Some elementary observations on Narayana polynomials and related topics II: q-Narayana polynomials

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Abstract

We show that $q-$Catalan numbers, $q-$central binomial coefficients and $q-$Narayana polynomials are moments of $q-$analogues of Fibonacci and Lucas polynomials and related polynomials.

1. Introduction

This note is a supplement to part I ([4]). Let

$$F_n(x)=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}(-1)^k\binom{n-k}{k}x^{n-2k}, \tag{1.1}$$

$n\in\mathbb{N},$ be Fibonacci polynomials and define a linear functional L by

$$L\left(F_n(x)\right)=[n=0]. \tag{1.2}$$

Then the moments $L\left(x^{2n}\right)$ are the Catalan numbers

$$L\left(x^{2n}\right)=C_n=\frac{1}{n+1}\binom{2n}{n}. \tag{1.3}$$

This well-known fact is the special case $t=1$ of the following result (cf. [4] and the literature cited there):

The Narayana polynomials

$$C_n(t)=\sum_{k\ge 0}\binom{n}{k}\binom{n}{k+1}\frac{1}{n}t^k \tag{1.4}$$

can be represented as moments $L\left(x^{2n}\right)$ of the linear functional L defined by $L\left(F_n(x,t)\right)=[n=0],$ where

$$F_n(x,t)=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}(-1)^k\sum_{j=0}^{k}\binom{\left\lfloor \frac{n}{2}\right\rfloor-j}{k-j}\binom{\left\lfloor \frac{n-1}{2}\right\rfloor-k+j}{j}t^jx^{n-2k} \tag{1.5}$$

are generalized Fibonacci polynomials which satisfy the recurrence

$$\begin{aligned} F_{2n}(x,t) &= xF_{2n-1}(x,t) - F_{2n-2}(x,t), \\ F_{2n+1}(x,t) &= xF_{2n}(x,t) - tF_{2n-1}(x,t). \end{aligned} \tag{1.6}$$

One of our purposes is to give a nice $q-$analogue of this result. More precisely we define nice polynomials $F_n(x,t,q)$ such that the linear functional defined by $L\left(F_n(x,t,q)\right) = [n=0]$ has as moments the $q-$Narayana polynomials

$$L\left(x^{2n}\right) = C_n(t,q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} n \\ j+1 \end{bmatrix} \frac{1}{[n]} q^{j^2+j} t^j, \tag{1.7}$$

which for $t=1$ reduce to the $q-$Catalan numbers

$$C_n(q) = C_n(1,q) = \frac{1}{[n+1]} \begin{bmatrix} 2n \\ n \end{bmatrix}. \tag{1.8}$$

We will always suppose that $0 < q < 1$ and use the notations $[n] = [n]_q = \dfrac{1-q^n}{1-q}$ and

$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \dfrac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$ for $0 \le k \le n.$

It is well known that for each sequence $(a_n)_{n \ge 0}$ with $a_0 = 1$ such that all Hankel determinants $\det\left(a_{i+j}\right)_{i,j=0}^{n} \ne 0$ there are uniquely defined monic polynomials $p_n(x)$ of degree $n,$ $n \in \mathbb{N},$ which are orthogonal with respect to the linear functional L defined by $L\left(x^n\right) = a_n.$ Orthogonality means that $L\left(p_n p_m\right) = 0$ for $n \ne m$ and $L\left(p_n^2\right) \ne 0.$ Since $p_0(x) = 1$ this implies that $L\left(p_n\right) = [n=0].$

Moreover there are uniquely determined numbers $a(n,k)$ such that

$$x^n = \sum_{k=0}^{n} a(n,k) p_k(x). \tag{1.9}$$

By applying L we see that $a(n,0) = L\left(x^n\right) = a_n.$

For $q=1$ the polynomials $F_n(x,t)$ are the orthogonal polynomials with moments $L\left(x^{2n}\right) = C_n(x,t)$ and $L\left(x^{2n+1}\right) = 0.$

Let now L be the linear functional defined by $L\left(x^{2n}\right) = C_n(t,q)$ and $L\left(x^{2n+1}\right) = 0.$

It would be tempting to consider also in this case the corresponding orthogonal polynomials, but it seems that there is no simple explicit formula or recurrence for them, not even for $t=1.$

In that case the first orthogonal polynomials are

$$1,\quad x,\quad x^2-1,\quad x^3-\left(1+q^2\right)x,\quad x^4-\left(q+q^2+q^4\right)x^2-1+q+q^4,$$

$$x^5-\frac{(1+q)\left(1-q+q^2\right)\left(-1+q^3+q^5+q^7\right)}{-1+q+q^4}x^3+\frac{q\left(1-q+q^2\right)\left(-1-q+q^5+q^6+q^7+q^8+q^9\right)}{-1+q+q^4}x.$$

The first numbers $a(n,k)$ are

1					
0	1				
1	0	1			
0	$1+q^2$	0	1		
$1+q^2$	0	$q\,(1+q+q^3)$	0	1	
0	$(1-q+q^2)\,(1+q+q^2+q^3+q^4)$	0	$\frac{(1+q)\,(1-q+q^2)\,(-1+q^3+q^5+q^7)}{-1+q+q^4}$	0	1

Fortunately there also exist "nice" polynomials with the same moments. Let us consider first the case $q=1.$ The orthogonal polynomials for the linear functional L defined by $L\left(x^{2n}\right)=C_n$ and $L\left(x^{2n+1}\right)=0$ are the Fibonacci polynomials $F_n(x)=\sum_{k=0}^{n}(-1)^k\binom{n-k}{k}x^{n-2k}.$

They satisfy $x^n=\sum_{k=0}^{n}a(n,k)F_k(x)$ with $a(2n+k,k)=\binom{2n+k}{n}-\binom{2n+k}{n-1}=\frac{k+1}{n+k+1}\binom{2n+k}{n}$ and $a(n,k)=0$ else.

Let me sketch how to find a nice $q-$analogue of this situation. It is easier to begin with $a(n,k).$ A natural $q-$analogue is

$a(2n+k,k,q)=\frac{1}{q^n}\left(\begin{bmatrix}2n+k\\ n\end{bmatrix}-\begin{bmatrix}2n+k\\ n-1\end{bmatrix}\right)=\frac{[k+1]}{[n+k+1]}\begin{bmatrix}2n+k\\ n\end{bmatrix}$ and $a(n,k,q)=0$ else. The first terms are

1					
0	1				
1	0	1			
0	$1+q$	0	1		
$1+q^2$	0	$1+q+q^2$	0	1	
0	$1+q+q^2+q^3+q^4$	0	$(1+q)\,(1+q^2)$	0	1

Now we are looking for the polynomials $F_n(x,q)$ such that

$$x^n=\sum_{k=0}^{n}a(n,k,q)F_k(x,q). \tag{1.10}$$

Their coefficients are given by the inverse matrix $\left(a(i,j,q)\right)^{-1}$. Fortunately this also turns out to be nice. The first terms are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1-q & 0 & 1 & 0 & 0 \\ q & 0 & -1-q-q^2 & 0 & 1 & 0 \\ 0 & q\left(1+q+q^2\right) & 0 & -(1+q)\left(1+q^2\right) & 0 & 1 \end{pmatrix}$$

Thus the sequence $\left(F_n(x,q)\right)_{n\geq 0}$ begins with

$1,\quad x,\quad x^2-1,\quad x^3-(1+q)x,\quad x^4-\left(1+q+q^2\right)x^2+q,$
$x^5-(1+q)\left(1+q^2\right)x^3+q\left(1+q+q^2\right)x,\cdots.$

$F_n(x,q)$ can of course also be computed inductively by $F_n(x,q)=x^n-\sum_{k=1}^{n}a(n,k,q)F_k(x,q).$

It is now easy to guess that in general

$$F_n(x,q)=\sum_{j=0}^{\left\lfloor \frac{n}{2}\right\rfloor}(-1)^j q^{\binom{j}{2}}\begin{bmatrix} n-j \\ j \end{bmatrix}x^{n-2j}. \tag{1.11}$$

Remark

Note that these $q-$Fibonacci polynomials, which have been considered in [2] and [3], are not orthogonal. There are also nice orthogonal $q-$analogues of the Fibonacci polynomials, i.e. the Carlitz $q-$Fibonacci polynomials $\sum_{j=0}^{\left\lfloor \frac{n}{2}\right\rfloor}(-1)^j q^{j^2}\begin{bmatrix} n-j \\ j \end{bmatrix}x^{n-2j}$. Their moments are the Carlitz $q-$Catalan numbers, but unfortunately these have no closed formula.

In this note we first recall some results about the above mentioned class of non- orthogonal $q-$Fibonacci and $q-$Lucas polynomials whose moments are $q-$Catalan numbers and $q-$central binomial coefficients and then propose "nice" $q-$analogues of the generalized Fibonacci and Lucas polynomials of part I such that the corresponding moments are $q-$Narayana polynomials $C_n(t,q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}\begin{bmatrix} n \\ j+1 \end{bmatrix}\frac{1}{[n]}q^{j^2+j}t^j$ and $M_n(t,q)=\sum_{j=0}^{n}q^{j^2}\begin{bmatrix} n \\ j \end{bmatrix}^2 t^j.$

2. Some background material

Let us first state some known results (cf. [1],[2],[3]). As already mentioned the $q-$Fibonacci polynomials

$$F_n(x,q)=\sum_{j=0}^{\left\lfloor \frac{n}{2}\right\rfloor}(-1)^j q^{\binom{j}{2}}\begin{bmatrix} n-j \\ j \end{bmatrix} x^{n-2j} \tag{2.1}$$

satisfy

$$x^n=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}\frac{1}{q^k}\left(\begin{bmatrix} n \\ k \end{bmatrix}-\begin{bmatrix} n \\ k-1 \end{bmatrix}\right)F_{n-2k}(x,q). \tag{2.2}$$

If we define a linear functional L by $L\left(F_n(x,q)\right)=[n=0]$ then we get

$$L\left(x^{2n}\right)=C_n(q)=\begin{bmatrix} 2n \\ n \end{bmatrix}\frac{1}{[n+1]}, \tag{2.3}$$

where $C_n(q)=\frac{1}{q^n}\left(\begin{bmatrix} 2n \\ n \end{bmatrix}-\begin{bmatrix} 2n \\ n-1 \end{bmatrix}\right)=\begin{bmatrix} 2n \\ n \end{bmatrix}\frac{1}{[n+1]}$ is a $q-$analogue of the Catalan numbers $C_n=\binom{2n}{n}\frac{1}{n+1}.$

The $q-$Lucas polynomials

$$L_n(x,q)=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}(-1)^k q^{\binom{k}{2}}\frac{[n]}{[n-k]}\begin{bmatrix} n-k \\ k \end{bmatrix} x^{n-2k} \tag{2.4}$$

for $n>0$ and $L_0(x,q)=1$ satisfy

$$x^n=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}\begin{bmatrix} n \\ k \end{bmatrix} L_{n-2k}(x,q). \tag{2.5}$$

If we define the linear functional M by $M\left(L_n(x,q)\right)=[n=0]$ then we get

$$M\left(x^{2n}\right)=M_n(q)=\begin{bmatrix} 2n \\ n \end{bmatrix} \tag{2.6}$$

is a central $q-$binomial coefficient.

It is easy to verify that

$$L_n(x,q)=F_n(x,q)-q^{n-1}F_{n-2}(x,q) \tag{2.7}$$

for $n>1.$ Moreover $L_0(x,q)=F_0(x,q)=1$ and $L_1(x,q)=F_1(x,q)=x.$

These results can be proved with an inversion formula by L. Carlitz [1]. Another proof is in [2] and [3]. Carlitz uses the fact that

$$c(n,k):=\sum_{j=0}^{\min(k,n-k)}(-1)^j q^{\binom{j+1}{2}}\begin{bmatrix}k\\j\end{bmatrix}\begin{bmatrix}n-j\\k\end{bmatrix}=1 \tag{2.8}$$

for $0\le k\le n.$

To prove this let U be the linear operator on $\mathbb{C}\left[q^x\right]$ defined by $U\begin{bmatrix}x\\k\end{bmatrix}=\begin{bmatrix}x-1\\k\end{bmatrix}$ for integers $k>0$ and $U1=0.$

Then $c(n,k):=\sum_{j=0}^{k}(-1)^j q^{\binom{j+1}{2}}\begin{bmatrix}k\\j\end{bmatrix}U^j\begin{bmatrix}n\\k\end{bmatrix}=\left(1-qU\right)\left(1-q^2U\right)\cdots\left(1-q^kU\right)\begin{bmatrix}n\\k\end{bmatrix}=\begin{bmatrix}n-k\\0\end{bmatrix}=1$

because $\left(1-q^kU\right)\begin{bmatrix}n\\k\end{bmatrix}=\begin{bmatrix}n\\k\end{bmatrix}-q^k\begin{bmatrix}n-1\\k\end{bmatrix}=\begin{bmatrix}n-1\\k-1\end{bmatrix}.$

Lemma (Carlitz [1], Theorem 7)

If $u(n)=\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{1}{q^k}\left(\begin{bmatrix}n\\k\end{bmatrix}-\begin{bmatrix}n\\k-1\end{bmatrix}\right)v(n-2k)$

then

$$v(n)=\sum_{j=0}^{\left\lfloor\frac{n}{2}\right\rfloor}(-1)^j q^{\binom{j}{2}}\begin{bmatrix}n-j\\j\end{bmatrix}u(n-2j).$$

Proof

$$\sum_{j=0}^{\left\lfloor\frac{n}{2}\right\rfloor}(-1)^j q^{\binom{j}{2}}\begin{bmatrix}n-j\\j\end{bmatrix}u(n-2j)=\sum_{j=0}^{\left\lfloor\frac{n}{2}\right\rfloor}(-1)^j q^{\binom{j}{2}}\begin{bmatrix}n-j\\j\end{bmatrix}\sum_{\ell=0}^{\left\lfloor\frac{n-2j}{2}\right\rfloor}\frac{1}{q^\ell}\left(\begin{bmatrix}n-2j\\\ell\end{bmatrix}-\begin{bmatrix}n-2j\\\ell-1\end{bmatrix}\right)v(n-2j-2\ell)$$

$$=\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}v(n-2k)\sum_{j=0}^{k}(-1)^j q^{\binom{j}{2}}\begin{bmatrix}n-j\\j\end{bmatrix}\frac{1}{q^{k-j}}\left(\begin{bmatrix}n-2j\\k-j\end{bmatrix}-\begin{bmatrix}n-2j\\k-j-1\end{bmatrix}\right)$$

$$=v(n)+\sum_{k=1}^{\left\lfloor\frac{n}{2}\right\rfloor}v(n-2k)\frac{1}{q^k}\left(\sum_{j=0}^{k}(-1)^j q^{\binom{j+1}{2}}\begin{bmatrix}k\\j\end{bmatrix}\begin{bmatrix}n-j\\k\end{bmatrix}-\sum_{j=0}^{k}(-1)^j q^{\binom{j+1}{2}}\begin{bmatrix}k-1\\j\end{bmatrix}\begin{bmatrix}n-j\\k-1\end{bmatrix}\right)=v(n)$$

by (2.8).

If we choose $u(n)=x^n$ we get (2.1).

By (2.7) and

$$q^k\begin{bmatrix}n\\k\end{bmatrix}-q^{n-k+1}\begin{bmatrix}n\\k-1\end{bmatrix}=\begin{bmatrix}n\\k\end{bmatrix}-\begin{bmatrix}n\\k-1\end{bmatrix}+\left(q^k-1\right)\begin{bmatrix}n\\k\end{bmatrix}-\left(q^{n-k+1}-1\right)\begin{bmatrix}n\\n-k+1\end{bmatrix}$$

$$=\begin{bmatrix}n\\k\end{bmatrix}-\begin{bmatrix}n\\k-1\end{bmatrix}+\left(q^n-1\right)\begin{bmatrix}n-1\\k-1\end{bmatrix}-\left(q^n-1\right)\begin{bmatrix}n-1\\n-k\end{bmatrix}=\begin{bmatrix}n\\k\end{bmatrix}-\begin{bmatrix}n\\k-1\end{bmatrix}$$

we get

$$\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\begin{bmatrix}n\\k\end{bmatrix}L_{n-2k}(x,q)=\sum\begin{bmatrix}n\\k\end{bmatrix}F_{n-2k}(x,q)-\sum\begin{bmatrix}n\\k\end{bmatrix}q^{n-1-2k}F_{n-2k-2}(x,q)$$

$$=\sum\left(\begin{bmatrix}n\\k\end{bmatrix}-q^{n-2k+1}\begin{bmatrix}n\\k-1\end{bmatrix}\right)F_{n-2k}(x,q)=\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{1}{q^k}\left(\begin{bmatrix}n\\k\end{bmatrix}-\begin{bmatrix}n\\k-1\end{bmatrix}\right)F_{n-2k}(x,q)=x^n$$

and thus also (2.5).

Remark

The simplest recursion for $F_n(x,q)$ for a fixed number x is (cf. [2])

$$F_n(x,q)=xF_{n-1}(x,q)-q^{n-2}xF_{n-3}(x,q)+q^{n-3}F_{n-4}(x,q). \qquad (2.9)$$

Let us also note that comparing coefficients gives the recursion

$$F_n(x,q)=xF_{n-1}(x,q)-\left(\sqrt{q}\right)^{n-2}F_{n-2}\left(\sqrt{q}x,q\right). \qquad (2.10)$$

We will also consider the polynomials

$$P_n(x,q)=F_{2n}\left(\sqrt{x},q\right)=\sum_{k=0}^{n}(-1)^{n-k}q^{\binom{n-k}{2}}\begin{bmatrix}n+k\\2k\end{bmatrix}x^k \qquad (2.11)$$

and

$$Q_n(x,q)=\frac{F_{2n+1}\left(\sqrt{x},q\right)}{\sqrt{x}}=\sum_{k=0}^{n}(-1)^{n-k}q^{\binom{n-k}{2}}\begin{bmatrix}n+k+1\\2k+1\end{bmatrix}x^k. \qquad (2.12)$$

For these polynomials we get

$$x^n=\sum_{k=0}^{n}\frac{1}{q^{n-k}}\left(\begin{bmatrix}2n\\n-k\end{bmatrix}-\begin{bmatrix}2n\\n-k-1\end{bmatrix}\right)P_k(x,q) \qquad (2.13)$$

and

$$x^n=\sum_{k=0}^{n}\frac{1}{q^{n-k}}\left(\begin{bmatrix}2n+1\\n-k\end{bmatrix}-\begin{bmatrix}2n+1\\n-k-1\end{bmatrix}\right)Q_k(x,q). \qquad (2.14)$$

If we define linear functionals L_0 and L_1 by

$$L_0\left(P_n(x,q)\right)=[n=0],\qquad L_1\left(Q_n(x,q)\right)=[n=0], \tag{2.15}$$

then (2.13) and (2.14) give

$$L_0\left(x^n\right)=C_n(q),\qquad L_1\left(x^n\right)=\frac{1+q}{1+q^{n+1}}C_{n+1}(q). \tag{2.16}$$

Analogously let

$$R_n(x,q)=L_{2n}\left(\sqrt{x},q\right)=\sum_{k=0}^{n}(-1)^{n-k}q^{\binom{n-k}{2}}\frac{[2n]}{[n+k]}\begin{bmatrix}n+k\\2k\end{bmatrix}x^k \tag{2.17}$$

and

$$S_n(x,q)=\frac{L_{2n+1}\left(\sqrt{x},q\right)}{\sqrt{x}}=\sum_{k=0}^{n}(-1)^{n-k}q^{\binom{n-k}{2}}\frac{[2n+1]}{[n+k+1]}\begin{bmatrix}n+k+1\\2k+1\end{bmatrix}x^k. \tag{2.18}$$

This implies

$$x^n=\sum_{k=0}^{n}\begin{bmatrix}2n\\n-k\end{bmatrix}R_k(x,q),\qquad x^n=\sum_{k=0}^{n}\begin{bmatrix}2n+1\\n-k\end{bmatrix}S_k(x,q). \tag{2.19}$$

Let $M_0\left(R_n(x,q)\right)=[n=0]$ and $M_1\left(S_n(x,q)\right)=[n=0].$ Then we get

$$M_0\left(x^n\right)=\begin{bmatrix}2n\\n\end{bmatrix},\qquad M_1\left(x^n\right)=\begin{bmatrix}2n+1\\n\end{bmatrix}. \tag{2.20}$$

By comparing coefficients we get

$$R_n(x,q)=Q_n(x,q)-q^{2n-2}Q_{n-2}(x,q). \tag{2.21}$$

3. q-Narayana polynomials as moments

In the following we extend the above results by introducing a new parameter t as in part I.

In [4] we have defined $F_n(x,t)$ by the recursions

$$\begin{aligned} F_{2n}(x,t) &= xF_{2n-1}(x,t) - F_{2n-2}(x,t), \\ F_{2n+1}(x,t) &= xF_{2n}(x,t) - tF_{2n-1}(x,t) \end{aligned} \tag{3.1}$$

and initial values $F_0(x,t)=1$ and $F_1(x,t)=x.$

If L denotes the linear functional defined by $L\left(F_n(x,t)\right)=[n=0],$ then we have

$$L\left(x^{2n}\right)=C_n(t) \text{ and } L\left(x^{2n+1}\right)=0 \tag{3.2}$$

where

$$C_n(t)=\sum_{k\geq 0}\binom{n-1}{k}\binom{n}{k}\frac{1}{k+1}t^k \tag{3.3}$$

is a Narayana polynomial.

Theorem 1

Let $\tau_{2n}(t)=1$ *and* $\tau_{2n+1}(t)=t$ *and define* $F_n(x,t,q)$ *by the recursion*

$$F_n(x,t,q)=xF_{n-1}(x,qt,q)-q^{\left\lfloor \frac{n-1}{2}\right\rfloor}\tau_{n-2}(t)F_{n-2}(x,t,q) \tag{3.4}$$

with initial values $F_0(x,t,q)=1$ *and* $F_1(x,t,q)=x.$ *These polynomials are explicitly given by*

$$\begin{aligned} F_{2n}(x,t,q) &= \sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\sum_{j=0}^{k}\begin{bmatrix} n-j \\ k-j \end{bmatrix}\begin{bmatrix} n-k+j-1 \\ j \end{bmatrix} q^{(n-k+1)j}t^j x^{2n-2k}, \\ F_{2n+1}(x,t,q) &= \sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\sum_{j=0}^{k}\begin{bmatrix} n-j \\ k-j \end{bmatrix}\begin{bmatrix} n-k+j \\ j \end{bmatrix} q^{(n-k+1)j}t^j x^{2n+1-2k}. \end{aligned} \tag{3.5}$$

If L *denotes the linear functional defined by* $L\left(F_n(x,t,q)\right)=[n=0],$ *then we have*

$$L\left(x^{2n}\right)=C_n(t,q) \text{ and } L\left(x^{2n+1}\right)=0 \tag{3.6}$$

where

$$C_n\left(t,q\right)=\frac{1}{[n]}\sum_{j=0}^{n}q^{j^2+j}\begin{bmatrix} n \\ j \end{bmatrix}\begin{bmatrix} n \\ j+1 \end{bmatrix}t^j. \tag{3.7}$$

is a $q-$*Narayana polynomial.*

Proof

To prove (3.5) we have only to show that (3.4) holds. This follows by comparing coefficients.

For $t=1$ we get by using $\begin{bmatrix}-a\\k\end{bmatrix}=\begin{bmatrix}a+k-1\\k\end{bmatrix}\left(-q^{a}\right)^{k}q^{-\binom{k}{2}}$ and the $q-$Vandermonde formula

$$\sum_{j=0}^{k}\begin{bmatrix}n-j\\k-j\end{bmatrix}\begin{bmatrix}n-k+j-1\\j\end{bmatrix}q^{(n-k+1)j}$$
$$=\sum_{j=0}^{k}\begin{bmatrix}j-n+k-j-1\\k-j\end{bmatrix}(-q^{n-j})^{k-j}q^{-\binom{k-j}{2}}\begin{bmatrix}-n+k-j+1+j-1\\j\end{bmatrix}q^{-\binom{j}{2}}\left(-q^{n-k+j-1}\right)^{j}q^{(n-k+1)j}$$
$$=(-1)^{k}q^{2nk-k^2-\binom{k}{2}}\sum_{j=0}^{k}\begin{bmatrix}-n+k-1\\k-j\end{bmatrix}\begin{bmatrix}-n+k\\j\end{bmatrix}q^{(k-j)(k-n-j)}=\begin{bmatrix}-2n+2k-1\\k\end{bmatrix}(-1)^{k}q^{2nk-k^2-\binom{k}{2}}=\begin{bmatrix}2n-k\\k\end{bmatrix}$$

and analogously

$$\sum_{j=0}^{k}\begin{bmatrix}n-j\\k-j\end{bmatrix}\begin{bmatrix}n-k+j\\j\end{bmatrix}q^{(n-k+1)j}=\begin{bmatrix}2n+1-k\\k\end{bmatrix}.$$

This implies that

$$F_n(x,1,q)=F_n(x,q). \tag{3.8}$$

To prove (3.6) let $a(n,k,t,q)$ be the uniquely defined numbers such that

$$\sum_{k=0}^{n}a(n,k,t,q)F_k(x,t,q)=x^n. \tag{3.9}$$

By (3.4) we get

$$a(n,k,t,q)=a(n-1,k-1,qt,q)+q^{\left\lfloor\frac{k+1}{2}\right\rfloor}\tau_k(t)a(n-1,k+1,qt,q) \tag{3.10}$$

with initial values $a(n,-1,t,q)=0$ and $a(0,k,t,q)=[k=0]$.

This implies that

$$\begin{aligned}a(2n+1,2k+1,t,q)&=\frac{1}{q^{n-k}}\sum_{j=0}^{n-k}q^{j^2+(k+1)j}\left(\begin{bmatrix}n\\j\end{bmatrix}\begin{bmatrix}n+1\\j+k+1\end{bmatrix}-\begin{bmatrix}n+1\\j\end{bmatrix}\begin{bmatrix}n\\j+k+1\end{bmatrix}\right)t^j,\\ a(2n,2k,t,q)&=\frac{1}{q^{n-k}}\sum_{j=0}^{n-k}q^{j^2+(k+1)j}\left(\begin{bmatrix}n-1\\j\end{bmatrix}\begin{bmatrix}n+1\\j+k+1\end{bmatrix}-\begin{bmatrix}n\\j\end{bmatrix}\begin{bmatrix}n\\j+k+1\end{bmatrix}\right)t^j\end{aligned} \tag{3.11}$$

and $a(n,k,t,q)=0$ else.

In order to show this we must verify that

$$a(2n,2k,t,q)=a(2n-1,2k-1,qt,q)+q^{k}a(2n-1,2k+1,qt,q) \tag{3.12}$$

and

$$a(2n+1,2k+1,t,q)=a(2n,2k,qt,q)+q^{k+1}ta(2n,2k+2,qt,q). \tag{3.13}$$

(3.12) follows from

$$a(2n-1,2k-1,qt,q)+q^k a(2n-1,2k+1,qt,q)=\frac{1}{q^{n-k}}\sum_{j=0}^{n-k} q^{j^2+kj}\left(\begin{bmatrix}n-1\\ j\end{bmatrix}\begin{bmatrix}n\\ j+k\end{bmatrix}-\begin{bmatrix}n\\ j\end{bmatrix}\begin{bmatrix}n-1\\ j+k\end{bmatrix}\right)q^j t^j$$

$$+\frac{q^k}{q^{n-k-1}}\sum_{j=0}^{n-k} q^{j^2+kj+j}\left(\begin{bmatrix}n-1\\ j\end{bmatrix}\begin{bmatrix}n\\ j+k+1\end{bmatrix}-\begin{bmatrix}n\\ j\end{bmatrix}\begin{bmatrix}n-1\\ j+k+1\end{bmatrix}\right)q^j t^j$$

$$=\frac{1}{q^{n-k}}\sum_{j=0}^{n-k} q^{j^2+kj+j}\left(\begin{bmatrix}n-1\\ j\end{bmatrix}\begin{bmatrix}n+1\\ j+k+1\end{bmatrix}-\begin{bmatrix}n\\ j\end{bmatrix}\begin{bmatrix}n\\ j+k+1\end{bmatrix}\right)t^j=a(2n,2k,t,q).$$

(3.13) follows from

$$a(2n,2k,qt,q)+q^{k+1}ta(2n,2k+2,qt,q)=\frac{1}{q^{n-k}}\sum_{j=0}^{n-k} q^{j^2+(k+1)j}\left(\begin{bmatrix}n-1\\ j\end{bmatrix}\begin{bmatrix}n+1\\ j+k+1\end{bmatrix}-\begin{bmatrix}n\\ j\end{bmatrix}\begin{bmatrix}n\\ j+k+1\end{bmatrix}\right)q^j t^j$$

$$+\frac{q^{k+1}}{q^{n-k-1}}\sum_{j=0}^{n-k} q^{j^2+(k+2)j}\left(\begin{bmatrix}n-1\\ j\end{bmatrix}\begin{bmatrix}n+1\\ j+k+2\end{bmatrix}-\begin{bmatrix}n\\ j\end{bmatrix}\begin{bmatrix}n\\ j+k+2\end{bmatrix}\right)q^j t^{j+1}$$

$$=\frac{1}{q^{n-k}}\sum_{j=0}^{n-k} q^{j^2+kj+2j}\left(\begin{bmatrix}n-1\\ j\end{bmatrix}\begin{bmatrix}n+1\\ j+k+1\end{bmatrix}-\begin{bmatrix}n\\ j\end{bmatrix}\begin{bmatrix}n\\ j+k+1\end{bmatrix}\right)t^j+\frac{1}{q^{n-k}}\sum_{j=0}^{n-k} q^{j^2+kj+j}\left(\begin{bmatrix}n-1\\ j-1\end{bmatrix}\begin{bmatrix}n+1\\ j+k+1\end{bmatrix}-\begin{bmatrix}n\\ j-1\end{bmatrix}\begin{bmatrix}n\\ j+k+1\end{bmatrix}\right.$$

$$=\frac{1}{q^{n-k}}\sum_{j=0}^{n-k} q^{j^2+kj+j}\left(\begin{bmatrix}n\\ j\end{bmatrix}\begin{bmatrix}n+1\\ j+k+1\end{bmatrix}-\begin{bmatrix}n+1\\ j\end{bmatrix}\begin{bmatrix}n\\ j+k+1\end{bmatrix}\right)t^j=a(2n+1,2k+1,t,q)$$

As special case we get

$$a(2n,0,t,q)=\frac{1}{q^n}\sum_{j=0}^{n} q^{j^2+j}\left(\begin{bmatrix}n-1\\ j\end{bmatrix}\begin{bmatrix}n+1\\ j+1\end{bmatrix}-\begin{bmatrix}n\\ j\end{bmatrix}\begin{bmatrix}n\\ j+1\end{bmatrix}\right)t^j=C_n(t,q). \qquad (3.14)$$

$C_n(t,q)$ is related to the $q-$Catalan numbers $c_n(\lambda;q)$ of J. Fürlinger and J. Hofbauer [5]. They have shown that

$$C_n(1,q)=C_n(q)=\frac{1}{[n+1]}\begin{bmatrix}2n\\ n\end{bmatrix}. \qquad (3.15)$$

This result follows again from Theorem 1 because of (3.8) and (2.3).

Remark

Let us also consider the polynomials $P_n(x,t,q)=F_{2n}\left(\sqrt{x},t,q\right)$ and

$$Q_n\left(x,t,q\right)=\frac{F_{2n+1}\left(\sqrt{x},t,q\right)}{\sqrt{x}}.$$

Corollary 1.1

Let

$$Q_n(x,t,q)=\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\sum_{j=0}^{k}\begin{bmatrix} n-j \\ k-j \end{bmatrix}\begin{bmatrix} n-k+j \\ j \end{bmatrix}\left(q^{n-k+1}t\right)^j x^{n-k} \tag{3.16}$$

and

$$B_{n,k}(t,q)=\frac{1}{q^{n-k}}\sum_{j=0}^{n-k}q^{j^2+j+kj}\left(\begin{bmatrix} n \\ j \end{bmatrix}\begin{bmatrix} n+1 \\ k+j+1 \end{bmatrix}-\begin{bmatrix} n+1 \\ j \end{bmatrix}\begin{bmatrix} n \\ k+j+1 \end{bmatrix}\right)t^j. \tag{3.17}$$

Then

$$\sum_{k=0}^{n}B_{n,k}(t,q)Q_k(x,t,q)=x^n \tag{3.18}$$

with

$$B_{n,0}(t,q)=\frac{1}{[n+1]}\sum_{j=0}^{n}q^{j^2}\begin{bmatrix} n+1 \\ j \end{bmatrix}\begin{bmatrix} n+1 \\ j+1 \end{bmatrix}t^j=C_{n+1}\left(\frac{t}{q},q\right). \tag{3.19}$$

Note that by [5] $C_n\left(\frac{1}{q},q\right)=\frac{1+q}{1+q^n}C_n(q)$ in accordance with (2.16).

$B_{n,k}(t,q)$ can also be written as $B_{n,k}(t,q)=\frac{[k+1]}{[n+1]}\sum_{j=0}^{n-k}q^{j^2+kj}\begin{bmatrix} n+1 \\ j \end{bmatrix}\begin{bmatrix} n+1 \\ k+j+1 \end{bmatrix}t^j$.

For $t=1$ $B_{n,k}(t,q)$ reduces to $B_{n,k}(1,q)=\frac{1}{q^{n-k}}\left(\begin{bmatrix} 2n+1 \\ n-k \end{bmatrix}-\begin{bmatrix} 2n+1 \\ n-k-1 \end{bmatrix}\right)$.

Corollary 1.2

Let

$$P_n(x,t,q)=\sum_{k=0}^{n}(-1)^{n-k}q^{\binom{n-k}{2}}\sum_{j=0}^{n}\begin{bmatrix} n-j \\ k \end{bmatrix}\begin{bmatrix} k+j-1 \\ j \end{bmatrix}q^{j(k+1)}t^j x^k \tag{3.20}$$

and

$$A_{n,k}(t,q)=\frac{1}{q^{n-k}}\sum_{j=0}^{n-k}q^{j^2+(k+1)j}\left(\begin{bmatrix} n-1 \\ j \end{bmatrix}\begin{bmatrix} n+1 \\ j+k+1 \end{bmatrix}-\begin{bmatrix} n \\ j \end{bmatrix}\begin{bmatrix} n \\ j+k+1 \end{bmatrix}\right)t^j. \tag{3.21}$$

Then

$$\sum_{k=0}^{n}A_{n,k}(t,q)P_k(x,t,q)=x^n \tag{3.22}$$

with

$$A_{n,0}(t,q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} n \\ j+1 \end{bmatrix} \frac{1}{[n]} q^{j^2+j} t^j = C_n(t,q). \tag{3.23}$$

The first terms of $A_{n,k}(t,q)$ are

1			
1	1		
$1+q^2 t$	$1+q+q^2 t$	1	
$1+q^2 t+q^3 t+q^4 t+q^6 t^2$	$1+q+q^2+q^2 t+q^3 t+2q^4 t+q^5 t+q^6 t^2$	$1+q+q^2+q^3 t+q^4 t$	1

2.1. q-Narayana polynomials of type B

In [4] we have seen that the orthogonal polynomials $L_n(x,t)$ whose moments are the Narayana polynomials $M_n(t) = \sum_{k=0}^{n} \binom{n}{k}^2 t^k$ of type B, satisfy the recurrence

$$L_n(x,t) = xL_{n-1}(x,t) - \tau_{n-2}(t) L_{n-2}(x,t) \tag{3.24}$$

with initial values $L_0(x,t) = 1$ and $L_1(x,t) = x.$

Here we have

$$\tau_0(t) = 1+t,$$
$$\tau_{2n}(t) = \frac{1+t^{n+1}}{1+t^n} \; for \; n > 0,$$
$$\tau_{2n+1}(t) = \frac{t\left(1+t^n\right)}{1+t^{n+1}}.$$

We now show that there exists a natural $q-$analogue of $L_n(x,t,q)$ with $L_n(x,t,1) = L_n(x,t)$ and which for $t=1$ reduces to the $q-$Lucas polynomials:

Theorem 2

Let

$$\begin{aligned} &\tau_0(t,q) = 1+qt, \\ &\tau_{2n}(t,q) = q^n \frac{1+q^{n+1}t^{n+1}}{1+q^n t^n}, \\ &\tau_{2n+1}(t,q) = q^{n+1} t \frac{1+t^n}{1+t^{n+1}} \end{aligned} \tag{3.25}$$

and define $L_n(x,t,q)$ *by*

$$L_n(x,t,q)=xL_{n-1}(x,qt,q)-\tau_{n-2}(t,q)L_{n-2}(x,t,q) \tag{3.26}$$

with $L_0(x,t,q)=1$ *and* $L_1(x,t,q)=x.$

Let M *denote the linear functional defined by* $M\left(L_n(x,t,q)\right)=[n=0],$ *then we get*

$$M\left(x^{2n}\right)=M_n(t,q)=\sum_{j=0}^{n}q^{j^2}\begin{bmatrix}n\\j\end{bmatrix}^2 t^j. \tag{3.27}$$

Proof

Let $a(n,k,t,q)$ satisfy

$$a(n,k,t,q)=a(n-1,k-1,qt,q)+\tau_k(t,q)a(n-1,k+1,qt,q) \tag{3.28}$$

with $a(n,-1,t,q)=0$ and $a(0,k,t,q)=[k=0].$

Then (3.26) implies that

$$\sum_{k=0}^{n}a(n,k,t,q)L_k(x,t,q)=x^n. \tag{3.29}$$

By induction it is easy to verify that

$$\begin{aligned}
a(2n,2k,t,q)&=\sum_{j=0}^{n-k}q^{j(j+k)}\begin{bmatrix}n\\j\end{bmatrix}\begin{bmatrix}n\\j+k\end{bmatrix}t^j,\\
a(2n+1,2k+1,t,q)&=\frac{1}{1+t^{k+1}}\sum_{j=0}^{n-k}\begin{bmatrix}n\\k+j\end{bmatrix}\begin{bmatrix}n+1\\j\end{bmatrix}\left(q^{j(j+k)}t^j+q^{(n+1-j)(n-j-k)}t^{n+1-j}\right)\\
&=\frac{1}{1+t^{k+1}}\left(\sum_{j=0}^{n-k}\begin{bmatrix}n\\k+j\end{bmatrix}\begin{bmatrix}n+1\\j\end{bmatrix}q^{j(j+k)}t^j+\sum_{j=k+1}^{n+1}\begin{bmatrix}n\\j-k-1\end{bmatrix}\begin{bmatrix}n+1\\j\end{bmatrix}q^{j(j-k-1)}t^j\right)
\end{aligned} \tag{3.30}$$

and $a(n,k,t,q)=0$ else.

If $M\left(L_n(x,t,q)\right)=[n=0]$ then (3.29) implies

$$M\left(x^{2n}\right)=a(2n,0,t,q)=\sum_{j=0}^{n}q^{j^2}\begin{bmatrix}n\\j\end{bmatrix}^2 t^j=M_n(t,q).$$

By $q-$Vandermonde we see that $a(2n,2k,1,q)=\begin{bmatrix}2n\\n-k\end{bmatrix}$ and $a(2n+1,2k+1,1,q)=\begin{bmatrix}2n+1\\n-k\end{bmatrix}.$

Comparing with (2.19) we conclude that $L_n(x,1,q)$ is a $q-$Lucas polynomial. Further it is clear that $L_n(x,t,1)=L_n(x,t).$

Let $R_n(x,t,q)=L_{2n}\left(\sqrt{x},t,q\right).$

In order to get a formula for $R_n(x,t,q)$ observe that

$$B_{n,k}(t,q)=D_{n,k}(t,q)-q^{2k+2}tD_{n,k+2}(t,q), \tag{3.31}$$

which is equivalent with the easily verified identity

$$\frac{[k+1]}{[n+1]}\begin{bmatrix}n+1\\j\end{bmatrix}\begin{bmatrix}n+1\\k+j+1\end{bmatrix}=\begin{bmatrix}n\\j\end{bmatrix}\begin{bmatrix}n\\k+j\end{bmatrix}-q^{k+1}\begin{bmatrix}n\\j-1\end{bmatrix}\begin{bmatrix}n\\k+j+1\end{bmatrix}.$$

By (3.18) this implies

$$\sum_{k=0}^{n}D_{n,k}(t,q)\left(Q_k(x,t,q)-q^{2k-2}tQ_{k-2}(x,t,q)\right)$$
$$=\sum_{k=0}^{n}D_{n,k}(t,q)Q_k(x,t,q)-\sum_{k=0}^{n}q^{2k+2}tD_{n,k+2}(t,q)Q_k(x,t,q)=\sum_{k=0}^{n}B_{n,k}(t,q)Q_k(x,t,q)=x^n.$$

Comparing with (3.37) we see that

$$R_n(x,t,q)=Q_n(x,t,q)-q^{2n-2}tQ_{n-2}(x,t,q) \tag{3.32}$$

if we let $Q_{-2}(x,t,q)=Q_{-1}(x,t,q)=0.$

The first terms of $\left(R_n(x,t,q)\right)_{n\geq 0}$ are

$\{1,\ -1-q\,t+x,\ q+q^3\,t^2-x-q\,x-q^2\,t\,x-q^3\,t\,x+x^2,$
$-q^3-q^6\,t^3+q\,x+q^2\,x+q^3\,x+q^3\,t\,x+q^4\,t\,x+q^5\,t\,x+q^5\,t^2\,x+$
$q^6\,t^2\,x+q^7\,t^2\,x-x^2-q\,x^2-q^2\,x^2-q^3\,t\,x^2-q^4\,t\,x^2-q^5\,t\,x^2+x^3\}$

From (3.32) and (3.16) we get the formula

$$R_n(x,t,q)=\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}c(n,k,t)x^{n-k} \tag{3.33}$$

with

$$c(n,k,t)=\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}q^{(n+1-k)j}\frac{\begin{bmatrix}n+j-k-1\\j\end{bmatrix}}{\begin{bmatrix}n-1\\j\end{bmatrix}}t^j \quad \text{for } k<n,$$
$$c(n,n,t)=1+q^nt^n. \tag{3.34}$$

This can also be written as

$$R_k(x,t,q)=(-1)^kq^{\binom{k}{2}}\left(1+q^kt^k\right)+\sum_{\ell=1}^{k}(-1)^{k-\ell}q^{\binom{k-\ell}{2}}\begin{bmatrix}k\\\ell\end{bmatrix}x^\ell\sum_{j=0}^{k-\ell}q^{(\ell+1)j}\begin{bmatrix}k-\ell\\j\end{bmatrix}\frac{\begin{bmatrix}\ell+j-1\\j\end{bmatrix}}{\begin{bmatrix}k-1\\j\end{bmatrix}}t^j. \tag{3.35}$$

By (3.29) and (3.30) we get

Corollary 2 .1

Let

$$D_{n,k}(t,q)=a(2n,2k,t,q)=\sum_{j=0}^{n-k} q^{j(j+k)}\begin{bmatrix} n \\ j \end{bmatrix}\begin{bmatrix} n \\ k+j \end{bmatrix} t^j. \qquad (3.36)$$

Then

$$\sum_{k=0}^{n} D_{n,k}(t,q)R_k(x,t,q)=x^n. \qquad (3.37)$$

Let M_0 *be the linear functional defined by* $M_0\left(R_n(x,t,q)\right)=[n=0].$ *Then*

$$M_0\left(x^n\right)=D_{n,0}(t,q)=B_n(t,q)=\sum_{j=0}^{n} q^{j^2}\begin{bmatrix} n \\ j \end{bmatrix}^2 t^j. \qquad (3.38)$$

Let now $S_n(x,t,q)=\dfrac{L_{2n+1}\left(\sqrt{x},t,q\right)}{\sqrt{x}}.$

From (3.26) we get for $n>0$

$$S_n(x,t,q)=\frac{1}{x\left(1+t^n\right)}\left(\left(1+t^n\right)R_{n+1}\left(x,\frac{t}{q},q\right)+q^n\left(1+t^{n+1}\right)R_n\left(x,\frac{t}{q},q\right)\right). \qquad (3.39)$$

Corollary 2.2

Let

$$E_{n,k}(t,q)=a(2n+1,2k+1,t,q)=\frac{1}{1+t^{k+1}}\sum_{j=0}^{n-k}\begin{bmatrix} n \\ k+j \end{bmatrix}\begin{bmatrix} n+1 \\ j \end{bmatrix}\left(q^{j(j+k)}t^j+q^{(n+1-j)(n-j-k)}t^{n+1-j}\right).$$

(3.40)

Then

$$\sum_{k=0}^{n} E_{n,k}(t,q)S_k(x,t,q)=x^n. \qquad (3.41)$$

Let M_1 *be the linear functional defined by* $M_1\left(S_n(x,t,q)\right)=[n=0].$ *Then*

$$M_1\left(x^n\right)=E_{n,0}(t,q)=\frac{1}{1+t}\sum_{j=0}^{n+1} q^{j^2-j}\begin{bmatrix} n+1 \\ j \end{bmatrix}^2 t^j=\frac{M_{n+1}\left(\frac{t}{q},q\right)}{1+t}. \qquad (3.42)$$

Remark

For the numbers $D_{n,k}(t,q)$ there exists an analogue of the Catalan-Stieltjes matrix for orthogonal polynomials:

$$D_{n,0}(t,q)=(1+q^n t)D_{n-1,0}(qt,q)+q\left(1+q^n\right)tD_{n-1,1}(qt,q) \qquad (3.43)$$

and

$$D_{n,k}(t,q)=D_{n-1,k-1}(qt,q)+q^k\left(1+q^n t\right)D_{n-1,k}(qt,q)+q^{n+2k+1}tD_{n-1,k+1}(qt,q). \qquad (3.44)$$

Let us mention some curious conjectures: Let $\Delta_{q,t}f(t)=\dfrac{f(t)-f(qt)}{(1-q)t}$ be the $q-$differential operator with respect to the variable $t.$ Then

$$\sum_{k=0}^{n}\Delta_{q,t}^{m}\left(D_{n,k}(x,t)\right)R_k(x,q^m t,q)=q^{m^2}[m]!\begin{bmatrix}n\\m\end{bmatrix}\sum_{j=0}^{n-m}q^{mj}c_{n-m-j}\left(q^{2j}t,m,q\right)x^j. \qquad (3.45)$$

Here $c_n(t,m,q)=0$ for $n<0,\quad c_0(t,m,q)=1$ and for $n>0$

$$c_n(t,m,q)=\sum_{k=0}^{n-1}\begin{bmatrix}n-1\\k\end{bmatrix}\begin{bmatrix}n+m\\k+m\end{bmatrix}\frac{[m]}{[n+m]}q^{2km+k^2+k}t^k. \qquad (3.46)$$

For $m=1$ we get

$$c_n(t,1,q)=\sum_{k=0}^{n-2}\begin{bmatrix}n-2\\k\end{bmatrix}\begin{bmatrix}n\\k+1\end{bmatrix}\frac{1}{[n]}q^{2k+k^2+k}t^k=C_{n-1}\left(q^2t\right)$$

Thus

$$\sum_{k=0}^{n}\Delta_{q,t}\left(D_{n,k}(x,t)\right)R_k(x,qt,q)=\begin{bmatrix}n\\1\end{bmatrix}\sum_{j=0}^{n-1}q^{j+1}C_{n-j-1}\left(q^{2j+2}t,q\right)x^j. \qquad (3.47)$$

It should be noted that the numbers $c_n(t,m,1)=\sum_{k=0}^{n-m-1}\binom{n-1}{k}\binom{n+m}{k+m}\frac{m}{n+m}t^k$

are the coefficients of the powers $C(x,t)^m$ of the generating function

$C(x,t)=\sum_{n\geq 0}C_n(t)x^n$ of the Narayana polynomials (cf. [4]).

Other such identities are

$$\sum_{k=0}^{n}\Delta_{q,t}A_{n,k}(t,q)P_k(x,qt,q)=\sum_{j=2}^{n}q^j\begin{bmatrix}j-1\\1\end{bmatrix}C_{n-j}\left(q^{2j}t,q\right)x^{j-1}. \qquad (3.48)$$

$$\sum_{k=0}^{n}\Delta_{q,t}B_{n,k}(t,q)Q_k(x,qt,q)=\sum_{j=1}^{n}q^j\begin{bmatrix}j\\1\end{bmatrix}C_{n-j}\left(q^{2j+1}t,q\right)x^{j-1}. \qquad (3.49)$$

$$\sum_{k=0}^{n}\Delta_{q,t}^{m}\left(A_{n,k}(x,t)\right)P_k(x,q^m t,q)=q^{m^2+m}\prod_{j=1}^{m-1}[n-j]\sum_{j=0}^{n-m}q^{mj}\left[j+1\right]c_{n-m-1-j}\left(q^{2j+2}t,m,q\right)x^{j+1}. \tag{3.50}$$

$$\sum_{k=0}^{n}\Delta_{q,t}^{m}\left(B_{n,k}(x,t)\right)Q_k(x,q^m t,q)=q^{m^2}\prod_{j=1}^{m-1}[n+1-j]\sum_{j=0}^{n-m}q^{mj}\left[j+1\right]c_{n-m-j}\left(q^{2j+1}t,m,q\right)x^{j}. \tag{3.51}$$

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