# NOTES ON THE CONFORMAL METHOD

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Abstract. These notes are an introduction to the *conformal method*, which is a technique for producing solutions to the constraint equations that arise when one wishes to solve the vacuum Einstein equations. They are based on Section 1.7 of [2] and, to a lesser extent, on [1, 4, 5], where more details may be found.

## 1. The constraints

Let $(M, \mathbf{g})$ be a Lorentzian manifold of dimension $(n+1)$. We assume, for simplicity, that $\mathbf{g}$ satisfies the vacuum Einstein equations $\mathbf{Ric}_{\mathbf{g}} = 0$. (The methods we discuss can be developed more generally.) If $\Sigma$ is a space-like hypersurface in $M$ with first fundamental form (or induced metric) $\mathbf{h}$ and second fundamental form (or extrinsic curvature) $\mathbf{k}$, then the 00 and $0i$ components of the vacuum Einstein equations tell us that these tensor fields must satisfy the equations

$$s_{\mathbf{h}} = |\mathbf{k}|^2_{\mathbf{h}} - (\operatorname{tr}\mathbf{k})^2, \tag{1.1a}$$

$$\nabla_i k^i{}_j = \nabla_j (\operatorname{tr}\mathbf{k}) \tag{1.1b}$$

on $\Sigma$. These equations are *constraints* that $(\mathbf{h}, \mathbf{k})$ must satisfy on $\Sigma$. In particular, given a Riemannian manifold $(\Sigma, \mathbf{h})$ and a symmetric $(0, 2)$ tensor field on $\Sigma$, we would like to solve the vacuum Einstein equations to construct a Lorentzian $(n+1)$ manifold $(M, \mathbf{g})$ such that $\Sigma$ may be viewed as a space-like hypersurface in $M$ with $\mathbf{h}$ and $\mathbf{k}$ being the first and second fundamental forms of $\Sigma$ induced from the embedding in $M$. Since (1.1) follow from the imposition of the vacuum Einstein condition on $\mathbf{g}$, it follows that these constraints being satisfied on $\Sigma$ is a minimal condition necessary for $(\Sigma, \mathbf{h}, \mathbf{k})$ to arise from such an $(M, \mathbf{g})$. If we wish to study the initial value formulation of the vacuum Einstein equations, it is therefore necessary to study methods of solving the constraint equations.

### 1.1. The conformal method.

The best-known general technique for producing solutions of the constraint equations is the *conformal method*. Let $(M, \mathbf{g})$ be a Riemannian manifold of dimension $n \geqslant 3$ and $\mathbf{k}$ a symmetric $(0, 2)$ tensor field on $M$.[1] We assume that $\mathbf{g}$ and $\mathbf{k}$ satisfy the equations

$$s_{\mathbf{g}} = |\mathbf{k}|^2_{\mathbf{g}} - (\operatorname{tr}\mathbf{k})^2, \tag{1.2a}$$

$$\nabla_i k^i{}_j = \nabla_j (\operatorname{tr}\mathbf{k}). \tag{1.2b}$$

Here $s_{\mathbf{g}}$ denotes the scalar curvature of the metric $\mathbf{g}$ and $\nabla$ denotes the Levi-Civita connection. Given a smooth, non-vanishing function $\varphi$ on $M$, we define the conformally related metric $\widehat{\mathbf{g}}$ by

$$\mathbf{g} = \varphi^{\frac{4}{n-2}} \widehat{\mathbf{g}}.$$

**Lemma 1.1.** *The Christoffel symbols of the metric $\widehat{\mathbf{g}}$ are related to those of $\mathbf{g}$ by*

$$\widehat{\Gamma}^i{}_{jk} - \Gamma^i{}_{jk} = -\frac{2}{(n-2)\varphi}\left[\varphi_j \delta^i_k + \varphi_k \delta^i_j - g^{il} g_{jk} \varphi_l\right]. \tag{1.3}$$

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[1] Note that I have changed notation.

*Proof.* In local coordinates, we have $g_{ij} = \varphi^{\frac{4}{n-2}} \widehat{g}_{ij}$. Taking inverses, we also have $g^{ij} = \varphi^{-\frac{4}{n-2}} \widehat{g}^{ij}$. We then have

$$\begin{aligned}\widehat{\Gamma}^i{}_{jk} &= \frac{1}{2}\widehat{g}^{il}\left[\partial_j \widehat{g}_{kl} + \partial_k \widehat{g}_{jl} - \partial_l \widehat{g}_{jk}\right] \\ &= \Gamma^i{}_{jk} + \frac{1}{2}\varphi^{\frac{4}{n-2}} g^{il}\left[\left(\partial_j \varphi^{-\frac{4}{n-2}}\right) g_{kl} + \left(\partial_k \varphi^{-\frac{4}{n-2}}\right) g_{jl} - \left(\partial_l \varphi^{-\frac{4}{n-2}}\right) g_{jk}\right].\end{aligned}$$

Simplifying gives (1.3). □

An additional calculation, using this result, gives the following result, the proof of which is left as an exercise.

**Lemma 1.2.** *The scalar curvature of* $\widehat{\mathbf{g}}$ *is related to that of* $\mathbf{g}$ *by the relation*

$$s_{\mathbf{g}} \varphi^{\frac{n+2}{n-2}} = \left[-a\Delta_{\widehat{\mathbf{g}}} + s_{\widehat{\mathbf{g}}}\right]\varphi, \tag{1.4}$$

*where* $a := \frac{4(n-1)}{n-2}$ *and* $\Delta_{\widehat{\mathbf{g}}}$ *denotes the Laplacian for the metric* $\widehat{\mathbf{g}}$*.*

The final preliminary result that we will require is the following.

**Lemma 1.3.** *Let* $L^{ij}$ *be a symmetric, trace-free (i.e.* $g_{ij}L^{ij} = 0$*) tensor field on* $M$*. Let*

$$\widehat{L}^{ij} := \varphi^{\frac{2(n+2)}{n-2}} L^{ij}.$$

*(Note that* $\widehat{L}$ *is still trace-free with respect to* $\mathbf{g}$ *and* $\widehat{\mathbf{g}}$*.) Then*

$$\widehat{\nabla}_i \widehat{L}^{ij} = \varphi^{\frac{2(n+2)}{n-2}} \nabla_i L^{ij}, \tag{1.5}$$

*where* $\widehat{\nabla}$ *denotes the Levi-Civita covariant derivative corresponding to* $\widehat{\mathbf{g}}$*.*

*Proof.*

$$\begin{aligned}\widehat{\nabla}_i \widehat{L}^{ij} &= \nabla_i \widehat{L}^{ij} + \left(\widehat{\Gamma}^i{}_{ik} - \Gamma^i{}_{ik}\right)\widehat{L}^{kj} + \left(\widehat{\Gamma}^j{}_{ik} - \Gamma^j{}_{ik}\right)\widehat{L}^{ik} \\ &= \nabla_i \widehat{L}^{ij} - \frac{2}{(n-2)\varphi}\left[\varphi_i \delta^i_k + \varphi_k \delta^i_i - g^{il} g_{ik}\varphi_l\right]\varphi^{\frac{2(n+2)}{n-2}} L^{kj} \\ &\qquad - \frac{2}{(n-2)\varphi}\left[\varphi_i \delta^j_k + \varphi_k \delta^j_i - g^{jl} g_{ik}\varphi_l\right]\varphi^{\frac{2(n+2)}{n-2}} L^{ik} \\ &= \nabla_i \widehat{L}^{ij} - \frac{2(n+2)}{(n-2)\varphi}\varphi_k \varphi^{\frac{2(n+2)}{n-2}} L^{kj} \\ &= \nabla_i\left(\varphi^{\frac{2(n+2)}{n-2}} L^{ij}\right) - \frac{2(n+2)}{(n-2)}\varphi_i \varphi^{\frac{2(n+2)}{n-2}-1} L^{ij} \\ &= \varphi^{\frac{2(n+2)}{n-2}} \nabla_i L^{ij}.\end{aligned}$$

□

We decompose the tensor field $\mathbf{k}$ into a trace part $\tau := \operatorname{tr}\mathbf{k}$ and a trace-free part

$$L^{ij} := k^{ij} - \frac{\tau}{n} g^{ij}.$$

We define the conformally transformed metric $\widehat{\mathbf{g}}$ and $\widehat{L}^{ij}$ as above.

The main point of these calculations is the following.

**Theorem 1.4.** *The fields* $(\mathbf{g}, \mathbf{k})$ *satisfy the constraint equations* (1.2) *if and only if the fields* $(\widehat{\mathbf{g}}, \widehat{\mathbf{L}}, \tau, \varphi)$ *satisfy the relations*

$$\widehat{\nabla}_i \widehat{L}^{ij} = \frac{n-1}{n}\varphi^{\frac{2n}{n-2}} \widehat{\nabla}^j \tau, \tag{1.6a}$$

$$\left[-a\Delta_{\widehat{\mathbf{g}}} + s_{\widehat{\mathbf{g}}}\right]\varphi = \varphi^{\frac{2-3n}{n-2}} |\widehat{L}|^2_{\widehat{\mathbf{g}}} - \frac{n-1}{n}\tau^2 \varphi^{\frac{n+2}{n-2}}. \tag{1.6b}$$

*Proof.* Using Lemma 1.3 and the second constraint equation in the form $\nabla_i k^{ij} = \nabla^j \tau$, we have

$$\begin{aligned}
\widehat{\nabla}_i \widehat{L}^{ij} &= \varphi^{\frac{2(n+2)}{n-2}} \nabla_i L^{ij} \\
&= \varphi^{\frac{2(n+2)}{n-2}} \nabla_i \left[ k^{ij} - \frac{\tau}{n} g^{ij} \right] \\
&= \varphi^{\frac{2(n+2)}{n-2}} \left[ \nabla^j \tau - \frac{1}{n} \nabla^j \tau \right] \\
&= \frac{n-1}{n} \varphi^{\frac{2(n+2)}{n-2}} \nabla^j \tau. \\
&= \frac{n-1}{n} \varphi^{\frac{2n}{n-2}} \widehat{\nabla}^j \tau.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\left[ -a\Delta_{\widehat{\mathbf{g}}} + s_{\widehat{\mathbf{g}}} \right] \varphi &= s_{\mathbf{g}} \varphi^{\frac{n+2}{n-2}} \\
&= \left[ |\mathbf{k}|^2_{\mathbf{g}} - \tau^2 \right] \varphi^{\frac{n+2}{n-2}} \\
&= \left[ g_{ij} g_{kl} (L^{ik} + \frac{\tau}{n} g^{ik})(L^{jl} + \frac{\tau}{n} g^{jl}) - \tau^2 \right] \varphi^{\frac{n+2}{n-2}} \\
&= \left[ \varphi^{-\frac{4n}{n-2}} \widehat{g}_{ij} \widehat{g}_{kl} \widehat{L}^{ik} \widehat{L}^{jl} + \frac{1}{n}\tau^2 - \tau^2 \right] \varphi^{\frac{n+2}{n-2}} \\
&= \varphi^{\frac{2-3n}{n-2}} |\widehat{L}|^2_{\widehat{\mathbf{g}}} - \frac{n-1}{n} \tau^2 \varphi^{\frac{n+2}{n-2}}.
\end{aligned}$$

□

1.2. **The constant mean curvature equations.** In Theorem 1.4, the equations that we need to solve have decoupled to some extent. This is most easily seen if we specialise to the case of a *constant mean curvature* (CMC) foliation, where we fix the $t$ coordinate of our Lorentzian metric so that $\operatorname{tr} \mathbf{k}$ is constant on the leaves.[2] In this case, $\tau$ is constant, so equations (1.6) above simplify giving

$$\widehat{\nabla}_i \widehat{L}^{ij} = 0, \tag{1.7a}$$

$$\left[ -a\Delta_{\widehat{\mathbf{g}}} + s_{\widehat{\mathbf{g}}} \right] \varphi = \varphi^{\frac{2-3n}{n-2}} |\widehat{L}|^2_{\widehat{\mathbf{g}}} - \frac{n-1}{n} \tau^2 \varphi^{\frac{n+2}{n-2}}. \tag{1.7b}$$

We view $\widehat{\mathbf{g}}$ as being a given Riemannian metric on $M$ and $\tau$ a given constant. We then attempt to find a solution $\widehat{L}^{ij}$ of (1.6a). Given $(\widehat{\mathbf{g}}, \tau, \widehat{L}^{ij})$, we attempt to find a solution $\varphi$ of (1.6b). From this data, we can then reconstruct $(\mathbf{g}, \mathbf{k})$ that satisfy the original constraint equations (1.2).[3]

There is a standard method for solving equation (1.7a). We define the linear differential operator $\mathbb{L}$ that maps a vector field $\mathbf{X} \in \mathfrak{X}(M)$ to the trace-free, symmetric $(2,0)$ tensor field

$$(\mathbb{L}\mathbf{X})^{ij} := \nabla^i X^j + \nabla^j X^i - \frac{2}{n} (\nabla \cdot \mathbf{X}) g^{ij}.$$

*Remark* 1.5. The kernel of the operator $\mathbb{L}$ consists of *conformal Killing vector fields* of the metric $\mathbf{g}$, i.e. vector fields with the property that the Lie derivative of the metric $\mathscr{L}_{\mathbf{X}} \mathbf{g}$ is proportional to the metric $\mathbf{g}$. Such vector fields generate diffeomorphisms $\phi \colon M \to M$ such that $\phi^* \mathbf{g} = \lambda \mathbf{g}$ for some function $\lambda$.

Let $S^{ij}$ be an arbitrary symmetric, trace-free tensor field on $M$. We now let $L^{ij} = S^{ij} + (\mathbb{L}\mathbf{X})^{ij}$, and wish to solve for a vector field $\mathbf{X}$ so that the constraint (1.7a) is satisfied. We therefore require that $\mathbf{X}$ satisfy

$$\operatorname{div} \mathbb{L}\mathbf{X} = -\operatorname{div} \mathbf{S},$$

where the divergence of a $(2,0)$ tensor field is defined by $(\operatorname{div} \mathbf{S})^j := \nabla_i S^{ij}$. We will require some properties of the differential operator $L := \operatorname{div} \circ \mathbb{L} \colon \mathfrak{X}(M) \to \mathfrak{X}(M)$.

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[2]Note that one of the limitations of the conformal method is that this is not always possible.

[3]For notational convenience, from now on we will drop the hats on all of the conformally transformed quantities.

**Assumption.** In order to simplify some analytical issues (e.g. integration by parts formulae, elliptic operator techniques, etc) we will assume, from now on, that $M$ is compact without boundary.

**Proposition 1.6.** *$L$ is (formally) self-adjoint with respect to the $L^2$ inner product on vector fields. More specifically, let $\mathbf{X}, \mathbf{Y} \in \mathfrak{X}(M)$, then $\langle \mathbf{X}, \mathbf{Y} \rangle_{L^2} := \int_M \mathbf{g}(\mathbf{X}, \mathbf{Y})\, d\mu_{\mathbf{g}}$. Then, for all compactly supported, smooth vector fields $\mathbf{X}, \mathbf{Y}$, we have*

$$\langle \mathbf{X}, L\mathbf{Y} \rangle_{L^2} = \langle L\mathbf{X}, \mathbf{Y} \rangle_{L^2} = -\frac{1}{2} \langle \mathbb{L}\mathbf{X}, \mathbb{L}\mathbf{Y} \rangle_{L^2} \tag{1.8}$$

*Proof.*

$$\begin{aligned}
\langle \mathbf{X}, L\mathbf{Y} \rangle_{L^2} &= \int_M X_i \nabla_j \left[ \nabla^i Y^j + \nabla^j Y^i - \frac{2}{n} (\nabla \cdot \mathbf{Y})\, g^{ij} \right] \\
&= -\int_M \nabla_j X_i \left[ \nabla^i Y^j + \nabla^j Y^i - \frac{2}{n} (\nabla \cdot \mathbf{Y})\, g^{ij} \right] \\
&= -\frac{1}{2} \int_M \left[ \nabla_i X_j + \nabla_j X_i - \frac{2}{n} (\nabla \cdot \mathbf{X})\, g_{ij} \right] \left[ \nabla^i Y^j + \nabla^j Y^i - \frac{2}{n} (\nabla \cdot \mathbf{Y})\, g^{ij} \right] \qquad (1.9) \\
&= -\int_M \left[ \nabla_i X_j + \nabla_j X_i - \frac{2}{n} (\nabla \cdot \mathbf{X})\, g_{ij} \right] \nabla^i Y^j \\
&= \int_M \nabla^i \left[ \nabla_i X_j + \nabla_j X_i - \frac{2}{n} (\nabla \cdot \mathbf{X})\, g_{ij} \right] Y^j \\
&= \langle L\mathbf{X}, \mathbf{Y} \rangle_{L^2}.
\end{aligned}$$

This gives the first part of (1.8), and (1.9) yields the second identity. □

**Corollary 1.7.** *A vector field $\mathbf{X}$ satisfies $L\mathbf{X} = 0$ if and only if it is a conformal Killing vector field.*

*Proof.* If $L\mathbf{X} = 0$, then we deduce that

$$0 = \langle \mathbf{X}, L\mathbf{X} \rangle_{L^2} = -\frac{1}{2} \langle \mathbb{L}\mathbf{X}, \mathbb{L}\mathbf{X} \rangle_{L^2} = -\frac{1}{2} \int_M |\mathbb{L}\mathbf{X}|^2_{\mathbf{g}}.$$

The right-hand-side is zero if and only if $\mathbb{L}\mathbf{X} = 0$, i.e. $\mathbf{X}$ is a conformal Killing vector field. □

Recall that an order $m$ linear differential operator $L = \sum_{l \leqslant m} a^{i_1 \dots a_l} \nabla_{i_1} \dots \nabla_{i_l}$ that takes sections of vector bundle $E$ to sections of vector bundle $F$ is *elliptic* if the principal symbol $a(p) := a^{i_1 \dots a_m} p_{i_1} \dots p_{i_m}$, where $p \in T^*M$ is an isomorphism for all $p \neq 0$.[4]

**Proposition 1.8.** *The operator $L \colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ is elliptic.*

*Proof.* Since $L$ is self-adjoint, it suffices to show that $\sigma(p)$ is injective for $p \neq 0$. In particular, if $p \neq 0$ and $\sigma(p)\mathbf{Y} = 0$, then $\mathbf{Y} = 0$. From the explicit form of $L$, we deduce that the symbol $\sigma(p)$ acting on $\mathbf{Y} \in \mathfrak{X}(M)$ takes the form

$$(\sigma(p)\mathbf{Y})^i = p_j \left[ p^i Y^j + p^j Y^i - \frac{2}{n} \langle p, \mathbf{Y} \rangle g^{ij} \right] = |p|^2 Y^i + \left( 1 - \frac{2}{n} \right) p^i \langle p, \mathbf{Y} \rangle. \tag{1.10}$$

If $\sigma(p)\mathbf{Y} = 0$, then contracting this expression with $p_i$ implies that

$$\left( 2 - \frac{2}{n} \right) |p|^2 \langle p, \mathbf{Y} \rangle = 0.$$

Since $n \geqslant 3$ and $|p|^2 \neq 0$, it follows that we must have $\langle p, Y \rangle = 0$. From (1.10), we deduce that $Y^i = 0$. Therefore, $\sigma(p)$ is injective for $p \neq 0$. □

We now require the following result from the theory of elliptic partial differential equations.

[4] More precisely, we would like something like $L \colon H^{m+k}(M, E) \to H^k(M, F)$, but we shall not concern ourselves with analytical subtleties.

**Theorem 1.9.** *Let $L$ be an elliptic partial differential operator of order $m$ on a compact manifold $M$. Assume that the equations $Lu = 0$, $L^\dagger v = 0$ have no non-trivial solutions. Then $L$ defines an isomorphism $L\colon : H^{k+m} \to H^k$. In particular, given $Z \in H^k$, the equation $LX = Z$ has a unique solution $X \in H^{k+m}$.*

**Corollary 1.10.** *If $M$ admits no non-trivial conformal Killing vector fields, then the equation $\operatorname{div} \mathbb{L}X = -\operatorname{div} S$ has a unique solution for any given trace-free, symmetric tensor field $S$.*

*Proof.* $L$ is self-adjoint, and the kernel of $L$ may be identified with the collection of conformal Killing vector fields. Hence the kernel of $L$ and its adjoint is trivial. An application of the theorem then gives the result. □

The result also holds if $M$ does admit a non-trivial conformal Killing vector fields. We may identify the image of $L$ with the orthogonal complement of the collection of conformal Killing vector fields.[5] We wish to solve $LX = -\operatorname{div} S$. Therefore, if we can show that $Z := -\operatorname{div} S$ is orthogonal to all conformal Killing vector fields, then we deduce that $Z \in \operatorname{Im} L$ and hence that there exists an $X$ with the properties that we require. We therefore calculate

$$\begin{aligned}\langle Z, Y\rangle &= \int_M Y_i Z^i = -\int_M Y_i \nabla_j S^{ij} = \int_M (\nabla_j Y_i)\, S^{ij} = \frac{1}{2}\int_M \left(\nabla_j Y_i + \nabla_i Y_j - \frac{2}{n} g_{ij} \nabla \cdot Y\right) S^{ij} \\ &= \frac{1}{2}\langle \mathbb{L}Y, S\rangle,\end{aligned}$$

where we have used the facts that $S^{ij}$ is symmetric and trace-free in the penultimate equality. It follows that if $\mathbb{L}Y = 0$, i.e. $Y$ is a conformal Killing vector field, then $Z$ is orthogonal to $Y$, as required.

We therefore have the following result.

**Theorem 1.11.** *Given any trace-free, symmetric tensor field $S$, there exists a vector field $X$ such that $\operatorname{div} \mathbb{L}X = -\operatorname{div} S$. The trace-free, symmetric tensor field $L^{ij} := S^{ij} + (\mathbb{L}X)^{ij}$ then satisfies the constraint* (1.7a).

As such, we may solve the first of the constraint equations (1.7a). Solving (1.7b) is more involved, and well beyond our scope. We will simply state some results. The problem uses the solution of the Yamabe problem [3] and, with this in mind, we define the *Yamabe invariant*

$$\lambda_{\mathbf{g}} := \inf_{\substack{f \in C^\infty(M) \\ f \not\equiv 0}} \frac{\int_M \left[a|\nabla f|^2_{\mathbf{g}} + s_{\mathbf{g}} f^2\right] d\mu_{\mathbf{g}}}{\left(\int_M |f|^{\frac{2n}{n-2}}\, d\mu_{\mathbf{g}}\right)^{\frac{n-2}{n}}}.$$

We restate (1.7b) (without hats) in the form

$$\left[-a\Delta_{\mathbf{g}} + s_{\mathbf{g}}\right]\varphi = \varphi^{\frac{2-3n}{n-2}} |L|^2_{\mathbf{g}} - \frac{n-1}{n}\tau^2 \varphi^{\frac{n+2}{n-2}}.$$

It is then known that this equation is solvable if and only if one of the following conditions holds:

(1) $|L|_{\mathbf{g}} \not\equiv 0$ and $\tau \neq 0$;
(2) $\lambda_{\mathbf{g}} > 0$, $|L|_{\mathbf{g}} \not\equiv 0$ and $\tau = 0$;
(3) $\lambda_{\mathbf{g}} = 0$, $|L|_{\mathbf{g}} \equiv 0$ and $\tau = 0$;
(4) $\lambda_{\mathbf{g}} < 0$, $|L|_{\mathbf{g}} \equiv 0$ and $\tau \neq 0$;

## References

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[5] Proof: Let $L\colon H_1 \to H_2$ be a map between Hilbert spaces with closed image. Let $L^\dagger\colon H_2 \to H_1$ be its adjoint. Then $y \in (\operatorname{Im} L)^\perp$ if and only if $\langle y, Lx\rangle_{H_2} = 0$ for all $x \in H_1$. This holds if and only if $\langle L^\dagger y, x\rangle_{H^1} = 0$ for all $x \in H_1$. This implies that $L^\dagger y = 0$ in $H_1$, i.e. $y \in \ker L^\dagger$. Hence $(\operatorname{Im} L)^\perp = \ker L^\dagger$. In our case, the kernel of $L^\dagger$ consists of the collection of conformal Killing vector fields on $M$.

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