The generally covariant locality principle and an extension to gauge symmetries

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We review the framework of locally covariant field theories and present an extension that also includes gauge symmetries.

The first part is based on


The second part is based on

1. The generally covariant locality principle

2. Extension to gauge symmetries
   - Geometric background
   - Extended covariance

3. Conclusion
Outline

1 The generally covariant locality principle

2 Extension to gauge symmetries
   - Geometric background
   - Extended covariance

3 Conclusion
Algebraic quantum field theory (AQFT) provides a framework for quantum field theories on Minkowski space. Its basic object is a net of $C^*$ algebras, i.e., to each open, bounded region $\mathcal{O}$ one associates a $C^*$ algebra $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}$. The basic requirement is locality, i.e.,

$$[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = \{0\}$$

for $\mathcal{O}_1$ and $\mathcal{O}_2$ spacelike. Such a notion only refers to the causal structure of the spacetime. Other axioms, however, explicitly refer to Minkowski space, namely Poincaré covariance and stability.

From general relativity, we know that spacetime is curved. It is desirable to be able to define a quantum field theory not only on special, but on generic spacetimes:

- One wants to know what is the same theory on different spacetimes.
- One wants to compare a theory on different spacetimes, for example to study the effect of metric perturbations.

The generally covariant locality principle provides such a framework for quantum field theories on generic spacetimes. A quantum field theory is defined on all spacetimes, in a coherent way.
It is physically reasonable to require that spacetimes are time-oriented, oriented, connected and \textit{globally hyperbolic} manifolds. That means that there is a smooth foliation $M \simeq \mathbb{R} \times \Sigma$ by Cauchy surfaces. A Cauchy surface is a subset that each inextendable causal curve (piecewise differentiable) intersects exactly once.

The essential point is that normally hyperbolic wave operators, i.e., second order differential operators whose second order part is, in any trivialization, given by $\mathbb{1} g^{\mu \nu} \partial_\mu \partial_\nu$, have a well-posed Cauchy problem.
A category is a collection of objects and morphisms (or arrows) between them. I.e., a morphism $\chi : A \to B$ has source $A$ and target $B$, where $A$ and $B$ are objects. Two morphisms $\chi : A \to B$ and $\psi : B \to C$ can be composed to a morphism $\psi \circ \chi : A \to C$. The following axioms are satisfied:

- For each object $A$, there is an identity morphism $\iota_A : A \to A$ s.t.
  $$\iota_B \circ \chi = \chi = \chi \circ \iota_A.$$ 
- $(\chi \circ \psi) \circ \phi = \chi \circ (\psi \circ \phi)$.

Some examples:

**Vec** The objects are locally convex vector spaces. The morphisms are injective continuous linear maps.

**Alg** The objects are unital $C^*$-algebras. The morphisms are the unital injective $*$-homomorphisms.

A covariant functor $\mathcal{F}$ from $\textbf{Cat}_1$ to $\textbf{Cat}_2$ assigns to each object (morphism) of $\textbf{Cat}_1$ an object (morphism) of $\textbf{Cat}_2$, s.t.

- If $\chi : A \to B$, then $\mathcal{F}(\chi) : \mathcal{F}(A) \to \mathcal{F}(B)$.
- $\mathcal{F}(\iota_A) = \iota_{\mathcal{F}(A)}$.
- $\mathcal{F}(\chi \circ \psi) = \mathcal{F}(\chi) \circ \mathcal{F}(\psi)$. 

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The spacetime category

The basic category we will be considering is the category of spacetime manifolds: Man The objects are four-dimensional, (time-) oriented, connected, globally hyperbolic manifolds. The morphisms are (time-) orientation preserving isometric embeddings $\chi : M \to M'$, which are diffeomorphisms on their range, and which respect the causal structure in the following way: For all $x, y \in M$ all causal curves in $M'$ connecting $\chi(x)$ and $\chi(y)$ lie in $\chi(M)$.

An example for a covariant functor involving Man is the functor $\mathcal{D}$ from Man to Vec, which assigns to each $M$ the vector space $\mathcal{D}(M) = C_\infty^c(M)$, and to each morphism $\chi$ the pushforward morphism $\mathcal{D}(\chi) = \chi_*$, defined as

$$\chi_*(f)(x') = \begin{cases} f(\chi^{-1}(x')) & x' \in \chi(M), \\ 0 & x' \notin \chi(M). \end{cases}$$
A locally covariant field theory is a covariant functor $\mathcal{A}$ from $\text{Man}$ to $\text{Alg}$. It is causal, if
\[ [\mathcal{A}(\chi_1)(\mathcal{A}(M_1)), \mathcal{A}(\chi_2)(\mathcal{A}(M_2))] = \{0\} \]
whenever there are morphisms $\chi_i : M_i \to M$ such that $\chi(M_1)$ and $\chi(M_2)$ are spacelike. It fulfills the time-slice axiom, if
\[ \mathcal{A}(\chi)(\mathcal{A}(M)) = \mathcal{A}(M'), \]
whenever $\chi(M)$ contains a Cauchy surface of $M'$. 
An example: The Klein-Gordon field

The Klein-Gordon field is characterized by the wave equation

\[ P\varphi = (g^{\mu\nu}\nabla_\mu \partial_\nu + m^2 + \xi R)\varphi = 0. \]

The operator \( P \) has unique retarded and advanced propagators \( \Delta_{r/a} \), and one defines the causal propagator \( E = \Delta_a - \Delta_r \). It is a map \( E : C_c^\infty (M) \to C^\infty (M) \). It is important to note that, for a morphism \( \chi : M \to M' \), \( E = \chi^* E' \), where \( E' \) is the causal propagator on \( M' \). Denote by \( R \) the range of \( E \) and define, for \( Ef, Eh \in R \),

\[ \sigma(Ef, Eh) \doteq \int_M f(x) (Eh)(x) d_g x. \]

Then \( (R, \sigma) \) is a symplectic space, to which one associates the corresponding CCR algebra \( \mathcal{A}(M) \doteq CCR(\mathcal{R}, \sigma) \), i.e., the C*-algebra generated by \( W(Ef) \), subject to

\[ W(Ef)W(Eh) = W(Ef + Eh)e^{-\frac{i}{2}\sigma(Ef, Eg)}, \quad W(Ef)^* = W(-Ef). \]

One also sets \( \mathcal{A}(\chi)(W(Ef)) \doteq W(E'\chi^* f) \). This defines a locally covariant field theory. The theory is causal, as \( E(f, h) = 0 \) for \( f, h \) with spacelike support. It fulfills the time-slice axiom, as \( Ef \) is characterized by \( Ef, \partial_t (Ef) \) on an arbitrary Cauchy surface.
Recovering the Haag-Kastler axioms

One can recover the Haag-Kastler axioms on a fixed globally hyperbolic spacetime $M$. Define $K(M)$ as the set of relatively compact causal subsets. For each $O \in K(M)$, there is a canonical morphism $\chi_O : O \to M$. Given a locally covariant field theory $\mathcal{A}$, we can then define a net $\mathcal{A}$ by

$$\mathcal{A}(O) = \mathcal{A}(\chi_O)(\mathcal{A}(O)).$$

If $\mathcal{A}$ is causal, then $\mathcal{A}$ is local. If there is a group $G$ of isometric diffeomorphisms of $M$, then there is a representation on $\mathcal{A}$ given by

$$\alpha_g(A) = \mathcal{A}(\chi_g)(A),$$

where $\chi_g : M \to M$ is the morphism corresponding to the group element $g \in G$. It fulfills

$$\alpha_g(\mathcal{A}(O)) = \mathcal{A}(gO).$$

Hence, up to the spectrum (stability) condition, one recovers the Haag-Kastler axioms.
Fields I

Up to now, we considered covariant assignments of algebras to spacetimes. However, we would also like to have distinguished elements of these algebras that are mapped into each other. These are the fields. As these are typically unbounded, one has to allow for topological $\star$-algebras instead of $C^*$-algebras.

A natural transformation $\Phi$ between two functors $\mathcal{F}$, $\mathcal{G}$ from $\text{Cat}_1$ to $\text{Cat}_2$ assigns to each object $A$ of $\text{Cat}_1$ a morphism $\Phi_A : \mathcal{F}(A) \to \mathcal{G}(A)$ of $\text{Cat}_2$ such that, if $\chi : A \to B$ is a morphism of $\text{Cat}_1$, one has

$$\Phi_B \circ \mathcal{F}(\chi) = \mathcal{G}(\chi) \circ \Phi_A.$$ 

A topological $\star$-algebra is also a locally convex vector space, so $\mathcal{A}$ can also be seen as a covariant functor from $\text{Man}$ to $\text{Vec}$. Fields are defined as natural transformations between the functors $\mathcal{D}$ and $\mathcal{A}$, i.e., to each $M$ one associates a linear map $\Phi_M : \mathcal{D}(M) \to \mathcal{A}(M)$. Roughly, linear fields are given by

$$\Phi_M(f) := \left. \frac{d}{dt} \right|_{t=0} W(tEf).$$

Properly, one could define the fields via the Borchers-Uhlmann algebra or in the functorial approach of perturbative algebraic quantum field theory.
If one also wants fields for Wick powers involving derivatives, such as $\partial_\mu \phi \partial_\nu \phi$, one has to extend the setting. Instead of $\mathcal{D}$ one uses the covariant functor $\mathcal{T}_c$ between $\text{Man}$ and $\text{Vec}$, defined by

$$\mathcal{T}_c \doteq \{(t_0, \ldots, t_j, 0, \ldots) | t_k \in \Gamma_c^\infty (M, \text{Sym}^k T \otimes M)\},$$

$$T \otimes M \doteq \bigoplus_{i=0}^\infty \text{Sym}^i TM.$$

Such fields would be necessary to define the stress-energy tensor. However, this requires renormalization, which is beyond the scope of the present talk.
The substitute for the requirement of stability is the condition that there exist quasifree Hadamard states, i.e., states $\omega$ given by

$$\omega(W(Ef)) = e^{-\omega_2(f,f)},$$

whose two-point function $\omega_2$ fulfills the microlocal spectrum condition

$$WF(\omega_2) \subset \{(x, x'; k, -k') \in T^*M^2 \setminus \{0\} | (x, k) \sim (x', k'), k \in \bar{V}_x^+ \}.$$ 

Here $(x, k) \sim (x', k')$ if there exists a lightlike geodesic from $x$ to $x'$ to which $k$ and $k'$ are co-tangent and co-parallel.

The set of states that are locally quasiequivalent to any quasifree Hadamard state is then a covariant state space (defined as a contravariant functor).
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Motivation

Up to now, the background consisted only of the metric, and the morphisms corresponded to isometries. We want to extend the framework to allow for other non-trivial background fields (in particular gauge fields) and to incorporate also gauge transformations into the categorial framework. This allows to describe gauge covariant fields, such as charged Klein-Gordon or Dirac fields. This is also the natural framework for describing dynamical Yang-Mills gauge fields.
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A Lie group $G$ is a group that is at the same time a smooth manifold, such that multiplication and inverse are smooth maps. We denote by $L_g$ the left translation map acting on $f \in C^\infty(G)$ by $(L_g f)(h) = f(g^{-1}h)$. A vector field $\Xi$ on $G$ is left-invariant, if $(L_g \circ \Xi)(f) = (\Xi \circ L_g)(f)$ for all $f$.

The corresponding Lie algebra $\mathfrak{g}$ is the tangent space $T_e G$ at the identity, with Lie bracket given by the Lie bracket of tangent vectors, $[\eta, \xi] f = \eta \xi f - \xi \eta f$. It is isomorphic to the algebra of left-invariant vector fields on $G$. The adjoint action of $g \in G$ on $\mathfrak{g}$ is defined by $\text{ad}_g \xi = \xi \circ \text{Ad}_g$, where $\text{Ad}_g$ acts on $f \in C^\infty(G)$ by $(\text{Ad}_g f)(h) = f(g^{-1}hg)$.

A representation $\rho$ of $G$ on a vector space $V$ is a smooth group homomorphism $\rho : G \to \text{Aut}(V)$, i.e. $\rho(g) : V \to V$ is linear and $\rho(gh) = \rho(g) \circ \rho(h)$. 
Let $G$ be a Lie group. A principal $G$-bundle $P$ over a base space $M$ is a manifold $P$, a smooth map $\pi : P \to M$, a right $G$ action on $P$ fulfilling $\pi(pg) = \pi(p)$, and a set local trivializations, i.e., an open covering $\{U_i\}_{i \in I}$ of $M$ and diffeomorphisms $\phi_i : U_i \times G \to \pi^{-1}(U_i)$ such that $\pi(\phi_i(x, g)) = x$ and $\phi_i(x, gg') = \phi_i(x, g)g'$.

For any base space $M$ the Cartesian product $M \times G$ is a principal $G$-bundle. If the base space has a non-trivial topology, non-trivial principal $G$-bundles are possible. A famous example is the Hopf bundle, in which $P = S^3$, $M = S^2$, $G = S^1 = U(1)$.

There is a one-to-one correspondence between trivializations $\phi_i$ and sections $s_i \in \Gamma^\infty(U_i, P)$ given by $s_i(x) = \phi_i(x, e)$.

The action of $G$ on $P$ induces an action on $C^\infty(P)$, given by $(\sigma_g f)(p) = f(pg)$.

This induces a map $(\sigma_g)_* : T_{pg^{-1}}P \to T_pP$ by $(\sigma_g v)(f) = v(\sigma_g f)$.

For $x \in M$, the set $\pi^{-1}(x)$ is called the fiber at $x$. The action of $G$ on $P$ determines, at each $p \in P$, the vertical subspace $\text{Ver}_p$ of $T_pP$ corresponding to movements along the fiber. Any $\xi \in g$ determines a vertical vector field $\xi^\#$ on $P$. 
A **connection** on a principal $G$ bundle $P$ is a $g$-valued one-form $A$, i.e., a linear map $A : TP \to g$, fulfilling

- $A(\xi^\#(p)) = \xi$ for all $\xi \in g$, $p \in P$,
- $\text{ad}_{g^{-1}} \circ A = A \circ \sigma_g$.

The choice of a connection corresponds to an equivariant choice of **horizontal subspaces** $\text{Hor}_p = \ker A_p$ of $T_pP$ such that $T_pP = \text{Ver}_p \oplus \text{Hor}_p$.

Given a curve $c : \mathbb{R} \to M$ and a starting point $p_0 \in \pi^{-1}(c(0))$, there is a unique lift to a curve $\tilde{c} : \mathbb{R} \to P$ such that $\tilde{c}(0) = p_0$, $\pi(\tilde{c}(t)) = c(t)$ and $\dot{\tilde{c}}(t) \in \text{Hor}_{\tilde{c}(t)}$.

Given a section $s_i \in \Gamma^\infty(U_i, P)$, one may define $A_i \in \Omega^1(U_i, g)$ by the pull-back $A_i \doteq s_i^* A$. Usually, one identifies gauge transformations with changes of the section.
Let $V$ be a vector space carrying a representation $\rho$ of $G$. One defines the associated vector bundle $B_\rho \doteq P \times_\rho V$ as the manifold $P \times V$ modulo the equivalence relation $(pg, v) \sim (p, \rho(g)v)$ and the base projection map $\pi_\rho : B_\rho \to M$ by $\pi_\rho(p, v) = \pi(p)$. The equivalence classes are denoted by $[p, v]$. A trivialization $\phi_i : U_i \to \pi^{-1}(U_i)$ of $P$ induces a trivialization $\tilde{\phi}_i : U_i \times V \to \pi^{-1}_\rho(U_i)$ of $B_\rho$ by $\tilde{\phi}_i(x, v) = [\phi_i(x, e), v]$.

Given a connection $A$ on $P$ and a curve $c : \mathbb{R} \to M$, one defines the parallel transport of $[p, v] \in B_\rho|_{c(0)}$ along this curve as $[p(t), v]$, where $p(t)$ is the lift of the curve $c$ with starting point $p$. The corresponding covariant derivative $\nabla : \Gamma^\infty(M, B_\rho) \otimes TM \to B_\rho$ is, in a local trivialization, given by $(X \in T_x M)$

$$\nabla_X[s_i, v] = [s_i, (\partial_X + \rho(A_i(X)))v](x).$$

Sections of associated bundles describe fields charged under the gauge group $G$. The covariant derivative (and hence the connection) appears in the wave operator for these fields. Hence, not only the gravitational, but also gauge background fields determine the wave operator. It is thus natural to put them on equal footing with the gravitational background.
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We now propose to consider the following category:

**GMan** The objects are tuples \((P, A)\) where \(P\) is a principal \(G\)-bundles whose base space is an object of \(\text{Man}\), and \(A\) is a connection on \(P\). The morphisms are smooth maps \(\eta : P \to P'\), which fulfill:

- \(\eta(pg) = \eta(p)g\),
- \(\chi = \pi \circ \eta\) is a morphism of \(\text{Man}\),
- \(A = \eta^* A'\).

An example for a morphism is the map \(\eta : P \to P\), which in a trivialization, is given by \(\eta_i(x, g) = (x, h_i g)\), for some smooth \(h_i \in C^\infty(U_i, G)\). This corresponds to a gauge transformation.

A locally gauge covariant theory is then a covariant functor from \(\text{GMan}\) to \(\text{Alg}\).
For the charged Klein-Gordon field, we take $G = U(1)$ and $\rho$ the fundamental representation on $V = \mathbb{C}$. The wave operator is

$$P = g^{\mu\nu} \nabla_\mu \nabla_\nu + m^2 + \xi R,$$

where $\nabla$ is the covariant derivative. The corresponding causal propagator is a map $E : \Gamma_\infty^c(M, B_\rho) \to \Gamma_\infty(M, B_\rho)$, which is covariant.

In order to include the antifield, we also consider the dual bundle, i.e., we use the representation $\rho \times \rho^*$ on $\mathbb{C} \times \mathbb{C}^*$. Analogously to the above, we have a wave operator $P : \Gamma_\infty(M, B_\rho \times \rho^*) \to \Gamma_\infty(M, B_\rho \times \rho^*)$ and the corresponding causal propagator $E : \Gamma_\infty^c(M, B_\rho \times \rho^*) \to \Gamma_\infty(M, B_\rho \times \rho^*)$.

On the fiber $\mathbb{C} \times \mathbb{C}^*$, the pairing $\mathbb{C}^* \times \mathbb{C} \to \mathbb{C}$ induces an inner product

$$\langle (v, v'), (w, w') \rangle = \langle v', w \rangle + \langle w', v \rangle.$$

This induces a pairing $\Gamma_\infty(M, B_\rho \times \rho^*) \times \Gamma_\infty(M, B_\rho \times \rho^*) \to \mathcal{C}_\infty(M)$. 
We can now define a symplectic space as $R \doteq \mathcal{E}\Gamma_{c}^{\infty}(M, B_{\rho \times \rho^{*}})$ with the symplectic form

$$\sigma(Ef, Eh) \doteq \int_{M} \langle f, Eh \rangle (x) dg x.$$  

The corresponding CCR algebra is denoted by $\mathcal{A}(P, A)$. The assignment $(P, A) \mapsto \mathcal{A}(P, A)$ is then a locally covariant gauge theory.
For the definition of gauge covariant bosonic fields we may define the following space of test sections:

$$\mathcal{T}_c \doteq \{(t_0, \ldots, t_j, 0, \ldots) | t_k \in \Gamma_c^\infty(M, \text{Sym}^k(B_{\rho \times \rho^*} \otimes T \otimes M))\}.$$  

This is a covariant functor between the categories $\text{GMan}$ and $\text{Vec}$. We can now define gauge covariant fields as natural transformations between $\mathcal{T}_c$ and $\mathcal{A}$. 

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The concept is also useful on Minkowski space, as it allows for the definition of a locally and covariantly constructed current for charged fields in the presence of an electromagnetic background field. This is can be applied to study backreaction effects, such as spontaneous pair creation.

The proposed framework seems to be useful also for the quantization of Yang-Mills fields. The space of connections is not a linear space, but is affine: After the choice of a background connection $\bar{A}$, all other connections are parameterized by elements of $\Gamma^\infty(M, P \times_{\text{ad}} \mathfrak{g} \otimes \Omega^1)$. This is a linear space. The background connection induces a covariant derivative on this space, which allows to formulate the Lorentz gauge in a covariant way as $\nabla^\mu A^\mu = 0$. 
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Summary

- Locally covariant field theory as a framework for the coherent definition of a quantum field theory on all spacetimes.
- Extension to gauge backgrounds, i.e., a framework for the coherent definition of a charged quantum field theory on all spacetimes and gauge potentials.
- Uses the language of connections on principal bundles.
Thank you for your attention!