The renormalized locally covariant Dirac field

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I present a recent construction of the renormalized locally covariant Dirac field [arXiv:1210:4031]. With respect to earlier approaches the main progress is

- a covariant treatment of Wick powers,
- the definition of time-ordered products,
- the inclusion of gauge and Yukawa background fields.

The latter is achieved by a generalization of the generally covariant locality principle, so that gauge backgrounds and gauge transformations are treated on equal footing with gravitational backgrounds and isometric embeddings. This framework seems to be very well suited to the treatment of Yang-Mills fields.
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Spinorial field are an essential ingredient of the standard model. Hence, it is highly desirable to formulate them as local, covariant fields in the framework of the generally covariant locality principle [Brunetti, Fredenhagen, Verch 03]. For linear fields, this was accomplished by Sanders. Non-linear fields (Wick powers) were treated by Dappiaggi, Hack, and Pinamonti (with some caveats).

The spinorial fields occurring in the standard model are gauged under some compact gauge group $G$, i.e., they live in a vector bundle associated to some principal $G$ bundle. We want to account for that fact and to allow for nontrivial background connections. Furthermore, we want to define covariant Wick powers and time-ordered products, in order to allow for the description of interactions.
The generally covariant locality principle

One defines the following categories:

**Vec**<sub>(i)</sub> The objects are locally convex vector spaces. The morphisms are (injective) continuous linear maps.

**Alg** The objects are unital topological ∗-algebras. The morphisms are the unital injective ∗-homomorphisms.

**Man** The objects are \(n\)-dimensional, (time-) oriented, connected, globally hyperbolic manifolds \((-\), \(+\), \(\ldots\), \(+\)) \((−, +, \ldots, +)\). The morphisms are (time-) orientation preserving isometric embeddings \(χ : M \rightarrow M'\), which are diffeomorphisms on their range, and which respect the causal structure in the following way: For all \(x, y \in M\) all causal curves in \(M'\) connecting \(χ(x)\) and \(χ(y)\) lie in \(χ(M)\).

A locally covariant field theory is a covariant functor \(\mathfrak{A}\) from **Man** to **Alg**. Let \(\mathfrak{F}_c(M) = Γ_c^\infty(M, T^{\oplus}M)\). This assignment is a covariant functor from **Man** to **Vec**<sub>(i)</sub>, where an isometric embedding is mapped to the push-forward. A locally covariant field is a natural transformation from \(\mathfrak{F}_c\) to \(\mathfrak{A}\). Examples are \((f \in C_c^\infty(M), t \in Γ_c^\infty(M, T^2M))\)

\[
Φ_M(f) = \int_M Φ(x)f(x)dgx,
Φ'_M(t) = \int_M :∂_μΦ∂_νΦ:(x)t^{μν}(x)dgx.
\]
The spinor representation

Let us recall some basic facts about the Spin group. Denote by $\text{Cl}(n)$ the real Clifford algebra for the standard bilinear form on $\mathbb{R}^n$ with signature $(-, +, \ldots, +)$. Identifying $\mathbb{R}^n$ with a subspace of $\text{Cl}(n)$, the $\text{Spin}(n)$ group is defined as

$$\text{Spin}(n) \doteq \{ s \in \text{Cl}(n) \mid s = u_1 \ldots u_{2k}, u_i \in \mathbb{R}^n, u_i^2 = \pm 1 \}$$

and $\text{Spin}_0(n)$ is its connected component. By interpreting $s$ as a succession of reflections at the hyperplanes orthogonal to $u_i$, there is a canonical homomorphism $\lambda$ to $SO_0(n-1,1)$.

$\text{Cl}^c(n)$ is isomorphic to $\text{Mat}_\mathbb{C}(2^{[n/2]})$ for even $n$ and to $\text{Mat}_\mathbb{C}(2^{[n/2]}) \oplus \text{Mat}_\mathbb{C}(2^{[n/2]})$ for odd $n$. Restricting to the first summand for odd $n$, one obtains an irreducible representation of $\text{Cl}^c(n)$ on $\mathbb{C}^{[n/2]}$, with inner product given by

$$(v, w) = -i \langle v, e^0 w \rangle_{\mathbb{C}^{[n/2]}}.$$ 

The restriction of this representation to $\text{Spin}_0(n)$ is the spinor representation. One defines the adjoint $^+ : \mathbb{C}^{[n/2]} \to \mathbb{C}^{[n/2]}^*$ by

$$v^+ \doteq -iv^* e^0.$$
Spin structure and Dirac bundles

A spin structure \( SM \) over \( M \) is a principal \( \text{Spin}_0 \) bundle over \( M \) with a projection \( \pi_S : SM \to FM \) to the frame bundle, which intertwines the action of \( \text{Spin}_0 \):

\[
\pi_S \circ S = \lambda(S) \circ \pi_S.
\]

The basis for the definition of the uncharged locally covariant Dirac field is the following category [Sanders 10]:

**SpMan** The objects are spin structures \( SM \) whose base spaces \( M \) are objects of \( \text{Man} \). The morphisms are principal \( \text{Spin}_0 \) bundle morphisms \( \chi : SM \to SM' \), covering a morphism \( \psi \) of \( \text{Man} \), such that \( \pi'_S \circ \chi = \psi_* \circ \pi_S \).

**Definition:** A principal \( G \) bundle morphism \( \eta : P \to P' \) is smooth map \( \eta : P \to P' \), which is \( G \)-equivariant, \( \eta(pg) = \eta(p)g \).

The associated vector bundle \( DM \cong SM \times_\sigma \mathbb{C}^{[n/2]} \) corresponding to the spinor representation \( \sigma \) is the standard Dirac bundle. Using this assignment, one defines a covariant functor \( \mathcal{A} \) from \( \text{SpMan} \) to \( \text{Alg} \), specifying a spinorial field theory. Defining the functor \( \mathcal{D}(SM) \cong \Gamma_c^\infty(M, DM) \), one can define the linear Dirac field as a natural transformation from \( \mathcal{D} \) to \( \mathcal{A} \) [Sanders 10].
Introduction of the gauge group

In order to allow for fields charged under a gauge group $G$, in the presence of a connection and a Yukawa field, we define

$\textbf{GSpMan}$ The objects are quadruples $(SM, P, A, m)$, where $SM$ is an object of $\textbf{SpMan}$, $P$ a principal $G$ bundle over $M$, $A$ a connection on $P$, and $m \in C^\infty(M)$. Morphisms

$\chi : (SM, P, A, m) \rightarrow (SM', P', A', m')$ are tuples $(\chi_{SM}, \chi_P)$, where $\chi_{SM}$ is a morphism of $\textbf{SpMan}$, and $\chi_P$ is a principal $G$ bundle morphism $\chi_P : P \rightarrow P'$, which covers the same morphism $\psi$ of $\textbf{Man}$ as $\chi_{SM}$. Furthermore, $m = \psi^*m'$, $A = \chi_P^*A'$.

- With the spin connection, we have a unique connection on the principal $\text{Spin}_0 \times G$ bundle $SM + P$ over $M$.
- Taking $SM' = SM$, $P' = P$, $\chi_{SM} = \text{id}$ and, in a local trivialization, $\chi_P(x, g) = (x, h(x)g)$ for some $h \in C^\infty(M, G)$, we can describe gauge transformations within the categorial framework.
The charged Dirac bundle I

Given a representation $\rho$ of $G$ on a finite-dimensional $\mathbb{C}$ vector space $V$, we define the associated Dirac bundle as

$$D_{\rho} M \doteq (SM + P) \times_{\sigma \times \rho} \mathbb{C}^{[n/2]} \otimes V.$$ 

The corresponding dual bundle is denoted by $D_{\rho}^* M$ and the double spinor bundle by $D_{\rho} \oplus M \doteq D_{\rho} M \oplus D_{\rho}^* M$. We define the vector spaces

$$\mathfrak{C}(SM, P) \doteq \Gamma^\infty(M, D_{\rho}^* M),$$
$$\mathfrak{D}(SM, P) \doteq \Gamma^\infty(M, D_{\rho}^* M),$$
$$\mathfrak{C}(SM, P) \doteq \Gamma^\infty(M, D_{\rho} \oplus M),$$
$$\mathfrak{D}(SM, P) \doteq \Gamma^\infty_c(M, D_{\rho} \oplus M).$$

$\mathfrak{C}/^*/\oplus$ is a contravariant functor from $\text{GSpMan}$ to $\text{Vec}$ and $\mathfrak{D}/^*/\oplus$ is a covariant functor from $\text{GSpMan}$ to $\text{Vec}$. 

We define a bilinear pairing $\mathfrak{C}(SM, P) \times \mathfrak{C}(SM, P) \rightarrow \mathcal{C}^\infty(M)$ by

$$\langle (f, f'), (g, g') \rangle \doteq \langle g', f \rangle + \langle f', g \rangle, \quad f, g \in \mathfrak{C}(SM, P), f', g' \in \mathfrak{C}^*(SM, P).$$
As $G$ is compact, there is a non-degenerate sesquilinear form on $V$ that is conserved under $\rho$. Define the anti-linear map $+ : V \to V^*$ by $v^+(w) = \langle v, w \rangle_V$. The conjugation $+ : D_\rho M \to D^*_\rho M$ defined by

$[p, z \otimes v]^+ \doteq [p, z^+ \otimes v^+]$, \quad p \in SM + P, z \in \mathbb{C}^{[n/2]}, v \in V$

yields the antilinear conjugation maps

$+ : \mathcal{E}^*/(SM, P) \to \mathcal{E}^*/(SM, P), \quad + : \mathcal{E}^\oplus(SM, P) \to \mathcal{E}^\oplus(SM, P)$. 
The Dirac operator

The connection on $SM + P$ determines covariant derivatives $\nabla(\cdot)$ on $\mathcal{E}(\cdot)(SM, P)$ and $\mathcal{D}(\cdot)(SM, P)$. We may then define Dirac operators

$$D \doteq -\gamma^\mu \nabla_\mu + m, \quad D^* \doteq \gamma^\mu \nabla^*_\mu + m.$$  

These are covariant, i.e., $D(\cdot)$ intertwines the action of $\mathcal{E}(\cdot)(\chi)$ and $\mathcal{D}(\cdot)(\chi)$. The square of $D(\cdot)$ is a normally hyperbolic operator, so there are unique retarded/advances propagators $\Delta_{r/a}(\cdot) : \mathcal{D}(\cdot)(SM, P) \to \mathcal{E}(\cdot)(SM, P)$. Define

$$S_{r/a}(\cdot) \doteq D(\cdot) \circ \Delta_{r/a}(\cdot).$$  

One can show that not only $D(\cdot) \circ S_{r/a}(\cdot) = \text{id}$, but also $S_{r/a}(\cdot) \circ D(\cdot) = \text{id}$ [Dimock 82].

One defines $S(\cdot) \doteq S_{r}(\cdot) - S_{a}(\cdot)$ and

$$D^\oplus \doteq D \oplus -D^*, \quad S^\oplus \doteq S \oplus -S^*.$$  

The construction is functorial, i.e., $S(\cdot) = \mathcal{E}(\cdot)(\chi) \circ S'(\cdot) \circ \mathcal{D}(\cdot)(\chi)$. We may also see $S^\oplus$ as a distribution, $S^\oplus \in \Gamma^\infty_c(M^2, D^\oplus \rho M \boxtimes D^\oplus \rho M)'$, by

$$S^\oplus(u, v) \doteq \int \langle u, Sv \rangle(x) d_g x.$$
In the framework of perturbative algebraic quantum field theory, quantization consists in the deformation quantization of a (graded) commutative algebra of functionals on configuration space. For fermionic fields, it is natural to consider functionals on the space of antisymmetrized configurations \([\text{Rejzner 11}]\), i.e., on

\[
\wedge^{\oplus} \mathcal{E} \oplus (SM, P) = \bigoplus_{k=0}^{\infty} \wedge^k \mathcal{E} \oplus (SM, P),
\]

\[
\wedge^k \mathcal{E} \oplus (SM, P) \equiv \{B_k \in \Gamma^\infty(M^k, (D^\oplus M)^k) | B_k \text{ antisymmetric}\}.
\]

Functionals are now continuous linear maps \(F : \wedge^{\oplus} (SM, P) \rightarrow \mathbb{C}\), whose restriction to \(\wedge^k \mathcal{E} \oplus (SM, P)\) is denoted by \(F_k\). Furthermore, \(|F_k| \equiv k\).

The regular functionals, \(\mathcal{F}_{\text{reg}}(SM, P)\) are those of the form

\[
F_k(B) = \int \langle f_k, B_k \rangle (x_1, \ldots, x_k) d_g x_1 \ldots d_g x_k,
\]

with \(f_k \in \Gamma^\infty_c(M^k, D^\oplus M^k)\), \(f_k\) antisymmetric. We call \(f_k\) the kernel of \(F_k\).
We introduce an antisymmetric product $\wedge$ on $\mathcal{F}_{\text{reg}}(SM, P)$, by defining the kernel of the product $F \wedge G$ as

$$(f \wedge g)_k(x_1, \ldots, x_k)$$

$$= \sum_{l=0}^{k} \frac{1}{l!(k-l)!} \sum_{\pi \in S_k} (-1)^{|\pi|} f_l(x_{\pi(1)}, \ldots, x_{\pi(l)}) g_{k-l}(x_{\pi(l+1)}, \ldots, x_{\pi(k)}).$$

An involution on $\mathcal{F}_{\text{reg}}(SM, P)$ is defined as

$$F^*(B) \doteq F(B^+).$$

In microcausal functionals $\mathcal{F}(SM, P)$, the kernels are allowed to be compactly supported distributions, fulfilling $\text{WF}(f_k) \cap (\bar{V}_+^k \cup \bar{V}_-^k) = \emptyset$. This space can be equipped with a locally convex topology [Brunetti, Dütsch, Fredenhagen 09]. In local functionals $\mathcal{F}_{\text{loc}}(SM, P)$ the kernels $f_k$ are supported on the diagonal $D_k$ and fulfill $\text{WF}(f_k) \perp TD_k$. $\mathcal{F}_{\text{loc}}$ is a covariant functor from $\text{GSpMan}$ to $\text{Alg (Vec}_i)$. 
Denote by $\mathfrak{F}_0(SM, P, A, m)$ the ideal of functionals that vanish on all configurations $B \in \wedge \mathfrak{c}^\oplus(SM, P)$ fulfilling $D^\oplus B = 0$. The algebra of on-shell functionals is defined as

$$\mathfrak{F}^S(SM, P, A, m) \doteq \mathfrak{F}(SM, P)/\mathfrak{F}_0(SM, P, A, m).$$

Again, this is a covariant functor from $\text{GSpMan}$ to $\text{Alg}$.

One defines a functional derivative as [Rejzner 11]

$$F^{(1)}(B)(u) \doteq F(u \wedge B), \quad B \in \wedge \mathfrak{c}^\oplus(SM, P), u \in \mathfrak{c}^\oplus(SM, P).$$

Hence, $F^{(1)}(B)$ can be interpreted as a compactly supported distributional section of $D^\oplus_{\rho} M$. We denote its integral kernel by $F^{(1)}(B)(x)$. For $F \in \mathfrak{F}_{\text{reg}}$, this is even a smooth section.
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Quantization of regular functionals

One defines the Peierls bracket of two observables $F, G \in \mathcal{F}$ as

$$[F, G] \doteq (-1)^{|F|+1} \int F^{(1)}(x) \wedge G^{(1)}(y) S^{\oplus}(x, y) d_g x d_g y.$$ 

This is also well-defined on on-shell functionals. In deformation quantization, one aims at finding a product $\star$, fulfilling

$$F \star G = F \wedge G + \mathcal{O} (\hbar), \quad F \star G - (-1)^{|F||G|} G \star F = i\hbar [F, G] + \mathcal{O}(\hbar^2),$$

in the sense of formal power series in $\hbar$. For regular functionals, define

$$\Gamma_{\frac{i}{2} S} \otimes (F \otimes G) \doteq (-1)^{|F|+1} \frac{i}{2} \int F^{(1)}(x) \otimes G^{(1)}(y) S^{\oplus}(x, y) d_g x d_g y,$$

$$F \star G \doteq \wedge \exp (\hbar \Gamma_{\frac{i}{2} S}) F \otimes G.$$

Here the wedge denotes the wedge product, $\wedge (F \otimes G) \doteq F \wedge G$. The assignment

$$(SM, P, A, m) \mapsto (\mathcal{O}_{\text{reg}}(SM, P, A, m)[[\hbar]], \star)$$

is a covariant functor from $\text{GSpMan}$ to $\text{Alg}$.
Hadamard functions

Hadamard two-point functions are distributional sections \( \omega \in \Gamma^\infty_c(M^2, D^\oplus \rho M^2)' \) s.t.

\[
\omega(D^\oplus u, v) = 0,
\]

\[
\omega(u, v) + \omega(v, u) = iS^\oplus(u, v), \quad \text{(AC)}
\]

\[
\omega(u, v) = \omega(v^+, u^+), \quad \text{(CON)}
\]

\[
\WF(\omega) \subset C_+, \quad \text{(WF)}
\]

where \( u, v \in \Gamma^\infty_c(M, D^\oplus \rho M) \) and

\[
C_\pm \doteq \{(x_1, x_2; k_1, -k_2) \in T^*M^2 \setminus \{0\}| (x_1; k_1) \sim (x_2; k_2), k_1 \in \bar{V}_x^\pm\}.
\]

\((x_1; k_1) \sim (x_2; k_2)\) if there is a lightlike geodesic joining \( x_1 \) and \( x_2 \) to which \( k_1 \) and \( k_2 \) are co-tangent and co-parallel. For \( x_1 = x_2, k_1, k_2 \) are lightlike and coinciding.

Proposition: Two Hadamard two-point functions \( \omega, \omega' \) differ by a smooth section.

Proof: Set \( \omega_{s/a}(u, v) \doteq \frac{1}{2}(\omega(u, v) \pm \omega(v, u)). \) If \( \Xi \in \WF(\omega_{s/a} - \omega'_{s/a}) \), then also \(-\Xi. \) By (AC), \( \WF(\omega_s - \omega'_s) = \emptyset. \) By (WF) \( \WF(\omega - \omega') \subset C_+ \), and as \( C_+ \cap C_- = \emptyset \) it follows that \( \WF(\omega_a - \omega'_a) = \emptyset \) and thus \( \WF(\omega_a - \omega'_a) = \emptyset. \)
Extension to microcausal functionals

We define

$$\Gamma_{\omega} F \doteq \int d_g x d_g y \, \omega_a(x, y) F^{(2)}(x, y),$$

$$F \ast_{\omega} G \doteq \exp(\hbar \Gamma_{\omega}) \left( \exp(-\hbar \Gamma_{\omega}) F \right) \ast \exp(-\hbar \Gamma_{\omega}) \left( \exp(\hbar \Gamma_{\omega}) G \right).$$

$\ast_{\omega}$ is equivalent to $\ast$ and well-defined on $\mathcal{F}$ and $\mathcal{F}^S$. There is no covariant choice of $\omega$. In order to restore covariance, consider the set $\text{Had}(SM, P, A, m)$ of Hadamard two-point functions and define $\mathcal{A}(SM, P, A, m)$ as the space of families

$$F = \{F_\omega\}_{\omega \in \text{Had}(SM, P, A, m)}, \quad F_\omega \in \mathcal{F}(SM, P)[[\hbar]]$$

fulfilling

$$F_{\omega'} = \exp(\hbar \Gamma_{\omega'} - \omega_a) F_\omega.$$

$F \in \mathcal{A}(SM, P, A, m)$ is specified by $F_\omega$ for a single $\omega \in \text{Had}(SM, P, A, m)$. Define

$$(F \ast G)_\omega \doteq F_\omega \ast_{\omega} G_\omega.$$

The assignment $(SM, P, A, m) \mapsto (\mathcal{A}(SM, P, A, m), \ast)$ is a covariant functor from $GSpMan$ to $Alg$, mapping a morphism $\chi$ to the morphism $\chi_{\ast}$ defined by

$$(\chi_{\ast} F)_\omega \doteq \chi_{\ast}(F_\omega|_{M \times M}).$$
**Existence of Hadamard two-point functions**

**Proposition:** For each \((SM, P, A, m)\), Hadamard two-point functions exist.

**Proof (sketch):** We adapt the deformation argument [Fulling, Narcowich, Wald 81]: Fix a Cauchy surface \(\Sigma\) of \(M\) and deform the metric on \(J^- (\Sigma)\) such that, for a Cauchy surface \(\Sigma' \subset J^- (\Sigma)\) (w.r.t. the deformed metric), \(J^- (\Sigma')\) is ultrastatic, i.e., there are coordinates \((t, x)\) s.t. \(g = -dt^2 + h(x)_{ij} dx^i dx^j\). We also deform \(A\) s.t. \(J^- (\Sigma') \cong I \times \Sigma'\) can be covered by open sets \(U_i = I \times V_i\), with sections \(s_i \in \Gamma^\infty (U_i, P)\) s.t. \(A_i \equiv s_i^* A\) fulfills

\[
A_{i,0} = 0, \quad A_{i,a}(t, x) = A_{i,a}(x).
\]

One also requires \(m\) to be time-independent on \(J^- (\Sigma')\). The Dirac equation reads

\[
i \partial_t \psi + K \psi = 0, \quad K \psi \equiv i \gamma^0 \gamma^a (\partial_a - i \sigma_a - i A_a) \psi - i \gamma^0 m \psi.
\]

\(K\) is an essentially self-adjoint operator on \(L^2(\Sigma', D^\rho J^- (\Sigma')|_{\Sigma'})\) [Chernoff 73]. This yields a CAR algebra for \(J^- (\Sigma')\) [Araki 70] and the corresponding two-point functions is Hadamard [Sahlmann, Verch 00]. Using the equation of motion, one transports it to \(J^+ (\Sigma)\) and from there to all of \(M\), where it is Hadamard [Sanders 10].

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The linear field

In order to define fields, we need an appropriate space of test sections.

$$\mathcal{T}_c(SM, P) \doteq \Gamma_c^\infty(M, \wedge(D^\rho \oplus M \otimes T^\oplus M)), \quad T^\oplus M \doteq \bigoplus_k \text{Sym}^k TM.$$ 

$\wedge$ denotes the exterior tensor product. This is a covariant functor from $\text{GSpMan}$ to $\text{Vec}_i$. $\mathcal{T}_c^j A$ denotes the subspace where the $j$th exterior power is taken, and $A \in \mathbb{N}_0^i$ counts the tensor power corresponding to $T^\oplus M$ in each of the factors. For example, $\mathcal{T}_c^{10} = \mathcal{D}^\oplus$.

Fields are natural transformations from $\mathcal{T}_c$ to $\mathcal{A}_{loc}$. An example is

$$\psi_{(SM,P)}(u)(B) \doteq \int \langle u, B_1 \rangle(x) dg x, \quad u \in \mathcal{T}_c^{10}(SM, P),$$

which is a natural transformations $\mathcal{T}_c^{10} \rightarrow \mathcal{A}_{loc}$. It fulfills

$$\psi(u)^* = \psi(u^+) \quad \psi(u) \star \psi(v) + \psi(v) \star \psi(u) = i\hbar S^\oplus(u, v).$$

By choosing $u$ to be a pure cospinor (spinor), one obtains the usual spinor (cospinor) fields, which, in abuse of notation, we occasionally denote by $\psi (\psi^+)$. 

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Nonlinear fields can be defined if one has a covariantly defined distribution $H$ s.t. $\omega - H$ is smooth [Hollands, Wald 01]. These are the parametrices.

**Definition:** A parametrix is a covariant assignment

$$(SM, P, A, m) \mapsto H \in \Gamma_c^\infty(U, D^\oplus_{\rho} M \boxtimes D^\oplus_{\rho} M)'$$

where $U$ is a neighborhood of the diagonal of $M \times M$, such that (AC), (CON), (WF) hold. Covariance here means that for $\chi : D^\oplus_{\rho} M \to D^\oplus_{\rho} M'$ the bundle morphism corresponding to a morphism $\chi : (SM, P, A, m) \to (SM', P', A', m')$ we have that $H - \chi^* H'$ is smooth on the common domain and vanishing at the diagonal, together with all the derivatives.

**Proposition:** $H - \omega$ is smooth on $U$ for any parametrix $H$ and any Hadamard two-point function $\omega$. 

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Given a parametrix $H$, we may define a natural transformation $\alpha_H$ between $\mathcal{F}_{1oc}$ and $\mathcal{A}_{1oc}$, by

$$(\alpha_H F)\omega \doteq \exp(\hbar \Gamma_{\omega - H}) F.$$ 

This is well-defined, as we only act on local functionals.

There is a natural transformation $\Psi : \mathcal{T}_c \to \mathcal{F}_{1oc}$, given by

$$\Psi_{(SM,P)}(t)(B) \doteq \sum_{k=0}^{\infty} \int \left\langle t^{\mu_1...\mu_k}, \nabla^\oplus_{(\mu_1)} ... \nabla^\oplus_{(\mu_k)} B_k \right\rangle (x_1, ..., x_k) d_g x_1 ... d_g x_k,$$

where $\nabla^\oplus_{(\mu_i)}$ denotes the symmetrized covariant derivative on the $i$th coordinate, with $\nabla^\oplus \doteq \nabla \oplus \nabla^*$. Composition, $\alpha_H \circ \Psi|_{\mathcal{T}_c^{jA}}$, yields fields, the Wick powers.

As $H$ is not unique (one may always add smooth, covariant, locally constructed terms), the Wick powers are not unique.
Existence of parametrices

Definition: A pre-parametrix $H^\pm$ is a covariant assignment

$$(SM, P, A, m) \rightarrow H^\pm \in \Gamma_c(\mathcal{U}, D^*_\rho \mathcal{M} \boxtimes D_\rho \mathcal{M})$$

such that

$$H^+ - H^- - iS \in \Gamma^\infty,$$

$$\text{WF}(H^\pm) \subset C^\pm.$$ 

Given a pre-parametrix $H'$, we define a parametrix $H$ by setting

$$H(f', f) = \frac{1}{2} \left( H^+(f', f) + \overline{H^+(f^+, f'^+)} \right) - \frac{1}{4} \left( r(f', f) + \overline{r(f^+, f'^+)} \right),$$

$$H(f, f') = -\frac{1}{2} \left( H^-(f', f) + \overline{H^-(f^+, f'^+)} \right) - \frac{1}{4} \left( r(f', f) + \overline{r(f^+, f'^+)} \right),$$

$$H(f, f) = 0,$$

$$H(f', g') = 0,$$

where $f, g \in \mathcal{D}(SM, P), f', g' \in \mathcal{D}^*(SM, P)$, and $r$ is the remainder term in ($\ast$).

Proposition: On suitably chosen neighborhoods $U$ of the total diagonal, pre-parametrices exist.
Defining $\tilde{D} = -\gamma^\mu \nabla_\mu - m$, the operator $D\tilde{D}$ on $\Gamma^\infty(M, D\rho M)$ is normally hyperbolic, with vanishing first order component. On each causal domain $\Omega$, there are Hadamard coefficients $V_k \in \Gamma^\infty(\Omega \times \Omega, D\rho M \boxtimes D^*\rho M)$, recursively defined by

$$\nabla_{\nabla^\Gamma} V_k - \left(-\frac{1}{2} \nabla^\mu \partial_\mu \Gamma - n + 2k\right) V_k = 2kPV_{k-1},$$

with the initial condition $V_0(x, x) = \text{id}_{D\rho M_x}$. Here $\Gamma(x, x')$ is the negative of the squared geodesic distance along the unique geodesic connecting $x$ and $x'$. There are retarded/advances propagators $\Delta_{r/a}$ for $D\tilde{D}$, which can be approximated up to regularity $\Gamma^k$ by [Bär, Ginoux, Pfäffle 07]

$$\Delta^k_{r/a} \coloneqq \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - 1 + k} V_j R_{r/a}(2j + 2)$$

in terms of Riesz distributions $R_{r/a}(j)$. The approximation of the retarded/advances propagators for $D$ are then

$$S^k_{r/a} \coloneqq \tilde{D} \Delta^{k+1}_{r/a}.$$
Existence of parametrices: Sketch of proof II

There are $T_{\pm}(j)$, s.t. for $j \in \{2, 4, 6, \ldots \},$

\[
T_{+}(j) - T_{-}(j) = 2\pi i(R_{r}(j) - R_{a}(j)), \quad \text{WF}(T_{\pm}(j)) \subset C_{\pm}.
\]

For $n$ even, these are given by

\[
 T_{\pm}(j) \doteq \lim_{\varepsilon \to +0} C'(j, n)(-\Gamma \mp i\varepsilon\theta_{0} + \varepsilon^{2})^{\frac{j-n}{2}}
\]

for $j < n,$

\[
 T_{\pm}(j) \doteq \lim_{\varepsilon \to +0} C(j, n)\Gamma^{\frac{j-n}{2}} \log(-\Gamma \mp i\varepsilon\theta_{0} + \varepsilon^{2})/\Lambda^{2}
\]

for $j \geq n.$

For a given $\Omega' \subseteq \Omega$ and a bump function $\sigma,$ there is a sequence $\{\varepsilon_{i}\}, \varepsilon_{i} > 0,$ s.t.

\[
\sum_{j=n/2-1}^{\infty} C(2j + 2, n)\Gamma^{j-(n/2-1)}\sigma(\Gamma/\varepsilon_{j})V_{j}
\]

converges to a smooth section on $\Omega' \times \Omega'$ [Bär, Ginoux, Pfäffle 07]. Then

\[
h_{\pm} = \frac{1}{2\pi} \left( \sum_{j=0}^{n/2-2} V_{j} T_{\pm}(2j + 2) + \sum_{j=n/2-1}^{\infty} \chi(\Gamma/\varepsilon_{j})V_{j} T_{\pm}(2j + 2) \right)
\]

converges to a parametrix for $D\tilde{D}$ on $\Omega' \times \Omega'$. Glue these together to cover a nbh $U$ of the diagonal and set $H_{\pm} = \tilde{D}h_{\pm}.$
An application: The current

For electrodynamics \((G = U(1), \rho \text{ the fundamental representation})\), the current is

\[
j^\mu = \text{tr} \, \psi^+ \gamma^\mu \psi.
\]

Assume that \(\partial_\mu j^\mu\) is not weakly vanishing, i.e., not vanishing on all on-shell configurations. As \(\partial_\mu j^\mu\) vanishes weakly classically, the violation must be of \(O(\hbar)\) and a c-number. Let us write the violation as

\[
\partial_\mu j^\mu = \partial_\mu Q^\mu.
\]

As the construction was local and covariant, \(Q^\mu\) must be a local and covariant co-vector. Hence, we may simply use a new parametrix

\[
H'(x, x') = H(x, x') - 2^{-[n/2]} \gamma^\mu Q_\mu(s(x, x')),
\]

where \(s(x, x')\) is half-way on the unique geodesic from \(x\) to \(x'\). This ensures current conservation.

Further redefinitions are only possible with a covariant, conserved covector. For this, only \(J^\mu\), the current responsible for the background field, is available. Hence, \(j^\mu\) is unique up to multiples of \(J^\mu\).
Outline

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2. Quantization
   - Locally covariant Dirac field theory
   - Fields

3. Application to Yang-Mills theories

4. Conclusion
Existing approaches to locally covariant Yang-Mills fields require the existence of a flat background connection \[\text{[Hollands 07; Fredenhagen, Rejzner 11]}\]. This is not satisfactory for two reasons:

- One might want to perturb around non-flat connections.
- On generic principal $G$ bundles, there are no flat connections.

The space of connections is an affine space. Hence, after fixing a background connection $\bar{A}$, one obtains a linear space. Thus, the proposed framework seems to be the natural setting. In the following, a few first steps are taken.
After fixing a background connection $\tilde{A}$, the difference to all other connections is parametrized by

$$A \in E^1, \quad E^k \equiv \Gamma^\infty(M, P \times_{\text{ad}} g \otimes \Omega^k).$$

$E = \bigoplus_k E^k$ is a $C^\infty(M)$ module, $E = \Gamma^\infty(M, P \times_{\text{ad}} g) \otimes_{C^\infty(M)} \Gamma^\infty(M, \Omega(M))$. On $E^0$, the background connection induces a covariant derivative $\tilde{\nabla}$. For $a \omega \in E^k$, $a \in E^0$, $\omega \in \omega^k$ define

$$\tilde{d}(a \omega) = \tilde{d}a \wedge \omega + a d\omega, \quad (\tilde{d}a)(X) = \tilde{\nabla}_X a.$$ \hfill (1)

Similarly,

$$(a \omega) \wedge (b \nu) = [a, b] \omega \wedge \nu, \quad * (a \omega) = a * \omega.$$ \hfill (2)

The Killing form on $g$ induces a pairing $\langle \cdot, \cdot \rangle_k : E^0 \times E^0 \to C^\infty(M)$. This gives a pairing $E^k \times E^l \to C^\infty(M)$ by

$$\langle a \omega, b \nu \rangle \equiv \langle a, b \rangle_k \langle \omega, \nu \rangle_g.$$
Composition of the pairing with integration yields a scalar product on $E$. W.r.t. this, the operators $\bar{d}$ and $(a\omega) \wedge \cdot$ have adjoints

$$\bar{\delta} \doteq (-1)^{n(k+1)+1} \ast \circ \bar{d} \circ \ast, \quad (a\omega) \rhd (b\nu) \doteq [b, a]\omega \rhd \nu.$$  

The curvature corresponding to $\bar{A} + A$ is $F = \bar{F} + \bar{d}A + \frac{1}{2}A \wedge A$. The Yang-Mills action

$$\int \langle F, F \rangle \, dg \, x$$

is invariant under the gauge transformation

$$\gamma A = \bar{d}c + A \wedge c, \quad c \in E^0.$$  

A covariant gauge condition generalizing the Lorentz gauge is $\bar{\delta}A = 0$. The free equation of motion is $\bar{\delta}\bar{d}A + A \rhd \bar{F} = 0$. This would be the starting point for a perturbative quantization in the Batalin–Vilkovisky framework [Fredenhagen, Rejzner 11].
Open problems

- Kinematical setup (take anticommutativity of forms into account?).
- Existence of gauge invariant Hadamard states for the free theory.
- Cohomological analysis in the presence of a background connection.
- Background independence?
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Extension of the framework of locally covariant field theory to gauge theories, such that gauge backgrounds and gauge transformations are treated on equal footing with gravitational backgrounds and isometries.

Definition of the charged locally covariant Dirac field, including a covariant definition of Wick powers.

Right framework for the treatment of locally covariant Yang-Mills fields?
Thank you for your attention!