1}(x,s,q)\right)z^{2n+1}}{[2n+1]_q!}\right)=\frac{1}{e_q(xz)}\left(1+\frac{e_q(xz)-e_q(-xz)}{e_q(xz)+e_q(-xz)}\right)$$

$$=\frac{2}{e_q(xz)+e_q(-xz)}=:\sum_{n\ge 0}\frac{(-1)^n E_{2n}(q)x^{2n}}{[2n]_q!}z^{2n}.$$