Supplementary online materials

This document provides the calculation of equilibria and proofs for claims in the paper "Voter Motivation and the Quality of Democratic Choice" by Lydia Mechtenberg and Jean-Robert Tyran (August 13, 2018).

1. A model of selfish democrats

Consider an odd number n of individuals forming a group that has to choose whether they want some expert decision-maker to decide for them between two alternative policies or whether they want to clamor democracy in order to make this decision themselves, by vote. For instance, citizens of a town choose whether they want to let their mayor decide which firm to mandate with an important construction project or whether they want to claim this right for themselves, signing a petition in favor of having a vote on which firm to mandate. Alternatively, the faculty of a currently reforming university department must choose whether they want to make a specific type of decisions themselves, by committee voting, or whether they want to delegate this type of decisions to the dean. Generally speaking, the group makes a collective constitutional choice between direct and expert judgment.

Under any of these two constitutions, the ultimate aim is to select the correct policy among two alternatives, P_A and P_B. To give meaning to the idea of a correct policy, define the state of the world, ω, that has two possible realizations, A or B. Realizations are drawn by nature with equal probability and cannot be directly observed. What it means for a policy to be correct is implied by the group's monetary interests which are as follows: Each individual earns a fixed payoff normalized to 1 if the policy matches the state of the world and zero otherwise. Hence, the individuals' interests in the outcome of the decision-making process are perfectly aligned; and the policy that matches the state of the world is the correct policy. Formally, if u_m denotes monetary payoffs, then $u_m(P_A \mid A) = u_m(P_B \mid B) = 1$ and $u_m(P_A \mid B) = u_m(P_B \mid A) = 0$.

A policy $P \in \{P_A, P_B\}$ must be implemented either by an expert decision-maker (expert judgment) or by the group itself (democracy). The default constitution implies expert judgment; but if at least $\nu \in [1, n]$ individuals are in favor of democracy, the constitution is changed accordingly.

If democracy is imposed, the procedure is as follows: First, at the informational stage, all group members individually decide whether to acquire a private signal $s_i \in \{A^*, B^*\}$ indicating the state of the world at individual costs $c > 0$. Signals are imperfect but informative and uncorrelated across subjects: $Pr\{\omega = A \mid s_i = A^*\} = Pr\{\omega = B \mid s_i = B^*\} = p \in$

$(\frac{1}{2}, 1)$ for all i. Second, at the voting stage, the group members individually and simultaneously decide whether to vote for P_A or P_B or abstain. The policy that gets a simple majority of votes is implemented (if there is a tie the policy is chosen randomly, both policies with equal probability); and the resulting payoff is realized.

If, by contrast, the constitution remains unchanged, i.e., under expert judgment, the expert acts in the group's monetary interests with probability $q \in [p, 1)$, then implementing the correct policy, and fails to do so with the remaining probability. The parameter q measures the quality of expert judgment and can be taken to refer to either the quality of the incumbent's information or the probability with which his preference is to serve the "common good"; it may also represent the quality of unmodelled decision processes under expert judgment to which the incumbent has to submit. By setting $q \geq p$, we assume that expert judgment works at least as well as democracy with one informed voter.[21]

The sequence of events is as follows: First, the group chooses its constitution (constitutional stage). Second, if the constitution prescribes democracy, the individuals privately decide whether to get informed prior to the vote (informational stage). Third, nature draws the state of the world and the signal realization(s). Fourth, the policy is chosen, either by a simple majority vote under democracy (voting stage) or by expert (policy-making stage), depending on the group's constitution. Finally, payoffs are realized.

As a benchmark, we assume that individuals' preferences are given by expected monetary payoffs $E[u_m \mid \sigma]$; i.e., individuals are selfish. We test this assumption in the experimental part of the paper.

Let $\hat{\pi}(\sigma)$ denote the probability of a "correct" policy choice, given the strategy profile σ, and assume risk neutrality. Then, the individuals' preferences are fully described by $\hat{\pi}(\sigma)$; and the game can be solved by backward induction. The relevant equilibrium concept is subgame-perfect Nash equilibrium. In the SOM, section 2 (SOM 2), we restrict our equilibrium analysis to pure-strategy equilibria. Off-equilibrium improvements are analyzed in SOM 3. An additional analysis of symmetric mixed equilibria for the specification of our model that we implement in our experiment is relegated to SOM 4. SOM 5 contains a QRE analysis. All proofs are relegated to SOM 6.

Pure-strategy equilibria. Any pure-strategy equilibrium of the game involves a number $k \in \mathbb{N}_0^{\leq n}$ of players who acquire information and a number $n - k$ who do not. As a first step toward identifying the strategy profiles that can become equilibria, we make explicit two

[21] Dropping this assumption does not change the main results.

sets of conditions on equilibrium behavior. The first set of conditions characterizes the behavior of informed voters. The second set of conditions defines in which way uninformed voting can occur in equilibrium. The former are straightforward:

Lemma 1: Informed Voters *(a) Individuals get informed only if they can be pivotal at the voting stage. (b) Conditional on being informed and having a strictly positive pivot probability, individuals vote for the policy that is indicated by their signal.*

The corresponding conditions on uninformed voting are not equally obvious. First, let us distinguish between strategy profiles in which all uninformed voters cast the same vote, for instance for policy P_A, and such strategy profiles in which they cast different votes. We will refer to the first as "homogeneous uninformed voting" and to the second as "heterogeneous uninformed voting". Second, consider all strategy profiles in which, for all draws of nature, the voting outcome is always the policy indicated by the majority of signals or, if no such majority exists, the outcome of a random draw of nature. Note that any strategy profile in this class is, in expectation, outcome-equivalent to a profile in which the same number of informed voters vote and the uninformed abstain. Define this latter class of strategy profiles as "let the experts decide" strategy profiles. Then, we get the following restrictions on uninformed voting in equilibrium:

Lemma 2: Uninformed Voters *Let the number of voters who do not abstain be denoted by $m \leq n$. (a) Homogenous uninformed voting occurs in a pure-strategy equilibrium if and only if all m voters stay uninformed. (b) Heterogeneous uninformed voting occurs in a pure-strategy equilibrium only if the numbers of uninformed voters for either policy are exactly equal and the number of informed voters k and hence the total number of voters m is odd.*

As a consequence of these two Lemmata, we can pin down the two most important properties that all non-trivial pure-strategy equilibria must have in common:

Definition: *A strategy profile exhibits "informational efficiency" at the voting stage if, for any draw of nature, the policy implemented by the majority vote is indicated by the signal realization most often received in the subgroup of informed individuals.*

Proposition 1: Non-trivial equilibria. *All pure-strategy equilibria in which any player can be pivotal are outcome-equivalent to "let the experts decide" equilibria and thus exhibit informational efficiency at the voting stage.*

Hereafter, we shall assume that only non-trivial equilibria are played; i.e., equilibria in which players can be pivotal. We will hence drop explicit reference to "let the experts decide" equilibria whenever possible, intending, when we speak of pure-strategy equilibria in general, to refer to such equilibria that are outcome-equivalent to "let the experts decide" equilibria. As a consequence, we can define classes of outcome-equivalent equilibria by defining the precise number k of informed voters. Each k entails one "let the experts decide" strategy profile and corresponding outcome-equivalent strategy profiles, all involving the k informed voters voting in favor of the policy indicated by their private signal. The "let the experts decide" strategy profile implies abstention for all $n-k$ uninformed voters. The outcome-equivalent strategy profiles allowing for uninformed voting involve $\frac{1}{2}(m-k)$ uninformed voters voting for policy P_A and $\frac{1}{2}(m-k)$ uninformed voters voting for policy P_B, thereby offsetting each others' votes, with m ranging from k to n. A class of equilibria is hence defined by an *odd* number k, as implied by Lemma 2, part (b), and an information cost c.

In SOM 2 and 3, respectively, we give a full characterization of the set of pure-strategy equilibria and describe possible off-equilibrium improvements in our model. The picture that emerges from this characterization provides us with four main insights: First, within a given interval of information costs, there are classes of equilibria that differ in the number of informed voters k^* and can be Pareto-ranked, their rank increasing in k^*. Second, generally a group under democracy could improve by having more informed voters than attainable in equilibrium. Third, and relatedly, groups that are more able to coordinate on high numbers of informed voters under democracy tend more to select into democracy. Fourth, however, there is no decision rule ν at the constitutional stage that forestalls the group choosing the "wrong constitution". Such a wrong choice is made if, for instance, the group chooses democracy despite the fact that the sum of expected net earnings would be higher under expert judgment or, conversely, the group chooses expert judgment although it would (or at least could) fare better under democracy.

2. Pure-strategy equilibria of the model

Consider the probability that a majority vote of k informed voters identifies the correct policy, i.e., the policy that matches the state of the world. We refer to this probability as the "success probability" (SP) and define it as

$$\pi(k) = \sum_{l=0}^{\frac{k-1}{2}} \binom{k}{\frac{k+1}{2}+l} p^{\frac{k+1}{2}+l} (1-p)^{\frac{k-1}{2}+l}$$

Note that $\pi(0) < \pi(1) = \pi(2) < \pi(3) = \pi(4) < \ldots < \pi(n)$, $\log_{n\to\infty} \pi(n) = 1$, and that $\pi(2x+1) - \pi(2x-1)$ is strictly decreasing and strictly convex in $x \in \{1, 2, 3, \ldots, \frac{n-1}{2}\}$. Hence, if the strategy profile σ contains only pure strategies, any given individual i can increase the SP by acquiring information if and only if the number of other informed voters is even. Moreover, starting from a given odd number of informed voters and adding two additional informed voters always increases the success probability, but at a decreasing rate.

Let c_0 denote the information cost that makes a given individual exactly indifferent between getting informed to vote his signal and remaining uninformed (and hence abstain), given that the other individuals stay uninformed and abstain. Hence, only cost levels below c_0 allow for informed voting. With regard to these cost levels, let c_x denote the cut-off information costs that make a given individual exactly indifferent between getting informed to vote his signal, on the one hand, and remaining uninformed to abstain, on the other hand, given that x other individuals get informed to vote in line with their signal, while the remaining individuals abstain, with $x \in \{1, 2, 3, \ldots, k-1\}$. Then, we get

Proposition 1 *(a) There are the following pure-strategy subgame equilibria in the voting game: (i) No individual is informed for $c \geq c_0$. (ii) There are intervals $(c_x, c_{x-1}]$, with $x \in \{1, 2, 3, \ldots, \frac{n+1}{2}\}$ and $k \in \{1, 3, \ldots, n\}$ such that k individuals get informed for $c_x < c \leq c_{x-1}$, with $c_{\frac{n+1}{2}} = 0$. (b) Suppose that the equilibrium played in the subgame involves k^* informed voters. Then, the group will select into voting in the equilibrium of the entire game if and only if either (i) $\pi(k^*) - c \geq q$ or (ii) $\pi(k^*) \geq q$ and $n - k^* \geq \nu$.*

Intuitively, moving the information costs from a high-cost interval to the neighbouring lower-cost interval brings into existence one more equilibrium; and in this equilibrium, two more individuals get informed and vote what their signal indicates. Thus, n equilibria exist in the lowest cost interval and only one equilibrium – with one informed voter – in the highest (closed) cost interval. For (1) = q, this equilibrium would be in weakly dominated strategies.

Corollary 1 *Let $\bar{k}$ denote the highest number of informed voters attainable in a pure-strategy subgame equilibrium under democracy, for given c. Then, all pure-strategy subgame equilibria under democracy with $k \leq \bar{k} - 2$, if existent (i.e., if $\bar{k} \geq 3$), can be Pareto-ranked for any given c; the rank increases with the number k of informed voters. Moreover, moving from the equilibrium with $\bar{k} - 2$ informed voters to the equilibrium with $\bar{k}$ informed voters weakly increases everybody's expected net payoff.*

Corollary 1 implies that, once democracy has been chosen in the constitutional stage, it is socially beneficial to coordinate on the most informative equilibrium, i.e., the equilibrium with the highest number of informed voters. In general, the picture emerging from the model at this stage suggests that democracy works well when the issue at hand is easy to solve (low c and/or high p) but tends to fail in its purpose when a reasonably good decision requires intense aggregation of hard-to-get information (high c and low p).

This insight opens up the question whether the group succeeds in making the socially optimal constitutional choice in the face of known information costs and informational quality. To give a precise meaning to the concept of a socially optimal constitution, we will henceforth speak of one constitution "socially dominating" the other if the former generates a strictly higher expected net group payoff $n\pi(k^*) - kc$ in equilibrium.

Proposition 2 *(a) Let $n\pi(k^*) - q > k^*c$ in the subgame under democracy. Then, democracy socially dominates expert judgment but is chosen if and only if either $\pi(k^*) - q > c$ or $\nu \leq n - k$. (b) Let $0 < n\pi(k^*) - q) < k^*c$ and $\pi(k^*) - \pi(k^* - 1) \geq c$. Then, if $\nu \leq n - k^*$, democracy is chosen although socially dominated by expert judgment. (c) If $\pi(k^*) > q$, then democracy is the socially optimal constitutional choice for large n. (d) If $\pi(k^*) < q$, then expert judgment is the socially optimal constitutional choice for large n.*

As Proposition 2 reveals, the question which of the two opposite constitutional choices is optimal for the group has no straightforward theoretical answer but depends on parameters and group coordination under democracy.

3. Off-equilibrium improvements

The fact that the subgame equilibria under democracy can be Pareto-ranked, their rank increasing with the odd number k of informed voters, suggests that the group would be able to improve its expected net payoff even beyond the bounds of equilibrium if it could increase the number of informed voters above the highest that is attainable in equilibrium. Remember that this number is denoted by $\bar{k}$ and let $\triangle\,\pi(k)$ denote the difference in success probabilities $\pi(k+2) - \pi(k)$. Moreover define the socially optimal number of informed voters k^{**} by

$$k^{**} = arg \max_{k, k \leq n} \{n\pi(k) - kc\}.$$

Then, we get:

Corollary 2. *(a) For large n, $k^{**} > \bar{k}$. (b) For any finite group size n, the socially optimal number of informed voters is n if $\frac{1}{2}\triangle\,\pi(n) > \frac{c}{n}$ and some k^{**} with $k \leq k^{**} < n$ otherwise.*

Intuitively, if the group size converges to infinity, then the positive externality that two additional informed voters generate outweighs their costs of information acquisition, at least at the limit. Moreover, if in a group of finite size everyone gets informed and would still profit from two more additional votes, it is obvious that the best what they can do is to get all informed. If, by contrast, the information costs are so high that at some point, the increased SP would not be sufficient to outweigh the additional information costs, then there is an interior solution to the problem of finding the socially optimal number of informed voters. This solution may well lie above the highest number attainable in equilibrium.

4. Symmetric mixed-strategy equilibria in the example

Let EU_i^I denote the expected utility of i if i gets informed and votes his signal with probability $r \in [0,1]$ and remains uninformed and abstains with probability $(1-r)$, given that the other six individuals apply the same mixed strategy. Let EU_i^{NI} denote the expected utility of i if i remains uninformed and abstains, given that the other six individuals each get informed and vote their signal with probability $r \in [0,1]$ and remain uninformed and abstain with probability $(1-r)$. Note that

$$EU_i^I = \sum_{k=0}^{6} \frac{6!}{k!\,(6-k)!} r^k (1-r)^{6-k} \pi(k+1) 25 - c$$

$$EU_i^{NI} = \sum_{k=0}^{6} \frac{6!}{k!\,(6-k)!} r^k (1-r)^{6-k} \pi(k) 25$$

in our example. The indifference condition required for i to apply the mixed strategy described amounts to $EU_i^I - EU_i^{NI} = 0$, i.e.,

$$25(1.366r^6 - 4.516r^5 + 6.338r^4 - 4.88r^3 + 2.22r^2 - 0.6r + 0.1) - c = 0.$$

Solving for the different cost levels $0.1, 0.9$, and 1.7, and denoting the equilibrium probability of getting informed by r_l, r_m, and r_h for low, medium and high costs, we get $r_l = 1, r_m = 0.221$, and $r_h = 0.068$. Denoting the mixing probability by r_c, the expected utility of the mixed subgame equilibrium with voting amounts to:

$$EU = \sum_{k=0}^{7} \left(\frac{7!}{(7-k)!k!} r_c^k (1-r_c)^{7-k} \pi(k) 25 \times 7 - kc \right).$$

Plugging in the mixing probabilities for the three cost levels reveals that the expected utility from the mixed strategy equilibria with voting lies strictly below 105, the expected utility from delegation, if costs are high or medium. Hence, for these two cost levels, individuals will not sign the petition in equilibrium. For low costs, the result from the pure-strategy equilibrium analysis applies.

5. Log-QRE

We now present the existence condition for the symmetric quantal response equilibrium (QRE; McKelvey and Palfrey 1995) at the information-acquisition stage, using the logit-specification of QRE (Goeree and Holt 2005). Let μ denote a noise parameter with $\mu \geq 0$. Then, the existence condition for a log-QRE is

$$\mu\left(-\ln\left(\frac{1-r}{r}\right)\right) = EU_i^I(r) - EU_i^{NI}(r),$$

with r, $EU_i^I(r)$, and $EU_i^{NI}(r)$ defined analogously to above (SOM, section 4). Since the derivation of this condition is analogous to the derivation of equation (1) in Großer and Seebauer (2016), we skip it here.

Inserting the respective definitions for $EU_i^I(r)$ and $EU_i^{NI}(r)$, we get

$$\mu\left(-\ln\left(\frac{1-r}{r}\right)\right) = 25(1.366r^6 - 4.516r^5 + 6.338r^4 - 4.88r^3 + 2.22r^2 - 0.6r + 0.1) - c.$$

Großer and Seebauer (2016) estimate three different values of μ for their voluntary-voting game with 7 voters and endogenous information acquisition, $\hat{\mu} = 0.05$ for the second half of periods, $\hat{\mu} = 0.06$ for all periods, and $\hat{\mu} = 0.07$ for the first half of periods. In a first step, we insert these for μ into the above existence condition, generating three different existence conditions for the log-QRE per cost level c.[22] The table below summarizes the resulting predictions for the individual information-acquisition probability r. With "n.a." we denote the non-existence of a log-QRE for the given values of c and μ.

c / μ	$\mu = 0.05$	$\mu = 0.06$	$\mu = 0.07$
$c = 0.1$	n.a.	n.a.	n.a.
$c = 0.9$	0.24389	0.24822	0.25249
$c = 1.7$	0.082397	0.085165	0.087913

To test whether our observed information-acquisition rates differ significantly from the above QRE predictions, we proceeded as follows: We simulated information acquisition in a group of 7 voters according to the above predicted values for r. We then tested our observed information-acquisition distributions against the simulated ones, using Wilcoxon-signed rank tests. We did this multiple times. In all cases (i.e., for $c = 0.9$ and $c = 1.7$), our

[22] We are grateful to an anonymous referee for suggesting this type of approach.

observed information-acquisition distributions differ significantly from the simulated ones ($p = 0.0022$, see Table SOM 1 below). We obtain an analogous result when testing the observed average frequencies of information acquisition on the group level against the predicted values for r (degenerate distribution, see Table SOM 2 below).

Hence, for the three estimates of μ $(0.05,\ 0.06,\ \text{and}\ 0.07)$ taken from Großer and Seebauer (2016), we do not find that log-QRE can explain our observed rates of information acquisition.

Since we did not find a game more similar to ours than the one in Großer and Seebauer (2016) for which estimates of μ exist, we refrain from inserting further out-of-sample estimates for μ. Instead, we implement the reverse procedure: We insert our observed average frequencies of information acquisition (per treatment, taken from Table 2) for r into the existence condition for the log-QRE and solve for μ. The resulting solutions for μ are summarized in the table below.

c / treatment	Exo	Endo
$c = 0.1$	$\mu = 0.26939$	$\mu = 0.16077$
$c = 0.9$	$\mu = -0.73152$	$\mu = -0.38307$
$c = 1.7$	$\mu = -6.2587$	$\mu = -1.4286$

Since the predicted values for μ for cost levels $c = 0.9$ and $c = 1.7$ violate the condition that $\mu \geq 0$, we conclude that log-QRE cannot explain the (off-equilibrium) overinvestment into information that we find for medium and high cost levels. Note that log-QRE explains *under*investment in information in Großer and Seebauer (2016) and can also explain the underinvestment in information acquisition that we observe for low costs, where the predicted information-acquisition rate in the symmetric mixed- and pure-strategy Nash equilibria are 100%.

Table SOM 1: Wilcoxon signed-rank tests, averages calculated by group, simulation

μ	0.05				0.06				0.07			
Endo / Exo	Endo		Exo		Endo		Exo		Endo		Exo	
Infocost	0.9	1.7	0.9	1.7	0.9	1.7	0.9	1.7	0.9	1.7	0.9	1.7
QRE prediction*	0.2439	0.0824	0.2439	0.0824	0.2482	0.0852	0.2482	0.0852	0.2525	0.0879	0.2525	0.0879
Z	3.061	3.061	3.062	3.062	3.059	3.061	3.061	3.061	3.059	3.062	3.061	3.059
Prob > |z|	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022

Table SOM 2: Wilcoxon signed-rank tests, averages calculated by group

μ	0.05				0.06				0.07			
Endo / Exo	Endo		Exo		Endo		Exo		Endo		Exo	
Infocost	0.9	1.7	0.9	1.7	0.9	1.7	0.9	1.7	0.9	1.7	0.9	1.7
QRE prediction*	0.2439	0.0824	0.2439	0.0824	0.2482	0.0852	0.2482	0.0852	0.2525	0.0879	0.2525	0.0879
Z	3.065	3.062	3.065	3.065	3.065	3.062	3.065	3.065	3.065	3.062	3.065	3.065
Prob > $\|z\|$	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022

* No QRE predictions for $c = 0.1$ available due to non-existence of log-QRE for these specific parameter combinations.

6. Proofs

Proof of Lemma 1. Let π_{-i} denote the probability with which the group votes for the correct policy if group member i abstains from the vote and let π_{Σ_i} denote the corresponding probability if i participates in the vote according to his strategy Σ_i. The expected utility of i is $\pi_{-i} - c$ if i plans to acquire information and either abstain or participate without ever being pivotal and π_{-i} if i plans to abstain without information or if i plans to participate without information but cannot be pivotal. Since $c > 0$, $\pi_{-i} > \pi_{-i} - c$. Hence, i will buy information only if he plans to participate and has a positive pivot probability. This proves part (a) of the Lemma. Consider now part (b) and let Σ_i denote i's strategy to get informed and vote in line with the signal. By contrast, let $\Sigma_i^{'}$ denote i's strategy to get informed and vote the opposite of what the signal indicates. Since conditional on being informed, i has a positive pivot probability (part (a)), and since the quality of the signal is $p > 0.5$, we have $\pi_{\Sigma_i} > \pi_{\Sigma_i^{'}}$. Thus, conditional on being informed, i has the highest expected utility when voting in line with his signal. This proves part (b).

Proof of Lemma 2. *Step 1*: Let $m \leq n$ denote the *odd* number of group members participating in the vote. Among those, k group members are informed and vote in line with their signal; and $m-k$ cast identical uninformed votes. W.l.o.g., we assume this vote to be in favor of policy P_A. The remaining group members (if there are any) abstain. Let $\hat{k}$ denote the number of signals that indicate state of the world A. Policy P_A wins the vote if and only if it gets more votes than the alternative, P_B, i.e., if and only if $m-k+\hat{k} > k-\hat{k}$. Consider now an uninformed voter i. He is pivotal if and only if $\hat{k}+m-k = k-\hat{k}+1$ with $\hat{k} \geq 0$. This is equivalent to $\hat{k} = k - \frac{1}{2}(m-1) \geq 0$ which we will refer to as condition (i). Condition (i) implies that a number $k \geq \frac{1}{2}(m-1)$ group members get informed to vote in line with their signal.

We will now prove the following *claim*: Conditional on pivotality of an uninformed voter voting for P_A, the number $\hat{k}$ of signals in favor of state of the world A is strictly lower than the number $k-\hat{k}$ of signals in favor of B, which implies that the uninformed voter wants to deviate to voting for P_B.

Proof of the claim: Suppose for the sake of the argument that the opposite holds true, i.e., $\hat{k} \geq k-\hat{k}$, while the uninformed voters are pivotal. Due to condition (i), this implies $2\left(k-\frac{1}{2}(m-1)\right) \geq k$ or $k \geq m-1$. This is equivalent to saying that either $k=m-1$ or $k=m$. Since we are assuming that a strictly positive number of group members are uninformed voters, we have $k<m$ and hence conclude that $k=m-1$, i.e., we have one single uninformed voter voting for P_A. Since m is odd, this implies that k be even. Hence, pivotality of the uninformed voter implies that half of the informed voters vote (according to their signals) for P_A while the other half vote (according to their signals) for P_B. Consider one of the informed voters with a signal in favor of A. He is pivotal among the informed voters if and only if this signal creates a tie. Hence, conditional on his pivotality, he is indifferent between both policies; and in all other cases, he wants the policy to win that is indicated by the majority of signals received by the other $k-1$ informed voters. Hence, he can strictly improve his expected payoff by deviating to remaining uninformed (saving c) and voting in favor of P_B, thereby offsetting the other uninformed vote and making sure that the policy indicated by the majority of $k-1$ signals always wins the vote. Therefore, $m=k-1$ cannot be an equilibrium. In sum, we have shown by contradiction that our claim must be true.

Step 2: Let $m \leq n-1$ denote the *even* number of group members participating in the vote. Among those, k group members are informed and vote in line with their signal; and $m-k$ place identical uninformed votes. Without loss of generality, we assume this vote to be in favor of policy P_A. The remaining group members (if there are any) abstain. Let $\hat{k}$ denote the number of signals that indicate state of the world A. Policy P_A wins the vote if and only if it either wins a tie or gets at least two more votes than the alternative, P_B. Hence, uninformed voter i is pivotal if and only if there is a tie without him, i.e., $m - k + \hat{k} = k - \hat{k}$ with $\hat{k} \geq 0$ which amounts to $\hat{k} = k - \frac{1}{2}m$. We will refer to this equality as condition (ii).

We will now prove the following *claim*: Conditional on pivotality of an uninformed voter voting for P_A, the number $\hat{k}$ of signals in favor of state of the world A is strictly lower than the number $k - \hat{k}$ of signals in favor of B, which implies that the uninformed voter wants to deviate to voting for P_B.

Proof of the claim: Suppose for the sake of the argument that the opposite holds true, i.e., $\hat{k} \geq k - \hat{k}$, while the uninformed voters are pivotal. Then, $\hat{k} \geq k - \hat{k}$ and condition (ii) imply that $k \geq m$, i.e., that there are no uninformed voters. This proves our claim by contradiction.

Hence, we conclude from steps 1 and 2 that there is no pure-strategy equilibrium in which uninformed group members place identical votes and have a positive pivot probability. This proves part (a) of Lemma 2.

Consider now **part (b)** of the Lemma, i.e., uninformed voters who need not place identical votes (heterogeneous uninformed voting). Again, we will provide the proof in two steps, first assuming an odd number and then an even number of voters.

Step 1: Let $m \leq n$ denote the *odd* number of group members participating in the vote. Among those, k group members are informed and vote in line with their signal; $m - k - j$ stay uninformed and vote for P_A; and j stay uninformed and vote for P_B. The remaining group members (if there are any) abstain. Let $\hat{k}$ denote the number of signals that indicate state of the world A.

Consider now an uninformed voter i who votes for P_A. He is pivotal if and only if $m - k - j + \hat{k} = k - \hat{k} + j + 1$ which is equivalent to $\hat{k} = k + j - \frac{1}{2}(m - 1)$. We refer to this equation as condition (i'). The uninformed voter i does not want to deviate to abstaining or voting for P_B if and only if conditional on his pivotality, "A" is not the signal received by a strict minority of the k informed group members. Formally, $\hat{k} \geq k - \hat{k}$, which we refer to as condition (ii'). Conditions (i') and (ii') imply that $j \geq \frac{1}{2}(m - k - 1)$ which we refer to as condition (iii'). The difference between uninformed votes in favor of P_A and uninformed votes in favor of P_B is $m - k - 2j$. Condition (iii') implies that this difference does not exceed 1: $m - k - 2j \leq 1$. This, together with condition (iii'), implies that

$$j = \begin{cases} \frac{1}{2}(m-k) & \textit{iff } k \textit{ is odd} \\ \frac{1}{2}(m-k-1) & \textit{iff } k \textit{ is even} \end{cases}$$

which we refer to as condition (iv').

Consider now an uninformed voter h who votes for P_B. An argument analogous to the one above (replacing $\hat{k}$ by $k-\hat{k}$ and $m-k-j$ by j) yields that h does not want to deviate, conditional on his pivotality, if and only if

$$m-k-j = \begin{cases} \frac{1}{2}(m-k) & \textit{iff } k \textit{ is odd} \\ \frac{1}{2}(m-k-1) & \textit{iff } k \textit{ is even} \end{cases}$$

which we refer to as condition (v'). It is easy to see that conditions (iv') and (v) are consistent if and only if $j = m-k-j = \frac{1}{2}(m-k)$. Hence, heterogeneous uninformed voting with an odd number of voters occurs in equilibrium only if the numbers of uninformed votes for P_A and P_B are exactly equal, implying that k be odd.

Step 2: Let $m \leq n$ denote the *even* number of group members participating in the vote. Among those, k group members are informed and vote in line with their signal; $m-k-j$ stay uninformed and vote for P_A; and j stay uninformed and vote for P_B. The remaining group members (if there are any) abstain. Let $\hat{k}$ denote the number of signals that indicate state of the world A.

Consider now an uninformed voter i who votes for P_A. He is pivotal if and only if his vote creates a tie: $m-k-j+\hat{k} = j+k-\hat{k}$, which is equivalent to $\hat{k} = k+j-\frac{1}{2}m$, with $j < m-k$. We refer to this equality as condition (i"). The uninformed voter i does not want to deviate to abstaining or voting for P_B if and only if conditional on his pivotality, "A" is not the signal received by a strict minority of the k informed group members. Formally, $\hat{k} \geq k-\hat{k}$, which we refer to as condition (ii"). Conditions (i") and (ii") imply that $j \geq \frac{1}{2}(m-k)$ which we refer to as condition (iii').

Consider now an uninformed voter h who votes for P_B. An argument analogous to the one above (replacing $\hat{k}$ by $k-\hat{k}$ and $m-k-j$ by j) yields that h does not want to deviate, conditional on his pivotality, if and only if $m-k-j \geq \frac{1}{2}(m-k)$ which we refer to as condition (iv'). Conditions (iii') and (iv') imply that $j = m-k-j = \frac{1}{2}(m-k)$. Hence, heterogeneous uninformed voting with an even number of voters occurs in equilibrium only if the numbers of uninformed votes for P_A and P_B are exactly equal, implying both that k be odd and that the number of uninformed voters is even. This, however, is a contradiction, because we assumed that the total number of voters m is even.

Hence, step 1 and 2 of this proof imply that heterogeneous uninformed voting occurs in equilibrium only if the numbers of uninformed votes for P_A and P_B are exactly equal, implying that both k and m be odd. (Note that this also captures the case in which the numbers of uninformed votes for P_A and P_B are both zero, as in the "let the experts decide" equilibria.) This proves part (b) of the Lemma.

Proof of Proposition 1. Lemma 2 implies that in any pure-strategy equilibrium in which all voters can be pivotal, the probability of any policy being implemented would remain the same if only the votes of the informed group members were counted. Hence, all pure-strategy equilibria are outcome-equivalent to "let the experts decide" profiles. Two claims remain to be shown: first, that "let the experts decide" profiles and the corresponding profiles with heterogeneous uninformed voting can be equilibria if the number k of informed voters is odd, and, second, that these equilibria exhibit informational efficiency at the voting stage.

Consider first a strategy profile with heterogeneous uninformed voting (half of the uninformed voters vote for P_A and the other half for P_B) and an odd number k of informed voters. We have established the no-deviation conditions for the uninformed voters in the proof of Lemma 2. Hence, it remains to be shown that no informed voter has an incentive to deviate on the voting stage or on the informational stage. Since the signal quality is $p > 0.5$, and since any voter can be pivotal, no informed voter wants to deviate to abstaining or voting the opposite of what his signal indicates. Consider now the informational stage. Deviating to remaining uninformed implies that the number of informed voters becomes even. This strictly decreases the probability that the correct policy is implemented after the vote (a tie replaces the win of the majority signal by one more vote). Hence, if the information costs are sufficiently small, it is strictly better not to deviate but to acquire the signal and vote what it indicates.

Consider now a strategy profile with the same k informed voters in which all uninformed group members abstain. Since this "let the experts decide" profile is outcome-equivalent to the profile considered above, the argument why no informed voter wants to deviate if c is sufficiently small remains the same. Hence, it only has to be shown that no uninformed voter wants to deviate. Consider the uninformed voter i. If he deviates to, say, voting for P_A, he is pivotal if and only if he creates a tie. He creates a tie if and only if there has been one more signal indicating B than indicating A. Hence, conditional on his pivotality, i would vote for the policy that is less likely than its alternative to be correct. Thereby, i would strictly decrease his expected payoff. This proves Proposition 1.

Proof of Proposition 2. Lemma 1 implies that individuals either stay uninformed or get informed and vote in line with their signal. Hence, whoever gets informed then votes in line with his signal. Let k denote the number of informed voters and $\pi(k)$ the probability that a simple majority of the k informed voters makes the correct decision. Then, given Lemma 1 and 2, the difference in individual payoffs from casting an informed vote and remaining uninformed amounts to $\pi(k) - \pi(k-1) - c$. Next, we argue that $\pi(k) - \pi(k-1) = 0$ for all k even. To see this, assume that the total number of informed voters is even. Then, a voter is pivotal if and only if he creates a tie. Hence, conditional on pivotality, half the signals are in favor of state of the world A and half in favor of state of the world B, and the voter is indifferent between abstaining, voting for P_A, and voting for P_B. Hence, the voter is indifferent between voting his signal and abstaining: $\pi(k) - \pi(k-1) = 0$ for all k even. Hence, k must be odd. Next, we know from Condorcet's Jury theorem as formally stated in Berg (1996), Theorem 1, that $\pi(k)$ monotonously increases in k and $\lim_{k\longrightarrow\infty}\pi(\mathrm{k}) = 1$. Thus, $\pi(k) - \pi(k-1)$ must be monotonously decreasing in k (strictly decreasing for odd k). Define the cut-off cost level $c(k) = \pi(k) - \pi(k-1)$. Then, $c(k+2) < c(k)$ for all odd k, $k = 0$ for $c > c(1)$, $k = 1$ for $c(3) < c < c(1)$, and generally k voters get informed in equilibrium iff $c(k+2) < c < c(k)$. Define $x = \frac{k+1}{2}$ and $c_x = c(k+2)$. Then, we have shown part (a) of Proposition 2. Consider now part (b) and note that the payoff of an informed voter under democracy is $\pi(k^*) - c$, while the payoff of an uninformed individual under democracy is $\pi(k^*)$, with k^* denoting the equilibrium number of informed voters. The payoff of everyone under expert judgment is q. Then, part (b) of Proposition 2 immediately follows.

Proof of Corollary 1. Let k be the number of informed voters in a strict "let the expert decide" equilibrium of the subgame under democracy. Then, $\pi(k) - c > \pi(k-1)$, otherwise, the equilibrium would not be strict or the informed voters would want to deviate to remaining uninformed and abstain. Since $\pi(k-1) = \pi(k-2)$ for any k, given that k must be odd, we get $\pi(k) - \pi(k-2) > c$, which, in its turn, implies that the two additional informed voters required for the equilibrium with k rather than $k-2$ strictly prefer the former over the latter equilibrium. Then, this holds true for the uninformed, too. Now let k be the number of informed voters in a "let the expert decide" equilibrium of the subgame under democracy that is not strict. Then, $\pi(k) - c = \pi(k-1)$, which implies that k is the highest number of informed voters attainable in equilibrium: $k = \bar{k}$. Hence, all equilibria with $k < \bar{k}$ are strict equilibria and can be Pareto-ranked as stated in the Corollary. For $k = \bar{k}$, the strict inequalities must be replaced by the corresponding weak inequalities: $\pi(\bar{k}) - c \geq \pi(\bar{k} - 1)$ and $\pi(\bar{k}) - \pi(\bar{k} - 2) \geq c$, which implies that everyone's expected net payoffs weakly increase if one moves from the equilibrium with $\bar{k} - 2$ informed voters to the equilibrium with $\bar{k}$ informed voters.

Proof of Proposition 3. Parts (a) and (b) of Proposition 3 are obvious implications of the simple payoff comparisons stated there; hence, we skip the proof. Now consider part (c) and let $\pi(k^*) > q$ as stated there. The choice of democracy is socially beneficial if and only if $\pi(k^*) - q \geq \frac{k^*}{n}c$. For any given k^*, this inequality is always fulfilled for sufficiently large n, which proves part (c). The proof of part (d) is analogous.

Proof of Corollary 2. Consider the difference in expected net group payoffs of having $k + 2$ informed voters and $n - k - 2$ abstainers, and having k informed voters and $n - k$ abstainers, with $k \geq \bar{k}$; and let this difference be denoted by $\Delta E(U_m(k) \mid \sigma^{LTED})$. Similarly, let $\triangle \pi(k)$ denote the difference $\pi(k+2) - \pi(k)$ and remember that $k^{**} = \arg\max_{k, k \leq n}\{n\pi(k) - kc\}$. Then, simple algebra provides uns with the following condition:

$$\Delta E(U_m(k) \mid \sigma^{LTED}) \geq 0 \text{ iff } \tfrac{1}{2} \triangle \pi(k) \geq \tfrac{c}{n}.$$

Since $\lim_{n \longrightarrow \infty}\left(\frac{1}{2} \triangle \pi(k) - \frac{c}{n}\right) = \frac{1}{2} \triangle \pi(k) > 0$ for any given k, we have that $k^{**} \in [\bar{k}, n]$ for large n, which proves (a). Now consider part (b) and let $< k \longrightarrow n - 2 >$ denote the convergence of the *odd* numbers $k \in \{0,1,3 \ldots n-2\}$ toward $n - 2$. Moreover, let $x \in \mathbb{N}^{>0}$ and note that $\triangle \pi(0) > 0$. Then, we have

$$\lim_{n \to \infty} \lim_{\substack{x \longrightarrow \frac{n-k}{2} \\ <k \longrightarrow n-2>}} \frac{1}{2}\big(\pi(k + 2x) - \pi(k)\big) = \lim_{n \to \infty} \frac{1}{2}(\pi(\mathrm{n}) - \pi(\mathrm{n} - 2)) = 0$$

However, at the same time, $\lim_{n \to \infty} \frac{c}{n} = 0$, too. Hence, if $\frac{1}{2}(\pi(n) - \pi(n-2)) \geq \frac{c}{n}$, we have the corner solution $k^{**} = n$; and if $\frac{1}{2}(\pi(n) - \pi(n-2)) < \frac{c}{n}$, there exists some interior solution $k^{**} \in (0, n)$ that maximizes net expected group payoffs.

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