THE WAVE EQUATION ON SINGULAR SPACE-TIMES

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ABSTRACT. We prove local unique solvability of the wave equation for a large class of weakly singular, locally bounded space-time metrics in a suitable space of generalised functions.

1. INTRODUCTION

The notion of a singularity in general relativity significantly differs from that in other field theories. In the absence of a background metric, one has to detect the presence of singularities by showing that the space-time is "incomplete" in some sense. In the standard approach to singularities (see, e.g., Hawking and Ellis [12, Ch. 8]), a singularity is regarded as an obstruction to extending geodesics. However, this definition does not correspond very closely to ones physical intuition and classifies many space-times that have been used to model physically reasonable scenarios as being "singular". Such "weakly singular" space-times have long been used to describe, for example, impulsive gravitational waves, shell-crossing singularities and thin cosmic strings. Typically these space-times admit a metric that is locally bounded but its differentiability is below $\mathcal{C}^{1,1}$ (i.e., the first derivative locally Lipschitz) — the largest differentiability class where standard differential geometric properties, such as existence and uniqueness of geodesics, remain valid. For a recent review on the use of metrics of low regularity in general relativity, see [24].

This set of problems has stimulated considerations of whether physical objects would be subjected to unbounded tidal forces on approaching the singularity and was formulated mathematically in terms of strong curvature conditions. Unfortunately, it is hard to model the behaviour of real physical objects in a strong gravitational field. This led Clarke [2] to suggest that one consider the behaviour of physical fields (for which one has a precise mathematical description) near the singularity instead. According to this philosophy of "generalised hyperbolicity" one should regard singularities as obstructions to the Cauchy development of these fields rather than as an obstruction to the extension of geodesics.

However, the weak singularities mentioned above are obstructions if one formulates the Cauchy problem for the wave equation in the standard theory of distributions. More precisely, there is no generally valid distributional solution concept for the wave equation on a space-time with a non-smooth metric. The equation, although linear, involves coefficients of low regularity that cannot be multiplied with the distributional solution.

To resolve this problem in the case of shell-crossing singularities, Clarke [3] introduced a specific weak solution concept (called $\Box$-*global hyperbolicity*) to prove unique solvability of the wave equation, hence showing that these space-times, indeed, satisfy the conditions of generalised hyperbolicity. On the other hand, Vickers and Wilson [25] used the setting of Colombeau algebras [4, 5] to arrive at a valid formulation of the Cauchy problem for the wave equation on conical space-times (modelling a thin cosmic string) and showed the existence and uniqueness of solutions in a suitable algebra, $\mathcal{G}$, of generalised functions. Hence they showed that conical space-times are generalised hyperbolic or, more precisely, $\mathcal{G}$-hyperbolic. Vickers and Wilson also showed that their unique generalised solution corresponds to the "forbidden" distributional solution expected on physical grounds (via the concept of association — see Section 2.1 below). Their key tool is a refinement

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of the energy estimates for hyperbolic PDEs (see, e.g., [12, Sec. 7.4], [1, Sec. 4.4]), which makes them applicable in the new situation.

In this paper, we generalise this method to a much wider class of weakly singular space-times and prove $\mathcal{G}$-hyperbolicity for this class. Since our approach is based on regularisation of the singular metric by sequences of smooth ones, we must put restrictions on the growth of the sequence with respect to the regularisation parameter ε. Essentially we shall assume (see Section 2.3) asymptotic local uniform boundedness with respect to ε. Recall that the space-times of interest here typically possess a locally bounded metric. In particular, our class includes impulsive *pp*-waves (in the Rosen form), expanding spherical impulsive waves, and conical space-times (thereby generalising the results of Vickers and Wilson [25]).

This work is organised in the following way. In Section 2 we fix our notation, recall some facts on the geometric theory of generalised functions and define our class of weakly singular space-times (Section 2.3). We state our main result in Theorem 3.1 of Section 3: given a point p in a weakly singular space-time, there exists a neighbourhood, V, of p such that the initial value problem for the wave equation admits a unique solution in $\mathcal{G}(V)$. The proof is split into several steps: (generalised) higher order energy integrals are introduced and proved to be equivalent to suitable Sobolev norms in Section 4. The energy estimates are provided in Section 5, while some auxiliary estimates are proved in Section 6. Finally these results are collected to provide the proof of the main theorem in Section 7. We end with some concluding remarks.

2. Prerequisites

In this section, we give a precise definition of the class of weakly singular metrics that we are going to consider in the sequel. Prior to that, and for the convenience of the reader, we give a brief summary of the geometric theory of generalised functions in the sense of Colombeau. Our main reference for the latter is [8, Sec. 3.2] and we adopt most notational conventions from there. For an overview of the use of these constructions in general relativity, we refer to [24].

2.1. Geometric theory of generalised functions.

The basic idea of Colombeau's approach to generalised functions [4, 5] is regularisation by sequences (nets) of smooth functions and the use of asymptotic estimates in terms of a regularisation parameter ε. Let M be a separable, smooth, orientable, Hausdorff manifold of dimension n, and let $\mathfrak{X}(M)$ denote the space of smooth vector fields on M. Let $(u_\varepsilon)_{\varepsilon\in(0,1]}$ with $u_\varepsilon \in \mathcal{C}^\infty(M)$ for all ε. The (special) algebra of generalised functions on M is defined as the quotient $\mathcal{G}(M) := \mathcal{E}_M(M)/\mathcal{N}(M)$ of the moderate nets modulo the negligible nets, where the respective notions are defined by the following asymptotic estimates:

$$\begin{aligned}
\mathcal{E}_M(M) \quad &:= \quad \{(u_\varepsilon)_\varepsilon :\ \forall K \subset\subset M,\, \forall k \in \mathbb{N}_0\, \exists N \in \mathbb{N} \\
&\qquad \forall \boldsymbol{\eta}_1, \dots, \boldsymbol{\eta}_k \in \mathfrak{X}(M):\ \sup_{p\in K} |\mathscr{L}_{\boldsymbol{\eta}_1} \dots \mathscr{L}_{\boldsymbol{\eta}_k} u_\varepsilon(p)| = O(\varepsilon^{-N})\}, \\
\mathcal{N}(M) \quad &:= \quad \{(u_\varepsilon)_\varepsilon :\ \forall K \subset\subset M,\, \forall k, q \in \mathbb{N}_0 \\
&\qquad \forall \boldsymbol{\eta}_1, \dots, \boldsymbol{\eta}_k \in \mathfrak{X}(M):\ \sup_{p\in K} |\mathscr{L}_{\boldsymbol{\eta}_1} \dots \mathscr{L}_{\boldsymbol{\eta}_k} u_\varepsilon(p)| = O(\varepsilon^{q})\}.
\end{aligned}$$

Elements of $\mathcal{G}(M)$ are denoted by $u = [(u_\varepsilon)_\varepsilon] = (u_\varepsilon)_\varepsilon + \mathcal{N}(M)$. With component-wise operations, $\mathcal{G}(M)$ is a fine sheaf of differential algebras with respect to the Lie derivative with respect to classical vector fields defined by $\mathscr{L}_{\boldsymbol{\eta}} u := [(\mathscr{L}_{\boldsymbol{\eta}} u_\varepsilon)_\varepsilon]$. The spaces of moderate resp. negligible sequences and hence the algebra itself may be characterised locally, i.e., $u \in \mathcal{G}(M)$ iff $u \circ \psi_\alpha \in \mathcal{G}(\psi_\alpha(V_\alpha))$ for all charts (V_α, ψ_α), where, on the open set $\psi_\alpha(V_\alpha) \subset \mathbb{R}^n$, Lie derivatives are replaced by partial derivatives in the respective estimates. Smooth functions are embedded into $\mathcal{G}$ simply by the "constant" embedding σ, i.e., $\sigma(f) := [(f)_\varepsilon]$. On open sets of $\mathbb{R}^n$, compactly supported distributions are embedded into $\mathcal{G}$ via convolution with a mollifier $\rho \in \mathscr{S}(\mathbb{R}^n)$ with unit integral satisfying $\int \rho(x)x^\alpha dx = 0$ for all $|\alpha| \geq 1$; more precisely setting $\rho_\varepsilon(x) = (1/\varepsilon^n)\rho(x/\varepsilon)$, we have $\iota(w) := [(w * \rho_\varepsilon)_\varepsilon]$. In the case where $\mathrm{supp}(w)$ is non-compact, one uses a sheaf-theoretical construction which can be lifted to the manifold using a partition of unity. From the explicit formula, it is clear that the embedding commutes with partial differentiation. This embedding, however, is not canonical since it depends on the mollifier as well as the partition of unity. A canonical embedding of $\mathscr{D}'$ *is* provided by the so-called full version of the construction (see [9], resp. [10] for the tensor case). However, since we will model our weakly singular metrics in generalised functions from the start (see Sec. 2.2 and the discussion at the end of Section 2.3 below) we have chosen to work in the so-called special setting which is technically more accessible. Note that this is in contrast to [25].

Inserting $p \in M$ into $u \in \mathcal{G}(M)$ yields a well-defined element of the ring of constants (also called generalised numbers) $\mathcal{K}$ (corresponding to $\mathbb{K} = \mathbb{R}$ resp. $\mathbb{C}$), defined as the set of moderate nets of numbers ($(r_\varepsilon)_\varepsilon \in \mathbb{K}^{(0,1]}$ with $|r_\varepsilon| = O(\varepsilon^{-N})$ for some N) modulo negligible nets ($|r_\varepsilon| = O(\varepsilon^m)$ for each m). Finally, generalised functions on M are characterised by their generalised point values, i.e., by their values on points in $\tilde{M}_c$, the space of equivalence classes of compactly supported nets $(p_\varepsilon)_\varepsilon \in M^{(0,1]}$ with respect to the relation $p_\varepsilon \sim p'_\varepsilon :\Leftrightarrow d_h(p_\varepsilon, p'_\varepsilon) = O(\varepsilon^m)$ for all m, where d_h denotes the distance on M induced by any Riemannian metric.

As is evident from the definitions, all estimates are only required to hold for ε small enough, that is there exists ε_0 such that for all $\varepsilon < \varepsilon_0$ the respective statement holds. However, in order not to unnecessarily complicate our formulations we will notationally suppress this fact most of the time.

The $\mathcal{G}(M)$-module of generalised sections in vector bundles — especially the space of generalised tensor fields $\mathcal{G}^r_s(M)$ — is defined along the same lines using analogous asymptotic estimates with respect to the norm induced by any Riemannian metric on the respective fibers. However, it is more convenient to use the following algebraic description of generalised tensor fields

$$\mathcal{G}^r_s(M) = \mathcal{G}(M) \otimes \mathcal{T}^r_s(M)\,, \tag{2.1}$$

where $\mathcal{T}^r_s(M)$ denotes the space of smooth tensor fields and the tensor product is taken over the module $\mathcal{C}^\infty(M)$. Hence generalised tensor fields are just given by classical ones with generalised coefficient functions. Many concepts of classical tensor analysis carry over to the generalised setting [14], in particular Lie derivatives with respect to both classical and generalised vector fields, Lie brackets, exterior algebra, etc. Moreover, generalised tensor fields may also be viewed as $\mathcal{G}(M)$-multilinear maps taking generalised vector and covector fields to generalised functions, i.e., as $\mathcal{G}(M)$-modules we have

$$\mathcal{G}^r_s(M) \cong L_{\mathcal{G}(M)}(\mathcal{G}^0_1(M)^r, \mathcal{G}^1_0(M)^s; \mathcal{G}(M)). \tag{2.2}$$

Finally, in light of the Schwartz impossibility result [22], the setting introduced above gives a minimal framework within which tensor fields may be subjected to nonlinear operations, while maintaining consistency with smooth geometry and allowing an embedding of the distributional geometry as developed in [16, 19]. Moreover, the interplay between generalised functions and distributions is most conveniently formalised in terms of the notion of *association*. A generalised function $u \in \mathcal{G}(M)$ is called associated to zero, $u \approx 0$, if one (hence any) representative $(u_\varepsilon)_\varepsilon$ converges to zero weakly. The equivalence relation $u \approx v :\Leftrightarrow u - v \approx 0$ gives rise to a linear quotient of $\mathcal{G}$ that extends distributional equality. Moreover, we call a distribution $w \in \mathscr{D}'(M)$ the *distributional shadow* or *macroscopic aspect of* u and write $u \approx w$ if, for all compactly supported n-forms $\boldsymbol{\nu}$ and one (hence any) representative $(u_\varepsilon)_\varepsilon$, we have

$$\lim_{\varepsilon \to 0} \int_M u_\varepsilon \boldsymbol{\nu} = \langle w, \boldsymbol{\nu} \rangle,$$

where $\langle\, ,\, \rangle$ denotes the distributional action. By (2.1), the concept of association extends to generalised tensor fields in a natural way.

2.2. **Elements of Lorentzian geometry.** A *generalised pseudo-Riemannian metric* is defined to be a symmetric, generalised $(0,2)$-tensor field $\mathbf{g}$ with a representative $\mathbf{g}_\varepsilon$ that is a smooth pseudo-Riemannian metric for each ε such that the determinant $\det(\mathbf{g})$ is invertible in the generalised sense. The latter condition is equivalent to the following notion called *strictly nonzero on compact sets*: for any representative $\left(\det(\mathbf{g}_\varepsilon)\right)_\varepsilon$ of $\det(\mathbf{g})$ we have $\forall K \subset\subset M\ \exists m \in \mathbb{N}:\ \inf_{p\in K} |\det(\mathbf{g}_\varepsilon)| \geq \varepsilon^m$. This notion captures the intuitive idea of a generalised metric as a net of classical metrics approaching a singular limit in the following precise sense: $\mathbf{g}$ is a generalised metric iff on every relatively compact open subset V of M there exists a representative $(\mathbf{g}_\varepsilon)_\varepsilon$ of $\mathbf{g}$ such that, for fixed ε, $\mathbf{g}_\varepsilon$ is a classical metric and its determinant, $\det(\mathbf{g})$, is invertible in the generalised sense, i.e., does not go to zero too fast as $\varepsilon \to 0$. Note that we work exclusively with representatives of generalised metrics that are classical metrics for each ε. If $\mathbf{g}$ is Lorentzian, i.e., there exists a representative which is Lorentzian, we call the pair $(M, \mathbf{g})$ a *generalised space-time*.

A generalised metric induces a $\mathcal{G}(M)$-linear isomorphism from $\mathcal{G}^1_0(M)$ to $\mathcal{G}^0_1(M)$. The inverse of this isomorphism gives a well-defined element of $\mathcal{G}^2_0(M)$ (i.e., independent of the representative $(\mathbf{g}_\varepsilon)_\varepsilon$). This is the "inverse metric", which we denote by $\mathbf{g}^{-1}$, with representative $\left(\mathbf{g}_\varepsilon^{-1}\right)_\varepsilon$. The generalised covariant derivative, as well as the generalised Riemann-, Ricci- and Einstein tensors, of a generalised metric is defined by the usual formulae on the level of representatives. For further details see [15].

Next, we review the concept of causality in the generalised framework. Let $\boldsymbol{\xi} \in \mathcal{G}^1_0(M)$ be a generalised vector field on M. Then, by (2.2), $\mathbf{g}(\boldsymbol{\xi}, \boldsymbol{\xi}) \in \mathcal{G}(M)$. For functions $f \in \mathcal{G}(M)$ we have the following notion of strict positivity:

$$f > 0 \;:\Longleftrightarrow \forall K \subset\subset M,\, \exists m \in \mathbb{N} : \inf_{p \in K} f_\varepsilon(p) \geq \varepsilon^m, \quad \text{as } \varepsilon \to 0$$

and we define time-like, null and space-like for $\boldsymbol{\xi}$ by demanding $\mathbf{g}(\boldsymbol{\xi}, \boldsymbol{\xi}) < 0$, $\mathbf{g}(\boldsymbol{\xi}, \boldsymbol{\xi}) = 0$, respectively $\mathbf{g}(\boldsymbol{\xi}, \boldsymbol{\xi}) > 0$. (See [17] for details, as well as for a general account of basic Lorentzian geometry in the present setting.)

2.3. **A class of metrics.** We are now ready to define the class of metrics that we will study. Let $(M, \mathbf{g})$ be a generalised space-time, and $\mathbf{g}_\varepsilon$ a representative of the generalised metric. Let $p \in M$, U a relatively compact open neighbourhood of p, and let $t : U \to \mathbb{R}$ be a smooth map with the properties that $t(p) = 0$, $dt \neq 0$ on U. We assume that there exists an $M_0 > 0$ with $\mathbf{g}_\varepsilon^{-1}(dt, dt) \leq -1/M_0^2$, as $\varepsilon \to 0$ on U. Therefore the level sets of the function t, $\Sigma_\tau := \{q \in U : t(q) = \tau\}$, are space-like hyper-surfaces with respect to the representative metrics, $\mathbf{g}_\varepsilon$, uniformly as $\varepsilon \to 0$. We define the normal covector field to these hyper-surfaces $\boldsymbol{\sigma} := -dt \in \Omega^1(U)$ which, via the constant embedding, may also be viewed as a generalised covector field on U. We define the corresponding generalised normal vector field, $\boldsymbol{\xi}$, by its representative $\boldsymbol{\xi}_\varepsilon \in \mathfrak{X}(U)$, given, for each ε, by $\boldsymbol{\sigma} = \mathbf{g}_\varepsilon(\boldsymbol{\xi}_\varepsilon, \cdot)$. We now define the generalised function, V, on U by its representative $V_\varepsilon : U \to \mathbb{R}^+$, given by

$$V_\varepsilon^2 = -\mathbf{g}_\varepsilon(\boldsymbol{\xi}_\varepsilon, \boldsymbol{\xi}_\varepsilon).$$

We will also require the corresponding normalised versions of the generalised normal vector field, $\widehat{\boldsymbol{\xi}} = [(\widehat{\boldsymbol{\xi}}_\varepsilon)_\varepsilon] = [(\boldsymbol{\xi}_\varepsilon / V_\varepsilon)_\varepsilon]$, and covector field, $\widehat{\boldsymbol{\sigma}} = [(\widehat{\boldsymbol{\sigma}_\varepsilon})_\varepsilon] = \mathbf{g}(\widehat{\boldsymbol{\xi}}, \cdot)$. Observe that, although $\boldsymbol{\sigma}$ does not depend on ε, the quantities derived from it, i.e., $\widehat{\boldsymbol{\sigma}_\varepsilon}$, $\boldsymbol{\xi}_\varepsilon$ and $\widehat{\boldsymbol{\xi}}_\varepsilon$ necessarily do, since we are dealing with a generalised metric. Using these quantities, one may construct a positive-definite metric associated with the generalised space-time (cf. [17, Sec. 4]). In particular, we define

$$\mathbf{e}_\varepsilon := \mathbf{g}_\varepsilon + 2\widehat{\boldsymbol{\sigma}_\varepsilon} \otimes \widehat{\boldsymbol{\sigma}_\varepsilon},$$

which clearly, for each fixed ε, is a Riemannian metric on U. Additionally, the resulting class $\mathbf{e} = [(\mathbf{e}_\varepsilon)_\varepsilon]$ defines a generalised Riemannian metric on U ([17, Prop. 4.3]).

We denote by $\Sigma := \Sigma_0$ the three-dimensional space-like hypersurface through p. Let $\mathbf{m}$ be a background Riemannian metric on U and denote by $\| \, \|_{\mathbf{m}}$ the norm induced on the fibers of the respective tensor bundle on U. We demand the following conditions.

(A) For all K compact in U, for all orders of derivative $k \in \mathbb{N}_0$ and all k-tuples of vector fields $\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_k \in \mathfrak{X}(U)$ and for any representative $(\mathbf{g}_\varepsilon)_\varepsilon$ we have:
  - $\sup_K \left\| \mathscr{L}_{\boldsymbol{\eta}_1} \ldots \mathscr{L}_{\boldsymbol{\eta}_k} \mathbf{g}_\varepsilon \right\|_{\mathbf{m}} = O(\varepsilon^{-k}) \qquad (\varepsilon \to 0)$;
  - $\sup_K \left\| \mathscr{L}_{\boldsymbol{\eta}_1} \ldots \mathscr{L}_{\boldsymbol{\eta}_k} \mathbf{g}_\varepsilon^{-1} \right\|_{\mathbf{m}} = O(\varepsilon^{-k}) \qquad (\varepsilon \to 0)$.

  In particular, this implies (for $k = 0$) that the metrics $\mathbf{g}_\varepsilon$ and their inverses $\mathbf{g}_\varepsilon^{-1}$ are locally uniformly bounded with respect to ε.

(B) For all K compact in U, we have

$$\sup_K \|\nabla^\varepsilon \boldsymbol{\xi}_\varepsilon\|_{\mathbf{m}} = O(1), \qquad (\varepsilon \to 0), \tag{2.3}$$

where ∇^ε denotes the covariant derivative with respect to the Lorentzian metric $\mathbf{g}_\varepsilon$.

(C) For each representative $(\mathbf{g}_\varepsilon)_\varepsilon$ of the metric $\mathbf{g}$ on U, Σ is a past-compact space-like hypersurface such that $\partial J^+_\varepsilon(\Sigma) = \Sigma$. Here $J^+_\varepsilon(\Sigma)$ denotes the topological closure of the future emission $D^+_\varepsilon(\Sigma) \subset U$ of Σ with respect to $\mathbf{g}_\varepsilon$. Moreover, there exists a nonempty open set $A \subseteq \mathcal{M}$ and an ε_0 such that

$$A \subseteq \bigcap_{\varepsilon<\varepsilon_0} J^+_\varepsilon(\Sigma).$$

(Note that we are here following the notation of Friedlander [6]. The set $D^+_\varepsilon(\Sigma)$ would be denoted $I^+_\varepsilon(\Sigma)$ in the conventions of, for example, Hawking and Ellis [12].)

A generalised metric with the above properties will be referred to as a *weakly singular metric*.

Some comments are in order:

- Condition (A) is independent of the background Riemannian metric, $\mathbf{m}$, chosen on U, and may be rephrased in terms of a fixed but arbitrary coordinate system, $\{x^a\}$, on a neighbourhood of p as follows. With $k = 0$, Condition (A) states that the components of the metric $\mathbf{g}_\varepsilon$ and its inverse are locally uniformly bounded on U. In particular, the Lorentzian norm of the generalised normal vector field $\boldsymbol{\xi}_\varepsilon$ may be assumed to satisfy
$$\frac{1}{M_0} \le \sqrt{-g_\varepsilon(\boldsymbol{\xi}_\varepsilon, \boldsymbol{\xi}_\varepsilon)} = V_\varepsilon \le M_0. \tag{2.4}$$
For $k > 0$, Condition (A) states that there exists M_k such that
$$\left|\frac{\partial^k g^\varepsilon_{ab}}{\partial x^{a_1}\cdots\partial x^{a_k}}\right| \le \frac{M_k}{\varepsilon^k}, \qquad \left|\frac{\partial^k g^{ab}_\varepsilon}{\partial x^{a_1}\cdots\partial x^{a_k}}\right| \le \frac{M_k}{\varepsilon^k}.$$
- Conditions (A) and (B) imply that the generalised Riemannian metric, $\mathbf{e}$, obeys the asymptotic condition
$$\left\|\nabla^\varepsilon \mathbf{e}^{-1}_\varepsilon\right\|_{\mathbf{m}} = O(1) \qquad \text{as} \qquad \varepsilon \to 0.$$
From condition (A), it follows that $\|\mathbf{e}_\varepsilon\|_{\mathbf{m}}, \left\|\mathbf{e}^{-1}_\varepsilon\right\|_{\mathbf{m}} = O(1)$. (This can most easily be deduced from the form of the metric given, below, in (3.2).) Therefore, by the Cauchy-Schwarz inequality for the inner product induced by $\mathbf{m}$ on the bundle of $(2,1)$ tensors on M, we have
$$\left\|\nabla^\varepsilon \mathbf{e}^{-1}_\varepsilon\right\|_{\mathbf{e}_\varepsilon} = O(1) \qquad \text{as} \qquad \varepsilon \to 0. \tag{2.5}$$
Similarly, "lowering the index" on $\boldsymbol{\xi}_\varepsilon$ in equation (2.3) implies that $\sup_K \|\nabla^\varepsilon \boldsymbol{\sigma}\|_{\mathbf{m}} = O(1)$ as $\varepsilon \to 0$, where we have again used the fact that $\|\mathbf{g}_\varepsilon\|_{\mathbf{m}} = O(1)$. Taking the symmetric part of $\nabla^\varepsilon\boldsymbol{\sigma}$ implies that $\left\|\mathscr{L}_{\boldsymbol{\xi}_\varepsilon}\mathbf{g}_\varepsilon\right\|_{\mathbf{m}} = O(1)$ as $\varepsilon \to 0$. Finally, again using the Cauchy-Schwarz inequality, we deduce that
$$\left\|\mathscr{L}_{\boldsymbol{\xi}_\varepsilon}\mathbf{g}_\varepsilon\right\|_{\mathbf{e}_\varepsilon} = O(1) \qquad \text{as} \qquad \varepsilon \to 0. \tag{2.6}$$
These estimates will be required in Section 5.
- Condition (C) is necessary to ensure existence of smooth solutions on the level of representatives on a common domain (cf. Step 1 in the proof of Theorem 3.1 in Section 7).

Remark 2.1. Conditions (A) and (B) are given in terms of the ε-asymptotics of the generalised metric. There is, however, the following close connection to the classical situation. Assume that we are given a space-time metric that is locally bounded but not necessarily $\mathcal{C}^{1,1}$ or of Geroch-Traschen class [7] (i.e., the largest class that allows a "reasonable" distributional treatment). We may then embed this metric into the space of generalised metrics by convolution with a standard mollifier (cf. Section 2.1). From the explicit form of the embedding it is then clear that condition (A) holds.

We recall that in the special version of Colombeau's construction the embedding is non-geometric and we could – at the price of technical complications – resort to the full version where a geometric embedding is available (as was done in [25]). Nevertheless, in the full construction generalised functions that are embeddings of locally bounded functions still display the

ε-asymptotics of condition (A). Moreover our approach using the special version offers more flexibility: Whenever one succeeds, e.g. by using some physically motivated procedure, to model a singular metric by a sequence of classical metrics obeying (A)-(C), then our results apply.

Condition (B) on the other hand demands somewhat better asymptotics of the derivatives of the $(0, 0)$-component of the metric in adapted coordinates (see also (3.4) below). This is a technical condition that is satisfied by several relevant examples (see below).

As to condition (C), the only part that exceeds the classical condition for existence and uniqueness of solutions is the existence of the non-empty open set A. Geometrically, this means that the light-cones of the metric $\mathbf{g}_\varepsilon$ do not collapse as $\varepsilon \to 0$. In terms of regularisations of classical metrics, this condition will always be satisfied if the classical metric is non-degenerate.

Examples. To begin with we discuss the conical space-times of [25]. They fall into our class since estimates (6) and (7) in [25] for the embedded metric imply our condition (A), while (B) is immediate from the staticity of the metric.

The metric of impulsive *pp*-waves (in "Rosen form") fall into our class. For simplicity we only consider plane waves of constant linear polarisation, i.e,

$$-dudv + (1 + u_+)^2\, dx^2 + (1 - u_+)^2\, dy^2,$$

where $u_+ := uH(u)$ denotes the kink function. This metric is locally bounded (actually continuous) and, since the non-trivial behaviour involves simply the spatial part of the metric, will therefore obey Conditions (A) and (B) when embedded with a standard mollifier, or – more generally – if we use any other regularisation that converges at least locally uniformly to the original metric.

Similarly, in [21], metrics for expanding spherical impulsive waves of the form

$$2dudv + 2v^2 \left| dz + \frac{u_+}{2v} \overline{H} d\overline{z} \right|^2$$

were studied, where $H(z)$ is the Schwartzian derivative of any arbitrary analytic function $h(z)$. Again, this metric is continuous and the non-trivial behaviour occurs in only the spatial directions. So we obtain conditions (A) and (B) as for the above case.

In all of these examples, the discussion at the end of Remark 2.1 imply that Condition (C) is also satisfied.

3. The main result

We are interested in the initial value problem for the wave equation

$$\begin{aligned} \Box u &= 0 \\ u|_\Sigma &= v \\ \mathscr{L}_{\widehat{\boldsymbol{\xi}}} u|_\Sigma &= w \end{aligned} \tag{3.1}$$

on the subset U of a weakly singular space-time $(M, \mathbf{g})$ (i.e., $\mathbf{g}$ subject to the assumptions (A)–(C) of Section 2.3). Here $\Sigma := \Sigma_0$ denotes the level set $\{q \in U : t(q) = 0\}$ of the function $t : U \to \mathbb{R}$ introduced in Section 2.3. The initial conditions are defined by v and w, which are given functions in $\mathcal{G}(\Sigma)$. Note that this, in particular, includes the case of arbitrary distributional initial data. We are interested in finding a local solution $u \in \mathcal{G}$ on U resp. an open subset thereof.

A general strategy to solve PDEs in $\mathcal{G}$ is the following. First, solve the equation for fixed ε in the smooth setting and form the net $(u_\varepsilon)_\varepsilon$ of smooth solutions. This will be a candidate for a solution in $\mathcal{G}$, but particular care has to be taken to guarantee that the u_ε share a common domain of definition. In the second step, one shows that the solution candidate $(u_\varepsilon)_\varepsilon$ is a moderate net, hence obtaining existence of the solution $[(u_\varepsilon)_\varepsilon]$ in $\mathcal{G}$. Finally, to obtain uniqueness of solutions, one has to prove that changing representatives of the data or the metric leads to a solution that is still in the class $[(u_\varepsilon)_\varepsilon]$. Note that this amounts to an additional stability of the equation with respect to negligible perturbations of the initial data and the metric.

According to this strategy, given the point p in Σ we may, without loss of generality, assume that $(U, \{x^a\})$, with $(x^a)_{a=0,1,2,3} = (t, x^i)$ is a coordinate neighbourhood of p, and formulate the initial value problem (3.1) in terms of representatives on U. To this end, given a representative $(\mathbf{g}_\varepsilon)_\varepsilon$ of the metric $\mathbf{g}$, there exist functions $h^\varepsilon_{ij}, N^i_\varepsilon$ on U such that

$$\mathbf{g}_\varepsilon = -V_\varepsilon^2 dt^2 + h^\varepsilon_{ij}\left(dx^i - N^i_\varepsilon dt\right) \otimes \left(dx^j - N^j_\varepsilon dt\right). \tag{3.2}$$

We further choose representatives $(v_\varepsilon)_\varepsilon$, $(w_\varepsilon)_\varepsilon$ of the data and a negligible net $(f_\varepsilon)_\varepsilon$ on U. We then consider the initial value problem

$$\begin{aligned} \Box^\varepsilon u_\varepsilon &= f_\varepsilon \\ u_\varepsilon(t=0,x^i) &= v_\varepsilon(x^i) \\ \mathscr{L}_{\hat{\boldsymbol{\xi}}_\varepsilon} u_\varepsilon(t=0,x^i) &= w_\varepsilon(x^i), \end{aligned} \tag{3.3}$$

where $\Box^\varepsilon$ is the d'Alembertian derived from our particular representative $\mathbf{g}_\varepsilon$, i.e.,

$$\begin{aligned} \Box^\varepsilon u_\varepsilon &= |g_\varepsilon|^{-1/2}\partial_a\left(|g_\varepsilon|^{1/2} g^{ab}_\varepsilon \partial_b u_\varepsilon\right) \\ &= -\frac{1}{V_\varepsilon^2}\partial_t^2 u_\varepsilon - \frac{2}{V_\varepsilon^2}N^i\partial_t\partial_i u_\varepsilon + \left(h^{ij}_\varepsilon - \frac{1}{V_\varepsilon^2}N^i_\varepsilon N^j_\varepsilon\right)\partial_i\partial_j u_\varepsilon - g^{ab}_\varepsilon\,\Gamma[\mathbf{g}_\varepsilon]^c{}_{ab}\,\frac{\partial u_\varepsilon}{\partial x^c}, \end{aligned}$$

and h^{ij}_ε are the components of the inverse of h^ε_{ij}, $g_\varepsilon := \det \mathbf{g}_\varepsilon$, and $\Gamma[\mathbf{g}_\varepsilon]^c{}_{ab}$ denote the Christoffel symbols of the metric $\mathbf{g}_\varepsilon$. Note that, by Conditions (A) and (B) of Section 2.3, the following asymptotic estimates hold for the components of the metric in the above coordinate system

$$\left.\begin{aligned} &V_\varepsilon, h^\varepsilon_{ij}, N^i_\varepsilon = O(1) \\ &\partial_a V_\varepsilon = O(1) \\ &\partial^\alpha V_\varepsilon, \partial^\alpha h^\varepsilon_{ij}, \partial^\alpha N^i_\varepsilon = O(\varepsilon^{-|\alpha|}) \text{ for all multi-indices } \alpha \text{ with } |\alpha| \geq 1 \end{aligned}\right\} \quad \text{as } \varepsilon \to 0. \tag{3.4}$$

Following the general strategy outlined above, we will prove local unique solvability of (3.1) by showing that the smooth solutions, $(u_\varepsilon)_\varepsilon$, of (3.3) form a moderate net, and hence determine a class in $\mathcal{G}$, and that this class is independent of the choice of representatives of v, w and $\mathbf{g}$. More precisely, our main result is the following:

Theorem 3.1 (Local existence and uniqueness of generalised solutions)**.** *Let $(M,\mathbf{g})$ be a generalised space-time and assume that Conditions (A)–(C) of Section 2.3 hold. Then, for each $p \in \Sigma$, there exists an open neighbourhood V on which the initial value problem for the wave equation (3.1) has a unique solution in $\mathcal{G}(V)$.*

We split the proof in a series of arguments, the core of which are higher order energy estimates. To prepare for these, we first introduce suitable energy tensors and energy integrals.

4. Energy integrals

By assumption, we have a point $p \in M$ and an open neighbourhood of p, U, and a map $t : U \to \mathbb{R}$ with $t(p) = 0$ such that U is foliated by the level sets of the function t, $\Sigma_\tau := \{q \in U : t(q) = \tau\}, \tau \in [-\gamma,\gamma]$, for some $\gamma > 0$. Moreover, the level sets Σ_τ are space-like with respect to the generalised metric $\mathbf{g}$. We now consider solving the forward in time initial value problem for the wave equation on U, i.e., with $\tau \geq 0$ (see Figure 1).

Given $p \in \Sigma = \Sigma_0$, let Ω be a neighbourhood of p with the properties that $\overline{\Omega} \subset U$, and such that the boundary of the region $\Omega \cap \{q \in U : t(q) \geq 0\}$ is space-like[1]. We denote by $S_\tau := \Sigma_\tau \cap \Omega$ and by Ω_τ the open part of Ω between Σ and Σ_τ. We denote the part of the boundary of Ω_τ with $0 \leq t \leq \tau$ by $S_{\Omega,\tau}$, so that $\partial\Omega_\tau = S_0 \cup S_\tau \cup S_{\Omega,\tau}$.

Notation. In order to simplify calculations, from now on we will adopt abstract index notation for (generalised) tensorial objects (see, e.g., [20]). In particular, representatives of the metric $\mathbf{g}_\varepsilon$ and its inverse will be denoted by g^ε_{ab} and g^{ab}_ε, respectively, and similarly for the corresponding Riemannian metric $\mathbf{e}_\varepsilon$. We denote the representative of the generalised normal vector field, $\boldsymbol{\xi}$, by ξ^a_ε, and the corresponding generalised covector field, $\boldsymbol{\sigma}$, by ξ^ε_a. In addition, to simplify the

[1] The existence of such a set, Ω, follows from the fact that $\|\mathbf{g}_\varepsilon^{-1}\|_{\mathbf{m}} = O(1)$ as $\varepsilon \to 0$. Geometrically, this condition means that the collection of timelike directions at a given point is not collapsing to the empty set.

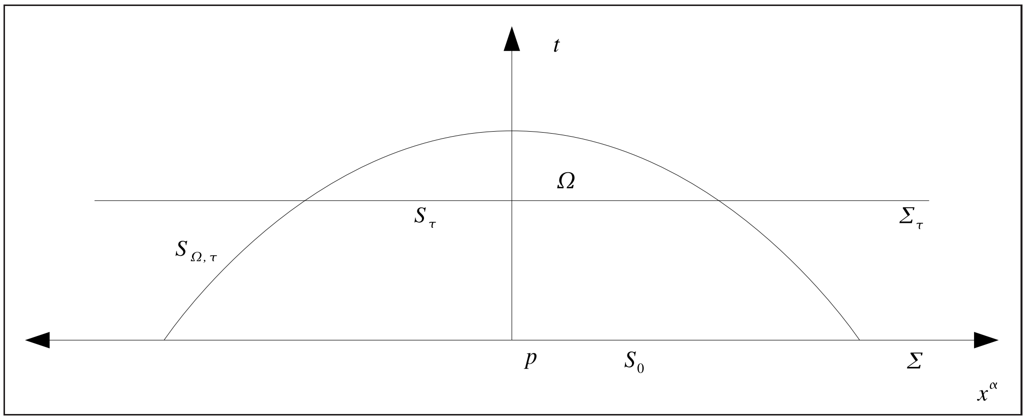

Figure 1. Local foliation of space-time

notation for tensors we are going to use capital letters to abbreviate tuples of indices, i.e., we will write T^I_J for $T^{p_1\dots p_r}_{q_1\dots q_s}$ with $|I| = r$, $|J| = s$. Also for I, J of equal length, say r, we write e_{IJ} for $e_{p_1 q_1} e_{p_2 q_2} \dots e_{q_r p_r}$.

We now use the Riemannian metric $\mathbf{e}_\varepsilon$ and the covariant derivative with respect to $\mathbf{g}_\varepsilon$ — which we have denoted by ∇^ε — to define ε-dependent Sobolev norms on U.

Definition 4.1 (Sobolev norms)**.** Let T^I_J be a smooth tensor field and u a smooth function on U, $\varepsilon > 0$, $0 \le \tau \le \gamma$ and $k, j \in \mathbb{N}_0$.

(1) We define the "pointwise" norm of T^I_J by
$$\|T^I_J\|^2_{e_\varepsilon} := e^\varepsilon_{KL} e_\varepsilon^{IJ} T^K_I T^L_J$$
and the "pointwise norm" of covariant derivatives of u by
$$|\nabla^{(j)}_\varepsilon u|^2 := ||\nabla^\varepsilon_{p_1} \dots \nabla^\varepsilon_{p_j} u||^2_{e_\varepsilon} = e_\varepsilon^{p_1 q_1} \dots e_\varepsilon^{p_j q_j} \left(\nabla^\varepsilon_{p_1} \dots \nabla^\varepsilon_{p_j} u\right) \left(\nabla^\varepsilon_{q_1} \dots \nabla^\varepsilon_{q_j} u\right).$$

(2) On Ω_τ we define Sobolev norms with respect to ∇^ε_a resp. partial derivatives by
$$\begin{aligned} {}^\nabla\|u\|^k_{\Omega_\tau,\,\varepsilon} &:= \left(\sum_{j=0}^k \int_{\Omega_\tau} |\nabla^{(j)}_\varepsilon(u)|^2 \mu^\varepsilon\right)^{\frac12} \\ {}^\partial\|u\|^k_{\Omega_\tau,\,\varepsilon} &:= \left(\sum_{\substack{p_1,\dots,p_j \\ 0\le j\le k}} \int_{\Omega_\tau} |\partial_{p_1} \dots \partial_{p_j} u|^2 \mu^\varepsilon\right)^{\frac12}, \end{aligned}$$
where μ^ε denotes the volume form derived from $\mathbf{g}_\varepsilon$.

(3) The respective "three-dimensional" Sobolev norms are defined by
$$\begin{aligned} {}^\nabla\|u\|^k_{S_\tau,\,\varepsilon} &:= \left(\sum_{j=0}^k \int_{S_\tau} |\nabla^{(j)}_\varepsilon(u)|^2 \mu^\varepsilon_\tau\right)^{\frac12} \\ {}^\partial\|u\|^k_{S_\tau,\,\varepsilon} &:= \left(\sum_{\substack{p_1,\dots,p_j \\ 0\le j\le k}} \int_{S_\tau} |\partial_{p_1} \dots \partial_{p_j} u|^2 \mu^\varepsilon_\tau\right)^{\frac12}, \end{aligned}$$
where μ^ε_τ is the unique three-form induced on S_τ by μ^ε such that $dt \wedge \mu^\varepsilon_\tau = \mu^\varepsilon$ holds on S_τ. Note that although the integration is performed over the three-dimensional manifold S_τ only, derivatives are not confined to directions tangential to S_τ.

Observe that, due to the use of a generalised metric, even the norms ${}^{\partial}\|u\|^{k}_{S_\tau,\varepsilon}$ depend on ε. However, due to Condition (A), with $k = 0$, they are equivalently to an ε-independent norm derived, for example, from the fixed background metric $\mathbf{m}$. In the following, we will provide suitable higher order energy estimates for nets of solutions of the wave equation. These estimates are best expressed in terms of energy momentum tensors and energy integrals, which we define following [25, Sec. 4]. For the "classical" case, see [12, Sec. 7.4], [1, Sec. 4.4] and [11] for a recent review.

Definition 4.2 (Energy momentum tensors and energy integrals)**.** Let $u \in \mathcal{C}^\infty(U)$ and $k \in \mathbb{N}_0$. On Ω we define

(1) the energy momentum tensors by $(k > 0)$

$$\begin{aligned} T_\varepsilon^{ab,0}(u) &:= -\frac{1}{2} g_\varepsilon^{ab} u^2, \\ T_\varepsilon^{ab,k}(u) &:= \left(g_\varepsilon^{ac} g_\varepsilon^{bd} - \frac{1}{2} g_\varepsilon^{ab} g_\varepsilon^{cd}\right) e_\varepsilon^{p_1 q_1} \dots e_\varepsilon^{p_{k-1} q_{k-1}} (\nabla_c^\varepsilon \nabla_{p_1}^\varepsilon \dots \nabla_{p_{k-1}}^\varepsilon u)(\nabla_d^\varepsilon \nabla_{q_1}^\varepsilon \dots \nabla_{q_{k-1}}^\varepsilon u), \end{aligned}$$

(2) the energy integrals by

$$E^k_{\tau,\varepsilon}(u) := \sum_{j=0}^{k} \int_{S_\tau} T_\varepsilon^{ab,j}(u) \xi_a \widehat{\xi}^\varepsilon_b \mu^\varepsilon_\tau, \qquad k \geq 0. \tag{4.1}$$

It may be verified, by direct calculation, that the tensor fields $T_\varepsilon^{ab,k}(u)$ satisfy the dominant energy condition. Indeed it suffices to observe that, for any future-directed time-like vector field $\mathbf{U}$, the expression $U^a U^b - \frac{1}{2}\mathbf{g}_\varepsilon(\mathbf{U},\mathbf{U}) g_\varepsilon^{ab}$ defines a Riemannian metric for fixed ε. For details, see Proposition 3.6 of [18]. For a generalised formulation of the dominant energy condition, see [17].

Remark 4.3. The energy momentum tensors introduced above are related to the super-energy tensors of Senovilla [23]. Omitting indices and ε's for the moment, we construct the super-energy tensor, S^k, of type $(0, 2k)$ (see Definition 3.1 in [23]). Then $T^k(u)$ are the $(2k-2)$-fold contraction of S^k with the time-like vector field $\boldsymbol{\xi}$. Theorem 4.1 of [23] then implies that the tensors $S^k (k \geq 0)$ satisfy the dominant super-energy property, from which it follows more elegantly that the $T^k(u)$ satisfy the dominant energy condition.

Remark 4.4. The energy integrals may be written in the more symmetrical form

$$E^k_{\tau,\varepsilon}(u) := \sum_{j=0}^{k} \int_{S_\tau} T_\varepsilon^{ab,j}(u) \widehat{\xi}^\varepsilon_a \widehat{\xi}^\varepsilon_b \widehat{\mu}^\varepsilon_\tau,$$

where we have defined the volume element $\widehat{\mu}^\varepsilon_\tau = V_\varepsilon^{-1} \mu^\varepsilon_\tau$ on S_τ. In terms of the decomposition of the metric given in equation (3.2), $\widehat{\mu}^\varepsilon_\tau$ is the volume form on S_τ defined by the three-dimensional metric $\mathbf{h}_\varepsilon := h^\varepsilon_{ij} dx^i \otimes dx^j$.

Since the part $S_{\Omega,\tau}$ of the boundary of Ω is space-like and $T_\varepsilon^{ab,j}(u)$ satisfies the dominant energy condition, an application of the Stokes theorem yields

$$\begin{aligned} \int_{\Omega_\tau} \nabla_a^\varepsilon \left(T_\varepsilon^{ab,j}(u)\xi_b\right) \mu_\varepsilon &= \int_{S_\tau} T_\varepsilon^{ab,j}(u)\xi_b \widehat{\xi}^\varepsilon_a \mu^\varepsilon_\tau - \int_{S_0} T_\varepsilon^{ab,j}(u)\xi_b \widehat{\xi}^\varepsilon_a \mu^\varepsilon_0 + \int_{S_{\Omega,\tau}} T_\varepsilon^{ab,j}(u)\xi_b n^\varepsilon_a dS_\varepsilon \\ &\geq \int_{S_\tau} T_\varepsilon^{ab,j}(u)\xi_b \widehat{\xi}^\varepsilon_a \mu^\varepsilon_\tau - \int_{S_0} T_\varepsilon^{ab,j}(u)\xi_b \widehat{\xi}^\varepsilon_a \mu^\varepsilon_0, \end{aligned}$$

where $\mathbf{n}^\varepsilon$ and dS_ε denote the unit normal and surface element on $\partial\Omega_\tau$, respectively. Hence summing over j we have the following energy inequality for each $\varepsilon > 0$ and each $0 \leq \tau \leq \gamma$

$$E^k_{\tau,\varepsilon}(u) \leq E^k_{\tau=0,\varepsilon}(u) + \sum_{j=0}^{k} \int_{\Omega_\tau} \left(\xi_b \nabla_a^\varepsilon T_\varepsilon^{ab,j}(u) + T_\varepsilon^{ab,j}(u) \nabla_a^\varepsilon \xi_b\right) \mu_\varepsilon. \tag{4.2}$$

Note that the energy integrals and foliation used here correspond closely to those used in [12, Sec. 4.3]. In [25, pp. 1341], due to a different choice of foliation, inequality (4.2) is replaced with an equality. This alternative foliation allows one to work without the explicit use of the dominant energy condition, but complicates some of the resulting energy estimates.

To end this section, we prove the equivalence of the Sobolev norms and the energy integrals. Note that this result is the analogue of Lemma 1 in [25] for our class of metrics, and is one of the key estimates in our approach.

Lemma 4.5 (Energy integrals and Sobolev norms)**.**

(1) *There exist constants A, A' such that for each $k \geq 0$*
$$A'({}^{\nabla}\|u\|^k_{S_\tau,\,\varepsilon})^2 \leq E^k_{\tau,\varepsilon}(u) \leq A({}^{\nabla}\|u\|^k_{S_\tau,\,\varepsilon})^2 \tag{4.3}$$
(2) *For each $k \geq 1$, there exist positive constants B_k, B'_k such that*
$$({}^{\nabla}\|u\|^k_{S_\tau,\,\varepsilon})^2 \quad \leq \quad B'_k \sum_{j=1}^{k} \frac{1}{\varepsilon^{2(k-j)}} ({}^{\partial}\|u\|^j_{S_\tau,\,\varepsilon})^2 \tag{4.4}$$
$$({}^{\partial}\|u\|^k_{S_\tau,\,\varepsilon})^2 \quad \leq \quad B_k \sum_{j=1}^{k} \frac{1}{\varepsilon^{2(k-j)}} ({}^{\nabla}\|u\|^j_{S_\tau,\,\varepsilon})^2 \tag{4.5}$$
For $k = 0$ we simply have $({}^{\nabla}\|u\|^0_{S_\tau,\,\varepsilon})^2 = ({}^{\partial}\|u\|^0_{S_\tau,\,\varepsilon})^2$.

Proof. (1): For $k = 0$ we have
$$T^{ab,0}_\varepsilon(u)\xi_a\widehat{\xi}^\varepsilon_b = -\frac{1}{2}g^{ab}_\varepsilon\xi_a\widehat{\xi}^\varepsilon_b u^2 = -\frac{1}{2}\sqrt{-\mathbf{g}_\varepsilon(\boldsymbol{\xi}_\varepsilon, \boldsymbol{\xi}_\varepsilon)}u^2 = \frac{V_\varepsilon}{2}u^2,$$
hence by (2.4), setting $A := M_0/2$ and $A' := 1/(2M_0)$, we obtain
$$A'u^2 \leq T^{ab,0}_\varepsilon(u)\xi_a\widehat{\xi}^\varepsilon_b \leq Au^2,$$
which upon integrating gives the result.

For the case $k > 0$ note that
$$(g^{ac}_\varepsilon g^{bd}_\varepsilon - \frac{1}{2}g^{ab}_\varepsilon g^{cd}_\varepsilon)\xi_a\widehat{\xi}^\varepsilon_b \quad = \quad \frac{1}{2}V_\varepsilon\left(g^{cd}_\varepsilon + \frac{2}{V^2_\varepsilon}\xi^c_\varepsilon\xi^d_\varepsilon\right) \; = \; \frac{1}{2}V_\varepsilon e^{cd}_\varepsilon.$$
Hence, we may write
$$\begin{aligned} T^{ab,j}_\varepsilon(u)\xi_a\widehat{\xi}^\varepsilon_b &= \frac{1}{2}V_\varepsilon e^{cd}_\varepsilon e^{p_1q_1}_\varepsilon \dots e^{p_{j-1}q_{j-1}}_\varepsilon(\nabla^\varepsilon_c\nabla^\varepsilon_{p_1}\dots\nabla^\varepsilon_{p_{j-1}}u)(\nabla^\varepsilon_d\nabla^\varepsilon_{q_1}\dots\nabla^\varepsilon_{q_{j-1}}u) \\ &= \frac{1}{2}V_\varepsilon|\nabla^{(j)}_\varepsilon u|^2. \end{aligned}$$
Using (A), this implies that
$$A'|\nabla^{(j)}_\varepsilon u|^2 \leq T^{ab,j}_\varepsilon(u)\xi_a\widehat{\xi}^\varepsilon_b \leq A|\nabla^{(j)}_\varepsilon u|^2,$$
which upon summation and integration establishes the claim.

(2) follows by (A) from the fact that on the compact closure of Ω the metrics $\mathbf{e}_\varepsilon$ and δ_{ab} are equivalent and the Christoffel symbols and its derivatives are bounded by the respective inverse powers of ε. ☐

5. Energy estimates

In this section, we establish the core estimates needed in the proof of our main theorem.

Proposition 5.1. *Let u_ε be a solution of (3.3) on U. Then, for each $k \geq 1$, there exist positive constants C'_k, C''_k, C'''_k such that for each $0 \leq \tau \leq \gamma$ we have*
$$\begin{aligned} E^k_{\tau,\varepsilon}(u_\varepsilon) \quad \leq \quad & E^k_{0,\varepsilon}(u_\varepsilon) + C'_k({}^{\nabla}\|f_\varepsilon\|^{k-1}_{\Omega_\tau,\,\varepsilon})^2 + C''_k\sum_{j=1}^{k-1}\frac{1}{\varepsilon^{2(1+k-j)}}\int_{\zeta=0}^{\tau} E^j_{\zeta,\varepsilon}(u_\varepsilon)d\zeta \\ & + C'''_k\int_{\zeta=0}^{\tau} E^k_{\zeta,\varepsilon}(u_\varepsilon)d\zeta. \end{aligned} \tag{5.1}$$

Before proving this statement, we draw the essential conclusions from it. Observe that the constant in front of the highest order term on the r.h.s. does not depend on ε, hence we obtain, by an application of Gronwall's lemma.

Corollary 5.2. *Let u_ε be a solution of (3.3) on U. Then, for each $k \geq 1$, there exist positive constants C_k', C_k'', C_k''' such that for each $0 \leq \tau \leq \gamma$,*

$$E^k_{\tau,\varepsilon}(u_\varepsilon) \leq \left(E^k_{0,\varepsilon}(u_\varepsilon) + C_k'({}^{\nabla}\|f_\varepsilon\|^{k-1}_{\Omega_\tau,\varepsilon})^2 + C_k'' \sum_{j=1}^{k-1} \frac{1}{\varepsilon^{2(1+k-j)}} \int\limits_{\zeta=0}^{\tau} E^j_{\zeta,\varepsilon}(u_\varepsilon) d\zeta \right) e^{C_k''' \tau} \tag{5.2}$$

This statement immediately implies the main result in this section.

Corollary 5.3. *Let u_ε be a solution of (3.3) on U. If, all $k \geq 1$, the initial energy $(E^k_{0,\,\varepsilon}(u_\varepsilon))_\varepsilon$ is a moderate resp. negligible net of real numbers, and $(f_\varepsilon)_\varepsilon$ is negligible then*

$$\sup_{0 \leq \tau \leq \gamma} (E^k_{\tau,\,\varepsilon}(u_\varepsilon))_\varepsilon$$

is moderate resp. negligible.

Proof of 5.1. We begin by estimating the second integrand on the r.h.s. of equation (4.2). Using the fact that the energy tensors are symmetric then, by an application of the Cauchy-Schwarz inequality to the inner product induced on the tensor bundle $\mathcal{T}^2_0(M)$ by the metric $\mathbf{e}_\varepsilon$, we deduce that

$$\left|T^{ab,j}_\varepsilon(u_\varepsilon)\nabla^\varepsilon_a \xi_b\right| \leq \|T^{ab,j}_\varepsilon(u_\varepsilon)\|_{\mathbf{e}_\varepsilon} \|\nabla^\varepsilon_{(a}\xi_{b)}\|_{\mathbf{e}_\varepsilon} = \frac{1}{2}\|T^{ab,j}_\varepsilon(u_\varepsilon)\|_{\mathbf{e}_\varepsilon} \|\mathscr{L}_{\boldsymbol{\xi}_\varepsilon}\mathbf{g}_\varepsilon\|_{\mathbf{e}_\varepsilon}. \tag{5.3}$$

Equation (2.6) implies that there exists a constant $K > 0$ such that $\|\mathscr{L}_{\boldsymbol{\xi}_\varepsilon}\mathbf{g}_\varepsilon\|_{\mathbf{e}_\varepsilon} \leq K$. In the case $j = 0$, we have

$$\|T^{ab,0}_\varepsilon(u_\varepsilon)\|^2_{\mathbf{e}_\varepsilon} = \left(-\frac{1}{2}\right)^2 u^4_\varepsilon e^\varepsilon_{ab} e^\varepsilon_{cd} g^{ac}_\varepsilon g^{bd}_\varepsilon = u^4_\varepsilon,$$

so

$$\|T^{ab,0}_\varepsilon(u_\varepsilon)\|_{\mathbf{e}_\varepsilon} = u^2_\varepsilon = |\nabla^{(0)}_\varepsilon u_\varepsilon|^2.$$

For $j \geq 1$, we have

$$\begin{aligned}
\|T^{ab,j}_\varepsilon(u_\varepsilon)\|^2_{\mathbf{e}_\varepsilon} &= e^\varepsilon_{aa'} e^\varepsilon_{bb'} \left(g^{ac}_\varepsilon g^{bd}_\varepsilon - \frac{1}{2} g^{ab}_\varepsilon g^{cd}_\varepsilon\right)\left(g^{a'c'}_\varepsilon g^{b'd'}_\varepsilon - \frac{1}{2} g^{a'b'}_\varepsilon g^{c'd'}_\varepsilon\right) \\
&\qquad \times \left(\nabla_c \nabla_I u_\varepsilon\right)\left(\nabla_d \nabla_J u_\varepsilon\right)\left(\nabla_{c'} \nabla_{I'} u_\varepsilon\right)\left(\nabla_{d'} \nabla_{J'} u_\varepsilon\right) e^{IJ}_\varepsilon e^{I'J'}_\varepsilon \\
&= e^{cc'}_\varepsilon e^{dd'}_\varepsilon \left(\nabla_c \nabla_I u_\varepsilon\right)\left(\nabla_d \nabla_J u_\varepsilon\right)\left(\nabla_{c'} \nabla_{I'} u_\varepsilon\right)\left(\nabla_{d'} \nabla_{J'} u_\varepsilon\right) e^{IJ}_\varepsilon e^{I'J'}_\varepsilon \\
&\leq A_j |\nabla^{(j)}_\varepsilon u_\varepsilon|^2,
\end{aligned}$$

where $A_j \sim 4^j$ are combinatorial constants. Letting $A_0 := 1$, we deduce that

$$\left|T^{ab,j}_\varepsilon(u_\varepsilon)\nabla^\varepsilon_a \xi_b\right| \leq \frac{1}{2} A_j K |\nabla^{(j)}_\varepsilon u_\varepsilon|^2, \quad \text{for } j \geq 0.$$

Letting $\tilde{A}_k := \max_{j=0,\ldots,k} A_k$, we therefore find that

$$\left| \sum_{j=0}^{k} \int_{\Omega_\tau} T^{ab,j}_\varepsilon(u_\varepsilon) \nabla^\varepsilon_a \xi_b \mu_\varepsilon \right| \leq \frac{1}{2} \tilde{A}_k K ({}^{\nabla}\|u_\varepsilon\|^k_{\Omega_\tau,\,\varepsilon})^2 \tag{5.4}$$

for $k \geq 0$.

We now consider the first integrand on r.h.s. of (4.2). Beginning with the case $k = 1$, the divergence terms that we require take the form

$$\begin{aligned}
\nabla^\varepsilon_a T_\varepsilon^{ab,0}(u_\varepsilon) &= -\frac{1}{2}(\nabla^\varepsilon_a g_\varepsilon^{ab})u_\varepsilon^2 - \left(\frac{1}{2}g_\varepsilon^{ab}\right)(2u_\varepsilon\nabla^\varepsilon_a u_\varepsilon) = -u_\varepsilon\nabla_\varepsilon^b u_\varepsilon \\
\nabla^\varepsilon_a T_\varepsilon^{ab,1}(u_\varepsilon) &= \left(g_\varepsilon^{ac}g_\varepsilon^{bd} - \frac{1}{2}g_\varepsilon^{ab}g_\varepsilon^{cd}\right)(\nabla^\varepsilon_a\nabla^\varepsilon_c u_\varepsilon\nabla^\varepsilon_d u_\varepsilon + \nabla^\varepsilon_c u_\varepsilon\nabla^\varepsilon_a\nabla^\varepsilon_d u_\varepsilon) \\
&= \nabla_\varepsilon^c\nabla^\varepsilon_c u_\varepsilon\nabla_\varepsilon^b u_\varepsilon = (\Box^\varepsilon u_\varepsilon)\nabla_\varepsilon^b u_\varepsilon = f_\varepsilon\nabla_\varepsilon^b u_\varepsilon.
\end{aligned}$$

Inserting this and the $k = 1$ form of (5.4) into (4.2) yields

$$\begin{aligned}
E^1_{\tau,\varepsilon}(u_\varepsilon) &\leq E^1_{0,\varepsilon}(u_\varepsilon) + \int_{\Omega_\tau}\xi_\varepsilon^a\nabla^\varepsilon_a u_\varepsilon(f_\varepsilon - u_\varepsilon)\mu_\varepsilon + \frac{1}{2}\tilde{A}_1 K({}^\nabla\|u_\varepsilon\|^1_{\Omega_\tau,\varepsilon})^2 \\
&\leq E^1_{0,\varepsilon}(u_\varepsilon) + \left(\int_{\Omega_\tau}(\xi_\varepsilon^a\nabla^\varepsilon_a u_\varepsilon)^2\mu_\varepsilon\right)^{\frac{1}{2}}\left(\int_{\Omega_\tau}|f_\varepsilon - u_\varepsilon|^2\mu_\varepsilon\right)^{\frac{1}{2}} + \frac{1}{2}\tilde{A}_1 K({}^\nabla\|u_\varepsilon\|^1_{\Omega_\tau,\varepsilon})^2 \\
&\leq E^1_{0,\varepsilon}(u_\varepsilon) + M_0\left(\int_{\Omega_\tau}|\nabla_\varepsilon^{(1)}u_\varepsilon|^2\mu_\varepsilon\right)^{\frac{1}{2}}\left(\left(\int_{\Omega_\tau}|f_\varepsilon|^2\mu_\varepsilon\right)^{\frac{1}{2}} + \left(\int_{\Omega_\tau}|u_\varepsilon|^2\mu_\varepsilon\right)^{\frac{1}{2}}\right) \\
&\quad + \frac{1}{2}\tilde{A}_1 K({}^\nabla\|u_\varepsilon\|^1_{\Omega_\tau,\varepsilon})^2 \\
&\leq E^1_{0,\varepsilon}(u_\varepsilon) + \frac{M_0}{2}\left(\int_{\Omega_\tau}\left(|\nabla_\varepsilon^{(1)}u_\varepsilon|^2 + |u_\varepsilon|^2\right)\mu_\varepsilon + \int_{\Omega_\tau}|\nabla_\varepsilon^{(1)}u_\varepsilon|^2\mu_\varepsilon + \int_{\Omega_\tau}|f_\varepsilon|^2\mu_\varepsilon\right) \\
&\quad + \frac{1}{2}\tilde{A}_1 K({}^\nabla\|u_\varepsilon\|^1_{\Omega_\tau,\varepsilon})^2 \\
&\leq E^1_{0,\varepsilon}(u_\varepsilon) + \frac{M_0}{2}\left({}^\nabla\|f_\varepsilon\|^0_{\Omega_\tau,\varepsilon}\right)^2 + \left(M_0 + \frac{1}{2}\tilde{A}_1 K\right)\left({}^\nabla\|u_\varepsilon\|^1_{\Omega_\tau,\varepsilon}\right)^2,
\end{aligned}$$

where we have repeatedly used the Cauchy-Schwarz inequality. Now we use (4.3) to obtain

$$\left({}^\nabla\|u_\varepsilon\|^1_{\Omega_\tau,\varepsilon}\right)^2 = \int_{\zeta=0}^{\tau}({}^\nabla\|u_\varepsilon\|^1_{S_\zeta,\varepsilon})^2 d\zeta \leq \frac{1}{A'}\int_{\zeta=0}^{\tau}E^1_{\zeta,\varepsilon}(u_\varepsilon)d\zeta.$$

Setting $C_1' := M_0/2$, $C_1'' = 0$, and $C_1''' := (M_0 + \frac{1}{2}\tilde{A}_1 K)/A' = 2M_0(M_0 + \frac{1}{2}\tilde{A}_1 K)$ yields the claim for $k = 1$.

We now turn to the case $k > 1$. We first derive an estimate for

$$\xi_b\nabla^\varepsilon_a T_\varepsilon^{ab,k}(u_\varepsilon) = I_1 + I_2 + I_3,$$

where we have defined

$$\begin{aligned}
I_1 &:= \left(g_\varepsilon^{ac}\xi_\varepsilon^d - \frac{1}{2}\xi_\varepsilon^a g_\varepsilon^{cd}\right)(\nabla^\varepsilon_a e_\varepsilon^{IJ})(\nabla^\varepsilon_c\nabla^\varepsilon_I u_\varepsilon)(\nabla^\varepsilon_d\nabla^\varepsilon_J u_\varepsilon) \\
I_2 &:= -2e_\varepsilon^{IJ}(\nabla^\varepsilon_d\nabla^\varepsilon_J u_\varepsilon)\left(\xi_\varepsilon^a g_\varepsilon^{cd}\nabla^\varepsilon_{[a}\nabla^\varepsilon_{c]}\nabla^\varepsilon_I u_\varepsilon\right) \\
I_3 &:= e_\varepsilon^{IJ}(\xi_\varepsilon^d\nabla^\varepsilon_d\nabla^\varepsilon_J u_\varepsilon)(g_\varepsilon^{ac}\nabla^\varepsilon_a\nabla^\varepsilon_c\nabla^\varepsilon_I u_\varepsilon)
\end{aligned}$$

The strategy is, again, to remove the terms involving derivatives of order $k+1$ using the wave equation. This requires interchanging the order of covariant derivatives, and therefore introduces additional curvature terms. We now calculate the moduli of the terms I_1, I_2, I_3 separately.

We begin by estimating $|I_1|$:

$$\begin{aligned}
|I_1| &= \left|\left(g_\varepsilon^{ac}\xi_\varepsilon^d - \frac{1}{2}\xi_\varepsilon^a g_\varepsilon^{cd}\right)(\nabla^\varepsilon_a e_\varepsilon^{IJ})(\nabla^\varepsilon_c\nabla^\varepsilon_I u_\varepsilon)(\nabla^\varepsilon_d\nabla^\varepsilon_J u_\varepsilon)\right| \\
&\leq \left\|\left(g_\varepsilon^{ac}\xi_\varepsilon^d - \frac{1}{2}\xi_\varepsilon^a g_\varepsilon^{cd}\right)\nabla^\varepsilon_a e_\varepsilon^{IJ}\right\|_{e_\varepsilon} \cdot \|(\nabla^\varepsilon_c\nabla^\varepsilon_I u_\varepsilon)(\nabla^\varepsilon_d\nabla^\varepsilon_J u_\varepsilon)\|_{e_\varepsilon} \\
&= \left\|\left(g_\varepsilon^{ac}\xi_\varepsilon^d - \frac{1}{2}\xi_\varepsilon^a g_\varepsilon^{cd}\right)\nabla^\varepsilon_a e_\varepsilon^{IJ}\right\|_{e_\varepsilon} \cdot |\nabla_\varepsilon^{(k)}u_\varepsilon|^2,
\end{aligned}$$

where the inequality in the second line results from applying the Cauchy-Schwarz inequality to the inner product induced on the tensor bundle $\mathcal{T}^{2k}_0(M)$ by the metric $\mathbf{e}_\varepsilon$. The square of the first term may then be evaluated as

$$\begin{aligned}\left\| \left(g_\varepsilon^{ac}\xi_\varepsilon^d - \frac{1}{2}\xi_\varepsilon^a g_\varepsilon^{cd} \right) \nabla_a^\varepsilon e_\varepsilon^{IJ} \right\|_{e_\varepsilon}^2 &= e^\varepsilon_{cc'} e^\varepsilon_{dd'} e^\varepsilon_{II'} e^\varepsilon_{JJ'} \left(g_\varepsilon^{ac}\xi_\varepsilon^d - \frac{1}{2}\xi_\varepsilon^a g_\varepsilon^{cd} \right) \left(g_\varepsilon^{a'c'}\xi_\varepsilon^{d'} - \frac{1}{2}\xi_\varepsilon^{a'} g_\varepsilon^{c'd'} \right) \\ &\qquad \times \left(\nabla_a^\varepsilon e_\varepsilon^{IJ} \right) \left(\nabla_{a'}^\varepsilon e_\varepsilon^{I'J'} \right) \\ &= \|\xi_\varepsilon\|_{e_\varepsilon}^2 \, e_\varepsilon^{aa'} e^\varepsilon_{II'} e^\varepsilon_{JJ'} \left(\nabla_a^\varepsilon e_\varepsilon^{IJ} \right) \left(\nabla_{a'}^\varepsilon e_\varepsilon^{I'J'} \right) = \|\xi_\varepsilon\|_{e_\varepsilon}^2 \cdot \left\| \nabla_a^\varepsilon e_\varepsilon^{IJ} \right\|_{e_\varepsilon}^2 .\end{aligned}$$

We now note that, by Condition (A) and equation (2.5) of Section 2.3, we have,

$$\left\| \nabla_a^\varepsilon e_\varepsilon^{IJ} \right\|_{e_\varepsilon} = O(1), \quad (\varepsilon \to 0).$$

In particular, on each compact set there exists a positive constant, C_k, such that $\left\| \nabla_a^\varepsilon e_\varepsilon^{IJ} \right\|_{e_\varepsilon} \le C_k$, as $\varepsilon \to 0$. Therefore, we have the following estimate for I_1:

$$|I_1| \le C_k \cdot \|\xi_\varepsilon\|_{e_\varepsilon} \cdot |\nabla_\varepsilon^{(k)} u_\varepsilon|^2 \le C_k M_0 \cdot |\nabla_\varepsilon^{(k)} u_\varepsilon|^2, \tag{5.5}$$

locally, as $\varepsilon \to 0$.

Next we turn to I_2. We then have

$$\begin{aligned}|I_2| = \left| 2e_\varepsilon^{IJ} \left(\nabla_d^\varepsilon \nabla_J^\varepsilon u_\varepsilon \right) \left(\xi_\varepsilon^a g_\varepsilon^{cd} \nabla_{[a}^\varepsilon \nabla_{c]}^\varepsilon \nabla_I^\varepsilon u_\varepsilon \right) \right| &= \left| e_\varepsilon^{IJ} \left(\nabla_d^\varepsilon \nabla_J^\varepsilon u_\varepsilon \right) \left(\xi_\varepsilon^a e_\varepsilon^{cd} \left[\nabla_a^\varepsilon, \nabla_c^\varepsilon \right] \nabla_I^\varepsilon u_\varepsilon \right) \right| \\ &= \left| \left(e_\varepsilon^{IJ} \xi_\varepsilon^a e_\varepsilon^{cd} \nabla_d^\varepsilon \nabla_J^\varepsilon u_\varepsilon \right) \left(\left[\nabla_a^\varepsilon, \nabla_c^\varepsilon \right] \nabla_I^\varepsilon u_\varepsilon \right) \right| \\ &\le \|\xi_\varepsilon^a \nabla_c^\varepsilon \nabla_I^\varepsilon u_\varepsilon\|_{e_\varepsilon} \cdot \left\| \left[\nabla_a^\varepsilon, \nabla_c^\varepsilon \right] \nabla_I^\varepsilon u_\varepsilon \right\|_{e_\varepsilon} \\ &= \|\xi_\varepsilon\|_{e_\varepsilon} \cdot |\nabla_\varepsilon^{(k)} u_\varepsilon| \cdot \left\| \left[\nabla_a^\varepsilon, \nabla_c^\varepsilon \right] \nabla_I^\varepsilon u_\varepsilon \right\|_{e_\varepsilon},\end{aligned}$$

where the equality in the first line follows from skew-symmetry in a, c, and the inequality on the third line follows from the Cauchy-Schwarz inequality. Moreover, from Condition (A), we have the following estimates for the curvature on compact sets

$$\|\nabla_{a_1}^\varepsilon \dots \nabla_{a_l}^\varepsilon R^\varepsilon_{ab}{}^c{}_d\|_{e_\varepsilon} \le \frac{F_l}{\varepsilon^{2+l}}, \qquad l \ge 0. \tag{5.6}$$

Using this estimate with $l = 0$ and the Ricci identity, we deduce the existence of a combinatorial constant N_k depending only on k such that

$$\left\| \left[\nabla_a^\varepsilon, \nabla_c^\varepsilon \right] \nabla_I^\varepsilon u_\varepsilon \right\|_{e_\varepsilon} \le N_k \frac{F_0}{\varepsilon^2} |\nabla_\varepsilon^{(k-1)} u_\varepsilon|.$$

Hence, we have

$$|I_2| \le N_k \frac{F_0}{\varepsilon^2} \|\xi_\varepsilon\|_{e_\varepsilon} \cdot |\nabla_\varepsilon^{(k)} u_\varepsilon| \cdot |\nabla_\varepsilon^{(k-1)} u_\varepsilon| \le \frac{N_k F_0 M_0}{2} \left(|\nabla_\varepsilon^{(k)} u_\varepsilon|^2 + \frac{1}{\varepsilon^4} |\nabla_\varepsilon^{(k-1)} u_\varepsilon|^2 \right). \tag{5.7}$$

on compact sets.

Finally, we consider the term I_3. We then have, by the Cauchy-Schwarz inequality

$$\begin{aligned}|I_3| = \left| e_\varepsilon^{IJ} \left(\xi_\varepsilon^d \nabla_d^\varepsilon \nabla_J^\varepsilon u_\varepsilon \right) \left(g_\varepsilon^{ac} \nabla_a^\varepsilon \nabla_c^\varepsilon \nabla_I^\varepsilon u_\varepsilon \right) \right| &\le \left\| \xi_\varepsilon^d \nabla_d^\varepsilon \nabla_I^\varepsilon u_\varepsilon \right\|_{e_\varepsilon} \cdot \|g_\varepsilon^{ac} \nabla_a^\varepsilon \nabla_c^\varepsilon \nabla_I^\varepsilon u_\varepsilon\|_{e_\varepsilon} \\ &\le P_k \|\xi_\varepsilon\|_{e_\varepsilon} \cdot |\nabla_\varepsilon^{(k)} u_\varepsilon| \cdot \|g_\varepsilon^{ac} \nabla_a^\varepsilon \nabla_c^\varepsilon \nabla_I^\varepsilon u_\varepsilon\|_{e_\varepsilon},\end{aligned}$$

where P_k is a combinatorial constant depending only on k. Again using the Ricci identities, and the fact that u_ε is a solution of (3.3), we may write

$$g_\varepsilon^{ac} \nabla_a^\varepsilon \nabla_c^\varepsilon \nabla_I^\varepsilon \, u_\varepsilon = \nabla_I^\varepsilon \, f_\varepsilon + \sum_{j=1}^{k-1} \left(\mathcal{R}_\varepsilon^{(k-1,j)} u_\varepsilon \right)_I,$$

where $\mathcal{R}_\varepsilon^{(k-1,j)}u_\varepsilon$ denotes a linear combination of contractions of the $(k-j-1)$'th covariant derivative of the Riemann tensor with the j'th covariant derivative of u_ε. A second appeal to (5.6) implies that on each compact set there exists a constant G_k such that

$$\left\| \left(\mathcal{R}_\varepsilon^{(k-1,j)}u_\varepsilon\right)_I \right\|_{e_\varepsilon} \leq \frac{G_k}{\varepsilon^{k-j+1}}|\nabla_\varepsilon^{(j)}u_\varepsilon|, \qquad (\varepsilon \to 0).$$

We therefore have

$$\begin{aligned} |I_3| &\leq P_k\, \|\xi_\varepsilon\|_{e_\varepsilon} \cdot |\nabla_\varepsilon^{(k)}u_\varepsilon| \left(|\nabla_\varepsilon^{(k-1)}f_\varepsilon| + G_k \sum_{j=1}^{k-1} \frac{1}{\varepsilon^{1+k-j}}|\nabla_\varepsilon^{(j)}u_\varepsilon| \right) \\ &\leq \frac{P_k M_0}{2} \cdot \left(k|\nabla_\varepsilon^{(k)}u_\varepsilon|^2 + |\nabla_\varepsilon^{(k-1)}f_\varepsilon|^2 + G_k^2 \sum_{j=1}^{k-1} \frac{1}{\varepsilon^{2(1+k-j)}}|\nabla_\varepsilon^{(j)}u_\varepsilon|^2 \right) \end{aligned} \tag{5.8}$$

Putting together (5.5), (5.7), and (5.8), we have

$$\left|\xi_b \nabla_a^\varepsilon T_\varepsilon^{ab,k}(u_\varepsilon)\right| \leq \alpha_k |\nabla_\varepsilon^{(k)}u_\varepsilon|^2 + \beta_k |\nabla_\varepsilon^{(k-1)}f_\varepsilon|^2 + \gamma_k \sum_{j=1}^{k-1} \frac{|\nabla_\varepsilon^{(j)}u_\varepsilon|^2}{\varepsilon^{2(1+k-j)}},$$

for positive constants $\alpha_k, \beta_k, \gamma_k$. Summation over $k = 1\ldots m$ and integration yields positive constants $\widetilde{\alpha}_m$, $\widetilde{\beta}_m$, $\widetilde{\gamma}_m$ such that

$$\left| \sum_{k=0}^m \int_{\Omega_\tau} \xi_b \nabla_a^\varepsilon T_\varepsilon^{ab,k}(u_\varepsilon)\mu_\varepsilon \right| \leq \widetilde{\alpha}_m ({}^\nabla\|u_\varepsilon\|^m_{\Omega_\tau,\,\varepsilon})^2 + \widetilde{\beta}_m ({}^\nabla\|f_\varepsilon\|^{m-1}_{\Omega_\tau,\,\varepsilon})^2 + \widetilde{\gamma}_m \sum_{j=1}^{m-1} \frac{1}{\varepsilon^{2(1+m-j)}} ({}^\nabla\|u_\varepsilon\|^j_{\Omega_\tau,\,\varepsilon})^2.$$

On substituting this inequality and (5.4) into equation (4.2), we deduce that

$$\begin{aligned} E^m_{\tau,\varepsilon}(u_\varepsilon) \quad &\leq \quad E^m_{0,\varepsilon}(u_\varepsilon) \\ &+ \left(\widetilde{\alpha}_m + \frac{1}{2}\tilde{A}_m K\right) ({}^\nabla\|u_\varepsilon\|^m_{\Omega_\tau,\,\varepsilon})^2 + \widetilde{\beta}_m ({}^\nabla\|f_\varepsilon\|^{m-1}_{\Omega_\tau,\,\varepsilon})^2 + \widetilde{\gamma}_m \sum_{j=1}^{m-1} \frac{1}{\varepsilon^{2(1+m-j)}} ({}^\nabla\|u_\varepsilon\|^j_{\Omega_\tau,\,\varepsilon})^2. \end{aligned} \tag{5.9}$$

As in the case with $k = 1$, we may use Lemma 4.5 to write

$$({}^\nabla\|u_\varepsilon\|^j_{\Omega_\tau,\,\varepsilon})^2 = \int_{\zeta=0}^\tau ({}^\nabla\|u_\varepsilon\|^j_{S_\zeta,\,\varepsilon})^2\, d\zeta \leq \frac{1}{A'} \int_{\zeta=0}^\tau E^j_{\zeta,\varepsilon}(u_\varepsilon)d\zeta,$$

for $j = 1, \ldots, m$. Substituting these relations into (5.9) yields the inequality (5.1), with $C'_m := \widetilde{\beta}_m$, $C''_m := \widetilde{\gamma}_m/A'$ and $C'''_m := (\widetilde{\alpha}_m + \frac{1}{2}\tilde{A}_m K)/A'$. □

Remark 5.4. As can be seen from the expression for $\nabla_a^\varepsilon T_\varepsilon^{ab,0}(u_\varepsilon)$, there is no estimate of the form (5.1) for $E^0_{\tau,\varepsilon}(u_\varepsilon)$. However, $E^0_{\tau,\varepsilon}(u_\varepsilon)$ is estimated in terms of $E^k_{\tau,\varepsilon}(u_\varepsilon)$, with $k \geq 1$; a fact that is implicit in Proposition 5.1.

6. Auxiliary estimates

In this section, we complement the energy inequalities derived in Section 5 with estimates that allow us to utilise the former in the proof of the main result. In particular, we shall prove that

(i) suitable bounds on the initial data give suitable bounds on the initial energies $E^k_{0,\varepsilon}(u_\varepsilon)$;
(ii) suitable bounds on the energies $E^k_{\tau,\varepsilon}(u_\varepsilon)$ give suitable bounds on the solution u_ε.

The existence as well as the uniqueness part of the proof of the main theorem will then use (i) combined with Corollary 5.3 and (ii) to establish moderateness resp. negligibility of the candidate solution.

Lemma 6.1 (Bounds on initial energies from initial data)**.** *Let u_ε be a solution of (3.3). If $(v_\varepsilon)_\varepsilon$, $(w_\varepsilon)_\varepsilon$ are moderate resp. negligible, then the initial energies $(E^k_{0,\varepsilon}(u_\varepsilon))_\varepsilon$, for each $k \geq 0$, are moderate resp. negligible nets of real numbers.*

Proof. The estimates for the spatial derivatives $\partial_{x^{i_1}} \dots \partial_{x^{i_k}} u_\varepsilon(0, x^i) = \partial_{x^{i_1}} \dots \partial_{x^{i_k}} v_\varepsilon(x^i)$ are immediate. To estimate $\partial_t \partial_{x^{i_1}} \dots \partial_{x^{i_k}} u_\varepsilon(0, x^i)$, we rewrite the initial conditions in equation (3.3) in the form

$$u_\varepsilon(t = 0, x^i) = v_\varepsilon(x^i)$$
$$\partial_t u_\varepsilon(t = 0, x^i) = \tilde{w}_\varepsilon(x^i),$$

where we define $\tilde{w}_\varepsilon := V_\varepsilon w_\varepsilon - N^i_\varepsilon \partial_{x^i} v_\varepsilon$. It is straightforward to show, using the asymptotic estimates (3.4), that $(v_\varepsilon, w_\varepsilon)$ being moderate resp. negligible implies moderateness resp. negligibility of $(v_\varepsilon, \tilde{w}_\varepsilon)$. Therefore moderateness resp. negligibility of $(v_\varepsilon, w_\varepsilon)$ implies moderateness resp. negligibility of $\partial_t \partial_{x^{i_1}} \dots \partial_{x^{i_k}} u_\varepsilon(0, x^i) \equiv \partial_{x^{i_1}} \dots \partial_{x^{i_k}} \tilde{w}_\varepsilon(x^i)$.

The estimates for higher (mixed) time derivatives follow inductively by rewriting the wave equation in the form

$$\partial_t^2 u_\varepsilon = -V_\varepsilon^2 \left(f_\varepsilon + \frac{2}{V_\varepsilon^2} N^i \partial_t \partial_i u_\varepsilon - \left(h^{ij}_\varepsilon - \frac{1}{V_\varepsilon^2} N^i_\varepsilon N^j_\varepsilon \right) \partial_i \partial_j u_\varepsilon + g^{ab}_\varepsilon \Gamma[\mathbf{g}_\varepsilon]^c{}_{ab} \frac{\partial u_\varepsilon}{\partial x^c} \right)$$

and using again the estimates (3.4) for V_ε, N^i_ε, h^{ij}_ε as well as f_ε, v_ε, w_ε. □

Lemma 6.2 (Bounds on solutions from bounds on energies)**.** *For $m > 3/2$ an integer, there exists a constant K and number N such that for all $u \in \mathcal{C}^\infty(\Omega_\tau)$ and for all $\zeta \in [0, \tau]$ we have*

$$\sup_{x \in \Omega_\tau} |\partial_{x^{a_1}} \cdots \partial_{x^{a_l}} u(x)| \le K \varepsilon^{-N} \sup_{0 \le \zeta \le \tau} E^{m+l}_{\zeta,\varepsilon}(u).$$

Remark 6.3. Note that the statement is for all $u \in \mathcal{C}^\infty(\Omega_\tau)$. In the proof of the main theorem, we will apply it to a solution, u_ε, of the wave equation.

Proof of 6.2. First we combine the standard Sobolev embedding theorem on S_τ with the fact that by assumption (A) the metric and hence the volume is $O(1)$ to obtain for $m > 3/2$

$$\sup_{x \in S_\zeta} |u(x)| \le K \; {}^\partial\|u\|^m_{S_\zeta,\,\varepsilon}. \tag{6.1}$$

Then we successively apply (4.5) and (4.3) to obtain

$$\sup_{x \in S_\zeta} |u(x)| \le \varepsilon^{-N} E^m_{\zeta,\varepsilon}(u).$$

Taking the supremum over $\zeta \in [0, \tau]$ on the right hand side gives the result for $l = 0$. To prove the general result, we replace u by the respective derivatives. In some more detail, note that time derivatives are not covered by the Sobolev embedding theorem since they are transversal to S_τ, i.e., we have to replace (6.1) by the estimate

$$\sup_{x \in S_\zeta} |\partial_{\rho_1} \dots \partial_{\rho_k} \partial_t^s u| \le K \; {}^\partial\|\partial_t^s u\|^{m+k}_{S_\zeta,\,\varepsilon} \le K \; {}^\partial\|u\|^{m+k+s}_{S_\zeta,\,\varepsilon},$$

where the last inequality holds because the norm ${}^\partial\| \;\|^m_{S_\zeta,\,\varepsilon}$, in addition, contains time derivatives. □

7. Proof of the main theorem

We finally prove the main result by putting together the estimates achieved so far.

Proof of 3.1.

Step 1: Existence of classical solutions. Due to assumption (C), classical theory provided us with smooth solutions for fixed ε. More precisely, by [6, Theorem 5.3.2], for ε fixed there exists a unique smooth function u_ε solving (3.3) on $A \subseteq \bigcap_{\varepsilon < \varepsilon_0} J^+_\varepsilon(\Sigma)$. Without loss of generality, we may assume that $\Omega_\gamma \subseteq A$.

Step 2: Existence of $\mathcal{G}$-solutions (moderateness estimates). We show that the net $(u_\varepsilon)_\varepsilon$ of Step 1 is moderate on Ω_γ: Moderate data $(v_\varepsilon)_\varepsilon$, $(w_\varepsilon)_\varepsilon$ translate, by means of Lemma 6.1, to moderate initial energies $(E^k_{0,\varepsilon}(u_\varepsilon))_\varepsilon$ for each $k \ge 1$. Moreover, by means of Corollary 5.3, moderate initial energies translate to moderate energies $(E^k_{\tau,\varepsilon}(u_\varepsilon))_\varepsilon$ $(k \ge 1)$ for all $0 \le \tau \le \gamma$. Finally, it follows from Lemma 6.2 that moderate energies $(E^k_{\tau,\varepsilon}(u_\varepsilon))_\varepsilon$ ($k \geq 1$, $0 \leq \tau \leq \gamma$) imply moderateness of $(u_\varepsilon)_\varepsilon$. Hence $u := [(u_\varepsilon)_\varepsilon]$ is a generalised solution on Ω_τ of the i.v.p. (3.3).

Step 3: Uniqueness of $\mathcal{G}$-solutions (negligibility estimates). We are left with showing that the solution u does not depend on the choice of representatives of $(f_\varepsilon)_\varepsilon$, $(v_\varepsilon)_\varepsilon$, $(w_\varepsilon)_\varepsilon$, and $(\mathbf{g}_\varepsilon)_\varepsilon$ of $f = 0$, v, w, and $\mathbf{g}$. Leaving the latter for Step 4, we observe that, to show independence of the choice of representatives of f, v, and w, it suffices to prove that if $(v_\varepsilon)_\varepsilon$ and $(w_\varepsilon)_\varepsilon$ are negligible, then the corresponding solution $(u_\varepsilon)_\varepsilon$ is also negligible. To establish this claim we argue as in Step 2 but using the negligibility parts of Lemma 6.1 and Corollary 5.3. We then observe that negligibility of the energies in Lemma 6.2 implies negligibility of $(u_\varepsilon)_\varepsilon$.

Step 4: Independence of the representative of the metric. We finally prove independence of the solution on representatives $(\mathbf{g}_\varepsilon)_\varepsilon$ of the metric. So let $(\widehat{\mathbf{g}}_\varepsilon)_\varepsilon$ be another representative of $\mathbf{g}$. Denoting the corresponding d'Alembertian by $\widehat{\Box}^\varepsilon$ we consider the initial value problem

$$\begin{aligned} \widehat{\Box}^\varepsilon \hat{u}_\varepsilon &= f_\varepsilon, \\ \hat{u}_\varepsilon(t=0,x^i) &= v_\varepsilon(x^i), \\ \partial_t \hat{u}_\varepsilon(t=0,x^i) &= w_\varepsilon(x^i). \end{aligned} \tag{7.1}$$

By Step 2, there exists a moderate net of solutions $(\hat{u}_\varepsilon)_\varepsilon$ of (7.1), and we only have to show that its difference with the unperturbed solution, $(\widetilde{u}_\varepsilon)_\varepsilon := (u_\varepsilon)_\varepsilon - (\hat{u}_\varepsilon)_\varepsilon$, is negligible on Ω_τ. This difference is a solution of the i.v.p.

$$\begin{aligned} \widehat{\Box}^\varepsilon \widetilde{u}_\varepsilon &= f_\varepsilon - \widehat{\Box}^\varepsilon u_\varepsilon \\ \widetilde{u}_\varepsilon(t=0,x^i) &= 0 \\ \partial_t \widetilde{u}_\varepsilon(t=0,x^i) &= 0. \end{aligned} \tag{7.2}$$

In view of Step 3, we only have to show that $f_\varepsilon - \widehat{\Box}^\varepsilon u_\varepsilon$ is negligible. To this end, we write

$$f_\varepsilon - \widehat{\Box}^\varepsilon u_\varepsilon = (f_\varepsilon - \Box^\varepsilon u_\varepsilon) + (\Box^\varepsilon u_\varepsilon - \hat{\Box}^\varepsilon u_\varepsilon) = \Box^\varepsilon u_\varepsilon - \widehat{\Box}^\varepsilon u_\varepsilon,$$

where we have used the fact that $(u_\varepsilon)_\varepsilon$ solves (3.3). Therefore, the problem is reduced to showing that $(\Box^\varepsilon u_\varepsilon - \widehat{\Box}^\varepsilon u_\varepsilon)_\varepsilon$ is negligible. This, however, is clear since $\Box$ is a well-defined differential operator on $\mathcal{G}$. □

8. Conclusion

We have proved unique local solvability of the wave equation for a large class of metrics of low regularity in the Colombeau algebra of generalised functions, hence establishing $\mathcal{G}$-hyperbolicity of these space-times in the sense of Vickers and Wilson [25]. (This, in itself, is a slight modification of Clarke's notion of generalised hyperbolicity [3], in the sense that we now consider solvability in $\mathcal{G}$ rather than $\mathscr{D}'$.) The essential assumption on this class of metrics is local boundedness: in particular, it includes conical space-times, and therefore generalises the results of Vickers and Wilson [25]. Our class of metrics also includes non-static examples such as impulsive *pp*-waves and expanding spherical impulsive waves.

Finally, we remark that the regularity assumptions (A) and (B) on the metric may be relaxed slightly. Indeed, we can replace the $O(1)$-asymptotics for the zeroth order derivative of the metric in Condition (A) as well as in Condition (B) by the condition that these quantities be $O(\log(1/\varepsilon))$. (This corresponds to generalised Hölder-Zygmund regularity of order zero of the metric as defined in [13].) Under these conditions, the constants A and A'^{-1} in (4.3) of Lemma 4.5 as well as of C'''_k in Proposition 5.1 and Corollary 5.2 have a growth behaviour of $O(\log(1/\varepsilon))$. However, Corollary 5.3 remains unchanged since the $O(\log(1/\varepsilon))$-growth together with Gronwall's lemma still yield moderateness resp. negligibility estimates. Therefore, given a classical metric which we regularise (either by convolution with a mollifier or by any physically motivated procedure) subject to these weaker asymptotic conditions, then our existence and uniqueness result still holds.

From these considerations, we also see that it is hard to imagine how the regularity assumptions for the metric could be further relaxed within our framework.

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