GLOBAL LORENTZIAN GEOMETRY II

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Lecture 1

Synopsis

The first part of the course gives an introduction to the Positive Mass/Energy theorem in General Relativity. In physical terms, the Positive Energy Theorem states that if a Lorentzian manifold obeys the Einstein field equations with "sensible" matter content (i.e. matter that satisfies the Dominant Energy Condition) then the energy of the system is necessarily non-negative. The original proof of this result is due to Schoen and Yau (see [25, 23, 27, 26]), but we follow a later argument due to Witten [29]. The original Positive Energy Theorem is a result concerning four-dimensional Lorentzian manifolds. We will, however, prove a simpler result (the Riemannian Positive Mass Theorem) that holds on Riemannian spin manifolds of any dimension (and non-spin manifolds of dimension $n \leq 7$). We then indicate the modifications required to prove the Positive Energy Theorem. Our approach is based on the papers of Parker and Taubes [19], Bartnik [6], and the Appendix of Lee and Parker's paper on the Yamabe problem [17]. A useful summary of the approaches to the Positive Energy Theorem around 1983 is the article by Choquet-Bruhat [10].

In the second part of the course, we outline Schoen's use of the Positive Mass Theorem to solve the Yamabe problem in dimensions $n \leq 5$. In the Yamabe problem, we are given a closed (i.e. compact without boundary), connected Riemannian manifold $(M, \mathbf{g})$ and we ask whether we can find a conformally rescaled metric $\widehat{\mathbf{g}} = f^2\mathbf{g}$, where f is a smooth, positive function on M, with the property that the scalar curvature of the metric $\widehat{\mathbf{g}}$ is constant. In dimension 2, all metrics are locally conformally flat, and the existence of such a metric is then essentially a corollary of the Uniformisation theorem. In particular, if $M = S^2$ then there exists a metric with scalar curvature 2 (hence the sectional curvature equals 1), if $M = T^2$ then there exists a flat metric with scalar curvature 0, and on a higher genus Riemann surface, there exists a metric with scalar curvature -2 that we may construct as the quotient under a discrete group of isometries of the upper half-plane with its standard hyperbolic metric. In higher dimensions, Yamabe considered the problem of finding a metric of constant scalar curvature within a conformal equivalence class as a first step towards proving the Poincaré conjecture. Although he claimed to have solved the problem in his original paper [30], there is an error in his paper (as pointed out by Trudinger [28]) and the general proof took some time to emerge. The proof in dimensions $n \geq 6$ in the case where the Yamabe invariant of $(M, \mathbf{g})$, $\lambda_{\mathbf{g}}$, is strictly less than that of the standard sphere, $\lambda(S^n)$, was completed by Aubin [4]. The proof in dimensions $n = 3, 4, 5$ and the remaining cases with $\lambda_{\mathbf{g}} = \lambda(S^n)$ for $n \geq 6$ are due to Schoen [22]. His solution involves the use of the Positive Mass theorem. For general information on the Yamabe problem, I would recommend the article by Lee and Parker [17] and, once you are more familiar with the material, perhaps Chapters 2 and 5 of Aubin's book [5].

If there is time, the third part of the course will present some recent work of Akutagawa, Ishida and LeBrun [1] (see also [2]) between the Yamabe invariant (or sigma constant) of a manifold, which arises in the proof of the Yamabe conjecture, and the $\overline{\lambda}$-invariant defined by Perelman in his proof of the Poincaré conjecture [20].

Lecture 2

Part 1. Witten's proof of the Positive Energy/Mass Theorem

Witten's proof of the Positive Mass/Energy Theorem uses spinors in a fundamental way. To understand spinor fields on manifolds, we first need to understand orthonormal coframes and Clifford algebras. To do this properly would require setting up a large amount of technical machinery that we do not have the time for (e.g. principal G-bundles (in particular, bundles of orthonormal (co)frames), connections on principal bundles, associated vector bundles, ...). As such, I will take a much more concrete, calculation-based approach which, unfortunately, is not the most aesthetically appealing approach from a mathematical point of view. For those of you that want to know about the abstract machinery, I will write an appendix that will give you the details.

1. Orthonormal frames and connections

Let $(M, \mathbf{g})$ be a semi-Riemannian manifold of dimension n, with $\mathbf{g}$ a semi-Riemannian metric of signature (p, q) on M (with $p + q = n$). An *orthonormal coframe* on an open set $U \subseteq M$ is a (continuous) basis $\{\boldsymbol{\epsilon}^1, \ldots, \boldsymbol{\epsilon}^n\}$ for the cotangent bundle T^*M (restricted to the set U) with the property that the metric $\mathbf{g}$ takes the form

$$\mathbf{g} = \boldsymbol{\epsilon}^1 \otimes \boldsymbol{\epsilon}^1 + \cdots + \boldsymbol{\epsilon}^p \otimes \boldsymbol{\epsilon}^p - \boldsymbol{\epsilon}^{p+1} \otimes \boldsymbol{\epsilon}^{p+1} - \cdots - \boldsymbol{\epsilon}^n \otimes \boldsymbol{\epsilon}^n. \tag{1.1}$$

It is often useful to define an $n \times n$ matrix η, with components $\eta_{ij} = \mathrm{diag}[1, \ldots, 1, -1, \ldots, -1]$ in terms of which the above can be written as

$$\mathbf{g} = \sum_{i,j=1}^{n} \eta_{ij} \boldsymbol{\epsilon}^i \otimes \boldsymbol{\epsilon}^j. \tag{1.2}$$

Remark 1.1. Given a semi-Riemannian manifold $(M, \mathbf{g})$, it will generally not be possible to define such a coframe globally on M. Such a global coframe will exists if and only if the cotangent bundle of M is trivial.

There is no unique way to construct an orthonormal basis. Given a point $p \in M$ and an orthonormal basis $\{\boldsymbol{\epsilon}^i\}$ for T_p^*M, consider another basis of T_p^*M, $\widetilde{\boldsymbol{\epsilon}}^i = \sum_{j=1}^n \Lambda^i{}_j \boldsymbol{\epsilon}^j$, where Λ is a non-singular $n \times n$ real matrix. Then, in order for $\widetilde{\boldsymbol{\epsilon}}^i$ to be orthonormal, we require

$$\mathbf{g} = \sum_{i,j} \eta_{ij} \widetilde{\boldsymbol{\epsilon}}^i \otimes \widetilde{\boldsymbol{\epsilon}}^j = \sum_{i,j,k,l} \eta_{ij} \Lambda^i{}_k \Lambda^j{}_l \boldsymbol{\epsilon}^k \otimes \boldsymbol{\epsilon}^l = \sum_{i,j} \eta_{ij} \boldsymbol{\epsilon}^i \otimes \boldsymbol{\epsilon}^j.$$

Comparing coefficients of $\boldsymbol{\epsilon}^i \otimes \boldsymbol{\epsilon}^j$, we require that

$$\sum_{i,j} \eta_{ij} \Lambda^i{}_k \Lambda^j{}_l = \eta_{kl},$$

or, in matrix notation, that

$$\Lambda^t \eta \Lambda = \eta. \tag{1.3}$$

The set of such matrices is the *orthogonal group* of η. We denote this group by

$$O_{p,q} = \{\Lambda \in GL_n(\mathbb{R}) : \Lambda^t \eta \Lambda = \eta\}.$$

As such, there is a natural action of $O(\eta)$ on the space of orthonormal coframes at each point $p \in M$. Taking the determinant of (1.3), we deduce that $(\det \Lambda)^2 = 1$, so $\det \Lambda = \pm 1$. We then define the *special orthogonal group*

$$\mathrm{SO}_{p,q} = \{\Lambda \in O(\eta) : \det \Lambda = 1\}.$$

This group acts on the space of orthonormal coframes, and preserves the orientation of the coframe (i.e. $\widetilde{\boldsymbol{\epsilon}}^1 \wedge \cdots \wedge \widetilde{\boldsymbol{\epsilon}}^n = \boldsymbol{\epsilon}^1 \wedge \cdots \wedge \boldsymbol{\epsilon}^n$).

To study the Lie algebra of $\mathrm{SO}_{p,q}$, we let $\Lambda = \exp sA$, where $A \in \mathbb{R}(n)$, and take d/ds of the condition (1.3) at $t = 0$, giving

$$0 = \frac{d}{ds} \left(\exp(sA^t) \eta \exp(sA) \right) \Big|_{t=0} = A^t \eta + \eta A.$$

In particular, if we consider $p = n, q = 0$, then we deduce that

$$\mathfrak{so}_n = \{A \in \mathbb{R}(n) : A^t = -A\}$$

is the Lie algebra of skew-symmetric $(n \times n)$ matrices. As such, we deduce that $\dim \mathfrak{so}_n = \frac{1}{2}n(n-1)$.

Example 1.2. We previously considered the constant curvature, two-dimensional metrics. Given $K \in \mathbb{R}$, we defined the functions

$$sn_K(r) := \begin{cases} \frac{1}{\sqrt{K}} \sin\left(\sqrt{K} r\right) & K > 0, \\ r & K = 0, \\ \frac{1}{\sqrt{|K|}} \sinh\left(\sqrt{|K|} r\right) & K < 0 \end{cases} \tag{1.4}$$

where $r \in [0, \pi/\sqrt{K}]$ for $K > 0$, and $r \in [0, \infty)$ for $K \leq 0$. We then define the metric

$$\mathbf{g}_K := dr^2 + sn_K(r)^2 d\theta^2, \tag{1.5}$$

where $\theta \in [0, 2\pi)$. In this case, one possible orthonormal coframe would be given by the basis

$$\boldsymbol{\epsilon}^1 := dr, \qquad \boldsymbol{\epsilon}^2 := sn_K(r) d\theta,$$

in terms of which

$$\mathbf{g} = \boldsymbol{\epsilon}^1 \otimes \boldsymbol{\epsilon}^1 + \boldsymbol{\epsilon}^2 \otimes \boldsymbol{\epsilon}^2.$$

Note, however, that we could equally well choose a coframe

$$\boldsymbol{\epsilon}^1_\alpha := \cos\alpha\, \boldsymbol{\epsilon}^1 + \sin\alpha\, \boldsymbol{\epsilon}^2, \qquad \boldsymbol{\epsilon}^2_\alpha := \cos\alpha\, \boldsymbol{\epsilon}^2 - \sin\alpha\, \boldsymbol{\epsilon}^1,$$

for any function $\alpha : M \to \mathbb{R}$, and we would still have

$$\mathbf{g} = \boldsymbol{\epsilon}^1_\alpha \otimes \boldsymbol{\epsilon}^1_\alpha + \boldsymbol{\epsilon}^2_\alpha \otimes \boldsymbol{\epsilon}^2_\alpha.$$

Given an orthonormal coframe $\{\boldsymbol{\epsilon}^i\}$, we may construct the *spin connection*, which is the analogue of the Levi-Civita connection. This is a collection of 1-forms ${\boldsymbol{\Gamma}^i}_j$, $i, j = 1, \ldots, n$, that obey *Cartan's first structure equation*

$$d\boldsymbol{\epsilon}^i = -\sum_{j=1}^n {\boldsymbol{\Gamma}^i}_j \wedge \boldsymbol{\epsilon}^j, \tag{1.6}$$

(which is the analogue of the vanishing torsion condition for the Levi-Civita connection) and the symmetry condition that

$$\sum_{k=1}^n \left(\eta_{ik} {\boldsymbol{\Gamma}^k}_j + \eta_{jk} {\boldsymbol{\Gamma}^k}_i\right) = 0. \tag{1.7}$$

This is the analogue of the metric property of the Levi-Civita connection[1]. In the same way as the Levi-Civita connection is uniquely determined by the conditions that it be torsion-free and metric, it turns out that equations (1.6) and (1.7) uniquely determine the 1-forms ${\boldsymbol{\Gamma}^i}_j$.

Given the connection $\boldsymbol{\Gamma}$, we define the curvature as the collection of 2-forms given by *Cartan's second structure equation*

$${\mathbf{R}^i}_j = d{\boldsymbol{\Gamma}^i}_j + \sum_{k=1}^n {\boldsymbol{\Gamma}^i}_k \wedge {\boldsymbol{\Gamma}^k}_j,$$

and has the symmetry property that

$$\sum_{k=1}^n \left(\eta_{ik} {\mathbf{R}^k}_j + \eta_{jk} {\mathbf{R}^k}_i\right) = 0.$$

The components of the curvature tensor are then defined by the relation

$${\mathbf{R}^i}_j = \frac{1}{2} \sum_{k,l} {R_{kl}{}^i}_j \boldsymbol{\epsilon}^k \wedge \boldsymbol{\epsilon}^l,$$

[1] The point is that we want the covariant derivative of $\mathbf{g}$ to vanish, and we want to organise the connection so that $\nabla \boldsymbol{\epsilon}^i = 0$. This implies that we require $\nabla \eta_{ij} = 0$, which then implies that $0 = \nabla \eta_{ij} = d\eta_{ij} - \sum_k {\boldsymbol{\Gamma}^k}_i \eta_{kj} - \sum_k {\boldsymbol{\Gamma}^k}_j \eta_{ik}$. Since the η_{ij} are constant, this implies equation (1.7).

and obey the symmetry condition

$$R_{kl}{}^{i}{}_{j} = -R_{lk}{}^{i}{}_{j}.$$

The components of the Ricci tensor with respect to the orthonormal frame are then defined by the sum

$$Ric_{jl} = \sum_i R_{il}{}^{i}{}_{j}.$$

Since the Ricci-tensor itself is tensorial, we deduce that

$$\mathrm{Ric} = \sum_{i,j} Ric_{ij} \boldsymbol{\epsilon}^i \otimes \boldsymbol{\epsilon}^j.$$

Finally, the scalar curvature is given by

$$s = \sum_{i,j} \eta_{ij} Ric_{ij}.$$

Example 1.3. We return to our example above, with $\eta = \mathrm{diag}[1, 1]$ and

$$\boldsymbol{\epsilon}^1 := dr, \qquad \boldsymbol{\epsilon}^2 := sn_K(r)\, d\theta.$$

Equation (1.7) implies that we have

$$\boldsymbol{\Gamma}^1{}_1 = \boldsymbol{\Gamma}^2{}_2 = 0, \qquad \boldsymbol{\Gamma}^1{}_2 + \boldsymbol{\Gamma}^2{}_1 = 0,$$

so the only non-trivial information on the connection is contained in $\boldsymbol{\Gamma}^1{}_2$, say. Equation (1.6) with $i = 1$ implies that

$$d\boldsymbol{\epsilon}^1 = d(dr) = 0 = -\boldsymbol{\Gamma}^1{}_2 \wedge \boldsymbol{\epsilon}^2.$$

Therefore $\boldsymbol{\Gamma}^1{}_2 = \lambda \boldsymbol{\epsilon}^2$, for some function λ. Equation (1.6) with $i = 2$ gives

$$d\boldsymbol{\epsilon}^2 = d(sn_K(r)d\theta) = sk_k'(r)dr \wedge d\theta = \frac{sk_k'(r)}{sk_k(r)} \boldsymbol{\epsilon}^1 \wedge \boldsymbol{\epsilon}^2 = -\boldsymbol{\Gamma}^2{}_1 \wedge \boldsymbol{\epsilon}^1 = \boldsymbol{\Gamma}^1{}_2 \wedge \boldsymbol{\epsilon}^1 = \lambda \boldsymbol{\epsilon}^2 \wedge \boldsymbol{\epsilon}^1.$$

Hence

$$\lambda = -\frac{sk_k'(r)}{sk_k(r)}.$$

and

$$\boldsymbol{\Gamma}^1{}_2 = -\frac{sk_k'(r)}{sk_k(r)} \boldsymbol{\epsilon}^2 = -sk_k'(r) d\theta.$$

Finally, the second Cartan structure equation implies that the only non-trivial curvature two-form is

$$\mathbf{R}^1{}_2 = d\boldsymbol{\Gamma}^1{}_2 = d(-sk_k'(r)d\theta) = -sk_k''(r)dr \wedge d\theta = -\frac{sk_k''(r)}{sk_k(r)} \boldsymbol{\epsilon}^1 \wedge \boldsymbol{\epsilon}^2.$$

Therefore, we have

$$R_{12}{}^1{}_2 = -\frac{sk_k''(r)}{sk_k(r)} = K,$$

where the final relationship follows from the form of the functions $sn_K(r)$. The non-vanishing components of the Ricci tensor are then

$$Ric_{11} = Ric_{22} = R_{12}{}^1{}_2 = K.$$

Therefore the Ricci tensor is of the form

$$\mathrm{Ric} = K\boldsymbol{\epsilon}^1 \otimes \boldsymbol{\epsilon}^1 + K\boldsymbol{\epsilon}^2 \otimes \boldsymbol{\epsilon}^2 = K\mathbf{g}.$$

and the scalar curvature is

$$s = Ric_{11} + Ric_{22} = 2K.$$

Lecture 3

2. Spinors

Our treatment in this section largely follows [21, Chapter 3]. For much more details on Clifford algebras, see, e.g., [16].

Let V be a finite-dimensional vector space over $\mathbb{R}$, with inner product $\mathbf{q}(\cdot,\cdot)$. A Clifford algebra for V is an algebra with unit, A, equipped with a map $\varphi : V \to A$ such that $\varphi(\mathbf{v})^2 = -\mathbf{q}(\mathbf{v},\mathbf{v})1$, for all $\mathbf{v} \in V$, that is universal among algebras equipped with such maps.

Remark 2.1. $\mathrm{Cl}(V,\mathbf{g})$ may be equivalently defined as the algebra generated by the vector space $V \subset \mathrm{Cl}(V,\mathbf{g})$ and the identity element 1 subject to the relation

$$\mathbf{v}\cdot\mathbf{v} = -\mathbf{g}(\mathbf{v},\mathbf{v})1, \qquad \forall \mathbf{v} \in V.$$

By polarisation, it follows that

$$\mathbf{v}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w} = -2\mathbf{g}(\mathbf{v},\mathbf{w})1, \qquad \forall \mathbf{v},\mathbf{w} \in V.$$

Remark 2.2. For any given $(V,\mathbf{q})$, such an algebra exists, and is unique up to isomorphism. We denote it by $\mathrm{Cl}(V,\mathbf{q})$.

Example 2.3. Consider $V = \mathbb{R}^n$ with the standard inner product

$$\mathbf{g}(\mathbf{x},\mathbf{y}) := x^1y^1 + \cdots + x^ny^n,$$

where $\mathbf{x} = (x^1,\ldots,x^n) \in \mathbb{R}^n$, etc. In this case, we denote the corresponding Clifford algebra by Cl_n (or $\mathrm{Cl}_{n,0}$).

$\boxed{n=1}$ In this case, $V = \mathbb{R}$ with $\mathbf{g}(x,y) = xy$ for $x,y \in \mathbb{R}$. We may then choose V to be spanned by the vector $\mathbf{x} = (1)$. Cl_1 is then the algebra generated by 1 and $\mathbf{v}$ (defined to be the image of $\mathbf{x}$ under the embedding given above) subject to the relation

$$\mathbf{v}^2 := \mathbf{v}\cdot\mathbf{v} = -\mathbf{g}(\mathbf{x},\mathbf{x}) = -1.$$

We therefore require the algebra generated by $\{1,\mathbf{v}\}$ with $\mathbf{v}^2 = -1$. As such, Cl_1 is isomorphic to the algebra of complex numbers $\mathbb{C}$.

$\boxed{n=2}$ In this case, $V = \mathbb{R}^2$, with $\mathbf{g}(\mathbf{x},\mathbf{y}) = x^1y^1 + x^2y^2$ for $\mathbf{x},\mathbf{y} \in \mathbb{R}^2$. We may then choose V to be spanned by the vector $\mathbf{x} = \begin{pmatrix}1\\0\end{pmatrix}$, $\mathbf{y} = \begin{pmatrix}0\\1\end{pmatrix}$. Cl_2 is then the algebra generated by $\{1,\mathbf{u},\mathbf{v}\}$ subject to the relations

$$\mathbf{u}^2 = \mathbf{v}^2 = -1, \qquad \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} = 0.$$

Defining $\mathbf{w} := \mathbf{u}\cdot\mathbf{v}$, we deduce that Cl_2 is the space generated by $\{1,\mathbf{u},\mathbf{v},\mathbf{w}\}$ subject to the relations

$$\mathbf{u}^2 = \mathbf{v}^2 = -1, \qquad \mathbf{u}\cdot\mathbf{v} = \mathbf{w} = -\mathbf{v}\cdot\mathbf{u}.$$

From these relations it follows that Cl_2 is isomorphic to the algebra of quaternions $\mathbb{H}$.

More generally, the Clifford algebras, Cl_n, with $n = 0,\ldots,7$ are given in Table 1 where, for $\mathbb{K} = \mathbb{R}/\mathbb{C}/\mathbb{H}$, the notation $\mathbb{K}(n)$ denotes the algebra of $(n\times n)$ matrices with entries in $\mathbb{K}$. In

n	1	2	3	4	5	6	7	8
Cl_n	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$

Table 1. Clifford algebras up to dimension 8.

dimensions $n > 8$, the Clifford algebras may be deduced from the isomorphism

$$\mathrm{Cl}_{n+8} \cong \mathrm{Cl}_n \otimes \mathbb{R}(16).$$

Representations: Let S be a complex vector space equipped with an $\mathbb{R}$-linear map $c : V \to \mathrm{End}_{\mathbf{C}}(S)$ with the property that $c(\mathbf{v})^2 = -\mathbf{q}(\mathbf{v}, \mathbf{v})$. S is then a Clifford module and we have the Clifford multiplication map $V \times S \to S : (\mathbf{v}, s) \mapsto c(\mathbf{v})s$, which we will also denote by $\mathbf{v} \cdot s$.

Bundles: If $(M, \mathbf{g})$ is a Riemannian manifold[2] then the fibres of TM are inner product spaces, so we form the Clifford bundle $\mathrm{Cl}(TM, \mathbf{g})$, the fibres of which are the Clifford algebras $\mathrm{Cl}(T_pM, \mathbf{g}_p)$, for $p \in M$. Similarly, we define the bundle of Clifford modules S, with fibre S_p defined as above.

We want to define a connection on the bundle S. Following [21], we have the following:

Definition 2.4. Let S be a bundle of Clifford modules over Riemannian manifold $(M, \mathbf{g})$. S is a *Clifford bundle* if it is equipped with a Hermitian metric $(\cdot, \cdot)$ and a compatible connection ∇ such that

i) the action of $\mathbf{v} \in T_pM$ on S_p is skew-adjoint with respect to $(\cdot, \cdot)$:

$$(\mathbf{v} \cdot s_1, s_2) + (s_1, \mathbf{v} \cdot s_2) = 0, \qquad \forall \mathbf{v} \in T_pM, \quad \forall s_1, s_2 \in S_p.$$

ii) The connection on S is compatible with the Levi-Civita connection

$$\nabla_{\mathbf{X}} (\mathbf{v} \cdot s) = (\nabla_{\mathbf{X}} \mathbf{v}) \cdot s + \mathbf{v} \cdot (\nabla_{\mathbf{X}} s), \qquad \forall \mathbf{X}, \mathbf{v} \in \mathfrak{X}(M), \quad \forall s \in C^\infty(S).$$

We then define the *Dirac operator*, D, on a Clifford bundle S to be the first order partial differential operator $C^\infty(S) \to C^\infty(S)$ defined as the composition

$$C^\infty(S) \xrightarrow{\nabla} C^\infty(T^*M \otimes S) \xrightarrow{\mathbf{g}} C^\infty(TM \otimes S) \xrightarrow{\cdot} C^\infty(S).$$

Here the first map is covariant differentiation, the second is contraction with the metric $\mathbf{g}$, and the third is Clifford multiplication. In terms of a local orthonormal frame $\{\mathbf{e}_i\}$ for TM, this may be written in the form

$$Ds = \sum_{i=1}^{n} \mathbf{e}_i \cdot \nabla_{\mathbf{e}_i} s$$

for $s \in C^\infty(S)$.

Example 2.5. Let $S = \Lambda^*(M)$, the exterior algebra on M. We define Clifford multiplication by $\mathbf{v} \in \mathfrak{X}(M)$ by

$$\mathbf{v} \cdot s = \mathbf{g}(\mathbf{v}, \cdot) \wedge s + \mathbf{v} \lrcorner s,$$

for all $s \in \Omega^*(M) \equiv C^\infty(S)$. In this case, the Dirac operator takes the form $D = d + d^*$, where d^* is the formal adjoint of d with respect to the inner product on S induced by the Riemannian metric.

For many purposes, we will need to know the square of the Dirac operator $D^2 : C^\infty(S) \to C^\infty(S)$. It is most straightforward to calculate this at a point $p \in M$ using an orthonormal coframe $\mathbf{e}_i$ defined on a neighbourhood of p that obeys the condition that $\nabla_{\mathbf{e}_i} \mathbf{e}_j \big|_p = 0$. Note that

[2] We will mention later the modifications for the semi-Riemannian case.

this condition implies that $[\mathbf{e}_i, \mathbf{e}_j]\big|_p = 0$. At p, we then have

$$
\begin{aligned}
D^2 s &= \sum_i \mathbf{e}_i \cdot \nabla_{\mathbf{e}_i} \left(\sum_j \mathbf{e}_j \cdot \nabla_{\mathbf{e}_j} s \right) \\
&= \sum_{i,j} \mathbf{e}_i \cdot \mathbf{e}_j \cdot \left(\nabla_{\mathbf{e}_i} \nabla_{\mathbf{e}_j} s \right) && \left(\text{using } \nabla_{\mathbf{e}_i} \mathbf{e}_j\big|_p = 0\right) \\
&= \sum_{i,j} \frac{1}{2} \left(\mathbf{e}_i \cdot \mathbf{e}_j + \mathbf{e}_j \cdot \mathbf{e}_i + \mathbf{e}_i \cdot \mathbf{e}_j - \mathbf{e}_j \cdot \mathbf{e}_i \right) \cdot \left(\nabla_{\mathbf{e}_i} \nabla_{\mathbf{e}_j} s \right) \\
&= \sum_{i,j} \frac{1}{2} \left\{ -2\mathbf{g}(\mathbf{e}_i, \mathbf{e}_j) \right\} \nabla_{\mathbf{e}_i} \nabla_{\mathbf{e}_j} s + \sum_{i,j} \frac{1}{2} \mathbf{e}_i \cdot \mathbf{e}_j \cdot \left[\nabla_{\mathbf{e}_i}, \nabla_{\mathbf{e}_j} \right] s \\
&= - \sum_i \nabla_{\mathbf{e}_i} \nabla_{\mathbf{e}_i} s + \sum_{i,j} \frac{1}{2} \mathbf{e}_i \cdot \mathbf{e}_j \cdot \left\{ \left[\nabla_{\mathbf{e}_i}, \nabla_{\mathbf{e}_j} \right] - \nabla_{[\mathbf{e}_i, \mathbf{e}_j]} \right\} s && \left(\text{using } [\mathbf{e}_i, \mathbf{e}_j]\big|_p = 0\right) \\
&= - \sum_i \nabla_{\mathbf{e}_i} \nabla_{\mathbf{e}_i} s + \sum_{i,j} \frac{1}{2} \mathbf{e}_i \cdot \mathbf{e}_j \cdot \mathbf{K}(\mathbf{e}_i, \mathbf{e}_j) s,
\end{aligned}
$$

where $\mathbf{K} \in \Omega^2(\mathrm{End}(S))$ is the curvature of the connection on S defined by

$$
\mathbf{K}(\mathbf{X}, \mathbf{Y}) = \left[\nabla_{\mathbf{e}_i}, \nabla_{\mathbf{e}_j} - \nabla_{[\mathbf{e}_i, \mathbf{e}_j]} \right] s, \qquad \forall \mathbf{X}, \mathbf{Y} \in \mathfrak{X}(M), \qquad \forall s \in C^\infty(S)
$$

We therefore have

$$
D^2 s = - \sum_i \nabla_{\mathbf{e}_i} \nabla_{\mathbf{e}_i} s + \sum_{i,j} \frac{1}{2} \mathbf{e}_i \cdot \mathbf{e}_j \cdot \mathbf{K}(\mathbf{e}_i, \mathbf{e}_j) s \tag{2.1}
$$

Lecture 4

Remark 2.6. The bundle $T^*M \otimes S$ has a natural (pointwise) inner product, defined using the Hermitian metric on S and the inner product on T^*M induced by the metric $\mathbf{g}$. Given $\varphi_1, \varphi_2 \in C^\infty(T^*M \otimes S)$, this takes the form

$$
(\varphi_1, \varphi_2)_{T^*M \otimes S} := \sum_{i,j} g^{ij} \left(\varphi_{1i}, \varphi_{2j} \right) \equiv \sum_i \left(\varphi_1(\mathbf{e}_i), \varphi_2(\mathbf{e}_i) \right),
$$

where the first expression is in local coordinates, the second is with respect to a local orthonormal frame, and $(\cdot, \cdot)$ is the inner product on S. We will also denote the inner product $(\cdot, \cdot)_{T^*M \otimes S}$ by $(\cdot, \cdot)$, when no confusion can arise. Integrating these inner products over M, we define the L^2 inner products

$$
\langle s_1, s_2 \rangle = \int_M (s_1, s_2) \, d\mathbf{vol}_{\mathbf{g}}, \qquad \langle \varphi_1, \varphi_2 \rangle = \int_M (\varphi_1, \varphi_2) \, d\mathbf{vol}_{\mathbf{g}},
$$

for $s_1, s_2 \in C^\infty(S)$, $\varphi_1, \varphi_2 \in C^\infty(T^*M \otimes S)$.

Given the operator $\nabla : C^\infty(S) \to C^\infty(T^*M \otimes S)$, we may define its formal adjoint $\nabla^* : C^\infty(T^*M \otimes S) \to C^\infty(S)$. Recall that we introduced an orthonormal frame $\{\mathbf{e}_i\}$ above with the property that $\nabla_{\mathbf{e}_i} \mathbf{e}_j\big|_p = 0$. In the next result, we also use the dual orthonormal coframe, denoted $\{\boldsymbol{\epsilon}^i\}$ which has the property that $\nabla_{\mathbf{e}_i} \boldsymbol{\epsilon}^j\big|_p = 0$

Lemma 2.7. *The operator $\nabla^* : C^\infty(T^*M \otimes S) \to C^\infty(S)$ is given in terms of the orthonormal coframe $\{\mathbf{e}_i\}$ by the formula*

$$
\nabla^* \left(\sum_i \boldsymbol{\epsilon}^i \otimes s_i \right) = - \sum_i \nabla_i s_i, \tag{2.2}
$$

where $s_i \in C^\infty(S)$. (From now on we will use the notation ∇_i to denote $\nabla_{\mathbf{e}_i}$.)

Proof. To show that ∇^* is the formal adjoint of ∇ with respect to the inner products $\langle \cdot, \cdot \rangle$, we need to show that

$$
\langle \varphi, \nabla s \rangle - \langle \nabla^* \varphi, s \rangle = \text{boundary term},
$$

for all $s \in C^\infty(S)$ and $\varphi \in C^\infty(T^*M \otimes S)$. (If we want to be more formal then, if M is, for example, an open set with compact closure, we show that the left-hand-side of the above is zero when s, φ have compact support contained in M.) To show that the assertion of the Lemma is correct, we need to calculate

$$(s, \nabla^* \varphi) - (\nabla s, \varphi)$$

for $s \in C^\infty(S)$, $\varphi \in C^\infty(T^*M \otimes S)$ with ∇^* defined as in (2.2), and show that this gives a boundary term when integrated over M. Working at point $p \in M$ with a basis $\mathbf{e}_i$ and dual basis $\{\boldsymbol{\epsilon}^i\}$ as above, we then, without loss of generality, let $\varphi = \sum_i \boldsymbol{\epsilon}^i \otimes s_i$, where $s_i \in C^\infty(S)$. We then calculate

$$\begin{aligned}(s, \nabla^* \varphi) - (\nabla s, \varphi) &= \left(s, \nabla^* \left(\sum_i \boldsymbol{\epsilon}^i \otimes s_i \right) \right) - \left(\nabla s, \sum_i \boldsymbol{\epsilon}^i \otimes s_i \right) \\ &= \left(s, -\sum_i \nabla_i s_i \right) - \sum_i (\nabla_i s, s_i) \\ &= -\sum_i \nabla_i (s, s_i) \\ &= -\operatorname{div} \mathbf{v},\end{aligned}$$

where $\mathbf{v} := \sum_i (s, s_i)\, \mathbf{e}_i$. Stokes' theorem then implies that

$$\langle s, \nabla^* \varphi \rangle = \langle \nabla s, \varphi \rangle - \int_{\partial M} \mathbf{v} \cdot d\mathbf{S}.$$

Therefore ∇^* is the formal adjoint of ∇ with respect to the L^2 inner product, as required. □

Remark 2.8. By similar methods, one can show that the Dirac operator is formally self-adjoint with respect to the L^2 inner product on S i.e. if $s_1, s_2 \in C^\infty(S)$, at least one of which has compact support, then

$$\langle s_1, D s_2 \rangle = \langle D s_1, s_2 \rangle.$$

To summarise: at this stage, equation (2.1) may be rewritten in the form

$$D^2 s = \nabla^* \nabla s + \sum_{i,j} \frac{1}{2} \mathbf{e}_i \cdot \mathbf{e}_j \cdot \mathbf{K}(\mathbf{e}_i, \mathbf{e}_j) s \tag{2.3}$$

We now aim to simplify the second term in this equation.

Proposition 2.9.

$$D^2 s = \nabla^* \nabla s + \frac{R}{4} s + \mathcal{F}^S s, \tag{2.4}$$

where R denotes the scalar curvature of the Riemannian metric $\mathbf{g}$ and

$$\mathcal{F}^S := \frac{1}{2} \sum_{i,j} c(\mathbf{e}_i) c(\mathbf{e}_j) \mathbf{F}(\mathbf{e}_i, \mathbf{e}_j) \in C^\infty(\operatorname{End}(S)),$$

with $\mathbf{F} \in \Omega^2(\operatorname{End}(S))$ commuting with Clifford multiplication.

We prove this Proposition in a sequence of smaller steps.

Lemma 2.10. *As endomorphisms of S,*

$$[\mathbf{K}(\mathbf{X}, \mathbf{Y}), c(\mathbf{Z})] = c(\mathbf{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z}),$$

for all $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z} \in \mathfrak{X}(M)$.

Proof. Since the connection is compatible with Clifford multiplication, we have

$$\begin{aligned}\left([\nabla_{\mathbf{X}}, \nabla_{\mathbf{Y}}] - \nabla_{[\mathbf{X},\mathbf{Y}]}\right)(\mathbf{Z}\cdot s) &= \nabla_{\mathbf{X}}\left((\nabla_{\mathbf{Y}}\mathbf{Z})\cdot s + \mathbf{Z}\cdot\nabla_{\mathbf{Y}}s\right) - \nabla_{\mathbf{Y}}\left((\nabla_{\mathbf{X}}\mathbf{Z})\cdot s + \mathbf{Z}\cdot\nabla_{\mathbf{X}}s\right)\\ &\qquad - \left(\left(\nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z}\right)\cdot s + \mathbf{Z}\cdot\nabla_{[\mathbf{X},\mathbf{Y}]}s\right)\\ &= (\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z})\cdot s + (\nabla_{\mathbf{Y}}\mathbf{Z})\cdot\nabla_{\mathbf{X}}s + (\nabla_{\mathbf{X}}\mathbf{Z})\cdot\nabla_{\mathbf{Y}}s + \mathbf{Z}\cdot\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}s\\ &\quad - (\nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{Z})\cdot s - (\nabla_{\mathbf{X}}\mathbf{Z})\cdot\nabla_{\mathbf{Y}}s - (\nabla_{\mathbf{Y}}\mathbf{Z})\cdot\nabla_{\mathbf{X}}s + \mathbf{Z}\cdot\nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}s\\ &\quad - \left(\nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z}\right)\cdot s - \mathbf{Z}\cdot\nabla_{[\mathbf{X},\mathbf{Y}]}s\\ &= \left(\left([\nabla_{\mathbf{X}}, \nabla_{\mathbf{Y}}] - \nabla_{[\mathbf{X},\mathbf{Y}]}\right)\mathbf{Z}\right)\cdot s + \mathbf{Z}\cdot\left(\left([\nabla_{\mathbf{X}}, \nabla_{\mathbf{Y}}] - \nabla_{[\mathbf{X},\mathbf{Y}]}\right)s\right)\\ &= \mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z}\cdot s + \mathbf{Z}\cdot\mathbf{K}(\mathbf{X},\mathbf{Y})s,\end{aligned}$$

for all $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z} \in \mathfrak{X}(M)$ and all $s \in C^\infty(S)$. Since the left-hand-side of this equation equals $\mathbf{K}(\mathbf{X},\mathbf{Y})(\mathbf{Z}\cdot s)$, we have the required result. □

Given the Riemann tensor $\mathbf{R}$ we define $\mathcal{R} \in \Omega^2(\mathrm{End}(S))$ by

$$\mathcal{R}(\mathbf{X},\mathbf{Y}) := \frac{1}{4}\sum_{k,l} c(\mathbf{e}_k)c(\mathbf{e}_l)\mathbf{g}(\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{e}_k, \mathbf{e}_l).$$

Lemma 2.11. *As endomorphisms of S,*

$$[\mathcal{R}(\mathbf{X},\mathbf{Y}), c(\mathbf{Z})] = c(\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z}),$$

for all $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z} \in \mathfrak{X}(M)$.

Proof. This is simply a calculation:

$$\begin{aligned}[\mathcal{R}(\mathbf{X},\mathbf{Y}), c(\mathbf{Z})] &= \frac{1}{4}\sum_{k,l}\left(\mathbf{e}_k\cdot\mathbf{e}_l\cdot\mathbf{Z} - \mathbf{Z}\cdot\mathbf{e}_k\cdot\mathbf{e}_l\right)\mathbf{g}(\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{e}_k,\mathbf{e}_l)\\ &= \frac{1}{4}\sum_{k,l}\left(\mathbf{e}_k\cdot(-\mathbf{Z}\cdot\mathbf{e}_l - 2\mathbf{g}(\mathbf{Z},\mathbf{e}_l)) - \mathbf{Z}\cdot\mathbf{e}_k\cdot\mathbf{e}_l\right)\mathbf{g}(\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{e}_k,\mathbf{e}_l)\\ &= \frac{1}{4}\sum_{k,l}\left(-(-\mathbf{Z}\cdot\mathbf{e}_k - 2\mathbf{g}(\mathbf{Z},\mathbf{e}_k))\cdot\mathbf{e}_l - 2\mathbf{g}(\mathbf{Z},\mathbf{e}_l)\mathbf{e}_k - \mathbf{Z}\cdot\mathbf{e}_k\cdot\mathbf{e}_l\right)\mathbf{g}(\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{e}_k,\mathbf{e}_l)\\ &= \frac{1}{4}\sum_{k,l}\left(2\mathbf{g}(\mathbf{Z},\mathbf{e}_k)\mathbf{e}_l - 2\mathbf{g}(\mathbf{Z},\mathbf{e}_l)\mathbf{e}_k\right)\mathbf{g}(\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{e}_k,\mathbf{e}_l)\\ &= \frac{1}{4}\sum_{k,l}\mathbf{g}(\mathbf{Z},\mathbf{e}_k)\mathbf{g}(\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{e}_k,\mathbf{e}_l)\mathbf{e}_l\\ &= \sum_{l}\mathbf{g}(\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z},\mathbf{e}_l)\mathbf{e}_l\\ &= \mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z},\end{aligned}$$

as required. □

Corollary 2.12. *As endomorphisms of S*

$$[\mathcal{R}(\mathbf{X},\mathbf{Y}), c(\mathbf{Z})] = [\mathbf{K}(\mathbf{X},\mathbf{Y}), c(\mathbf{Z})].$$

Hence

$$\mathbf{K}(\mathbf{X},\mathbf{Y}) = \mathcal{R}(\mathbf{X},\mathbf{Y}) + \mathbf{F}(\mathbf{X},\mathbf{Y}), \tag{2.5}$$

where $\mathbf{F} \in \Omega^2(\mathrm{End}(S))$ commutes with the action of the Clifford algebra.

Proof of Proposition 2.9. From (2.5), we have

$$\frac{1}{2}\sum_{i,j} c(\mathbf{e}_i)c(\mathbf{e}_j)\mathbf{K}(\mathbf{e}_i,\mathbf{e}_j) = \frac{1}{2}\sum_{i,j} c(\mathbf{e}_i)c(\mathbf{e}_j)\mathcal{R}(\mathbf{e}_i,\mathbf{e}_j) + \frac{1}{2}\sum_{i,j} c(\mathbf{e}_i)c(\mathbf{e}_j)\mathbf{F}(\mathbf{e}_i,\mathbf{e}_j).$$

The second term is the object denoted $\mathcal{F}^S$ in our Proposition. The first term, we can simplify. Note that

$$\begin{aligned}\frac{1}{2}\sum_{i,j} c(\mathbf{e}_i)c(\mathbf{e}_j)\mathcal{R}(\mathbf{e}_i,\mathbf{e}_j) &= \frac{1}{8}\sum_{i,j,k,l} c(\mathbf{e}_i)c(\mathbf{e}_j)\mathbf{g}(\mathbf{R}(\mathbf{e}_i,\mathbf{e}_j)\mathbf{e}_k,\mathbf{e}_l)c(\mathbf{e}_k)c(\mathbf{e}_l)\\ &= \frac{1}{8}\sum_{i,j,k,l} c(\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l)\mathbf{g}(\mathbf{R}(\mathbf{e}_i,\mathbf{e}_j)\mathbf{e}_k,\mathbf{e}_l).\end{aligned}$$

If i,j,k are distinct, then the expression $c(\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l)$ is skew-symmetric in i,j,k. Since we are summing, this means we would get a cyclic sum over the components $\mathbf{R}(\mathbf{e}_i,\mathbf{e}_j)\mathbf{e}_k$, which the Bianchi identity tells us is zero[3]. As such, the only contributions to the final sum are those with $i=k\neq j$ and $j=k\neq i$. We therefore want to calculate the quantity

$$\begin{aligned}\frac{1}{2}\sum_{i,j} c(\mathbf{e}_i)c(\mathbf{e}_j)\mathcal{R}(\mathbf{e}_i,\mathbf{e}_j) &= \frac{1}{8}\sum_{l}\left(\sum_{i=k\neq j}+\sum_{i=k\neq j}\right) c(\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l)\mathbf{g}(\mathbf{R}(\mathbf{e}_i,\mathbf{e}_j)\mathbf{e}_k,\mathbf{e}_l)\\ &= \frac{1}{4}\sum_{l}\sum_{i,j} c(\mathbf{e}_i\mathbf{e}_j\mathbf{e}_i\mathbf{e}_l)\mathbf{g}(\mathbf{R}(\mathbf{e}_i,\mathbf{e}_j)\mathbf{e}_i,\mathbf{e}_l)\\ &= \frac{1}{4}\sum_{l}\sum_{i,j} c(-\mathbf{e}_i\mathbf{e}_i\mathbf{e}_j\mathbf{e}_l)\mathbf{g}(\mathbf{R}(\mathbf{e}_i,\mathbf{e}_j)\mathbf{e}_i,\mathbf{e}_l)\\ &= \frac{1}{4}\sum_{l}\sum_{i,j} c(\mathbf{e}_j\mathbf{e}_l)\mathbf{g}(\mathbf{R}(\mathbf{e}_i,\mathbf{e}_j)\mathbf{e}_i,\mathbf{e}_l) \qquad \text{(since } c(\mathbf{e}_i)^2=-1)\\ &= -\frac{1}{4}\sum_{j,l} c(\mathbf{e}_j\mathbf{e}_l)\mathbf{Ric}(\mathbf{e}_j,\mathbf{e}_l)\\ &= -\frac{1}{4}\sum_{j,l} -\mathbf{g}(\mathbf{e}_j,\mathbf{e}_l)\,\mathbf{Ric}(\mathbf{e}_j,\mathbf{e}_l)\\ &= \frac{1}{4}R,\end{aligned}$$

where R denotes the Ricci scalar.

Therefore,

$$D^2 s = \nabla^*\nabla s + \frac{R}{4}s + \mathcal{F}^S s,$$

as required. □

Lecture 5

2.1. **Spin representations.** We have thus far worked with general Clifford modules and Clifford bundles. However, we now restrict ourselves to a particular class of modules and bundles that arise directly from the representation theory of Clifford algebras.

Claim: Let M be a spin manifold[4] of dimension n. If $n=2m$ is even, or $n=2m+1$ is odd, then there exists a complex vector bundle over M, Δ, over M of rank 2^m with the following property: Let S be a Clifford bundle over M. Then there exists a vector bundle V, equipped with a Hermitian metric and connection, with the property that $S\cong\Delta\otimes V$ (as Clifford bundles). Δ is referred to as the *spin bundle*. The curvature of the natural connection on S is then of the form $\mathbf{K}^\Delta\otimes 1+1\otimes\mathbf{K}^V$. In particular, the first term is the $\mathcal{R}$ of (2.5) and the second is the $\mathbf{F}$ term.

If you wish to see details of the proof of this result, see e.g. [16] or [21]. From now on, we will assume that we are working on the bundle Δ. In this case, we may explicitly write down (in local form) the compatible covariant derivative on Δ in terms of local bases, etc. If $\{\boldsymbol{\epsilon}^i\}$ is

[3]Recall $\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z}+\mathbf{R}(\mathbf{Y},\mathbf{Z})\mathbf{X}+\mathbf{R}(\mathbf{Z},\mathbf{X})\mathbf{Y}=0$.

[4]I will not formally define what this term means. It is basically the requirement that the bundles mentioned later in this paragraph are globally well-defined. If M is not a spin manifold, then the bundles Δ and V will not be globally defined, but the tensor product $\Delta\otimes V$ *will* be globally well-defined.

an orthonormal coframe for T^*M and $\{\mathbf{e}_i\}$ the dual orthonormal frame, then (in terms of a local trivialisation of Δ) we may write

$$\nabla\psi = d\psi + \frac{1}{2}\sum_{i,j=1}^{n} \mathbf{\Gamma}_{ij}\left(-\frac{1}{4}\left[c(\mathbf{e}_i), c(\mathbf{e}_j)\right]\right)\psi,$$

where $\psi \in C^\infty(\Delta)$ and $\mathbf{\Gamma}_{ij}$ are the components of the spin-connection that we introduced in Section 1[5]. The curvature of this connection then coincides with $\mathcal{R} \in \Omega^2(\mathrm{End}(\Delta))$ introduced above, and explicitly has $\mathcal{F}^S = 0$.

Remark 2.13. Letting $\Sigma_{ij} := -\frac{1}{4}\left[c(\mathbf{e}_i), c(\mathbf{e}_j)\right] \in C^\infty(\mathrm{End}(\Delta))$, it is straightforward to show that the Σ's define a representation of the orthogonal group SO_n in the sense that

$$[\Sigma_{ij}, \Sigma_{kl}] = -\delta_{ik}\Sigma_{jl} + \delta_{jk}\Sigma_{il} + \delta_{il}\Sigma_{jk} - \delta_{jl}\Sigma_{ik},$$

for $i, j, k, 1 = 1, \ldots, n$ where δ is the Kronecker symbol

$$\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

2.2. **The Lichnerowicz theorem.** [6]

In the case where the bundle S is the spin-bundle, Δ, then $\mathcal{F}^S = 0$ and we have the Lichnerowicz formula [18] for the square of the Dirac operator[7]

$$D^2\psi = \nabla^*\nabla\psi + \frac{s}{4}\psi. \tag{2.6}$$

for any section $\psi \in C^\infty(\Delta)$.

Proposition 2.14. *Let $(M, \mathbf{g})$ be a closed, Riemannian spin manifold with non-negative scalar curvature that is somewhere positive. Then there are no non-trivial sections $\psi \in C^\infty(\Delta)$ with $D\psi = 0$ on M.*

Proof. Let $\mathbf{g}$ be a metric of positive scalar curvature, and let ψ lie in the kernel of D (i.e. $D\psi = 0$). Then $D^2\psi = D(D\psi) = 0$. Taking the product of $D^2\psi$ with ψ and integrating over M, we deduce that

$$0 = \int_M \langle\psi, D^2\psi\rangle = \int_M \left\langle\psi, \nabla^*\nabla\psi + \frac{s}{4}\psi\right\rangle = \int_M \left(|\nabla\psi|^2 + \frac{s}{4}|\psi|^2\right).$$

Since the right-hand-side is a sum of non-negative terms, it follows that both terms must vanish. In particular, $\nabla\psi = 0$ (so ψ is necessarily parallel) and $\frac{s}{4}|\psi|^2 = 0$. Therefore, as long as there exists $p \in M$ at which $s(p) > 0$, then we deduce that $\psi = 0$. ☐

Remark 2.15. The Atiyah-Singer index theorem implies that, on a spin manifold, the index of the Dirac operator $\mathrm{ind}\, D = n_+ - n_-$ (where $n_\pm$ are non-negative integers) is given by the $\widehat{\mathbf{A}}$-genus $\widehat{\mathbf{A}}(M)$. For example, in four-dimensions this implies that

$$n_+ - n_- = -\frac{\tau}{8}, \tag{2.7}$$

where $\tau := b^{2+} - b^{2-}$ is an integer. For S^4, $\tau = 0$, for $\mathbb{C}\mathbf{P}^2$, $\tau = -1$, and for a $K3$ surface, $\tau = 16$. (Note that in the case of $\mathbb{C}\mathbf{P}^2$, the right-hand-side of (2.7) is not an integer. This is a consequence of the fact that $\mathbb{C}\mathbf{P}^2$ is not a spin manifold.)

Theorem 2.16. *Let M be a closed spin manifold. If M admits a Riemannian metric of positive scalar curvature, then $\widehat{\mathbf{A}}(M) = 0$.*

Proof. If M admits a metric of positive scalar curvature then, by the previous theorem, $n_+ = n_- = 0$. Therefore $\widehat{\mathbf{A}}(M) = 0$. ☐

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[5] We have lowered one of the indices but, since we are in Riemannian signature, this doesn't make any difference.

[6] This material actually appeared in Lecture 3, but it is more logically consistent to put it here.

[7] I have now returned to my usual notation where s denotes the scalar curvature of the metric $\mathbf{g}$.

Corollary 2.17. *Let M be a closed spin manifold with $\widehat{\mathbf{A}}(M) \neq 0$. Then M admits no Riemannian metric of positive scalar curvature.*

Remark 2.18. Note that the conditions that M be closed, a spin manifold, and that $\widehat{\mathbf{A}}(M) \neq 0$ are all topological statements[8]. As such, in the above result, topological restrictions have direct implications for the geometrical properties of the manifold.

If one considers manifolds that do not admit a spin structure, then the above results tell us nothing. In particular, it is known that if M is a closed, oriented, simply-connected manifold with $\dim M \geq 5$, then M admits a metric of positive scalar curvature (see, e.g., [8], where a more general result is proved).

On the other hand, there is no obstruction to negative scalar curvature metrics, in the sense that any closed manifold M with $\dim M \geq 3$ admits a metric of negative scalar curvature[9] [11, 15].

[8]Strictly speaking, $\widehat{\mathbf{A}}(M)$ is sensitive to the differentiable structure on M, rather than just the topological structure.

[9]In fact, such an M admits a metric with constant negative scalar curvature.

3. The positive energy/mass theorem

In [29], Witten used the properties of a particular type of Dirac operator to give a proof of the positive energy/mass theorem. We will give a proof of a simpler result, sometimes called the Riemannian positive mass theorem, based on the papers of Parker-Taubes [19] and the appendix of Lee-Parker [17]. We will then, briefly, mention the modifications that are needed to treat the full version of the positive energy/mass theorem. Details of the full proof are given in an Appendix. For additional background material and information on Witten's proof and the Schoen-Yau proof of the positive energy and mass theorems, see, e.g., the article by Choquet-Bruhat [10]. Bray's article [9] is an interesting introduction to the ideas involved in the positive mass theorem.

Let $(M, \mathbf{g})$ be a four-dimensional Lorentzian manifold. We assume that we are given a symmetric tensor field $\mathbf{T} \in \mathcal{T}^0_2(M)$, interpreted physically as the *energy-momentum tensor* of any matter that lives in our space-time. It is assumed that the Ricci curvature, $\mathbf{Ric}$, and the scalar curvature, s, of the metric $\mathbf{g}$ are related to $\mathbf{T}$ by the *Einstein equations*

$$\mathbf{Ric} - \frac{s}{2}\mathbf{g} = 8\pi \mathbf{T}.$$

(For those that know of such things, we will use units in which Newton's constant, G, and the speed of light, c, are equal to one.). It is assumed that the matter in our manifold has "non-negative local mass density". Mathematically speaking, this condition is imposed by asking that the energy momentum tensor, $\mathbf{T}$, satisfy the *dominant energy condition* [14]:

Definition 3.1. The energy momentum tensor, $\mathbf{T} \in \mathcal{T}^0_2(M)$ satisfies the *dominant energy condition* if, for every time-like vector field $\mathbf{V} \in \mathfrak{X}(M)$, we have $\mathbf{T}(\mathbf{V}, \mathbf{V}) \geq 0$ and the vector field corresponding to the one-form $\mathbf{T}(\mathbf{V}, \cdot)$ is causal.

Remark 3.2. This condition has the physical interpretation that the energy density measured by any local observer is non-negative, and the energy flow-vector is causal (i.e. non-space-like). In a local orthonormal frame $\{\mathbf{e}_a\}_{a=0}^3$, the dominant energy condition is equivalent to the condition that the components $T_{ab} := \mathbf{T}(\mathbf{e}_a, \mathbf{e}_b)$ obey the condition

$$T_{00} \geq |T_{ab}|, \qquad \text{for } a, b = 0, 1, 2, 3,$$

and that

$$T_{00}^2 \geq \sum_{i=1}^{3} (T_{0i})^2 .$$

This condition is satisfied for most sensible forms of matter (e.g. scalar, Dirac and Maxwell fields).

Definition 3.3. Let $\Sigma \subset M$ be a complete, oriented, 3-dimensional, space-like hypersurface. Σ is *asymptotically flat* if the following conditions are satisfied. There exists a compact set $K \subset \Sigma$ with the property that $\Sigma \setminus K$ is the disjoint union of a finite number of subsets $\Sigma_1, \dots, \Sigma_N$ of Σ – the *ends* of Σ – each of which is diffeomorphic to the complement of a contractible, compact set in $\mathbb{R}^3$. Under this diffeomorphism the metric on $\Sigma_l \subset M$ should be of the form

$$g_{ij} = \delta_{ij} + a_{ij},$$

in the standard coordinates, $\{x^i\}_{i=1}^3$, on $\mathbb{R}^3$, with

$$a_{ij} = O(r^{-1}), \quad \partial_i a_{jk} = O(r^{-2}), \quad \partial_i \partial_j a_{kl} = O(r^{-3}),$$

where $r := |\mathbf{x}| \equiv \sqrt{\sum_{i=1}^3 (x^i)^2}$. Furthermore, the components of the second fundamental form[10] of $\Sigma_l \subset M$, k_{ij}, should satisfy

$$k_{ij} = O(r^{-2}), \quad \partial_i k_{jk} = O(r^{-3}).$$

Remark 3.4. We will often, without comment, identify the ends Σ_l with the corresponding set in $\mathbb{R}^3$.

Lecture 6

[10]Denote the unit (time-like) normal vector field to Σ by $\mathbf{n} \in \Gamma(\Sigma, TM)$. Given vector fields tangent to Σ, we recall that the second fundamental form of Σ, $\mathbf{k}$, is defined by the formula $\mathbf{k}(\mathbf{v}, \mathbf{w}) := \mathbf{g}(\nabla_{\mathbf{v}}\mathbf{n}, \mathbf{w})$.

Example 3.5. If we, for the moment, drop the assumption of completeness, then the basic example of an asymptotically flat space-like hypersurface is a surface t = constant in the Schwarzschild solution

$$\mathbf{g} = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

where $(t, r, \theta, \phi) \in \mathbb{R} \times (2m, \infty) \times [0, \pi] \times [0, 2\pi)$. Letting Σ be the space-like hyper-surface $t = t_0$, then the induced metric on Σ is

$$\mathbf{h} = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

If we rewrite this in terms of Cartesian coordinates $(x, y, z) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta)$, then the metric takes the form

$$\begin{aligned}
\mathbf{h} &= |d\mathbf{x}|^2 + \left[\left(1 - \frac{2m}{r}\right)^{-1} - 1\right] \cdot (d|\mathbf{x}|)^2 \\
&= |d\mathbf{x}|^2 + \frac{2m}{|\mathbf{x}| - 2m} (d|\mathbf{x}|)^2 = |d\mathbf{x}|^2 + \frac{2m}{|\mathbf{x}| - 2m} \left(\frac{1}{|\mathbf{x}|}\mathbf{x} \cdot d\mathbf{x}\right)^2 \\
&= |d\mathbf{x}|^2 + \frac{2m}{(|\mathbf{x}| - 2m)|\mathbf{x}|^2} \sum_{i,j} x_i x_j dx_i \otimes dx_j.
\end{aligned}$$

Therefore the metric takes the required form with

$$a_{ij} = \frac{2m}{(|\mathbf{x}| - 2m)|\mathbf{x}|^2} x_i x_j.$$

Then $a_{ij} = O(|\mathbf{x}|^{-1})$ as $|\mathbf{x}| \to \infty$, etc, so the metric is asymptotically flat in the above sense.

If one adopts a Hamiltonian point of view to the Einstein field equations (i.e. one writes them as the Euler-Lagrange equations of an action functional, and then performs a Legendre transformation to go to a Hamiltonian version) then a by-product of such an approach is a definition of the total energy and the total momentum of an asymptotically flat manifold [3]. These quantities are defined in each asymptotic end Σ_l as limits over spheres $S_{r,l}$ of radius r in $\Sigma_l \subset \mathbb{R}^3$:

$$E_l := \frac{1}{16\pi} \lim_{r\to\infty} \int_{S_{r,l}} \sum_{i,j=1}^{3} (\partial_j g_{ij} - \partial_i g_{jj})\, d\Sigma^i, \tag{3.1a}$$

$$(\mathbf{p}_l)_k := \frac{1}{16\pi} \lim_{r\to\infty} \int_{S_{r,l}} 2\sum_{i=1}^{3} \left(k_{ik} - \delta_{ik} \sum_{j=1}^{3} k_{jj}\right) d\Sigma^i, \tag{3.1b}$$

where $d\Sigma^i$ denotes the area element of the sphere $S(r)$ in $\mathbb{R}^3$ of radius $r := |\mathbf{x}|$ (so $d\Sigma^i = \frac{x^i}{r} r^2 d\mathbf{vol}_{S^2}$). The justification of the definition of these quantities would require a lot of physics, so we will content ourselves with calculating a relevant example to show that the above give sensible answers[11].

[11]See Appendix A if you want a brief justification of these expressions.

Example 3.6. In the case of the Schwarzschild solution, with $\Sigma := \{t = t_0\}$, then $\mathbf{k} = 0$, so $\mathbf{p} = 0$. In terms of the above form of the metric, we find that

$$\begin{aligned}\sum_j (\partial_j g_{ij} - \partial_i g_{jj}) &= \sum_j (\partial_j a_{ij} - \partial_i a_{jj}) \\ &= \sum_j \left(\partial_j \left(\frac{2m}{(|\mathbf{x}| - 2m)|\mathbf{x}|^2} x_i x_j \right) - \partial_i \left(\frac{2m}{(|\mathbf{x}| - 2m)|\mathbf{x}|^2} x_j x_j \right) \right) \\ &= \frac{2m}{(|\mathbf{x}| - 2m)|\mathbf{x}|^2} \sum_j (\partial_j (x_i x_j) - \partial_i (x_j x_j)) \\ &= \frac{2m}{(|\mathbf{x}| - 2m)|\mathbf{x}|^2} \sum_j (\delta_{ij} x_j + x_i \delta_{jj} - (2\delta_{ij} x_j)) \\ &= \frac{2m}{(|\mathbf{x}| - 2m)|\mathbf{x}|^2} (x_i + 3x_i - 2x_i) \\ &= 4m \frac{x_i}{(|\mathbf{x}| - 2m)|\mathbf{x}|^2}.\end{aligned}$$

Therefore

$$\begin{aligned}E &= \frac{1}{16\pi} \lim_{|\mathbf{x}|\to\infty} \int_{S^2} \sum_{i=1}^{3} 4m \frac{x^i}{(|\mathbf{x}| - 2m)|\mathbf{x}|^2} \frac{x^i}{r} r^2 \, d\mathbf{vol}_{S^2} \\ &= \frac{1}{16\pi} \int_{S^2} 4m \, d\mathbf{vol}_{S^2} = \frac{1}{16\pi}(4m)(4\pi) = m.\end{aligned}$$

The physical intuition is that a gravitational system with non-negative matter density should have non-negative total energy. It is not clear, however, that the dominant energy condition and asymptotic flatness imply anything about the integrals (3.1).

Positive Energy Theorem. *Under the conditions of asymptotic flatness and the dominant energy condition, $E_l - |\mathbf{p}_l| \geq 0$ on each end Σ_l. If $E_l = 0$ for some l then Σ has only one end and M is flat along Σ.*

Remark 3.7. Viewing $\mathbf{P} = (E_l, \mathbf{p}_l)$ as a vector in $\mathbb{R}^{3,1}$, the Positive Energy Theorem implies that $\mathbf{P}$ is a future-directed, causal vector.

An important special case of the Positive Energy Theorem occurs if one assumes that the metric on Σ has the asymptotic form

$$g_{ij} = \left(1 + \frac{m_l}{2r}\right)^4 \delta_{ij} + p_{ij} \tag{3.2}$$

in the end Σ_l, with $p_{ij} = O(r^{-2})$, $\partial_k p_{ij} = O(r^{-3})$ and $\partial_l \partial_k p_{ij} = 0(r^{-4})$. In this case, the Positive Energy Theorem is equivalent to the following result:

Positive Mass Theorem. *Under the conditions of asymptotic flatness, the dominant energy condition and* (3.2)*, then $m_l \geq 0$ for each l, with equality if and only if Σ is flat along M.*

What we will prove, for the moment, is a simpler result. If we consider a space-like hypersurface Σ the extrinsic curvature of which is zero (i.e. $\mathbf{k} = 0$) then automatically we have $\mathbf{p}_l = 0$. Therefore the positive energy theorem is simply the statement that $E \geq 0$, under the condition that Σ be asymptotically flat, and the dominant energy condition hold on M. Such a condition is necessarily satisfied (via the Gauss-Codazzi relations) if the scalar curvature of the induced metric on Σ is non-negative.

Note that the asymptotically flat condition on Σ and the non-negativity of the scalar curvature are conditions that we require of Σ, viewed as an abstract Riemannian manifold (i.e. without reference to Σ being a space-like hyper-surface in a Lorentzian manifold M.) As such, we define a Riemannian manifold $(M, \mathbf{g})$ to be *asymptotically flat* if there exist ends, $M_1, \ldots, M_N$, of M, as in 3.3. We may then define the corresponding energies on the ends M_l by

$$E_l := \frac{1}{16\pi} \lim_{r\to\infty} \int_{S_{r,l}} \sum_{i,j=1}^{3} (\partial_j g_{ij} - \partial_i g_{jj})\, d\Sigma^i.$$

We then study the following version of the positive mass theorem:

Riemannian Positive Mass Theorem. *Let $(M, \mathbf{g})$ be a complete, Riemannian spin manifold of dimension n with non-negative scalar curvature that is asymptotically flat*[12]*. If $E_l < \infty$ for $l = 1, \dots, N$, then $E_l \geq 0$. Moreover, if $E_l = 0$ for some l then M has only one end and is flat.*

Remark 3.8. Note that the expressions for E_l depend only on the asymptotic information about the metric $\mathbf{g}$. This result implies that if we wish to localise the scalar curvature of our manifold in a small region and have the manifold sufficiently flat asymptotically that the energies E_l are zero, then this is not possible while maintaining non-negative scalar curvature. Similarly, if we have an asymptotically flat manifold for which $E_l < 0$, then it cannot be possible that the metric $\mathbf{g}$ is complete with positive scalar curvature.

Remark 3.9. This result is the version of the positive mass theorem that is required to solve the Yamabe problem.

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Definition 3.10. [13] Let $R \geq 1$ be large enough so that each end $M_l \subset \mathbb{R}^3$ contains the exterior of the ball B_R of radius R. For each l and each $r \geq R$, set $M_{l,r} := M_l \setminus B_r$ (considered either as a subset of $\mathbb{R}^3$ or of M). Fix a smooth function ρ on M with the following properties:

(i) $\rho \geq 1$;
(ii) $\rho = r$ in $M_{l,2R}$;
(iii) $\rho = 1$ in $M \setminus \left(\bigcup_l M_{l,R}\right)$.

For $1 \leq p < \infty$, $\delta \in \mathbb{R}$, we define the weighted Lebesgue spaces, $L^p_\delta(M)$, to be the completion of $C^\infty_c(M)$ with respect to the norm

$$\|u\|_{p,\delta} := \left(\int_M |u|^p \rho^{-p\delta - n} d\mathbf{vol_g}\right)^{1/p}.$$

We then define weighted Sobolev spaces, $W^{k,p}_\delta(M)$, as the completion of $C^\infty_c(M)$ with respect to the norm

$$\|u\|_{k,p,\delta} := \sum_{j=0}^{k} \left\|\partial^j u\right\|_{p,\delta-j}.$$

Similarly, given the spinor bundle Δ we define the weighted Lebesgue and Sobolev spaces of sections, $L^p_\delta(\Delta)$ and $W^{k,p}_\delta(\Delta)$ where, for the pointwise norm, we use where

$$\|\psi\|_p = \left(\int_\Sigma \langle\psi, \psi\rangle^{p/2}\right)^{1/p}$$

is the L^p norm.

Remark 3.11. The weighting, δ, gives a measure of the asymptotic growth properties of elements of $L^p_\delta(M)$. Given a smooth function, u with $\|u\|_{p,\delta} < \infty$, we find that $u = o(r^\delta)$ as $r \to \infty$.

We say that an asymptotically flat manifold M is *asymptotically flat of order τ* if there exists $q > n$ and $\tau \geq \frac{n-2}{2}$ such that, for each end M_l, we have

$$g_{ij} - \delta_{ij} \in W^{2,p}_{-\tau}(M_l).$$

We then study the following version of the positive mass theorem:

[12] We will actually need to modify, slightly, what we mean by the term "asymptotically flat" for the case of n arbitrary. We will do this in the next lecture.

[13] See, e.g., [6]

Riemannian Positive Mass Theorem. *Let $(M, \mathbf{g})$ be a complete, Riemannian spin manifold with non-negative scalar curvature $s \in L^1(M)$ that is asymptotically flat of order τ for some $\tau \geq \frac{n-2}{2}$. If $E_l < \infty$ for $l = 1, \dots, N$, then $E_l \geq 0$. Moreover, if $E_l = 0$ for some l then M has only one end and is flat.*

Remark 3.12. The result can also be proved with $\tau = n - 2$ in the case where $\dim M \leq 7$, and M is not assumed to be a spin manifold [24].

Remark 3.13. For $\tau \geq \frac{n-2}{2}$, the mass exists and is unique (i.e. independent of the choices of asymptotic coordinates, the choice of the function ρ, etc.). For $\tau > n - 2$, the mass is zero.

Remark 3.14. If you prefer not to think in terms of Sobolev spaces, then one has the same result if the metric $\mathbf{g}$ is smooth, and has asymptotic behaviour

$$g_{ij} - \delta_{ij} = O(r^{-\tau}), \qquad \partial g = O(r^{-\tau-1}), \qquad \partial^2 g = O(r^{-\tau-2}),$$

with $\tau > \frac{n-2}{2}$. (This is what Aubin [5], for example, does.) We will actually use this notation, although, strictly speaking, it should be interpreted as saying that a particular object lies in a particular weighted Sobolev space.

The proof of the Riemannian Positive Mass Theorem will be broken into several smaller steps. Recall from before that we have the Lichnerowicz formula for the square of the Dirac operator:

$$D^* D\psi = D^2\psi = \nabla^*\nabla\psi + \frac{s}{4}\psi. \tag{3.3}$$

Lemma 3.15. *Let $\psi \in C^\infty(\Delta)$. Then*

$$|\nabla\psi|^2 + \frac{s}{4}|\psi|^2 - |D\psi|^2 = \sum_{i=1}^{n} \nabla_i \left(\langle \psi, \nabla_i\psi + \mathbf{e}_i \cdot D\psi \rangle\right). \tag{3.4}$$

Given a subset $\Sigma \subseteq M$ with boundary $\partial\Sigma$, the integral form of (3.4) is

$$\int_\Sigma \left(|\nabla\psi|^2 + \frac{s}{4}|\psi|^2 - |D\psi|^2\right)\boldsymbol{\mu} = \sum_{i=1}^{n} \int_{\partial\Sigma} \left[\langle \psi, \nabla_i\psi + \mathbf{e}_i \cdot D\psi \rangle\, \mathbf{e}_i \lrcorner\, \boldsymbol{\mu}\right] \tag{3.5a}$$

$$= \frac{1}{2} \sum_{i,j=1}^{n} \int_{\partial\Sigma} \left\langle \psi, [\mathbf{e}_i\cdot, \mathbf{e}_j\cdot] \cdot \nabla_{\mathbf{e}_j}\psi \right\rangle \mathbf{e}_i \lrcorner\, \boldsymbol{\mu}, \tag{3.5b}$$

where $[\mathbf{e}_i\cdot, \mathbf{e}_j\cdot] := \mathbf{e}_i \cdot \mathbf{e}_j \cdot - \mathbf{e}_j \cdot \mathbf{e}_i\cdot$ is the commutator in the Clifford algebra, and

$$\boldsymbol{\mu} := \boldsymbol{\epsilon}^1 \wedge \cdots \wedge \boldsymbol{\epsilon}^n$$

is the volume form on M.

Proof. All that we need to do is work out the boundary terms that appear when computing the formal adjoints. Choosing our orthonormal frame $\{\mathbf{e}_i\}_{i=1}^n$ as before such that, at point $p \in M$, we have $\nabla_{\mathbf{e}_i}\mathbf{e}_j\big|_p = 0$. Therefore, as before, we have

$$D^2\psi = -\sum_{i=1}^{n} \nabla_i\nabla_i\psi + \frac{s}{4}\psi.$$

Taking the inner product with ψ, we require

$$\begin{aligned}
\langle\psi, D^2\psi\rangle &= \sum_{i,j=1}^{n} \langle\psi, \mathbf{e}_i \cdot \nabla_i \left(\mathbf{e}_j \cdot \nabla_j \psi\right)\rangle \\
&= \sum_{i,j=1}^{n} \nabla_i\langle\psi, \mathbf{e}_i \cdot (\mathbf{e}_j \cdot \nabla_j\psi)\rangle - \langle\nabla_i\psi, \mathbf{e}_i \cdot (\mathbf{e}_j \cdot \nabla_j\psi) \\
&= \sum_{i=1}^{n} \nabla_i\langle\psi, \mathbf{e}_i \cdot D\psi\rangle + \langle\nabla_i \mathbf{e}_i \cdot \psi, \mathbf{e}_j \cdot \nabla_j\psi \\
&= \sum_{i=1}^{n} \nabla_i\langle\psi, \mathbf{e}_i \cdot D\psi\rangle + \langle D\psi, D\psi\rangle
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sum_{i=1}^{n}\langle\psi, \nabla_i\nabla_i\psi\rangle &= \sum_{i=1}^{n} \left(\nabla_i\langle\psi, \nabla_i\psi\rangle - \langle\nabla_i\psi, \nabla_i\psi\rangle\right) \\
&= -\left|\nabla\psi\right|^2 + \sum_{i=1}^{n} \nabla_i\langle\psi, \nabla_i\psi\rangle.
\end{aligned}$$

Hence

$$\begin{aligned}
|\nabla\psi|^2 + \frac{s}{4}|\psi|^2 - |D\psi|^2 &= |\nabla\psi|^2 + \frac{s}{4}|\psi|^2 - \langle\psi, D^2\psi\rangle + \sum_{i=1}^{n} \nabla_i\langle\psi, \mathbf{e}_i \cdot D\psi\rangle \\
&= |\nabla\psi|^2 + \frac{s}{4}|\psi|^2 - \left\langle \psi, -\sum_{i=1}^{n} \nabla_i\nabla_i\psi + \frac{s}{4}\psi \right\rangle + \sum_{i=1}^{n} \nabla_i\langle\psi, \mathbf{e}_i \cdot D\psi\rangle \\
&= |\nabla\psi|^2 + \left\langle \psi, \sum_{i=1}^{n} \nabla_i\nabla_i\psi \right\rangle + \sum_{i=1}^{n} \nabla_i\langle\psi, \mathbf{e}_i \cdot D\psi\rangle \\
&= \sum_{i=1}^{n} \nabla_i \left\langle\psi, \nabla_i\psi\right\rangle + \sum_{i=1}^{n} \nabla_i\langle\psi, \mathbf{e}_i \cdot D\psi\rangle \\
&= \sum_{i=1}^{n} \nabla_i \left\langle\psi, \nabla_i\psi + \mathbf{e}_i \cdot D\psi\right\rangle,
\end{aligned}$$

as required.

Integrating this equality over Σ gives

$$\begin{aligned}
\int_\Sigma \left(|\nabla\psi|^2 + \frac{s}{4}|\psi|^2 - |D\psi|^2\right) \boldsymbol{\mu} &= \int_\Sigma \sum_{i=1}^{n} \nabla_i \left\langle\psi, \nabla_i\psi + \mathbf{e}_i \cdot D\psi\right\rangle \boldsymbol{\mu} \\
&= \int_{\partial\Sigma} \sum_{i=1}^{n} \left\langle\psi, \nabla_i\psi + \mathbf{e}_i \cdot D\psi\right\rangle \mathbf{e}_i \lrcorner \boldsymbol{\mu}
\end{aligned}$$

where the second equality has used the Stokes formula. To verify the use of the Stokes formula, calculating at $p \in M$ then $\nabla_{\mathbf{e}_i}\mathbf{e}_j\big|_p = 0$ implies that $d\boldsymbol{\epsilon}^i\big|_p = 0$. At p we therefore have

$$\begin{aligned} d\left[\sum_{i=1}^n \langle\psi, \nabla_i\psi + \mathbf{e}_i \cdot D\psi\rangle\, \mathbf{e}_i \lrcorner \boldsymbol{\mu}\right] &= d\left[\sum_{i=1}^n \langle\psi, \nabla_i\psi + \mathbf{e}_i \cdot D\psi\rangle\right] \mathbf{e}_i \lrcorner \boldsymbol{\mu} \\ &= \sum_{i,j=1}^n \nabla_j\{\langle\psi, \nabla_i\psi + \mathbf{e}_i \cdot D\psi\rangle\}\boldsymbol{\epsilon}^j \wedge (\mathbf{e}_i \lrcorner \boldsymbol{\mu}) \\ &= \sum_{i,j=1}^n \nabla_j\{\langle\psi, \nabla_i\psi + \mathbf{e}_i \cdot D\psi\rangle\}\,(\delta_{ij}\boldsymbol{\mu}) \\ &= \sum_{i=1}^n \nabla_i\{\langle\psi, \nabla_i\psi + \mathbf{e}_i \cdot D\psi\rangle\}\boldsymbol{\mu}, \end{aligned}$$

Lecture 8 as required.

Finally, equation (3.5b) follows using the following rearrangement

$$\begin{aligned} \nabla_{\mathbf{e}_i}\psi + \mathbf{e}_i \cdot D\psi &= \nabla_{\mathbf{e}_i}\psi + \sum_{j=1}^n \mathbf{e}_i \cdot \mathbf{e}_j \cdot \nabla_{\mathbf{e}_j}\psi \\ &= \nabla_{\mathbf{e}_i}\psi + \frac{1}{2}\sum_{j=1}^n \left(\mathbf{e}_i \cdot \mathbf{e}_j \cdot + \mathbf{e}_j \cdot \mathbf{e}_i \cdot + [\mathbf{e}_i\cdot, \mathbf{e}_j\cdot] \cdot \nabla_{\mathbf{e}_j}\right)\psi \\ &= \nabla_{\mathbf{e}_i}\psi - \sum_{j=1}^n \mathbf{g}(\mathbf{e}_i, \mathbf{e}_j)\nabla_{\mathbf{e}_j}\psi + \frac{1}{2}\sum_{j=1}^n [\mathbf{e}_i\cdot, \mathbf{e}_j\cdot]\,\nabla_{\mathbf{e}_j}\psi \\ &= \frac{1}{2}\sum_{j=1}^n [\mathbf{e}_i\cdot, \mathbf{e}_j\cdot]\,\nabla_{\mathbf{e}_j}\psi. \end{aligned}$$

□

The Weitzenböck formula (3.5) leads to a vanishing theorem discovered by Witten [29]. Namely, under the conditions of the Riemannian Positive Mass Theorem, if ψ is a spinor field on M that satisfies $D\psi = 0$ and that vanishes sufficiently quickly at infinity that the boundary term in (3.5) is zero, then $\psi \equiv 0$. This leads to the following result.

Proposition 3.16. *For $0 < \eta < n - 1$, the Dirac operator*

$$D : W^{2,p}_{-\eta}(\Delta) \to W^{1,p}_{-\eta-1}(\Delta)$$

is an isomorphism.

Sketch of Proof. Unfortunately, this result needs a fair bit of PDE technology to prove properly. Essentially, we need to show that D and its adjoint

$$D^* = D : W^{2,p'}_{\eta+1-n}(\Delta) \to W^{1,p'}_{\eta-n}(\Delta)$$

(where p' is defined by $\frac{1}{p} + \frac{1}{p'} = 1$) have trivial kernel. Let $\psi \in W^{2,p}_{-\eta}(\Delta)$ with $D\psi = 0$. Since D is an elliptic operator, elliptic regularity implies that ψ is smooth. We then have

$$\begin{aligned} \Delta|\psi|^2 &= \langle\Delta\psi, \psi\rangle + 2|\nabla\psi|^2 + \langle\psi, \Delta\psi\rangle \\ &= \left\langle \frac{s}{4}\psi, \psi \right\rangle + 2|\nabla\psi|^2 + \left\langle \psi, \frac{s}{4}\psi \right\rangle \\ &= 2\left(\frac{s}{4}\,|\psi|^2 + |\nabla\psi|^2\right) \\ &\geq 0, \end{aligned}$$

using the fact that $s \geq 0$. By the maximum principle, any maximum of $|\psi|^2$ will occur on the boundary of Σ. Since $|\psi|^2 \to 0$ at infinity (because $\eta > 0$), this implies that $|\psi|^2 = 0$, and hence $\psi = 0$. A similar argument shows that $\ker D^*$ is also trivial if $\eta < n - 1$.

For more information concerning elliptic regularity, the theory of Fredholm operators (which we are, implicitly, using above), maximum principles, see, e.g., [7, 12, 13]. □

Remark 3.17. In the case where the metric and connection are smooth, which we will generally assume, then elliptic regularity (see, e.g., [12, Chapter 6] or [13]) implies that if $\varphi \in W^{1,p}_{-\eta-1}(\Delta) \cap C^\infty(\Delta)$, then $D^{-1}\varphi \in W^{2,p}_{-\eta}(\Delta) \cap C^\infty(\Delta)$.

Theorem 3.18. *Let $(M, \mathbf{g})$ be asymptotically flat of order $\tau \geq \frac{n-2}{2}$ with non-negative scalar curvature $s \in L^1(M)$. Let $\{\psi_{0,l}\}_{l=1}^N$ be constant spinors defined in the asymptotic ends $\{M_l\}_{l=1}^N$. Then there exists a unique, smooth spinor field $\psi \in C^\infty(\Delta)$ on M with the properties that*

(i) $D\psi = 0$.
(ii) $\psi - \psi_{0,l} \in W^{2,p}_{-\tau}(\Delta)$ *in each end* M_l.

Proof. In each end l, fix a smooth function $0 \leq \beta_{R,l} \leq 1$ that is identically 1 in the end $M_{l,3R}$ and 0 inside $M_{l,2R}$. Let $\psi_0 \in C(M;\Delta)$ be the spinor

$$\psi_0 = \sum_{k=1}^{k} \psi_{0,l}\beta_{R,l}.$$

Since $d\psi_0 = 0$, we have

$$\begin{aligned} D\psi_0 &= \sum_{i=1}^{n} \mathbf{e}_i \cdot \left(\partial_{\mathbf{e}_i}\psi_0 - \frac{1}{8}\sum_{j,k=1}^{n} \Gamma_{ijk}\left[\mathbf{e}\cdot, \mathbf{e}_k\cdot\right]\psi_0 \right) \\ &\sim 0 + \Gamma\psi_0 \\ &\in W^{1,p}_{-\tau-1}(\Delta), \end{aligned}$$

since $\Gamma \sim \partial g \in W^{1,p}_{-\tau-1}(\Delta)$ and ψ_0 is smooth and asymptotically constant. Therefore $D\psi_0 \in W^{1,p}_{-\tau-1}(\Delta)$ so, by Proposition 3.16, we deduce that there exists a unique $\psi_1 \in W^{2,p}_{-\tau}(\Delta)$ with the property that

$$D\psi_1 = -D\psi_0.$$

Since ψ_0 is smooth, elliptic regularity implies that ψ_1 is also smooth. The spinor field $\psi := \psi_0 + \psi_1$ then has the required properties. □

Theorem 3.19. *Let $(M, \mathbf{g})$ asymptotically flat of order $\tau \geq \frac{n-2}{2}$ with non-negative scalar curvature $s \in L^1(M)$. Let $\{\psi_{0,l}\}_{l=1}^N$ and ψ be as in Theorem 3.18. Then*

$$0 \leq \int_\Sigma \left\{ |\nabla\psi|^2 + \frac{s}{4}|\psi|^2 \right\} \boldsymbol{\mu} = 4\pi \sum_{l=1}^{N} E_l \left\langle \psi_{0,l}, \psi_{0,l} \right\rangle. \tag{3.6}$$

Lecture 9

Proof. For the sake of simplicity, we assume, for the moment, that M only has one end, M_1. We then use the integration by parts formula (3.5) applied to the region $\Sigma \subset M$ interior to the asymptotic sphere, S_r, of radius $r > R$ in M_1. We need to work out the asymptotics of the boundary term (3.5) on S_r as $r \to \infty$.

Since $D\psi = 0$, we have

$$\int_\Sigma \left\{ |\nabla\psi|^2 + \frac{s}{4}|\psi|^2 \right\} \boldsymbol{\mu} = \sum_{i=1}^{n} \int_{S_r} \langle \psi, \nabla_i\psi \rangle \mathbf{e}_i \lrcorner \boldsymbol{\mu},$$

Since the left-hand-side is explicitly real, we can rewrite this in the form

$$\begin{aligned} \int_\Sigma \left\{ |\nabla\psi|^2 + \frac{s}{4}|\psi|^2 \right\} \boldsymbol{\mu} &= \mathrm{Re}\left\{ \sum_{i=1}^{n} \int_{S_r} \langle \psi, \nabla_i\psi \rangle \mathbf{e}_i \lrcorner \boldsymbol{\mu} \right\}, \\ &= \mathrm{Re}\sum_{i=1}^{n} \int_{S_r} \left[\langle \psi_0, \nabla_i\psi_0 \rangle + \langle \psi_0, \nabla_i\psi_1 \rangle + \langle \psi_1, \nabla_i\psi_0 \rangle + \langle \psi_1, \nabla_i\psi_1 \rangle \right] \mathbf{e}_i \lrcorner \boldsymbol{\mu} \end{aligned} \tag{3.7}$$

Furthermore, since $\psi_1 = o(\rho^{-\tau})$, $\nabla\psi_1 = o(\rho^{-\tau-1})$, and $\nabla\psi_0 = o(\rho^{-\tau-1})$, the third and fourth terms are $o(\rho^{-2\tau-1})$ as $r \to \infty$. As such, when integrated over S_r, they give contributions of order $r^{n-1} \cdot o(\rho^{-2\tau-1}) = o(r^{n-2\tau-2})$. Since $\tau \geq \frac{n-2}{2}$, it follows that $n - 2\tau - 2 \leq 0$, so the contributions of the third and fourth terms tend to zero as $r \to \infty$.

To calculate the contributions of the other terms, we need a little more information about the asymptotics of the connection. We can, asymptotically, take an orthonormal coframe $\boldsymbol{\epsilon}^i = dx^i + \frac{1}{2}\sum_j a_{ij} dx^j + o(\rho^{-2\tau})$. (Note that the $o(\rho^{-2\tau})$ terms is of the form a^2, and therefore its derivative is $o(\rho^{-2\tau} - 1)$.) To get the components of the spin connection, we consider the exterior derivative of the coframe

$$
\begin{aligned}
d\boldsymbol{\epsilon}^i &= \frac{1}{2}\sum_{j,k} \partial_k a_{ij} dx^k \wedge dx^j + o(\rho^{-2\tau-1}) \\
&= \frac{1}{2}\sum_{j,k} \left(\partial_k a_{ij} - \partial_i a_{jk}\right) dx^k \wedge dx^j + o(\rho^{-2\tau-1}) \\
&= -\frac{1}{2}\sum_{j,k} \left(\partial_k a_{ij} - \partial_i a_{jk}\right) \boldsymbol{\epsilon}^j \wedge \boldsymbol{\epsilon}^k + o(\rho^{-2\tau-1}) \\
&= -\sum_k \left(\frac{1}{2}\sum_j \left(\partial_k a_{ij} - \partial_i a_{jk}\right) \boldsymbol{\epsilon}^j + o(\rho^{-2\tau-1})\right) \wedge \boldsymbol{\epsilon}^k \\
&\equiv -\sum_k \boldsymbol{\Gamma}_{ik} \wedge \boldsymbol{\epsilon}^k.
\end{aligned}
$$

Hence[14]

$$
\boldsymbol{\Gamma}_{ik} = \frac{1}{2}\sum_j \left(\partial_k a_{ij} - \partial_i a_{jk}\right) \boldsymbol{\epsilon}^j + o(\rho^{-2\tau-1}) = \frac{1}{2}\sum_j \left(\partial_k g_{ij} - \partial_i g_{jk}\right) \boldsymbol{\epsilon}^j + o(\rho^{-2\tau-1}).
$$

If we now consider the first term in (3.7) then, since $\partial\psi_0 = 0$, we have

$$
\begin{aligned}
\mathrm{Re}\langle\psi_0, \nabla_i\psi_0\rangle &= -\frac{1}{8}\sum_{j,k} \frac{1}{2}\left(\partial_k g_{ij} - \partial_j g_{ik}\right) \mathrm{Re}\langle\psi_0, [\mathbf{e}_j\cdot, \mathbf{e}_k\cdot]\psi_0\rangle + o(\rho^{-2\tau-1}) \\
&= -\frac{1}{8}\sum_{j,k} \left(\partial_k g_{ij}\right) \mathrm{Re}\langle\psi_0, [\mathbf{e}_j\cdot, \mathbf{e}_k\cdot]\psi_0\rangle + o(\rho^{-2\tau-1})
\end{aligned}
$$

Using the fact that the inner product $\langle\cdot,\cdot\rangle$ is Hermitian, and the $\mathbf{e}_i\cdot$ are skew-Hermitian with respect to it, we deduce that

$$
\begin{aligned}
\overline{\langle\psi_0, [\mathbf{e}_j\cdot, \mathbf{e}_k\cdot]\psi_0\rangle} &= \langle[\mathbf{e}_j\cdot, \mathbf{e}_k\cdot]\psi_0, \psi_0\rangle \\
&= \langle\mathbf{e}_j \cdot \mathbf{e}_k \cdot \psi_0, \psi_0\rangle - \langle\mathbf{e}_k \cdot \mathbf{e}_j \cdot \psi_0, \psi_0\rangle \\
&= -\langle\mathbf{e}_k \cdot \psi_0, \mathbf{e}_j \cdot \psi_0\rangle + \langle\mathbf{e}_j \cdot \psi_0, \mathbf{e}_k \cdot \psi_0\rangle \\
&= \langle\psi_0, \mathbf{e}_k \cdot \mathbf{e}_j \cdot \psi_0\rangle - \langle\psi_0, \mathbf{e}_j \cdot \mathbf{e}_k \cdot \psi_0\rangle \\
&= -\langle\psi_0, [\mathbf{e}_j\cdot, \mathbf{e}_k\cdot]\psi_0\rangle.
\end{aligned}
$$

Therefore the first term in $\mathrm{Re}\langle\psi_0, \nabla_i\psi_0\rangle$ is zero, so $\mathrm{Re}\langle\psi_0, \nabla_i\psi_0\rangle = o(\rho^{-2\tau-1})$. Therefore, by the same argument as previously, the first term in (3.7) also vanishes as $r \to \infty$.

Lecture 10

We are therefore left with the second term in (3.7). We first define the operator

$$
L_i := \nabla_i + \mathbf{e}_i \cdot D = \frac{1}{2}\sum_j [\mathbf{e}_i\cdot, \mathbf{e}_j\cdot]\nabla_j.
$$

[14] Note that, since the $\boldsymbol{\Gamma}_{ik}$ that we have constructed is explicitly skew-symmetric under interchange of i and k, it follows that it is the (unique) spin-connection defined by skew-symmetry and the first Cartan structure equation.

We define the $(n-2)$ form

$$\boldsymbol{\alpha} := \sum_{i,j} \langle [\mathbf{e}_i\cdot, \mathbf{e}_j\cdot]\psi_0, \psi_1\rangle\, \mathbf{e}_i \lrcorner\, \mathbf{e}_j \lrcorner\, \boldsymbol{\mu}.$$

Calculating in our frame with $\nabla_{\mathbf{e}_i}\mathbf{e}_j\big|_p = 0$, then we deduce that, at p, we have

$$\begin{aligned}
d\boldsymbol{\alpha} &= \left(\sum_{i,j} d\left[\langle [\mathbf{e}_i\cdot, \mathbf{e}_j\cdot]\psi_0, \psi_1\rangle\right] \wedge (\mathbf{e}_i \lrcorner\, \mathbf{e}_j \lrcorner\, \boldsymbol{\mu}) \right) \\
&= \sum_{i,j,k} \left(\nabla_k\left[\langle [\mathbf{e}_i\cdot, \mathbf{e}_j\cdot]\psi_0, \psi_1\rangle\right]\right) \boldsymbol{\epsilon}^k \wedge (\mathbf{e}_i \lrcorner\, \mathbf{e}_j \lrcorner\, \boldsymbol{\mu}) \\
&= 2\sum_{i,j} \left(\nabla_i\left[\langle [\mathbf{e}_i\cdot, \mathbf{e}_j\cdot]\psi_0, \psi_1\rangle\right]\right) (\mathbf{e}_j \lrcorner\, \boldsymbol{\mu}) \\
&= 2\sum_{i,j} \left(\left[\langle [\mathbf{e}_i\cdot, \mathbf{e}_j\cdot]\nabla_i\psi_0, \psi_1\rangle + \langle [\mathbf{e}_i\cdot, \mathbf{e}_j\cdot]\psi_0, \nabla_i\psi_1\rangle\right]\right) (\mathbf{e}_j \lrcorner\, \boldsymbol{\mu}) \\
&= -4\sum_{j} \Big(\langle L_j\psi_0, \psi_1\rangle - \langle \psi_0, L_j\psi_1\rangle \Big)\, \mathbf{e}_j \lrcorner\, \boldsymbol{\mu}.
\end{aligned}$$

Therefore, by Stokes' theorem, and the fact that $D\psi_1 = -D\psi_0$, the second term in (3.7) is

$$\operatorname{Re}\sum_i \int_{S_r} \langle \psi_0, (L_i - \mathbf{e}_i\cdot D)\,\psi_1\rangle \mathbf{e}_i \lrcorner\, \boldsymbol{\mu} = \operatorname{Re}\sum_i \int_{S_r} \left[\langle L_i\psi_0, \psi_1\rangle + \langle \psi_0, e_i\cdot D\psi_0\rangle\right] \mathbf{e}_i \lrcorner\, \boldsymbol{\mu}. \tag{3.8}$$

As before, $\langle L_i\psi_0, \psi_1\rangle = o(\rho^{-2\tau-1})$, so the first term in the integral gives zero contribution as $r \to \infty$. Meanwhile the asymptotic form of the connection gives

$$\begin{aligned}
\mathbf{e}_i\cdot D\psi_0 &= -\frac{1}{8}\sum_{j,k,l} \left(\partial_k g_{lj} + o(\rho^{-2\tau-1})\right) \mathbf{e}_i\cdot\mathbf{e}_l\cdot[\mathbf{e}_j\cdot, \mathbf{e}_k\cdot]\psi_0 \\
&= -\frac{1}{8}\sum_{j,k,l} \left(\partial_k g_{lj} + o(\rho^{-2\tau-1})\right) \mathbf{e}_i\cdot\mathbf{e}_l\cdot(2\mathbf{e}_j\cdot\mathbf{e}_k + 2\delta_{jk})\cdot\psi_0 \\
&= -\frac{1}{4}\sum_{j,k} \left(\partial_j g_{kj} - \partial_k g_{jj} + o(\rho^{-2\tau-1})\right) \mathbf{e}_i\cdot\mathbf{e}_k\cdot\psi_0
\end{aligned}$$

Writing $\mathbf{e}_i\cdot\mathbf{e}_k\cdot = \frac{1}{2}[\mathbf{e}_i\cdot, \mathbf{e}_k\cdot] - \delta_{ik}$ and noting, as before, that $[\mathbf{e}_i\cdot, \mathbf{e}_k\cdot]$ is skew, we see that (3.8) becomes

$$\frac{1}{4}\sum_{i,j}\int_{S_r} \left(\partial_j g_{ij} - \partial_i g_{jj} + o(\rho^{-2\tau-1})\right) |\psi_0|^2 \mathbf{e}_i \lrcorner\, \boldsymbol{\mu}.$$

Putting this into (3.7) and letting $r \to \infty$ gives Witten's formula

$$\int_M \left(|\nabla\psi|^2 + \frac{s}{4}|\psi|^2\right)\boldsymbol{\mu} = \frac{1}{4}\lim_{r\to\infty}\sum_{i,j}\int_{S_r} (\partial_j g_{ij} - \partial_i g_{jj}))\, |\psi_0|^2 \mathbf{e}_i \lrcorner\, \boldsymbol{\mu} = 4\pi E|\psi_0|^2.$$

Finally, if M has more than one end, $M_1, \ldots, M_N$, then we apply the above argument on the interior of the set bounded by the sphere of radius r in each end. Our integral formula then becomes

$$\begin{aligned}
\int_M \left\{|\nabla\psi|^2 + \frac{s}{4}\,|\psi|^2\right\}\boldsymbol{\mu} &= \operatorname{Re}\left\{\sum_{l=1}^{N}\sum_{i=1}^{n}\int_{S_{r,l}} \langle\psi, \nabla_i\psi\rangle \mathbf{e}_i \lrcorner\, \boldsymbol{\mu}\right\} \\
&= 4\pi\sum_{l=1}^{N} E_l|\psi_{0,l}|^2.
\end{aligned}$$

The arguments then proceeds as above, giving (3.6). $\square$

Lemma 3.20. *Let ψ and $\{\psi_i\}$ be smooth spinor fields with $\nabla\psi = 0$ and $\nabla\psi_i = 0$ for each i.*

(a) *If* $\lim \psi(x) = 0$*, where this limit is taken along some path in one asymptotic end* M_l*, then* $\psi = 0$.

(b) *If* $\{\psi_i\}$ *are linearly independent in some end* M_l*, then they are linearly independent everywhere on* M.

Proof. (a) If $\nabla\psi = 0$, then we have

$$\nabla|\psi|^2 = \nabla\left(\langle\psi, \psi\rangle\right) = \langle\nabla\psi, \psi\rangle + \langle\psi, \nabla\psi\rangle = 0,$$

so $|\psi|^2$ is constant. Since, by assumption, $\lim \psi(x) = 0$, it follows that $|\psi|^2 = 0$, so $\psi = 0$.

(b) Suppose that there are constants c_i such that $\psi = \sum c_i \psi_i$ vanishes at some point $x_0 \in \Sigma$. Since $\nabla\psi = 0$ we can repeat the above argument to conclude that $\psi(x) = 0$, for any $x \in M$. This contradicts the assumption that the ψ_i are linearly independent in M_l. □

Proof of the Riemannian Positive Mass Theorem. Let $\{\psi_{0,l}\}_{l=1}^{N}$ be constant spinors on the asymptotic ends of M with $\psi_l = 0$ on each end except M_l. Theorem 3.18 then gives a field spinor ψ that satisfies $D\psi = 0$ and that asymptotically approaches $\psi_{0,l}$ in each end M_l. Substituting such a ψ into (3.6) then shows that $E_l \geq 0$. (For example, taking $\psi_{0,1} \neq 0$, $\psi_{0,l} = 0$ for $l \geq 2$ implies that $E_1 \geq 0$, and so on.)

Now suppose that the energy of some end, say M_1, is equal to zero. Choose a basis of constant spinors $\{\overline{\psi}^a\}_{a=1}^{r}$ on the region M_1, where $r := \operatorname{rank}\Delta$, and let $\{\psi^a\}$ be the solutions of $D\psi^a = 0$ with $\psi^a \to \overline{\psi}^a$ in M_1, and $\psi^a \to 0$ in M_l for $l \geq 2$. (Such ψ^a exist, by Theorem 3.18.) Our integral equality (3.6) then implies that, for $a = 1, \dots, r$

$$\int_M \left\{ |\nabla\psi^a|^2 + \frac{s}{4}\, |\psi^a|^2 \right\} \boldsymbol{\mu} = 0.$$

Since $s \geq 0$, this implies that $\nabla\psi^a = 0$. Therefore

$$\nabla|\psi^a|^2 = \nabla\left(\langle\psi^a, \psi^a\rangle\right) = \langle\nabla\psi^a, \psi^a\rangle + \langle\psi^a, \nabla\psi^a\rangle = 0,$$

so $|\psi^a|^2$ is constant. Since $\psi^a \to 0$ on each M_l, for $l = 2, \dots, N$, this implies that $\psi^a = 0$ if $N \geq 2$. If $N \geq 2$, this gives a contradiction to out theorem asserting the existence of non-trivial solutions of $D\psi = 0$ with constant boundary conditions. As such, we must have $N = 1$, so M_1 is the only end of M.

Finally, assuming that M has only one end, and $E_1 = 0$, we want to show that the metric $\mathbf{g}$ is flat. Letting $\{\psi^a\}$ be as in the previous paragraph, then we know that $\nabla\psi^a = 0$, $a = 1, \dots, r$. By assumption, the ψ^a are linearly independent on M_1. If they were not linearly independent on M, this would mean the existence of $p \in M$ and constants $a_1, \dots, a_r$ with

$$a_1\psi^1(p) + \cdots + a_r\psi^r(p) = 0.$$

However, since $\nabla\psi^a = 0$, we deduce that $\nabla\left(\sum_{a=1}^{r} a_a\psi^a\right) = 0$. By the same argument as in the previous paragraph, we then deduce that $\left|\left(\sum_{a=1}^{r} a_a\psi^a\right)\right|^2 = \left|\left(\sum_{a=1}^{r} a_a\psi^a\right)\right|^2\Big|_p = 0$, so $\sum_{a=1}^{r} a_a\psi^a = 0$. Therefore the ψ^a would be linearly dependent on M_1, contradicting our hypothesis. Therefore the ψ^a are linearly independent on M. Furthermore, $\nabla\psi^a = 0$, so in a local frame $\{\mathbf{e}_i\}$ of M,

$$0 = \left(\nabla_{\mathbf{e}_i}\nabla_{\mathbf{e}_j} - \nabla_{\mathbf{e}_j}\nabla_{\mathbf{e}_i} - \nabla_{[\mathbf{e}_i, \mathbf{e}_j]}\right)\psi^a = \mathcal{R}(\mathbf{e}_i, \mathbf{e}_j)\psi^a$$

for $i, j = 1, \dots, n$. Since $\{\psi_a\}$ are a basis of Δ (and, technically speaking, because Δ comes from a faithful representation of the spin group) this implies that $\mathcal{R} = 0$. From the definition of $\mathcal{R}$ in terms of the curvature tensor, it follows that this implies that the Riemann curvature tensor, $\mathbf{R}$, is also zero. Hence, $\mathbf{g}$ is flat. □

3.1. **Modifications required to prove the Positive Energy Theorem [Non-examinable].** The proof of the full Positive Energy Theorem on a Lorentzian manifold is, in its basic idea, very similar to the proof of the Riemannian Positive Mass Theorem. There are some important complications that should be mentioned, though. We now summarise the modifications required in the case where M is a four-dimensional Lorentzian manifold.

First of all, we need to consider spin-bundles and Dirac operators on Lorentzian manifolds. In the Lorentzian case, the compatibility conditions of the Hermitian inner product are modified:

$$(\mathbf{v} \cdot s_1, s_2) = (s_1, \mathbf{v} \cdot s_2), \qquad \forall \mathbf{v} \in T_pM, \quad \forall s_1, s_2 \in \Delta_p.$$

Given a space-like hyper-surface, $\Sigma \subset M$ with unit time-like normal vector $\mathbf{n}$, which we assume to be future-directed, then we consider the restriction of the spin-bundle $\Delta \to M$ to a vector bundle $\Delta|\,\Sigma \to \Sigma$. (We will also denote this bundle by $\Delta \to \Sigma$ when no confusion can occur.) Sections of the bundle $\Delta \to \Sigma$ are called *Dirac spinors along* Σ. The bundle $\Delta|\,\Sigma$ inherits the inner product $(\cdot,\cdot)$ from Δ. There is also a second Hermitian inner product on $\Delta|\,\Sigma$ defined by

$$\langle \phi, \psi \rangle := (\mathbf{n} \cdot \phi, \psi)\,.$$

The most important point is that the inner product $\langle \cdot, \cdot \rangle$ is definite, and (without loss of generality) may be chosen positive-definite.

The connection, ∇, on $\Delta \to M$ is compatible with the inner product $(,)$. On restriction to Σ, this determines a connection on the bundle $\Delta \to \Sigma$ that is compatible with the inner product $(,)$, which we also denote this by ∇. In general, this connection is *not* compatible with the inner product $\langle , \rangle$, since $\nabla \mathbf{n} \neq 0$.

We now define a Dirac operator acting on sections of $\Delta \to \Sigma$ using the connection ∇ inherited from the four-manifold M. We call this the *hypersurface Dirac operator* and denote it by $\mathscr{D}$. Intrinsically, $\mathscr{D}$ is the composition

$$\Gamma(\Delta) \xrightarrow{\nabla} \Gamma(T^*\Sigma \otimes \Delta) \xrightarrow{c} \Gamma(\Delta),$$

where c is Clifford multiplication. We adopt a local orthonormal frame $\{\mathbf{e}_\alpha\}_{\alpha=0}^3$ for M, where we take $\mathbf{e}_0 \equiv \mathbf{n}$. In this case, $\{\mathbf{e}_i\}_{i=1}^3$ becomes a local orthonormal frame for Σ along Σ. In terms of such an orthonormal frame, we have

$$\mathscr{D}\psi = \sum_{i=1}^{3} \mathbf{e}_i \cdot \nabla_{\mathbf{e}_i} \psi$$

for $\psi \in \Gamma(\Delta)$.

Denoting the dual orthonormal coframe by $\{\boldsymbol{\epsilon}^\alpha\}$, we define the volume form[15] $\boldsymbol{\mu} = \boldsymbol{\epsilon}^1 \wedge \boldsymbol{\epsilon}^2 \wedge \boldsymbol{\epsilon}^3$ on Σ. Using the volume $\boldsymbol{\mu}$, we may define the L^2 inner product

$$\langle \varphi, \psi \rangle_{L^2} := \int_\Sigma \langle \varphi, \psi \rangle\, \boldsymbol{\mu}, \qquad \varphi, \psi \in C^\infty(\Sigma, \Delta).$$

Similar to the discussion of the square of the Dirac operator on a Riemannian manifold, we have the following result.

Proposition 3.21. *Let $\mathscr{D}^*$ and ∇^* be the formal adjoints of $\mathscr{D}$ and ∇ under the inner product $\langle \cdot, \cdot \rangle_{L^2}$. Then we have the Weitzenböck formula*

$$\mathscr{D}^*\mathscr{D} = \mathscr{D}^2 = \nabla^*\nabla + \mathscr{R} \tag{3.9}$$

where

$$\mathscr{R} = \frac{1}{4}\left(s + 2\mathrm{Ric}(\mathbf{e}_0, \mathbf{e}_0) - 2\sum_{i=1}^{3} \mathrm{Ric}(\mathbf{e}_0, \mathbf{e}_i) c(\mathbf{e}_0) c(\mathbf{e}_i) \right) \in \mathrm{End}(S).$$

[15]Strictly speaking, what we require is $\boldsymbol{\mu} := i^*\left(\boldsymbol{\epsilon}^1 \wedge \boldsymbol{\epsilon}^2 \wedge \boldsymbol{\epsilon}^3\right) \in \Omega^3(\Sigma)$, where $i : \Sigma \to M$ is the embedding of Σ as a submanifold of M. Since $\boldsymbol{\epsilon}^i$, $i = 1, 2, 3$ are defined up to an SO(3) transformation, and such transformations do not change $\boldsymbol{\mu}$, it follows that $\boldsymbol{\mu}$ is uniquely defined once $\mathbf{e}_0$ is fixed.

The integral form of (3.9) *is*

$$\int_\Sigma \left(|\nabla\psi|^2 + <\psi, \mathscr{R}\cdot\psi> - |\mathscr{D}\psi|^2\right)\boldsymbol{\mu} = \frac{1}{2}\sum_{i,j=1}^{3}\int_\Sigma d\left[\langle\psi, [\mathbf{e}_i\cdot, \mathbf{e}_j\cdot]\cdot\nabla_{\mathbf{e}_j}\psi\rangle\,\mathbf{e}_i \lrcorner\,\boldsymbol{\mu}\right]$$
$$= \frac{1}{2}\sum_{i,j=1}^{3}\int_{\partial\Sigma} \langle\psi, [\mathbf{e}_i\cdot, \mathbf{e}_j\cdot]\cdot\nabla_{\mathbf{e}_j}\psi\rangle\,\mathbf{e}_i \lrcorner\,\boldsymbol{\mu}$$

for any Dirac spinor ψ *along* Σ.

Remark 3.22. The formal adjoint ∇^* is now (at a point $p \in \Sigma$) given by

$$\nabla^*\left(\sum_{i=1}^{3}\boldsymbol{\epsilon}^i \otimes s_i\right) = -\sum_{i=1}^{3}\nabla_{\mathbf{e}_i}s_i - \sum_{i,j=1}^{3}k_{ij}\mathbf{e}_0\cdot\mathbf{e}_i\cdot s_j.$$

in a local frame with

$$\nabla_{\mathbf{e}_i}\mathbf{e}_0\big|_p = \sum_{j=1}^{3}k_{ij}\mathbf{e}_j$$

and all other derivatives vanishing at p.

Remark 3.23. The endomorphism $\mathscr{R}$ may be rewritten in terms of the energy-momentum tensor $\mathbf{T}$ using the Einstein equations. We have

$$\mathscr{R} = 4\pi\left(T_{00} + \sum_{i=1}^{3}T_{0i}\mathbf{e}_0\cdot\mathbf{e}_i\cdot\right).$$

The dominant energy condition then implies that

$$\mathscr{R} \geq 4\pi\left[T_{00} - \left(\sum_{i=1}^{3}T_{0i}^2\right)^{1/2}\right] \geq 0.$$

Theorem 3.24. *Let* $(M, \mathbf{g})$ *be Lorentzian manifold with any matter present satisfying the dominant energy condition, and* Σ *a complete, asymptotically flat, space-like hypersurface. Let* $\{\psi_{0,l}\}_{l=1}^N$ *be constant spinors defined in the asymptotic ends* $\{\Sigma_l\}_{l=1}^N$. *Then there exists a unique, smooth spinor* ψ *on* Σ *that satisfies*

(i) $\mathscr{D}\psi = 0$.
(ii) $\psi - \psi_{0,l} \in W^{2,p}_{-1}(\Delta)$ *in each end* Σ_l.

Moreover, we have the following identity:

$$\int_\Sigma \left\{\langle\nabla\psi, \nabla\psi\rangle + \langle\psi, \mathscr{R}\cdot\psi\rangle\right\}\boldsymbol{\mu} = 4\pi\sum_{l=1}^{N}\left(E_l\,\langle\psi_{0,l}, \psi_{0,l}\rangle + \sum_{k=1}^{3}p_{l,k}\,\langle\psi_{0,l}, \partial_0\cdot\partial_k\cdot\psi_{0,l}\rangle\right).$$

Here $\{\partial_a\}$ *is the standard coordinate basis of* $T(\mathbb{R}^{3,1})$, *viewing* $\Sigma_l \subset \mathbb{R}^3 \subset \mathbb{R}^{3,1}$ *in each end.*

Remark 3.25. The proof of the first part of this theorem is essentially the same as for the Riemannian Positive Mass Theorem, with non-negativity of $\mathcal{R}$ playing the role of non-negative scalar curvature in the proof of existence of an inverse for $\mathscr{D}$. The extrinsic curvature terms in the integral equality arise from extra terms in the asymptotics of the connection due to the fact that we are working on a submanifold Σ, rather than the manifold M, and therefore need to consider components of the connection in the normal directions as well as the spatial directions.

The proof of the Positive Energy Theorem now follows from the above integral equality in the same way as the Riemannian Positive Mass Theorem, along with some use of Clifford multiplication identities.

Part 2. Appendices [Non-examinable]

Appendix A. Justification for definition of E and $\mathbf{P}$

For your peace of mind, we give here a brief derivation of the formulae (3.1) for the energy and momentum of an asymptotically flat space-like hypersurface. For more details, see, e.g., [10].

Let $(M, \mathbf{g})$ be a Lorentzian manifold, with the metric $\mathbf{g}$ satisfying the Einstein equations

$$\mathbf{G}[\mathbf{g}] := \mathrm{Ric}[\mathbf{g}] - \frac{1}{2}s[\mathbf{g}]\mathbf{g} = 8\pi\mathbf{T}.$$

Let $\boldsymbol{\eta}$ be another metric on M that satisfies the Einstein equations without any source term

$$\mathbf{G}[\boldsymbol{\eta}] = \mathrm{Ric}[\boldsymbol{\eta}] - \frac{1}{2}s[\boldsymbol{\eta}]\boldsymbol{\eta} = 0.$$

We then define the energy momentum tensor of $\mathbf{g}$ with respect to $\boldsymbol{\eta}$ as the linearisation

$$\mathbf{t} = \mathbf{G}'[\boldsymbol{\eta}] \cdot (\mathbf{g} - \boldsymbol{\eta}) - \mathbf{G}[\mathbf{g}],$$

where $\mathbf{G}'[\boldsymbol{\eta}]$ is the derivative of the map $\mathbf{g} \mapsto \mathbf{G}[\mathbf{g}]$ with respect to $\mathbf{g}$ evaluated at $\boldsymbol{\eta}$. In the case where $\boldsymbol{\eta}$ is flat and we adopt local flat coordinates $\{x^a\}_{a=0}^3$, then we find that the components of $\mathbf{G}'[\boldsymbol{\eta}] \cdot (\mathbf{g} - \boldsymbol{\eta})$ with respect to these local coordinates takes the form

$$\begin{aligned}(\mathbf{G}'[\boldsymbol{\eta}] \cdot \mathbf{h})_{ab} = \frac{1}{2}\Bigg(&-\sum_{c,d} \eta^{cd}\partial_c\partial_d H_{ab} + \sum_{c,d}\eta^{cd}\partial_a\partial_c H_{bd} \\ &+ \sum_{c,d}\eta^{cd}\partial_b\partial_c H_{ad} - \eta_{ab}\sum_{c,d,e,f}\eta^{cd}\eta^{ef}\partial_c\partial_d H_{df}\Bigg),\end{aligned}$$

where

$$H_{ab} := h_{ab} - \frac{1}{2}\left(\sum_{c,d}\eta^{cd}h_{cd}\right)\eta_{ab}.$$

We then consider the $(0, 2)$ tensor field on M

$$\frac{1}{8\pi}\left(\mathbf{t} + 8\pi\mathbf{T}\right) \equiv \frac{1}{8\pi}\mathbf{G}'[\boldsymbol{\eta}] \cdot \mathbf{a},$$

where we have defined

$$\mathbf{a} := \mathbf{g} - \boldsymbol{\eta}.$$

It follows from general theory that the above quantity is covariantly divergence-free with respect to $\boldsymbol{\eta}$:

$$\nabla^{\boldsymbol{\eta}} \cdot (\mathbf{t} + 8\pi\mathbf{T}) = 0.$$

If $\boldsymbol{\xi} \in \mathfrak{X}(M)$ is a Killing vector (i.e. $\mathscr{L}_{\boldsymbol{\xi}}\mathbf{g} = 0$), then the 1-form

$$\mathbf{p} := \frac{1}{8\pi}\left(\mathbf{t} + 8\pi\mathbf{T}\right)(\boldsymbol{\xi}, \cdot) \in \Omega^1(M)$$

is co-closed

$$d\left(\star\,\mathbf{p}\right) = 0, \tag{A.1}$$

where $\star : \Omega^p(M) \to \Omega^{n-p}(M)$ is the Hodge dual. Integrating this formula over a four-dimensional subset $S \subset M$, we deduce that

$$\int_{\partial S} \star\,\mathbf{p} = 0.$$

In particular, the flux of $\mathbf{p}$ over the surface of S is zero. In appropriate circumstances, this condition may be interpreted as the conservation of energy or momentum.

It follows from (A.1) that (locally) there exists a two-form $\mathbf{q} \in \Omega^2(M)$ with the property that $\star\,\mathbf{p} = d\left(\star\,\mathbf{q}\right)$. Integrating the three-form $\star\,\mathbf{p}$ over a three-dimensional set Σ, then we have

$$\int_\Sigma \star\,\mathbf{p} = \int_\Sigma d\left(\star\,\mathbf{q}\right) = \int_{\partial\Sigma} \star\,\mathbf{q}. \tag{A.2}$$

It turns out that such a $\mathbf{q}$ may be constructed globally on M. In particular, if we choose $\boldsymbol{\eta}$ to be the flat metric, and $\boldsymbol{\xi}$ to be a parallel vector field (with respect to $\boldsymbol{\eta}$), then, again in local coordinates,

$$q_{ab} = \sum_{c,d,e} \eta^{cd} \left(\partial_d K_{abce}\right) \xi^e, \tag{A.3}$$

where

$$\begin{aligned}\mathbf{K}(\mathbf{T},\mathbf{X},\mathbf{Y},\mathbf{Z}) = \frac{1}{16\pi} &\left(\eta(\mathbf{T},\mathbf{Z})\mathbf{H}(\mathbf{X},\mathbf{Y}) + \eta(\mathbf{X},\mathbf{Y})\mathbf{H}(\mathbf{T},\mathbf{Z})\right. \\ &\left. -\eta(\mathbf{T},\mathbf{Y})\mathbf{H}(\mathbf{X},\mathbf{Z}) - \eta(\mathbf{X},\mathbf{Z})\mathbf{H}(\mathbf{T},\mathbf{Y})\right).\end{aligned} \tag{A.4}$$

If we then consider the integral (A.2), where $\partial\Sigma$ is a sphere of large radius r in the surface $t = \text{constant}$, which we denote S_r, then we deduce that

$$P(\boldsymbol{\xi}) := \int_{S_r} \star\mathbf{q} = \int_{\partial\Sigma} q^{0i} dS_i.$$

Using the formula (A.3) for $\mathbf{q}$, the formula (A.4) for $\mathbf{K}$, and the fact that $\boldsymbol{\xi}$ is constant in the coordinate system $\{x^\alpha\}$, we find that

$$\lim_{r\to\infty} P(\boldsymbol{\xi}) = E\xi_0 + \sum_{i=1}^{3} p_i \xi_i,$$

where

$$\begin{aligned}E &= \frac{1}{16\pi} \sum_{i=1}^{3} \lim_{r\to\infty} \int_{S_r} \left(\sum_{j=1}^{3} \partial_j H_{ij} - \partial_i H_{00} \right) d\Sigma_i = \frac{1}{16\pi} \lim_{r\to\infty} \int_{S_r} \sum_{i,j=1}^{3} \left(\partial_j g_{ij} - \partial_i g_{jj}\right) d\Sigma^i, \\ p_i &= \frac{1}{16\pi} \sum_{k=1}^{3} \lim_{r\to\infty} \int_{S_r} \left(-\partial_0 H_{ik} + \delta_{ik} \left(\partial_0 H_{00} - \sum_{j=1}^{3} \partial_j H_{0j} \right) + \partial_k H_{0i} \right) d\Sigma_k \\ &= \frac{1}{16\pi} \lim_{r\to\infty} \int_{S_r} 2 \sum_{i=1}^{3} \left(k_{ik} - \delta_{ik} \sum_{j=1}^{3} k_{jj} \right) d\Sigma^i.\end{aligned}$$

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