An interesting class of Hankel determinants

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Abstract

For small r the Hankel determinants $d_r(n)$ of the sequence $\left(\binom{2n+r}{n}\right)_{n\geq 0}$ are easy to guess and show an interesting modular pattern. For arbitrary r and n no closed formulae are known, but for each positive integer r the special values $d_r(rn)$, $d_r(rn+1)$, and $d_r(rn+\lfloor\frac{r+1}{2}\rfloor)$ have nice values which will be proved in this paper.

0 Introduction

Let $(a_n)_{n\geq 0}$ be a sequence of real numbers with $a_0 = 1$. For each n consider the Hankel determinant

$$H_n = \det(a_{i+j})_{i,j=0}^{n-1}. \tag{1}$$

We are interested in the sequence $(H_n)_{n\geq 0}$ for the sequences $a_{n,r} = \binom{2n+r}{n}$ for some $r \in \mathbb{N}$. For $n = 0$ we let $H_0 = 1$.

Let

$$d_r(n) = \det\left(\binom{2i+2j+r}{i+j}\right)_{i,j=0}^{n-1}. \tag{2}$$

For $r = 0$ and $r = 1$ these determinants are well known and satisfy $d_0(n) = 2^{n-1}$ and $d_1(n) = 1$ for $n > 0$. Egecioglu, Redmond, and Ryavec [3] computed $d_2(n)$ and $d_3(n)$ and stated some conjectures for $r > 3$.

Many of these determinants are easy to guess and show an interesting modular

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pattern. For example

$$(d_0(n))_{n\geq 0} = (1,1,2,2^2,2^3,\dots), \tag{3}$$

$$(d_1(n))_{n\geq 0} = (1,1,1,1,1,\dots), \tag{4}$$

$$(d_2(n))_{n\geq 0} = (1,1,-1,-1,1,1,-1,-1,\dots), \tag{5}$$

$$(d_3(n))_{n\geq 0} = (1,1,-4,3,3,-8,5,5,-12,7,7,-16,\dots), \tag{6}$$

$$(d_4(n))_{n\geq 0} = (1,1,-8,8,1,1,-16,16,1,1,-24,24,\dots), \tag{7}$$

$$(d_5(n))_{n\geq 0} = (1,1,-13,-16,61,9,9,-178,-64,370,25,25,-695,-144,1127,\dots) \tag{8}$$

These and other computations suggest the following facts:

$$d_{2k+1}((2k+1)n) = d_{2k+1}((2k+1)n+1) = (2n+1)^k, \tag{9}$$

$$d_{2k+1}((2k+1)n+k+1) = (-1)^{\binom{k+1}{2}}4^k(n+1)^k, \tag{10}$$

$$d_{2k}(2kn) = d_{2k}(2kn+1) = (-1)^{kn}, \tag{11}$$

$$d_{2k}(2kn+k) = -d_{2k}(2kn+k+1) = (-1)^{kn+\binom{k}{2}}4^{k-1}(n+1)^{k-1}. \tag{12}$$

The purpose of this paper is to prove these conjectures. These methods seem to extend to the Hankel determinants of the sequences $\left(\binom{2n+r}{n-s}\right)_{n\geq 0}$, but we do not compute these here.

In Sections 1 and 2 we review some well-known facts from the theory of Hankel determinants. In particular we compute $d_1(n)$. In Sections 3 and 4 we introduce the matrices $\gamma^{(i)}$, α_n, and β_n, which serve as the basis of our method. In Section 5 we relate these matrices to $d_r(n)$, and in Sections 6 and 7 we use this information to compute $d_r(n)$ in the aforementioned seven cases.

1 Some background material

Let us first recall some well-known facts about Hankel determinants (cf. e.g. [1]). If $d_n = \det(a_{i+j})_{i,j=0}^{n-1} \neq 0$ for each n we can define the polynomials

$$p_n(x) = \frac{1}{d_n}\det\begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} & 1 \\ a_1 & a_2 & \cdots & a_n & x \\ a_2 & a_3 & \cdots & a_{n+1} & x^2 \\ \vdots & & & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n-1} & x^n \end{pmatrix}. \tag{13}$$

If we define a linear functional L on the polynomials by $L(x^n) = a_n$ then $L(p_n p_m) = 0$ for $n \neq m$ and $L(p_n^2) \neq 0$ (orthogonality).

By Favard's Theorem there exist s_n and t_n such that

$$p_n(x) = (x - s_{n-1})p_{n-1}(x) - t_{n-2}p_{n-2}(x). \tag{14}$$

For arbitrary s_n and t_n define numbers $a_n(j)$ by

$$\begin{aligned} a_0(j) &= [j = 0], \\ a_n(0) &= s_0 a_{n-1}(0) + t_0 a_{n-1}(1), \\ a_n(j) &= a_{n-1}(j-1) + s_j a_{n-1}(j) + t_j a_{n-1}(j+1). \end{aligned} \tag{15}$$

These numbers satisfy

$$\sum_{j=0}^{n} a_n(j) p_j(x) = x^n. \tag{16}$$

Let $A_n = (a_i(j))_{i,j=0}^{n-1}$ and D_n be the diagonal matrix with entries $d(i,i) = \prod_{j=0}^{i-1} t_j$. Then we get

$$(a_{i+j}(0))_{i,j=0}^{n-1} = A_n D_n A_n^{\top} \tag{17}$$

and

$$\det (a_{i+j}(0))_{i,j=0}^{n-1} = \prod_{i=1}^{n-1} \prod_{j=0}^{i-1} t_j. \tag{18}$$

If we start with the sequence $(a_n)_{n \geq 0}$ and guess s_n and t_n and if we also can guess $a_n(j)$ and show that $a_n(0) = a_n$ then all our guesses are correct and the Hankel determinant is given by the above formula.

There is a well-known equivalence with continued fractions, so-called J-fractions:

$$\sum_{n \geq 0} a_n x^n = \cfrac{1}{1 - s_0 x - \cfrac{t_0 x^2}{1 - s_1 x - \cfrac{t_1 x^2}{1 - \ddots}}}. \tag{19}$$

For some sequences this gives a simpler approach to Hankel determinants.

As is well known Hankel determinants are intimately connected with the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$. Consider for example the aerated sequence of Catalan numbers $(c_n) = (1, 0, 1, 0, 2, 0, 5, 0, 14, 0, \dots)$ defined by $c_{2n} = C_n$ and $c_{2n+1} = 0$. Since the generating function of the Catalan numbers

$$C(x) = \sum_{n \geq 0} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x} \tag{20}$$

satisfies

$$C(x) = 1 + xC(x)^2, \tag{21}$$

we get

$$C(x) = \frac{1}{1 - xC(x)} \tag{22}$$

and

$$C(x^2) = \frac{1}{1 - x^2 C(x^2)} = \cfrac{1}{1 - \cfrac{x^2}{1 - \cfrac{x^2}{1 - \ddots}}} \tag{23}$$

and therefore

$$\det(c_{i+j})_{i,j=0}^{n-1} = 1. \tag{24}$$

From $C(x) = 1 + xC(x)^2$ we get $C(x)^2 = 1 + 2xC(x)^2 + x^2 C(x)^4$ or

$$C(x)^2 = \frac{1}{1 - 2x - x^2 C(x)^2} = \cfrac{1}{1 - 2x - \cfrac{x^2}{1 - 2x - \cfrac{x^2}{1 - 2x - \ddots}}}. \tag{25}$$

The generating function of the central binomial coefficients $B_n = \binom{2n}{n}$ is

$$B(x) = \sum_{n \geq 0} B_n x^n = \frac{1}{\sqrt{1 - 4x}} = \frac{1}{1 - 2xC(x)} = \frac{1}{1 - 2x - 2x^2 C(x)^2}. \tag{26}$$

Therefore by (25) we get the J-fraction

$$B(x) = \frac{1}{1 - 2x - 2x^2 C(x)^2} = \cfrac{1}{1 - 2x - \cfrac{2x^2}{1 - 2x - \cfrac{x^2}{1 - 2x - \cfrac{x^2}{1 - 2x - \ddots}}}}. \tag{27}$$

Thus the corresponding numbers t_n are given by $t_0 = 2$ and $t_n = 1$ for $n > 0$ which implies $d_0(n) = 2^{n-1}$ for $n \geq 1$.

Let us also consider the aerated sequence (b_n) with $b_{2n} = B_n$ and $b_{2n+1} = 0$. Here we get

$$b(x) = B(x^2) = \frac{1}{1 - 2x^2 C(x)^2} = \cfrac{1}{1 - \cfrac{2x^2}{1 - \cfrac{x^2}{1 - \cfrac{x^2}{1 - \ddots}}}}. \tag{28}$$

In this case $s_n = 0$, $t_0 = 2$, and $t_n = 1$ for $n > 0$. Here we also get $\det(b_{i+j})_{i,j=0}^{n-1} = 2^{n-1}$ for $n > 0$. The corresponding orthogonal polynomials satisfy $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = xp_1(x) - 2$ and $p_n(x) = xp_{n-1}(x) - p_{n-2}(x)$ for $n > 2$. The first terms are $1, x, x^2 - 2, x^3 - 3x, \ldots$.

Now recall that the Lucas polynomials

$$L_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} \frac{n}{n-k} x^{n-2k} \tag{29}$$

for $n > 0$ satisfy $L_n(x) = xL_{n-1}(x) - L_{n-2}(x)$ with initial values $L_0(x) = 2$ and $L_1(x) = x$. The first terms are $2, x, x^2 - 2, x^3 - 3x, \ldots$. Thus $p_n(x) = \bar{L}_n(x)$, where $\bar{L}_n(x) = L_n(x)$ for $n > 0$ and $\bar{L}_0(x) = 1$.

For the numbers $a_n(j)$ we get

$$a_{2n}(2j) = \binom{2n}{n-j}, \tag{30}$$

$$a_{2n+1}(2j+1) = \binom{2n+1}{n-j}, \tag{31}$$

and $a_n(j) = 0$ else. Equivalently $a_n(n-2j) = \binom{n}{j}$ and $a_n(k) = 0$ else.

For the proof it suffices to verify (15) which reduces to the trivial identities $\binom{2n}{n} = 2\binom{2n-1}{n-1}$, $\binom{2n}{n-j} = \binom{2n-1}{n-j} + \binom{2n-1}{n-1-j}$, and $\binom{2n+1}{n-j} = \binom{2n}{n-j} + \binom{2n}{n-1-j}$. Identity (16) reduces to

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} \bar{L}_{n-2k} = x^n. \tag{32}$$

2 Some well-known applications of these methods

Now let us consider

$$d_1(n) = \det \binom{2i+2j+1}{i+j}. \tag{33}$$

The generating function of the sequence $\binom{2n+1}{n}$ is

$$\sum_{n \geq 0} \binom{2n+1}{n} x^n = \frac{1}{2} \sum_{n \geq 0} \binom{2n+2}{n+1} x^n = \frac{1}{2x} \left(\frac{1}{\sqrt{1-4x}} - 1 \right) = \frac{C(x)}{\sqrt{1-4x}}. \tag{34}$$

Now we have

$$\begin{aligned} \sqrt{1-4x} &= 1 - 2xC(x) = (C(x) - xC(x)^2) - 2xC(x) = C(x)(1 - 2x - xC(x)) \\ &= C(x)(1 - 2x - x(1 + xC(x)^2)) = C(x)(1 - 3x - x^2C(x)^2). \end{aligned} \tag{35}$$

Therefore

$$\frac{C(x)}{\sqrt{1-4x}} = \frac{1}{1-3x-x^2C(x)^2} = \cfrac{1}{1-3x-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-2x-\ddots}}}}. \tag{36}$$

The corresponding sequences s_n, t_n are $s_0 = 3$, $s_n = 2$ for $n > 0$ and $t_n = 1$. Thus $d_1(n) = 1$. The corresponding $a_i(j)$ are $a_i(j) = \binom{2i+1}{i-j}$.

To prove this we must verify (15) which reduces to

$$\binom{1}{-j} = [j = 0], \tag{37}$$

$$\binom{2n+1}{n} = 3\binom{2n-1}{n-1} + \binom{2n-1}{n-2}, \tag{38}$$

$$\binom{2n+1}{n-j} = \binom{2n-1}{n-j} + 2\binom{2n-1}{n-1-j} + \binom{2n-1}{n-2-j}. \tag{39}$$

The first line is clear. The right-hand side of the second line gives

$$\begin{aligned} 3\binom{2n-1}{n-1} + \binom{2n-1}{n-2} &= 2\binom{2n-1}{n-1} + \binom{2n}{n-1} \\ &= \binom{2n}{n} + \binom{2n}{n-1} = \binom{2n+1}{n}. \end{aligned} \tag{40}$$

For the third line we get

$$\binom{2n-1}{n-j} + 2\binom{2n-1}{n-1-j} + \binom{2n-1}{n-2-j} = \binom{2n}{n-j} + \binom{2n}{n-j-1} = \binom{2n+1}{n-j}. \tag{41}$$

By (17) we see that with

$$A(n) = \left(\binom{2i+1}{i-j}\right)_{i,j=0}^{n-1} \tag{42}$$

we get

$$A(n)A(n)^\top = \left(\binom{2i+2j+1}{i+j}\right)_{i,j=0}^{n-1}. \tag{43}$$

Let us give a direct proof of (43). Observe first that

$$\sum_{l=0}^{n-1}\binom{2i+1}{i-l}\binom{2j+1}{j-l} = \sum_{l=0}^{i}\binom{2i+1}{i-l}\binom{2j+1}{j-l} = \sum_{l=0}^{j}\binom{2i+1}{i-l}\binom{2j+1}{j-l} \tag{44}$$

and that

$$\sum_{l=0}^{i}\binom{2i+1}{i-l}\binom{2j+1}{j-l}=\sum_{l=0}^{i}\binom{2i+1}{i-l}\binom{2j+1}{j+1+l}$$
$$=\sum_{k=j+1}^{i+j+1}\binom{2i+1}{i+j+1-k}\binom{2j+1}{k} \tag{45}$$

and

$$\sum_{l=0}^{j}\binom{2i+1}{i-l}\binom{2j+1}{j-l}=\sum_{l=0}^{j}\binom{2i+1}{i+1+l}\binom{2j+1}{j+1+l}$$
$$=\sum_{k=0}^{j}\binom{2i+1}{i+j+1-k}\binom{2j+1}{k}. \tag{46}$$

Therefore

$$2\sum_{l=0}^{n-1}\binom{2i+1}{i-l}\binom{2j+1}{j-l}$$
$$=\sum_{k=0}^{j}\binom{2i+1}{i+j+1-k}\binom{2j+1}{k}+\sum_{k=j+1}^{i+j+1}\binom{2i+1}{i+j+1-k}\binom{2j+1}{k} \tag{47}$$
$$=\sum_{k=0}^{i+j+1}\binom{2i+1}{i+j+1-k}\binom{2j+1}{k}=\binom{2i+2j+2}{i+j+1}=2\binom{2i+2j+1}{i+j}.$$

Since $A(n)$ is a triangle matrix whose diagonal elements are $\binom{2i+1}{i-i}=1$ we get $\det(A(n)A(n)^{\top})=1$.

3 A new method

Let us consider the determinants of the Hankel matrices $B(n,k)=\left(\binom{2i+2j+2}{i+j+1-k}\right)_{i,j=0}^{n-1}$.

These have already been computed in [2], Theorem 21. There it is shown that

$$\det(B(i+j,k))_{i,j=0}^{km-1}=(-1)^{\binom{m}{2}k+m\binom{k}{2}} \tag{48}$$

and $\det(B(i+j),k)_{i,j=0}^{n-1}=0$ else.

Definition 3.1. *Let $\gamma^{(k)}=(c(i,j,k))_{i,j\geq 0}$ be the infinite matrix with $c(i,j,k)=1$ if $|i-j|=k$ or $i+j=k-1$. Let us also consider the finite truncations $\gamma^{(k)}|_n$, where $A|_n$ denotes the submatrix consisting of the first n rows and columns of a matrix A. We shall also write $\gamma^{(1)}=\gamma$ and $\gamma^{(k)}|_n=\gamma_n^{(k)}$.*

Theorem 3.2.

$$A(n)\gamma_n^{(k)}A(n)^\top = B(n,k). \tag{49}$$

Proof. Computer experiments suggested that

$$A(n)^{-1}B(n,k)(A(n)^\top)^{-1} = \gamma_n^{(k)} = (c(i,j,k))_{i,j=0}^{n-1}. \tag{50}$$

For example $\gamma_5^{(1)}$ and $\gamma_5^{(2)}$ are the following matrices:

$$\gamma_5^{(1)} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad \gamma_5^{(2)} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{51}$$

If we set $B(n,0) = 2I_n$, where I_n denotes the $n \times n$-identity matrix, then we already know that (49) holds for $k = 0$.

In the general case we have

$$\begin{aligned}
&\sum_{0 \le r,s \le n-1} A(n)(i,r)c(r,s,k)A(n)^\top(s,j) = \sum_{0 \le r,s \le n-1} \binom{2i+1}{i-r} c(r,s,k) \binom{2j+1}{j-s} \\
&= \sum_{s=0}^{n-k-1} \binom{2i+1}{i-(s+k)}\binom{2j+1}{j-s} + \sum_{s=k}^{n-1} \binom{2i+1}{i-(s-k)}\binom{2j+1}{j-s} \\
&\qquad\qquad + \sum_{s=0}^{k-1} \binom{2i+1}{i-(k-1-s)}\binom{2j+1}{j-s} \\
&= \sum_{s=0}^{i-k} \binom{2i+1}{i-s-k}\binom{2j+1}{j+1+s} + \sum_{s=k}^{j} \binom{2i+1}{i-k+s+1}\binom{2j+1}{j-s} \\
&\qquad\qquad + \sum_{s=0}^{k-1} \binom{2i+1}{i-k+1+s}\binom{2j+1}{j-s} \\
&= \sum_{s=j+1}^{i+j+1-k} \binom{2i+1}{i+j-k-s+1}\binom{2j+1}{s} + \sum_{s=0}^{j-k} \binom{2i+1}{i+j-k-s+1}\binom{2j+1}{s} \\
&\qquad\qquad + \sum_{s=j-k+1}^{j} \binom{2i+1}{i+j-k+1-s}\binom{2j+1}{s} \\
&= \sum_{s=0}^{i+j+1-k} \binom{2i+1}{i+j-k+1-s}\binom{2j+1}{s} = \binom{2i+2j+2}{i+j+1-k}.
\end{aligned} \tag{52}$$

The last identity follows from the Chu-Vandermonde formula. □

Lemma 3.3.

$$\det(\gamma_{2kn}^{(k)}) = (-1)^{kn} \tag{53}$$

$$\det(\gamma_{2kn+k}^{(k)}) = (-1)^{kn+\binom{k}{2}} \tag{54}$$

and all other determinants $\det(\gamma_n^{(k)})$ *vanish.*

Proof. By the definition of a determinant we have

$$\det(a_{i,j})_{i,j=0}^{n-1} = \sum_{\pi} \operatorname{sgn}(\pi) a_{0,\pi(0)} a_{1,\pi(1)} \cdots a_{n-1,\pi(n-1)} \tag{55}$$

where π runs over all permutations of the set $\{0, 1, \ldots, n-1\}$. The determinants of the matrices $\gamma_n^{(k)}$ either vanish or the sum over all permutations reduces to a single term $\operatorname{sgn}\pi_n c(0, \pi_n(0), k) c(1, \pi_n(1), k) \cdots c(n-1, \pi_n(n-1), k)$.

Let us first consider $k = 1$. The last row of $\gamma_n^{(1)}$ has only one non-vanishing element $c(n-1, n-2, 1)$. Thus each π which occurs in the determinant must satisfy $\pi(n-1) = n-2$. The next row from below contains two non-vanishing elements $c(n-2, n-3, 1)$ and $c(n-2, n-1, 1)$. The last element is the only element of the last column. Therefore we must have $\pi(n-2) = n-1$. The next row from below contains again two non-vanishing elements, $c(n-3, n-4)$ and $c(n-3, n-2)$. But since $n-2$ already occurs as image of π we must have $\pi(n-3) = n-4$. Thus the situation has been reduced to $\gamma_{n-2}^{(1)}$. In order to apply induction we need the two initial cases $\gamma_1^{(1)}$ and $\gamma_2^{(1)}$.

For $n = 1$ we get $\pi(0) = 0$ and for $n = 2$ $\pi(0) = 1$ and $\pi(1) = 0$ since

$$\gamma_2^{(1)} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{56}$$

If we write $\pi = \pi(0) \cdots \pi(n-1)$ we get in this way $\pi_1 = 0$, $\pi_2 = 10$, $\pi_3 = 021$, $\pi_4 = 1032, \ldots$. This gives $\operatorname{sgn}\pi_n = -\operatorname{sgn}\pi_{n-2}$ and thus by induction $\det \gamma_n^{(1)} = (-1)^{\binom{n}{2}}$, which agrees with (48).

For general k the situation is analogous. The last k rows and columns contain only one non-vanishing element. This implies $\pi(n-j) = n-j-k$ and $\pi(n-j-k) = n-j$ for $1 \le j \le k$. Now $\pi(n-2k-1) = n-3k-1$ since $n-k-1$ occurs already as image of π. Thus the determinant can be reduced to $\gamma_{n-2k}^{(k)}$ and we get $\det \gamma_n^{(k)} = (-1)^k \det \gamma_{n-2k}^{(k)}$ if $n \ge 2k$.

For $n = k$ $\gamma_k^{(k)}$ reduces to the anti-diagonal and thus $\det \gamma_k^{(k)} = (-1)^{\binom{k}{2}}$. For $0 < n < k$ the first row of $\gamma_n^{(k)}$ vanishes and thus $\det \gamma_n^{(k)} = 0$. For $k < n < 2k$ there are two identical rows because $c(k-1, 0, k) = c(k, 0, k) = 1$ and $c(k-1, j, k) =$

$c(k,j,k)=0$ for $0<j<n$. Thus we see by induction that

$$\det(\gamma^{(k)}_{2kn}) = (-1)^{kn} \tag{57}$$
$$\det(\gamma^{(k)}_{2kn+k}) = (-1)^{kn+\binom{k}{2}} \tag{58}$$

and all other determinants vanish. This is the same as (48) because $(-1)^{\binom{2n}{2}k+2n\binom{k}{2}} = (-1)^{kn}$ and $(-1)^{\binom{2n+1}{2}k+(2n+1)\binom{k}{2}} = (-1)^{kn+\binom{k}{2}}$. ☐

Theorem 3.4. *The matrices $\gamma^{(k)}$ satisfy $\gamma^{(k)} = \gamma\cdot\gamma^{(k-1)} - \gamma^{(k-2)}$ with initial values $\gamma^{(1)} = \gamma$ and $\gamma^{(0)} = 2I_\infty$.*

Proof. If $a = (a(i))$ is an arbitrary column vector then $(\gamma\cdot a)(0) = a_0 + a_1$ and $(\gamma\cdot a)(i) = a_{i-1} + a_{i+1}$ for $i \geq 1$. And $(\gamma^{(k)}\cdot a)(i) = a_{k-1-i} + a_{k+i}$ for $0 \leq i \leq k-1$ and $(\gamma^{(k)}\cdot a)(i) = a_{i-k} + a_{i+k}$ for $i \geq k$. This implies

$$(\gamma\cdot\gamma^{(k)}\cdot a)(0) = a_{k-2} + a_{k-1} + a_k + a_{k+1} \qquad (2 \leq i \leq k-2), \tag{59}$$
$$(\gamma\cdot\gamma^{(k)}\cdot a)(1) = a_{k-3} + a_{k-1} + a_k + a_{k+2}, \tag{60}$$
$$(\gamma\cdot\gamma^{(k)}\cdot a)(i) = a_{k-2-i} + a_{k-i} + a_{k+1-i} + a_{k+i+1}, \tag{61}$$
$$(\gamma\cdot\gamma^{(k)}\cdot a)(k-1) = a_0 + a_1 + a_{2k-2} + a_{2k}, \tag{62}$$
$$(\gamma\cdot\gamma^{(k)}\cdot a)(k) = a_0 + a_1 + a_{2k-1} + a_{2k+1}, \tag{63}$$
$$(\gamma\cdot\gamma^{(k)}\cdot a)(i) = a_{i-k-1} + a_{i-k+1} + a_{k+i-1} + a_{k+i+1} \qquad (i \geq k+1). \tag{64}$$

Now observe that $(\gamma^{(k-1)}\cdot a)(i) = a_{k-2-i} + a_{k+i-1}$ for $0 \leq i \leq k-2$ and $(\gamma^{(k+1)}\cdot a)(i) = a_{k-i} + a_{k+i+1}$ for $0 \leq i \leq k$. Therefore we have

$$(\gamma\cdot\gamma^{(k)}\cdot a)(i) = (\gamma^{(k-1)}\cdot a)(i) + (\gamma^{(k+1)}\cdot a)(i) \tag{65}$$

for $0 \leq i \leq k-2$. For $i = k-1$ we get $(\gamma^{(k-1)}\cdot a)(k-1) = a_0 + a_{2k-2}$ and $(\gamma^{(k+1)}\cdot a)(k-1) = a_1 + a_{2k}$. For $i = k$ we get $(\gamma^{(k-1)}\cdot a)(k) = a_1 + a_{2k-1}$ and $(\gamma^{(k+1)}\cdot a)(k) = a_0 + a_{2k+1}$, and for $i \geq k+1$ we have $(\gamma^{(k-1)}\cdot a)(i) = a_{i-k+1} + a_{i+k-1}$ and $(\gamma^{(k+1)}\cdot a)(i) = a_{i-k-1} + a_{i+k+1}$ and thus in all cases

$$(\gamma\cdot\gamma^{(k)}\cdot a)(i) = (\gamma^{(k-1)}\cdot a)(i) + (\gamma^{(k+1)}\cdot a)(i). \tag{66}$$

☐

By induction we see that each $\gamma^{(n)}$ is a polynomial in γ. Therefore all $\gamma^{(k)}$ commute. Theorem 3.4 shows that the matrices $\gamma^{(k)}$ are Lucas polynomials in γ. More precisely

$$\gamma^{(k)} = L_k(\gamma). \tag{67}$$

Therefore we can apply some theorems about Lucas polynomials to $\gamma^{(k)}$.

We have already mentioned the inversion theorem (32). In order to apply this let us define $\bar{\gamma}^{(k)} = \gamma^{(k)}$ for $k > 0$ and $\bar{\gamma}^{(0)} = I$. Let Φ be the algebra isomorphism from the polynomials in x to the polynomials in the matrix γ defined by $\Phi(p(x)) = p(\gamma)$. Then we get $\Phi(\bar{L}_n(x)) = \bar{L}_n(\gamma) = \bar{\gamma}^{(n)}$ and

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} \bar{\gamma}^{(n-2k)} = \gamma^n. \tag{68}$$

Thus we have e.g. $\gamma^{(2)} = \gamma \cdot \gamma^{(1)} - \gamma^{(0)} = \gamma^2 - 2I$ and $\gamma^2 = \binom{2}{0}\bar{\gamma}^{(2)} + \binom{2}{1}\bar{\gamma}^{(0)} = \gamma^{(2)} + 2I$.

Lemma 3.5. *For $i \geq n$ we have $\gamma^n(i,j) = 0$ for $j \leq i - n - 1$ and*

$$\gamma^n(i, i-n+2s) = \binom{n}{s}, \tag{69}$$

$$\gamma^n(i, i-n+2s+1) = 0. \tag{70}$$

For example,

$$\gamma_{12}^5 = \begin{pmatrix} 10 & 10 & 5 & 5 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 5 & 10 & 1 & 5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 5 & 10 & 1 & 10 & 0 & 5 & 0 & 1 & 0 & 0 & 0 & 0 \\ 5 & 1 & 10 & 0 & 10 & 0 & 5 & 0 & 1 & 0 & 0 & 0 \\ 1 & 5 & 0 & 10 & 0 & 10 & 0 & 5 & 0 & 1 & 0 & 0 \\ 1 & 0 & 5 & 0 & 10 & 0 & 10 & 0 & 5 & 0 & 1 & 0 \\ 0 & 1 & 0 & 5 & 0 & 10 & 0 & 10 & 0 & 5 & 0 & 1 \\ 0 & 0 & 1 & 0 & 5 & 0 & 10 & 0 & 10 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 & 0 & 5 & 0 & 10 & 0 & 10 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 5 & 0 & 10 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 5 & 0 & 9 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 & 0 & 5 & 0 \end{pmatrix}. \tag{71}$$

A curious observation:

The Lucas polynomials satisfy $L_k(x)^2 - L_{k-1}(x)L_{k+1}(x) = 4 - x^2$. Therefore we get

$$(\gamma^{(k)})^2 - \gamma^{(k-1)}\gamma^{(k+1)} = 4 - \gamma^2 = 2 - \gamma^{(2)}. \tag{72}$$

The matrices $2I_n - \gamma_n^{(2)}$ satisfy $\det(2I_n - \gamma_n^{(2)}) = n + 1$ and

$$A(n)(2I_n - \gamma_n^{(2)})A(n)^\top = (C_{i+j+2})_{i,j=0}^{n-1} \tag{73}$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is a Catalan number.

4 Two useful matrices

For the finite matrices $\gamma_n = \gamma|_n$ we have $\gamma_n^k \neq \gamma^k|_n$. In order to compute $\gamma^k|_n$ in the realm of $n \times n$-matrices we introduce two auxiliary matrices α_n and β_n.

Let v_n be the column vector of length n with entries $v_n(i) = [i = n-1]$. Then $v_n v_n^\top$ is the $n \times n$-matrix whose only nonzero entry is $v_n v_n^\top(n-1, n-1) = 1$.

Definition 4.1. *Let $\delta_{m,l}$ be the $m \times m$-matrix whose entries satisfy*

$$\delta_{m,l}(i, 2m-1-l-i+2s) = \binom{l}{s} \tag{74}$$

and $\delta_{m,l}(i,j) = 0$ else.

For example,

$$\delta_{6,5} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 5 & 0 & 10 \end{pmatrix}. \tag{75}$$

Theorem 4.2. *Let $\alpha_m = \gamma_m + v_m v_m^\top$ and $\beta_m = \gamma_m - v_m v_m^\top$. If $m \geq l$ then*

$$\frac{\alpha_m^l + \beta_m^l}{2} = \gamma^l|_m, \tag{76}$$
$$\frac{\alpha_m^l - \beta_m^l}{2} = \delta_{m,l}.$$

Proof. Observe that

$$\alpha_m^l - \beta_m^l = \gamma_m(\alpha_m^{l-1} - \beta_m^{l-1}) + v_m v_m^\top(\alpha_m^{l-1} + \beta_m^{l-1}), \tag{77}$$
$$\alpha_m^l + \beta_m^l = \gamma_m(\alpha_m^{l-1} + \beta_m^{l-1}) + v_m v_m^\top(\alpha_m^{l-1} - \beta_m^{l-1}). \tag{78}$$

Thus the theorem is equivalent with

$$\delta_{m,l} = \gamma_m \delta_{m,l-1} + r_{m,l}, \tag{79}$$
$$\gamma^l|_m = \gamma_m \gamma^{l-1}|_m + s_{m,l}$$

for $m \geq l$, where $r_{m,l}$ is the matrix whose last row is $(\gamma^{l-1}(m-1,0), \cdots, \gamma^{l-1}(m-1, m-1))$ and all other entries vanish, and $s_{m,l}$ is the matrix whose last row is $(\delta_{m,l-1}(m-1,0), \cdots, \delta^{m,l-1}(m-1,m-1))$ and all other entries vanish.

We now prove (79) by induction. It clearly holds for $l = 1$. Now suppose that (76) is true for $l - 1$.

Let us first prove the second assertion of (79). For $i < m - 1$ we have

$$\sum_{s \geq 0} \gamma(i,s)\gamma^{l-1}(s,j) = \sum_{0 \leq s \leq m-1} \gamma_m(i,s)\gamma^{l-1}(s,j) \tag{80}$$

because $\gamma(i,s) = 0$ for $s \geq m$. For $i = m - 1$ we get

$$\sum_{s \geq 0} \gamma(m-1,s)\gamma^{l-1}(s,j) = \sum_{0 \leq s \leq m-1} \gamma_m(m-1,s)\gamma^{l-1}(s,j) + \gamma^{l-1}(m,j). \tag{81}$$

By Lemma 3.5 we know that $\gamma^{l-1}(m, m-l+1+2s) = \binom{l-1}{s}$ and all other entries are 0. On the other hand the last row of $\delta_{m,l-1}$ is $\delta_{m,l-1}(m-1,j) = \binom{l-1}{s}$ if $j = m-l+1+2s$ and $\delta_{m,l-1}(m-1,j) = 0$ else. Thus the second line of (79) is true.

Now consider the first line. For $i < m - 1$ we have

$$\sum_r \gamma(i,r)\delta_{m,l-1}(r,j) = \delta_{m,l}(i,j). \tag{82}$$

This is equivalent with $\delta_{m,l-1}(i-1,j) + \delta_{m,l-1}(i+1,j) = \delta_{m,l}(i,j)$. For $(i,j) = (i, 2m-1-l-i+2s)$ we get $\binom{l-1}{s} + \binom{l-1}{s-1} = \binom{l}{s}$. For $i = m-1$ we get

$$\sum_r \gamma(m-1,r)\delta_{m,l-1}(r, m-l+2s) = \delta_{m,l-1}(m-2, m-l+2s) = \binom{l-1}{s-1}. \tag{83}$$

On the other hand for $(\gamma^{l-1}(m-1,0), \cdots, \gamma^{l-1}(m-1,m-1))$ we get by Lemma 3.5 that $\gamma^{l-1}(m-1, m-l+2s) = \binom{l-1}{s}$. Thus also in this case (79) is proved. □

5 Relating the determinant to the γ matrices

Let $g_n(x) = \det(xI - \gamma_n)$ with $g_0(x) = 1$. If we expand with respect to the last row we get $g_n(x) = xg_{n-1}(x) - g_{n-2}(x)$. The initial values are $g_1(x) = x - 1$ and $g_2(x) = x^2 - x - 1$. This gives $g_n(x) = \sum_{k=0}^{n}(-1)^k \bar{L}_{n-k}(x)$ and $g_n(x) + g_{n+1}(x) = L_{n+1}(x)$. Therefore we get

$$g_n(\gamma) = \sum (-1)^k \bar{\gamma}^{(n-k)}. \tag{84}$$

Let $b_n(x) = \det(xI - \beta_n)$. Then we get $b_n(x) = g_n(x) + g_{n-1}(x) = L_n(x)$ by cofactor expansion on the last row.

Note that $A(n)g_k(\gamma)A(n)^\top = (\binom{2i+2j+1}{i+j-k})_{i,j \geq 0}$. By (43) and Theorem 3.2, this holds for $k = 0$ and $k = 1$. Since $g_k(\gamma) = L_k(\gamma) - g_{k-1}(\gamma) = \gamma^{(k)} - g_{k-1}(\gamma)$, we get by induction

$$Ag_k(\gamma)A^\top = A\gamma^{(k)}A^\top - Ag_{k-1}(\gamma)A^\top = \left(\binom{2i+2j+2}{i+j+1-k} - \binom{2i+2j+1}{i+j+1-k}\right)_{i,j\geq 0}$$
$$= \left(\binom{2i+2j+1}{i+j-k}\right)_{i,j\geq 0} \tag{85}$$

We are interested in the Hankel determinants

$$\det\left(\binom{2i+2j+r}{i+j}\right)_{i,j=0}^{N}. \tag{86}$$

By Chu-Vandermonde we have

$$\binom{2n+r}{n} = \sum_k \binom{r-2}{k}\binom{2n+2}{n-k}. \tag{87}$$

This implies

$$\left(\binom{2i+2j+r}{i+j}\right)_{i,j=0}^{n-1} = \sum_k \binom{r-2}{k}\left(\binom{2i+2j+2}{i+j+1-(k+1)}\right)_{i,j=0}^{n-1} \tag{88}$$

or

$$\left(\binom{2i+2j+r}{i+j}\right)_{i,j=0}^{n-1} = \sum_{k\geq 0}\binom{r-2}{k}B(n,k+1). \tag{89}$$

This again implies that

$$\det\left(\binom{2i+2j+r}{i+j}\right)_{i,j=0}^{n-1} = \det\left(\sum_k \binom{r-2}{k}\gamma_n^{(k+1)}\right). \tag{90}$$

For $r=2$ we get

$$\det\left(\binom{2i+2j+2}{i+j}\right)_{i,j=0}^{n-1} = \det(\gamma_n^{(1)}). \tag{91}$$

There is a single 1 in the last row and column. If we expand first with respect to one and then with respect to the other we see that $\det(\gamma_n^{(1)}) = -\det(\gamma_{n-2}^{(1)})$. This gives $\det(\gamma_n^{(1)}) = (-1)^{\binom{n}{2}}$.

By (67) and (90), $d_r(n) = h(r)(\gamma)|_n$ for the polynomial $h(n) = \sum_k \binom{n-2}{k} L_{k+1}(x)$. Let us therefore obtain more information about $h(n)$. It satisfies $h(n) = (x+2)h(n-1) - (x+2)h(n-2)$ with $h(2) = x$, $h(3) = x^2+x-2 = (x+2)(x-1)$. This follows from

$$(x+2)\sum\left(\binom{n-1}{k}-\binom{n-2}{k}\right)L_{k+1}(x)-(x+2)\sum\binom{n-2}{k-1}L_{k+1}(x)$$
$$=\sum\binom{n-2}{k-1}(xL_{k+1}(x)+2L_{k+1}(x)) \tag{92}$$
$$=\sum\binom{n-2}{k-1}(L_{k+2}(x)+2L_{k+1}(x)+L_k(x))$$
$$=\sum\left(\binom{n-2}{k-2}+2\binom{n-2}{k-1}+\binom{n-2}{k}\right)L_{k+1}(x)=\sum\binom{n}{k}L_{k+1}(x).$$

Therefore we get

$$\begin{aligned} h(n) &= (x+2)h(n-1)-(x+2)h(n-2)\\ &= (x+2)((x+2)h(n-2)-(x+2)h(n-3))-(x+2)h(n-2)\\ &= (x+2)(x+1)h(n-2)-(x+2)(h(n-2)+(x+2)h(n-4))\\ &= (x+2)xh(n-2)-(x+2)^2h(n-4). \end{aligned} \tag{93}$$

Given the initial values $h(3)=(x+2)(x-1)$ and $h(5)=(x+2)^2(x^2-x-1)$, it follows that $h(2k+1)=(x+2)^kg_k(x)$. Given that $h(2)=x$ and $h(4)=(x+2)(x^2-2)$, it follows that $h(2k)=(x+2)^{k-1}b_k(x)$.

Combining this with (90) we get

Theorem 5.1. *For $r\geq 2$, let $k=\lfloor\frac{r}{2}\rfloor$ and $l=\lfloor\frac{r-1}{2}\rfloor$, and define the functions*

$$h_r(x)=\begin{cases} g_k(x) & \text{if } r=2k+1\\ b_k(x) & \text{if } r=2k\end{cases} \tag{94}$$

and $q_r(x)=(x+2)^lh_r(x)$. For $N\geq k+l$, by Theorem 4.2,

$$d_r(N)=\det\left(\sum_{j\geq 0}\binom{r-2}{j}\gamma_N^{(j+1)}\right)=\det(q_r(\gamma)|_N)=\det(\frac{1}{2}(q_r(\alpha_N)+q_r(\beta_N))). \tag{95}$$

6 Structure of the matrices

In this section we determine the structure of the matrices $(\beta_N+2)^{-1}$, $g_k(\alpha_N)$, $g_k(\beta_N)$, $b_k(\alpha_N)$, and $b_k(\beta_N)$, as well as the determinants of $g_k(\gamma)|_N$ and $b_k(\gamma)|_N$.

To determine $p(\alpha_N)$ and $p(\beta_N)$ for a polynomial p of degree less than N, we begin by writing $p(\gamma)$ as a sum of $\gamma^{(k)}$ matrices using the multiplicative formula of Theorem 3.4. We then apply Prop 6.2 to show that $p(\alpha_N)$ and $p(\beta_N)$ are the same as $p(\gamma)|_N$ on and above the anti-diagonal. The structure of $p(\alpha_N)$ follows from the symmetry of α_N across its anti-diagonal. The structure of $p(\beta_N)$ can be computed from $p(\alpha_N)$ and $p(\gamma)|_N$ with Theorem 4.2.

Proposition 6.1. *The determinant of a block matrix*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{96}$$

where A and D are square and D is invertible is $\det(D)\det(A - BD^{-1}C)$*.*

Proof. Note that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & 0 \\ -D^{-1}C & I \end{pmatrix} = \begin{pmatrix} A - BD^{-1}C & B \\ 0 & D \end{pmatrix}, \tag{97}$$

and that the determinant of a block-triangular matrix is the product of the determinants of its diagonal blocks. □

Proposition 6.2. *Let T be a N-by-N tridiagonal matrix and let p be a polynomial of degree d. Let v be the N-by-1 column vector with a 1 in its last entry and 0 elsewhere. Then the (i, j) entries of $p(T)$ and $p(T + vv^\top)$ agree when $i + j \le 2(N - 1) - d$.*

Proof. It suffices to prove this for $p(x) = x^d$. Call a N-by-N matrix "k-small" iff its entries (i, j) with $i + j \le 2(N - 1) - k$ are all 0. For instance, $vv^\top$ is 1-small.

Suppose a matrix M is k-small. For $i + j \le 2(N - 1) - k - 1$, the (i, j) entry of TM is $\sum_{l=0}^{N-1} T_{il}M_{lj} = T_{i,i-1}M_{i-1,j} + T_{i,i}M_{i,j} + T_{i,i+1}M_{i+1,j}$. Since M is k-small, its $(i-1, j)$, (i, j), and $(i+1, j)$ entries are 0, which implies that TM is $(k+1)$-small. Similarly, MT, $vv^\top M$, and $Mvv^\top$ are $(k + 1)$-small.

Consider $(T + vv^\top)^d - T^d$. Expanding the binomial product yields $2^d - 1$ terms, all of which are products of d T's and $vv^\top$'s and contain at least one $vv^\top$. It follows from the above that each of these terms is d-small, so $p(T + vv^\top) - p(T)$ is d-small. □

Lemma 6.3. *The inverse of $(\beta_N + 2)$ is $(\frac{1}{2}(-1)^{i+j}(2\min\{i, j\} + 1))_{i,j=0}^{N-1}$. The determinant of $(\beta_N + 2)$ is 2. For example,*

$$(\beta_5 + 2)^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ -1 & 3 & -3 & 3 & -3 \\ 1 & -3 & 5 & -5 & 5 \\ -1 & 3 & -5 & 7 & -7 \\ 1 & -3 & 5 & -7 & 9 \end{pmatrix}. \tag{98}$$

Proof. For $i \neq 0, N-1$ the row i of $(\beta_N + 2)$ is $(2\delta_{il} + \delta_{i,l-1} + \delta_{i,l+1})_{l=0}^{N-1}$. The product of this with column j of the claimed inverse is

$$\begin{aligned}
&\sum_{l=0}^{N-1}(2\delta_{il} + \delta_{i,l-1} + \delta_{i,l+1})\frac{1}{2}(-1)^{l+j}(2\min\{l,j\}+1) \\
&= \frac{1}{2}(-1)^{i+j}(4\min\{i,j\} + 2 - 2\min\{i+1,j\} - 1 - 2\min\{i-1,j\} - 1) \\
&= (-1)^{i+j}(2\min\{i,j\} - \min\{i+1,j\} - \min\{i-1,j\}).
\end{aligned} \tag{99}$$

This is 0 if $i+1 \leq j$ or $i-1 \geq j$ and is 1 if $i = j$.

The first row of $(\beta_N + 2)$ is $(3, 1, 0, \ldots, 0)$, and the last row is $(0, \ldots, 0, 1, 1)$. Column $j \neq 0, N-1$ of the claimed inverse begins and ends as

$$\frac{1}{2}((-1)^j, (-1)^{j+1}3, \ldots, (-1)^{j+N-2}(2j+1), (-1)^{j+N-1}(2j+1)), \tag{100}$$

so it kills the first and last rows of $(\beta_N + 2)$. Column 0 of the claimed inverse begins and ends as $\frac{1}{2}(1, -1, \ldots, (-1)^{N-2}, (-1)^{N-1})$ while column $N-1$ begins and ends as $\frac{1}{2}((-1)^{N-1}, (-1)^N 3, \ldots, -(2N-3), 2N-1)$. It's easy to verify that these columns have the correct products with rows of $(\beta_N + 2)$.

The determinant $\det(\beta + 2)$ is $(-1)^N b_N(-2)$, which can be computed with recurrence in Section 5 to be 2. ◻

Lemma 6.4. *For $k < N$, the (i,j) entry of $g_k(\alpha_N)$ is $(-1)^{i+j+k}$ if $k \leq i+j \leq 2N-k-2$ and $|i-j| \leq k$ and is 0 otherwise. The (i,j) entry of $g_k(\beta_N)$ is $(-1)^{i+j+k}$ if $k \leq i+j \leq 2N-k-2$ and $|i-j| \leq k$, is $2(-1)^{i+j+k}$ if $2N-k-1 \leq i+j$, and is 0 otherwise. For example,*

$$g_2(\beta_6) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & -2 \\ 0 & 0 & 0 & 1 & -2 & 2 \end{pmatrix}. \tag{101}$$

Proof. Recall that $g_j(\gamma) = \gamma^{(j)} - \gamma^{(j-1)} + \cdots \pm \gamma^{(1)} \mp 1$, by (84). Therefore $\frac{1}{2}(g_k(\alpha_N) + g_k(\beta_N)) = g_k(\gamma)|_N = \gamma_N^{(k)} - \gamma_N^{(k-1)} + \cdots \pm \gamma_N^{(1)} \mp 1$. From the definition of the $\gamma_N^{(j)}$, the (i,j) entry of $g_k(\gamma)|_N$ is $(-1)^{i+j+k}$ if $k \leq i+j$ and $|i-j| \leq k$ and is 0 otherwise.

Note that polynomials in α_N are symmetric about their anti-diagonal. Since the degree of g_k is $k < N$, Prop 6.2 says that $g_k(\alpha_N)$ agrees with $g_k(\gamma)|_N$ on and above its anti-diagonal. Thus, the (i,j) entry of $g_k(\alpha_N)$ is $(-1)^{i+j+k}$ if $k \leq$

$i+j \leq 2N-k-2$ and $|i-j| \leq k$ and is 0 otherwise. Similarly, the (i,j) entry of $g_k(\beta_N) = 2g_k(\gamma)|_N - g_k(\alpha_N)$ is $(-1)^{i+j+k}$ if $k \leq i+j \leq 2N-k-2$ and $|i-j| \leq k$, $2(-1)^{i+j+k}$ if $2N-k-1 \leq i+j$, and 0 otherwise. □

Lemma 6.5.

$$\det g_k(\gamma)|_N = \begin{cases} 1 & \textit{if } N = (2k+1)n \\ (-1)^{\binom{k+1}{2}} & \textit{if } N = (2k+1)n + k + 1 \\ 0 & \textit{otherwise.} \end{cases} \tag{102}$$

Proof. When $N = 0$ the determinant is vacuously 1. When $0 < N < k+1$, the first column is 0. When $N = k+1$ the matrix is 0 above its antidiagonal and 1 on its antidiagonal, so its determinant is $(-1)^{\binom{k+1}{2}}$. When $k+1 < N < 2k+1$, columns $k-1$ and $k+1$ are equal. Thus the claim holds for all $N < 2k+1$. We'll show that for $N \geq 2k+1$, $\det g_k(\gamma)|_N = \det g_k(\gamma)|_{N-2k-1}$.

Fix $N \geq 2k+1$ and let $M = g_k(\gamma)|_N$. Subdivide M into a block matrix consisting of the leading principal order-$N-1$ submatrix M_{11}, the bottom-right entry M_{22}, and the remainders of the last column and row M_{12} and M_{21}. The determinant of M is $\det(M_{22}) \det(M')$, where M' is the $N-1$-by-$N-1$ matrix $M_{11} - M_{12} M_{22}^{-1} M_{21}$ by Proposition 6.1.

We will perform cofactor expansion in the bottom right of M'. Since $M_{22} = (-1)^k$, the bottom right k-by-k submatrix of M' is the zero matrix. As a result, the only entry in the bottom row of M' is the 1 at $(N-2, N-k-2)$. After deleting its row and column, the only entry in the bottom row of M' is the 1 at $(N-3, N-k-3)$. This pattern continues up to the 1 at $(N-k-1, N-2k-1)$. Since M' is symmetric, a similar sequence of lone 1's can be removed in the last k columns.

After the last $2k$ rows and columns have been removed, M' has been reduced to $g_k(\gamma)|_{N-2k-1}$. The $2k$ removed 1's contribute a factor of $(-1)^k$ to the determinant, which comes from the parity of the permutation $(0\ k)(1\ k+1)\cdots(k-1\ 2k)$. This cancels with the sign of M_{22}. □

Lemma 6.6. *For $k < N$, the (i,j) entry of $b_k(\alpha_N)$ is 1 if $|i-j| = k$, $i+j = k-1$, or $i+j = 2(N-1)-(k-1)$ and is 0 otherwise. The (i,j) entry of $b_k(\beta_N)$ is 1 if $|i-j| = k$ or $i+j = k-1$, is -1 if $i+j = 2(N-1)-(k-1)$, and is 0 otherwise. In particular $b_k(\gamma) = \gamma^{(k)}$. Moreover,*

$$\det b_k(\gamma)|_N = \begin{cases} (-1)^{kn} & \textit{if } N = 2kn \\ (-1)^{kn+\binom{k}{2}} & \textit{if } N = 2kn + k \\ 0 & \textit{otherwise.} \end{cases} \tag{103}$$

Proof. The first set of claims follow from the Lemma 6.4 and the fact that $b_k(x) = g_k(x) + g_{k-1}(x)$. The determinant of $\gamma^{(k)}$ was calculated in Lemma 3.3. □

7 Calculation of the determinant

In this section we prove the seven formulas mentioned in the introduction. Recall Theorem 5.1 and its notation.

Let $\mu_i = \frac{1}{2}((\alpha_N + 2)^i h_r(\alpha_N) + (\beta_N + 2)^i h_r(\beta_N))$ for $0 \le i \le l$. From here on we'll suppress the subscripts on α_N and β_N. By Theorem 5.1, we're interested in calculating $d_r(N) = \det \mu_l$. Note that

$$\mu_{i+1} = \mu_i(\beta + 2) + (\alpha + 2)^i h_r(\alpha) v v^\top. \tag{104}$$

The results of the previous section give us control over μ_0. We will induct on the above equation to screw the smoothing operators $\alpha + 2$ and $\beta + 2$ into place, using the matrix determinant lemma to keep track of the determinants. In the seven cases proven here, the determinant or adjugate of μ_i is multiplied by a constant factor at each step.

Proposition 7.1 (Matrix determinant lemma)**.** *If A is an n-by-n matrix and u and v are n-by-1 column vectors, then*

$$\det(A + uv^\top) = \det(A) + v^\top \operatorname{adj}(A) u. \tag{105}$$

Proof. This is a polynomial identity in the entries of A, u, and v, so it suffices to prove it for the dense subset where A is invertible. Consider

$$\begin{pmatrix} I & 0 \\ v^\top & 1 \end{pmatrix} \begin{pmatrix} I + A^{-1}uv^\top & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & 0 \\ -v^\top & 1 \end{pmatrix} = \begin{pmatrix} I & u \\ 0 & 1 + v^\top A^{-1} u \end{pmatrix}, \tag{106}$$

which shows that $1 \cdot \det(I + A^{-1}uv^\top) \cdot 1 = \det(1 + v^\top A^{-1} u)$. Multiplying through by $\det A$ yields $\det(A + uv^\top) = \det(A)(1 + v^\top A^{-1} u) = \det(A) + v^\top \operatorname{adj}(A) u$. □

7.1 The case that μ_0 is invertible

Lemma 7.2. *Suppose there is an N-dimensional column vector w such that $\mu_0 w = h_r(\alpha_N) v$ and that the last $l - 1$ entries of $h_r(\beta_N) w$ are 0. Then*

$$\det(\mu_l) = \det(\mu_0) 2^l \left(1 + v^\top (\beta_N + 2)^{-1} w\right)^l. \tag{107}$$

Proof. By Prop 6.2, $(\alpha + 2)^i$ and $(\beta + 2)^i$ differ only in the last i columns. It follows from the second hypothesis that $(\beta + 2)^i h_r(\beta) w = (\alpha + 2)^i h_r(\beta) w$ for $0 \le i < l$. Thus

$$\mu_i w = (\alpha + 2)^i h_r(\alpha) v \tag{108}$$

and

$$\det(\mu_i) w = \operatorname{adj}(\mu_i)(\alpha + 2)^i h_r(\alpha) v \tag{109}$$

for $0 \le i < l$. By (104) and the matrix determinant lemma,

$$\begin{aligned}\det(\mu_{i+1}) &= \det(\beta+2)\left(\det(\mu_i) + v^\top(\beta+2)^{-1}\operatorname{adj}(\mu_i)(\alpha+2)^i h_r(\alpha)v\right) \\ &= \det(\beta+2)\left(\det(\mu_i) + v^\top(\beta+2)^{-1}\det(\mu_i)w\right).\end{aligned} \tag{110}$$

Hence

$$\det(\mu_{i+1}) = 2\det(\mu_i)\left(1 + v^\top(\beta_N+2)^{-1}w\right). \tag{111}$$

□

Theorem 7.3.

$$d_{2k+1}((2k+1)n) = (2n+1)^k \tag{112}$$

$$d_{2k+1}((2k+1)n+k+1) = (-1)^{\binom{k+1}{2}}4^k(n+1)^k \tag{113}$$

$$d_{2k}(2kn) = (-1)^{kn} \tag{114}$$

$$d_{2k}(2kn+k) = (-1)^{kn+\binom{k}{2}}4^{k-1}(n+1)^{k-1} \tag{115}$$

Proof. Given w, it is straightforward to verify the hypotheses and evaluate the final expression of Lemma 7.2 with the lemmas of Section 6. For the first formula, take w to be the $(2k+1)n$-dimensional column vector

$$w_1 = (-1)^{n-1}\left(\sum_{m=0}^{n-1}(-1)^m e_{(2k+1)m} - \sum_{m=0}^{n-1}(-1)^m e_{(2k+1)m+2k}\right) + e_{N-1}, \tag{116}$$

where $\{e_i\}_{i=0}^{N-1}$ is the standard basis. Then $g_k(\alpha)w_1 = g_k(\beta)w_1 = e_{N-k-1}$.

For the second formula, take w to be the $(2k+1)n+k+1$-dimensional column vector

$$w_2 = (-1)^n\left(\sum_{m=0}^{n}(-1)^m e_{(2k+1)m+k-1} - \sum_{m=0}^{n-1}(-1)^m e_{(2k+1)m+k+1}\right) + e_{N-1}, \tag{117}$$

which gives $g_k(\alpha)w_2 = e_{N-k-1} + e_{N-k}$ and $g_k(\beta)w_2 = e_{N-k-1} - e_{N-k}$.

For the third formula, take w to be the $2kn$-dimensional column vector

$$w_3 = (-1)^{n-1}\left(\sum_{m=0}^{n-1}(-1)^m e_{2km} - \sum_{m=0}^{n-1}(-1)^m e_{2km+2k-1}\right) + e_{N-1}, \tag{118}$$

which gives $b_k(\alpha)w_3 = b_k(\beta)w_3 = e_{N-k-1} + e_{N-k}$.

For the fourth formula, take w to be the $2kn+k$-dimensional column vector

$$w_4 = (-1)^n\left(\sum_{m=0}^{n}(-1)^m e_{2km+k-1} - \sum_{m=0}^{n-1}(-1)^m e_{2km+k+1}\right) + e_{N-1}, \tag{119}$$

which gives $b_k(\alpha)w_4 = e_{N-k-1} + 3e_{N-k}$ and $b_k(\beta)w_4 = e_{N-k-1} - e_{N-k}$. □

7.2 The case that μ_0 is singular

We will make use of the following fact about the adjugate matrix.

Proposition 7.4. *The rank of the adjugate* $\operatorname{adj}(M)$ *of an* n*-by-*n *matrix* M *satisfies*

$$\operatorname{rk}\operatorname{adj}(M) = \begin{cases} n & \text{if } \operatorname{rk} M = n \\ 1 & \text{if } \operatorname{rk} M = n-1 \\ 0 & \text{otherwise} \end{cases} \tag{120}$$

Proof. Recall that $\operatorname{adj}(M) \cdot M = \det(M) I$. If $\operatorname{rk} M = n$ then M is invertible with inverse $\frac{1}{\det(M)} \operatorname{adj}(M)$, which also has rank n.

If $\operatorname{rk} M = n-1$, then $\det(M) = 0$, in which case $\operatorname{adj}(M)$ must send all vectors into the kernel of M, which has rank 1. In this case M also has a nonzero order-$n-1$ minor, so $\operatorname{adj}(M)$ has rank 1.

If $\operatorname{rk} M \leq n-2$, then all order-$n-1$ minors of M are zero, so $\operatorname{adj}(M) = 0$. □

Lemma 7.5. *Suppose there is a nonzero* N*-dimensional column vector* w *such that* $\det(\mu_0) = 0$, $\det(\mu_0|_{N-1}) \neq 0$, $\mu_0 w = 0$, $v^\top w = 1$, $v^\top(\beta+2)^{-1} w \neq 0$, *and entries* $N-k-l$ *through* $N-3$ *of* w *are* 0. *Then*

$$\det(\mu_l) = \det(\mu_0|_{N-1}) \left(2v^\top(\beta_N+2)^{-1} w\right)^l \left(w^\top(\alpha+2)^{l-1} h_r(\alpha) v\right). \tag{121}$$

Proof. Let $c = \det(\mu_0|_{N-1})$. We will show by induction that

$$\operatorname{adj}(\mu_i) = c\left(2v^\top(\beta_N+2)^{-1} w\right)^i w w^\top, \tag{122}$$

for $0 \leq i < l$. For the base case of $i = 0$, note that the first two hypotheses imply that μ_0 has rank $N-1$. Since w generates the kernel and μ_0 is symmetric, Lemma 7.4 implies that $\operatorname{adj}(\mu_0)$ is a constant d times $ww^\top$. In fact $c = v^\top \operatorname{adj}(\mu_0) v = d v^\top w w^\top v = d$.

Suppose the claim holds for i. Since $\alpha+2$ is tridiagonal, the last hypothesis combined with Lemmas 6.4 and 6.6 imply that $w^\top(\alpha+2)^i h_r(\alpha) v = 0$. By (104) and the matrix determinant lemma,

$$\begin{aligned} \det(\mu_{i+1}) &= \det(\beta+2)\left(\det(\mu_i) + v^\top(\beta+2)^{-1}\operatorname{adj}(\mu_i)(\alpha+2)^i h_r(\alpha) v\right) \\ &= \det(\beta+2)\left(0 + c\left(2v^\top(\beta_N+2)^{-1} w\right)^i v^\top(\beta+2)^{-1} w w^\top(\alpha+2)^i h_r(\alpha) v\right) \\ &= 0, \end{aligned} \tag{123}$$

so μ_{i+1} has rank at most $n-1$. Since $(\alpha+2)^i h_r(\alpha)vv^\top$ doesn't affect the bottom-right cofactor,

$$\begin{aligned} v^\top \operatorname{adj}(\mu_{i+1})v &= v^\top \operatorname{adj}\left(\mu_i(\beta+2) + (\alpha+2)^i h_r(\alpha)vv^\top\right) v \\ &= v^\top \operatorname{adj}\left(\mu_i(\beta+2)\right) v \\ &= c\det(\beta+2)v^\top(\beta+2)^{-1}\left(2v^\top(\beta_N+2)^{-1}w\right)^i ww^\top v \\ &= c(2v^\top(\beta_N+2)^{-1}w)^{i+1}. \end{aligned} \tag{124}$$

This is nonzero by assumption, so $\operatorname{adj}(\mu_{i+1})$ is nonzero. By Prop 7.4, it is rank 1. The matrix μ_{i+1} is symmetric and w lies in its kernel:

$$w^\top \mu_{i+1} = w^\top \mu_i(\beta+2) + w^\top(\alpha+2)^i h_r(\alpha)vv^\top = 0 + 0, \tag{125}$$

so it is of the form $\operatorname{adj}(\mu_{i+1}) = c(2v^\top(\beta_N+2)^{-1}w)^{i+1}ww^\top$. This completes the induction.

The final μ_l has determinant

$$\begin{aligned} \det(\mu_l) &= \det(\beta+2)\left(\det(\mu_{l-1}) + v^\top(\beta+2)^{-1}\operatorname{adj}(\mu_{l-1})(\alpha+2)^{l-1}h_r(\alpha)v\right) \\ &= 2\left(0 + 2^{l-1}c(v^\top(\beta_N+2)^{-1}w)^l w^\top(\alpha+2)^{l-1}h_r(\alpha)v\right) \\ &= c\left(2v^\top(\beta_N+2)^{-1}w\right)^l\left(w^\top(\alpha+2)^{l-1}h_r(\alpha)v\right). \end{aligned} \tag{126}$$

□

Theorem 7.6.

$$d_{2k+1}((2k+1)n+1) = (2n+1)^k \tag{127}$$

$$d_{2k}(2kn+1) = (-1)^{kn} \tag{128}$$

$$d_{2k}(2kn+k+1) = -(-1)^{kn+\binom{k}{2}}4^{k-1}(n+1)^{k-1} \tag{129}$$

Proof. Given w, it is straightforward to verify the hypotheses and evaluate the final expression of Lemma 7.5 with the lemmas of Section 6.

For the first formula, take w to be

$$w_5 = (-1)^n\left(\sum_{m=0}^{n}(-1)^m e_{(2k+1)m} - \sum_{m=0}^{n-1}(-1)^m e_{(2k+1)m+2k}\right), \tag{130}$$

where $\{e_i\}_{i=0}^{N-1}$ is the standard basis.

For the second formula, w to be

$$w_6 = (-1)^n\left(\sum_{m=0}^{n}(-1)^m e_{2km} - \sum_{m=0}^{n-1}(-1)^m e_{2km+2k-1}\right). \tag{131}$$

For the third formula, use

$$w_7 = (-1)^{n-1} \left(\sum_{m=0}^{n} (-1)^m e_{2km+k-1} - \sum_{m=0}^{n} (-1)^m e_{2km+k} \right). \tag{132}$$

□

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