ON SELF-DUAL GRAVITY

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Abstract. We study the Ashtekar-Jacobson-Smolin equations that characterise four dimensional complex metrics with self-dual Riemann tensor. We find that we can characterise any self-dual metric by a function that satisfies a non-linear evolution equation, to which the general solution can be found iteratively. This formal solution depends on two arbitrary functions of three coordinates. We construct explicitly some families of solutions that depend on two free functions of two coordinates, included in which are the multi-centre metrics of Gibbons and Hawking.

1. Introduction

In four dimensions the Hodge duality operation takes two forms to two forms. Given a four dimensional metric, the most important two form associated with it is the curvature two form $R^a{}_b$. It is therefore natural to be interested in four dimensional metrics whose curvature form obeys the self-duality relation

$$R^a{}_b = {}^*R^a{}_b, \tag{1.1}$$

where * is the Hodge duality operator. We will refer to such metrics as "self-dual". Such metrics automatically have vanishing Ricci tensor, and so satisfy the vacuum Einstein equations with vanishing cosmological constant. If we restrict ourselves to real Lorentzian metrics then the only metric with self-dual Riemann tensor is the flat Minkowski metric. Therefore we drop this restriction and consider four dimensional complex metrics.

Physically, these metrics may be of interest in attempts to quantise gravity. It has been suggested that we may be able to interpret self-dual metrics as "one-particle states" in a quantised gravity theory [4]. Alternatively, in the path integral approach to Euclidean quantum gravity, such metrics will make large contributions to a path integral over metrics, since they are saddle points of the classical Einstein-Hilbert action [1]. Also, by analogy with Yang-Mills theory, we can look for "instanton" solutions – complete, non-singular solutions with curvature that dies away at large distances [2, 24]. Solutions that are asymptotically flat at spatial infinity, but periodic in time then contribute to a thermal canonical ensemble [3]. Solutions that are asymptotically flat in the four dimensional sense (asymptotically locally Euclidean), can be interpreted as tunnelling amplitudes between inequivalent gravitational vacua.

From a purely mathematical point of view these metrics are interesting since they are "hyper-Kähler". Hyper-Kähler manifolds are $4n$-dimensional Riemannian manifolds that admit three automorphisms, $\mathbf{J}^i$, of the tangent bundle which obey the quaternion algebra and are covariantly constant [5]. In other words

$$\nabla \mathbf{J}^i = 0, \, \mathbf{J}^i \mathbf{J}^j = -\delta_{ij} + \epsilon_{ijk} \mathbf{J}^k, \tag{1.2}$$

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where ∇ is the covariant derivative with respect to the Levi-Civita connection. In four dimensions, it turns out that for a metric, $\mathbf{g}$, to be hyper-Kähler it must have either self-dual, or anti-self-dual, curvature tensor [6].

The problem of constructing metrics with self-dual curvature tensor has been tackled in several ways. The most direct approach is to formulate the problem in terms of partial differential equations [7, 8]. An alternative approach is Penrose's 'Non-Linear Graviton' technique [4]. Here, the task of solving partial differential equations is replaced by that of constructing deformed twistor spaces, and holomorphic lines on them. Although in principle one can construct the general self-dual metric in this way, it is defined on twistor space and we cannot write down the metric in an explicit manner.

Here we concentrate on partial differential equations. We find a formulation which is similar to Plebański's First Heavenly equation [7], but which can be viewed as simply an evolution equation. This means that the free functions in our solution are just a field and its time derivative on some initial hypersurface i.e. two free functions of three coordinates. We construct, in a somewhat formal manner, the general solution to this equation. We then construct explicitly some infinite dimensional families of solutions to these equations, with a tri-holomorphic Killing vector. We also show that the theory has a sensible Lagrangian and Hamiltonian formulation, and admits an infinite dimensional family of conserved quantities. These quantities, however, are not in involution. Finally, in an appendix, we show how this formulation is equivalent to Plebański's First Heavenly equation.

2. Construction of Self-Duality Condition

In [9] the equations for complex self-dual metrics were reformulated in terms of the new Hamiltonian variables for general relativity introduced in [10]. By fixing the four manifold to be of the form $\mathcal{M} = \Sigma \times \mathbb{R}$ and using the coordinate T to foliate the manifold, they reduced the problem of finding self-dual metrics to that of finding a triad of complex vectors $\{\mathbf{V}_i : i = 1, 2, 3\}$ that satisfy the equations

$$\operatorname{Div} \mathbf{V}_i = 0, \tag{2.1}$$

$$\partial_T \mathbf{V}_i = \tfrac{1}{2}\epsilon_{ijk}\, [\mathbf{V}_j, \mathbf{V}_k]. \tag{2.2}$$

Defining the densitised inverse three metric

$$\hat{q}^{ab} \;=\; V_i^a\, V_j^b\, \delta_{ij}, \tag{2.3}$$

we recover the undensitised inverse three metric q^{ab} by the relation $q^{ab} = \hat{q}\, \hat{q}^{ab}$, where $\hat{q} = \det \hat{q}_{ab} = (\det \hat{q}^{ab})^{-1}$. If we now define the lapse function N by $N = (\det q_{ab})^{1/2}$, then we find that the metric defined by the line element

$$ds^2 = N^2\, dT^2 + q_{ab}\, dx^a\, dx^b \tag{2.4}$$

is self-dual.

Later, it was found that this triad of vectors could be related to the complex structures $\mathbf{J}^i$ that hyper-Kähler metrics admit [11]. Given a self-dual metric, we choose local coordinates (T, x^a) to put the line element in the form of equation (2.4). If we define the triad of vectors $\mathbf{V}_i = -\,\mathbf{J}^i(*, \partial_T)$, then these vectors will satisfy (2.1) and (2.2).

Here we will concentrate on the problem of finding local solutions to equations (2.1) and (2.2). In the usual spirit of gauge theories, we first fix a fiducial background metric on Σ, in terms of which we solve for our self-dual metric. Since we are interested in local properties of the metric, we will assume that locally we can use Cartesian coordinates (X, Y, Z) on Σ with the flat metric $ds^2 = dX^2 + dY^2 + dZ^2$, and the associated flat connection. Thus (2.1) becomes

$$\frac{\partial}{\partial x^a} V_i^a = 0, \tag{2.5}$$

where $\{x^a : a = 1, 2, 3\}$ denote the coordinates (X, Y, Z). The crucial step is to realise that we can write equation (2.2) as

$$[\frac{\partial}{\partial T}, \mathbf{V}_i] = \frac{1}{2}\,\epsilon_{ijk}\,[\mathbf{V}_j, \mathbf{V}_k]. \tag{2.6}$$

If we consider only Euclidean metrics, then we take the $\mathbf{V}_i$ to be real. In this case we define two complex vectors $\mathbf{A}, \mathbf{B}$ by

$$\mathbf{A} \;=\; \frac{\partial}{\partial T} \,+\, i\,\mathbf{V}_1, \qquad \mathbf{B} = \mathbf{V}_2 - i\,\mathbf{V}_3. \tag{2.7}$$

which, by virtue of (2.6), obey the Lie bracket algebra

$$[\mathbf{A}, \mathbf{B}] \,=\, 0, \qquad [\bar{\mathbf{A}}, \bar{\mathbf{B}}] \,=\, 0, \qquad [\mathbf{A}, \bar{\mathbf{A}}] \,+\, [\mathbf{B}, \bar{\mathbf{B}}] \,=\, 0, \tag{2.8}$$

where the bar denotes complex conjugate. We can generalise these equations by considering four complex vectors $\mathbf{U}, \mathbf{V}, \mathbf{W}$ and $\mathbf{X}$ that satisfy the relations

$$[\mathbf{U}, \mathbf{V}] \,=\, 0, \tag{2.9}$$

$$[\mathbf{W}, \mathbf{X}] \,=\, 0, \tag{2.10}$$

$$[\mathbf{U}, \mathbf{W}] \,+\, [\mathbf{V}, \mathbf{X}] \,=\, 0. \tag{2.11}$$

Here we are thinking of $\mathbf{W}$ and $\mathbf{X}$ as "generalised complex conjugates" of $\mathbf{U}$ and $\mathbf{V}$ respectively. By Frobenius' theorem, we can use (2.9) to define a set of coordinates (t, x) on the two (complex) dimensional surface defined by vectors $\mathbf{U}$ and $\mathbf{V}$, and take $\mathbf{U}$ and $\mathbf{V}$ to be

$$\mathbf{U} = \frac{\partial}{\partial t}, \qquad \mathbf{V} = \frac{\partial}{\partial x}. \tag{2.12}$$

We can now foliate our whole space using the coordinates (t, x, y, z). Eq. (2.11) then becomes $\partial_t \mathbf{W} + \partial_x \mathbf{X} = 0$. This means there exists a vector field $\mathbf{Y}$ such that $\mathbf{W} = \partial_x \mathbf{Y}, \mathbf{X} = -\partial_t \mathbf{Y}$. Thus we are only left with the problem of solving for vectors $\mathbf{Y}$ that satisfy $[\partial_t \mathbf{Y}, \partial_x \mathbf{Y}] = 0$[1]. We expand $\mathbf{W}$ and $\mathbf{X}$ as

$$\mathbf{W} = \partial_t + f_x \partial_y + g_x \partial_z, \tag{2.13}$$

$$\mathbf{X} = -f_t \partial_y - g_t \partial_z. \tag{2.14}$$

(The reason for the ∂_t term in $\mathbf{W}$ is, as alluded to above, we are thinking of $\mathbf{W}$ as a sort of complex conjugate of $\mathbf{U} = \partial_t$. Although this argument only seems sensible for t a real coordinate, we are still perfectly at liberty to expand $\mathbf{W}$ in this way if t is complex.) If, by analogy with (2.1), we impose $\frac{\partial}{\partial x^a} W^a = \frac{\partial}{\partial x^a} X^a = 0$, then we find that there exists a function $h(t, x, y, z)$ such that $f = h_z, g = -h_y$. Imposing (2.10), we find that there exists a function $\alpha(t, x)$ such that

$$h_{tt} + h_{xz} h_{ty} - h_{xy} h_{tz} = \alpha(t, x). \tag{2.15}$$

[1]It was only after this work was completed that I learned of reference [12], where the ideas developed so far were found independently. From here onwards, however, our treatments are different.

We can absorb the arbitrary function α into the function h, and conclude that we can form a self-dual metric for any function h that satisfies

$$h_{tt} + h_{xz}h_{ty} - h_{xy}h_{tz} = 0. \tag{2.16}$$

This is just an evolution equation. Thus we can arbitrarily specify data h and h_t on a $t =$ constant hypersurface and propagate it throughout the space according to (2.16) to get a solution. For example, if we expand h around the $t = 0$ hypersurface, and insist that h is regular on this surface, then h is of the form

$$h = a_0\,(x, y, z)\; +\; a_1\,(x, y, z)\, t\; +\; a_2\,(x, y, z)\,\frac{t^2}{2!}\; +\; a_3\,(x, y, z)\,\frac{t^3}{3!}\; + \ldots \tag{2.17}$$

Substituting this into (2.16) shows that a_0 and a_1 are arbitrary functions of x, y and z. a_2, $a_3 \ldots$ are then completely determined for chosen a_0 and a_1 by

$$a_2\; =\; a_{0xy}\, a_{1z}\; -\; a_{0xz}\, a_{1y}, \tag{2.18}$$

$$a_3\; =\; a_{0xy}\, a_{2z}\; -\; a_{0xz}\, a_{2y}\; +\; a_{1xy}\, a_{1z}\; -\; a_{1xz}\, a_{1y}, \tag{2.19}$$

and so on. Thus, in principle, we have a solution that depends on two arbitrary functions of three coordinates. It is interesting to compare our equation (2.16) with Plebański's First Heavenly equation

$$\Omega_{p\tilde{q}}\, \Omega_{\tilde{p}q}\; -\; \Omega_{p\tilde{p}}\, \Omega_{q\tilde{q}}\; =\; 1. \tag{2.20}$$

This is not an equation of Cauchy-Kowalesky form and so it is not obvious what our free data will be. (It is shown in the Appendix how to get equation (2.20) from our equation, showing that the two approaches are equivalent. Thus for *any* self-dual metric there will exist a corresponding function h that satisfies (2.16).)

From the work of [13] we know that the vectors $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$, $\mathbf{X}$ are proportional to a null tetrad that determines a self-dual metric. Indeed the tetrad is given by $\sigma_a = f^{-1}\mathbf{V}_a$, where $\mathbf{V}_a = (\mathbf{U}, \mathbf{V}, \mathbf{W}, \mathbf{X})$ for $a = 0, 1, 2, 3$ and $f^2 = \epsilon\,(\mathbf{U}, \mathbf{V}, \mathbf{W}, \mathbf{X})$, for ϵ the four dimensional volume form $dt \wedge dx \wedge dy \wedge dz$. In our case, $f^2 = -h_{tt}$ and our line element is

$$ds^2\; =\; dt\,(h_{ty}\, dy\; +\; h_{tz}\, dz)\; +\; dx\,(h_{xy}\, dy\; +\; h_{xz}\, dz)\; +\; \frac{1}{h_{tt}}\,(h_{ty}\, dy\; +\; h_{tz}\, dz)^2. \tag{2.21}$$

3. The Formal Solution

We now construct, at least formally, the general solution to (2.16). Instead of working with this equation directly, it is helpful to define two functions $A = h_t$, $B = h_x$, and rewrite (2.16) in the equivalent form

$$A_t + A_y B_z - A_z B_y = 0, \tag{3.1}$$

$$A_x = B_t. \tag{3.2}$$

If we just view B as some arbitrary function, then the solution to (3.1) is

$$A(t, x, y, z)\; =\; \exp\,[\int_0^t\; dt_1\, \zeta(t_1, x, y, z)]\, a_1(x, y, z), \tag{3.3}$$

where $a_1(x, y, z)$ is the value of A at $t = 0$ as in (2.17), and we have defined the vector

$$\zeta(t, x, y, z)\; =\; B_y(t, x, y, z)\, \partial_z\; -\; B_z(t, x, y, z)\, \partial_y. \tag{3.4}$$

The exponential in (3.3) is defined by its power series with the n'th term in this series being

$$\int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \, \zeta(t_1)\, \zeta(t_2) \ldots \zeta(t_n)\, a_1(x,y,z). \tag{3.5}$$

We now must impose (3.2) as a consistency condition on this solution. This tells us

$$\zeta(t) \;=\; \int_0^t dt_1 (A_{xy}(t_1)\, \partial_z - A_{xz}(t_1)\, \partial_y). \tag{3.6}$$

This in turn tells us that

$$A(t) \;=\; \exp\left[\int_0^t dt_1 \int_0^{t_1} dt_2 \, (A_{xy}(t_2)\, \partial_z - A_{xz}(t_2)\, \partial_y)\right] a_1(x,y,z). \tag{3.7}$$

Formally, this equation can now be solved iteratively. We can make successive approximations

$$A^{(0)} \;=\; a_1(x,y,z), \tag{3.8}$$

$$A^{(1)} \;=\; \exp\left[(t\, a_{0xy} \,+\, \frac{t^2}{2} a_{1xy})\, \partial_z \,-\, (t\, a_{0xz} \,+\, \frac{t^2}{2} a_{1xz})\, \partial_y\right] a_1(x,y,z), \tag{3.9}$$

$$A^{(n+1)} \;=\; \exp\,[\int_0^t dt_1 \int_0^{t_1} dt_2 \, (A^{(n)}_{xy}\, \partial_z \,-\, A^{(n)}_{xz}\, \partial_z\,)\,]\, a_1(x,y,z), \;\; \forall n \geq 1. \tag{3.10}$$

Then defining $A \,=\, \lim_{n\to\infty} A^{(n)}$ gives the formal solution for A. (It is beyond our scope here to show that the $A^{(n)}$ actually do converge to a well defined limit.) Integrating A with respect to t and imposing $h(t=0) = a_0(x,y,z)$ then gives us a solution of (2.16).

Finally, we note that (3.1) means that the quantity $A(t,x,\tilde{y},\tilde{z})$ is t independent, where $\tilde{y}$ and $\tilde{z}$ are defined implicitly by

$$\tilde{y}(t) \;=\; y \,+\, \int_0^t dt_1\, B_z(t_1, x, \tilde{y}(t_1), \tilde{z}(t_1)), \tag{3.11}$$

$$\tilde{z}(t) \;=\; z \,-\, \int_0^t dt_1\, B_y(t_1, x, \tilde{y}(t_1), \tilde{z}(t_1)). \tag{3.12}$$

This implies that $A(t,x,y,z) = a_1(x,y',z')$, where the coordinates y', z' are defined by $\tilde{y}(t,x,y',z') = y, \tilde{z}(t,x,y',z') = z$. Thus the dynamics are characterised by a coordinate transformation in the y,z plane.

4. Group Methods and Conserved Quantities

Several powerful techniques have been developed for the study of partial differential equations [14]. One of the most powerful is that of group analysis [15, 16]. By studying the Lie algebra under which a given system of partial differential equations is invariant, we can hopefully find new solutions to these equations. One method of doing so is to look for similarity solutions which are left invariant by the action of some sub-algebra of this symmetry algebra. This will reduce the number of independent variables present in the equation, possibly reducing a partial differential equation to an ordinary differential equation. However, such similarity solutions, by construction, will have some symmetries imposed upon them, so this method is not very useful if one is looking for the general solution to a system of equations.

A more powerful method of obtaining solutions is to exponentiate the infinitesimal action of the Lie algebra into a group action, which takes one solution of the equation to another. However, even if this is possible, it is unlikely that the group action can be used to find the general solution to the equation from any given solution.

Instead of attempting to find the symmetry algebra of (2.16), it is easier to work with the equivalent system (3.1) and (3.2). We find that (3.1) and (3.2) admit a symmetry group defined by the infinitesimal generators

$$\xi_1 = f_A \partial_t - f_x \partial_B, \tag{4.1}$$

$$\xi_2 = (t g_x + B g_A)_A \partial_t + g_A \partial_x - g_x \partial_A - (t g_x + B g_A)_x \partial_B, \tag{4.2}$$

$$\xi_3 = k t \partial_t + k x \partial_x + k y \partial_y, \tag{4.3}$$

$$\xi_4 = l_z \partial_y - l_y \partial_z, \tag{4.4}$$

where f and g arbitrary functions of x and A, l is an arbitrary function of y and z, and k is an arbitrary constant[2]. ξ_3 just generates dilations, whereas ξ_4 generates area preserving diffeomorphisms in the $y-z$ plane. Although ξ_4 gives a representation of W_∞ (modulo cocycle terms) [17], the transformations it generates are just coordinate transformations in the $y-z$ plane. However, we have two interesting symmetries, generated by ξ_1 and ξ_2.

It is possible to exponentiate the action of ξ_1 directly for an arbitrary function f. We find that if $A(t,x,y,z)$ and $B(t,x,y,z)$ are a solution of the system (3.1) and (3.2) then we can implicitly define a new solution, $\tilde{A}$ and $\tilde{B}$, by

$$\tilde{A} = A(t + f_A(x,\tilde{A}), x, y, z), \; \tilde{B} = B(t + f_A(x,\tilde{A}), x, y, z) + f_x(x,\tilde{A}), \tag{4.5}$$

for any function $f(x,A)$. Using this implicit form we can solve iteratively for the functions $\tilde{A}$ and $\tilde{B}$ given functions A, B and f. This means that given one solution of (2.16), we can form an infinite dimensional family of solutions depending on that solution. For a given function g we can also exponentiate the action of ξ_2, although its action cannot be exponentiated directly for a general function g.

Note that although both (4.3) and (4.4) give rise to infinite dimensional families of solutions from any given solution, they are not enough to derive a solution with arbitrary initial data from any given solution.

If we compute the commutators of generators $\xi_1(f_i)$ and $\xi_2(g_j)$ for arbitrary functions f_i and g_j we find that they obey the algebra

$$[\xi_1(f_1), \xi_1(f_2)] = 0, \tag{4.6}$$

$$[\xi_1(f), \xi_2(g)] = \xi_1(f_A g_x - f_x g_A), \tag{4.7}$$

$$[\xi_2(g_1), \xi_2(g_2)] = \xi_2(g_{1A} g_{2x} - g_{1x} g_{2A}). \tag{4.8}$$

If we define a basis for transformations

$${}^{(\alpha)}T_i^m = \xi_\alpha(x^{i+1} A^{m+1}) \tag{4.9}$$

for $\alpha = 1, 2$, where m and i are integers, then the above algebra becomes

$$[{}^{(1)}T_i^m, {}^{(1)}T_j^n] = 0, \tag{4.10}$$

$$[{}^{(1)}T_i^m, {}^{(2)}T_j^n] = ((m+1)(j+1) - (n+1)(i+1))\,{}^{(1)}T_{i+j}^{m+n}, \tag{4.11}$$

$$[{}^{(2)}T_i^m, {}^{(2)}T_j^n] = ((m+1)(j+1) - (n+1)(i+1))\,{}^{(2)}T_{i+j}^{m+n}. \tag{4.12}$$

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[2]Here we are treating t, x, y, z, A and B as independent coordinates on a "jet space" [15]. This means that expressions such as $f_x(x,A)$ should be interpreted as meaning the function $f_y(y,z)$ evaluated at the point $y = x$, $z = A$, as opposed to $\partial_x f(x,A)$, which in this notation would be $f_x(x,A) + f_A(x,A)\,A_x$.

The algebra (4.12) is the algebra of locally area preserving diffeomorphisms in two dimensions which, modulo cocycle terms, is again just the extended conformal algebra W_∞ [17]. Thus (4.10) – (4.12) represents some generalisation of W_∞. These are similar results to those found in [18, 19].

We now note that equation (2.16) can be derived from the Lagrangian

$$S \,=\, \int d^4x\{\frac{1}{2}\,h_t^2\,+\,\frac{1}{3}\,h_t\,(h_y\,h_{xz}\,-\,h_z\,h_{xy})\}. \tag{4.13}$$

The Hamiltonian is then

$$H \,=\, \frac{1}{2}\int_\Sigma d^3x\,(\pi - \frac{1}{3}(h_yh_{xz} - h_zh_{xy}))^2, \tag{4.14}$$

where $\pi = h_t + \frac{1}{3}(h_yh_{xz} - h_zh_{xy})$ is the momentum canonically conjugate to h. We now define the Poisson Bracket of functionals of h and π by

$$\{\alpha(t,\mathbf{x}'),\beta(t,\mathbf{x}'')\} = \int_\Sigma d^3x\left(\frac{\delta\alpha(t,\mathbf{x}')}{\delta h(t,\mathbf{x})}\frac{\delta\beta(t,\mathbf{x}'')}{\delta\pi(t,\mathbf{x})} - \frac{\delta\alpha(t,\mathbf{x}')}{\delta\pi(t,\mathbf{x})}\frac{\delta\beta(t,\mathbf{x}'')}{\delta h(t,\mathbf{x})}\right). \tag{4.15}$$

The algebra (4.10) – (4.12) now reflects the fact that, ignoring surface terms, we have two infinite dimensional families of conserved quantities of the form

$$I_1[f(x,A)] \,=\, \textstyle\int_\Sigma f(x,A)\,d^3x, \tag{4.16}$$

$$I_2[g(x,A)] \,=\, \textstyle\int_\Sigma (t\,g_x(x,A)\,+\,B\,g_A(x,A))\,d^3x. \tag{4.17}$$

The time independence of these quantities follows from the conservation equations

$$\partial_t\,(f)\,+\,\partial_y\,(f\,B_z)\,-\,\partial_z\,(f\,B_y)\,=\,0, \tag{4.18}$$

and

$$\begin{aligned}\partial_t\,(t\,g_x\,+\,B\,g_A)\,-\,\partial_x\,(g)\,+\,\partial_y\,(t\,g_x\,B_z\,+\,g_A\,B\,B_z)\\ -\,\partial_z\,(t\,g_x\,B_y\,+\,g_A\,B\,B_y)\,=\,0,\end{aligned} \tag{4.19}$$

which in turn follow from (3.1) and (3.2). Again this is similar to results in [18, 20] where an infinite hierarchy of conservation laws were constructed for the system. The quantities in (4.16) and (4.17) have Poisson Brackets which are non-vanishing. Explicitly we have

$$\{I_1[f_1],I_1[f_2]\} = \textstyle\int\, d^3x\,f_{2A}\,f_{1AA}\,[A_y\,B_z\,-A_z\,B_y]\,, \tag{4.20}$$

$$\{I_1[f],I_2[g]\} = \textstyle\int\, d^3x\,f_A\,[\partial_x\,g_A\,+\,(B_y\,\partial_z\,-B_z\,\partial_y)(t\,g_{xA}\,+\,B\,g_{AA})]\,, \tag{4.21}$$

and

$$\begin{aligned}\{I_2[g_1],I_2[g_2]\} \;=\; \int d^3x\,(t\,g_{1xA}\,+\,B\,g_{1AA})\,\partial_x\,g_{2A}\,-\,(t\,g_{2xA}\,+\,B\,g_{2AA})\,\partial_x\,g_{1A}\\ +(t\,g_{2xA}\,+\,B\,g_{2AA})(B_z\,\partial_y\,-B_y\,\partial_z)(t\,g_{1xA}\,+\,B\,g_{1AA}).\end{aligned} \tag{4.22}$$

This Poisson Bracket algebra now generates the algebra of symmetry vectors given in equations (4.10) to (4.12). In order to show that the system is classically integrable, we would need to show that we could find a set of conserved quantities that are a complete set of coordinates on the phase space, and which were in involution. This would involve a modification of the Hamiltonian formulation we have given (in order to make the conserved quantities be in involution), but must preserve the underlying equation of motion (2.16). It is not obvious that this is possible, and therefore self-dual gravity may not be integrable in the sense of having action-angle variables.

Nevertheless, the conserved quantities we have constructed seem to be the relationship between this formalism and the twistor approach to this problem [18, 19, 20, 21].

Finally, we note that we can generalise (4.13) and consider the field theory defined by the Lagrangian

$$S = \int d^4x \{ \frac{1}{2} h_t^2 + \alpha h_t (h_y h_{xz} - h_z h_{xy}) \}, \tag{4.23}$$

where α is an arbitrary parameter. Since h now has dimensions of (length)$^{-1}$, we see that the coupling constant α has dimensions of (length)^3. Thus it would seem that if we were to attempt to quantise this system perturbatively, by the standard techniques of non-relativistic field theory, the theory would not be renormalisable[3].

5. Explicit Solutions

We begin by looking for solutions that admit a *tri-holomorphic* Killing vector, ξ. This means the three complex structures $\mathbf{J}^i$ are invariant under the action of ξ, i.e. $\mathcal{L}_\xi \mathbf{J}^i = 0$, where $\mathcal{L}$ is the Lie derivative. Using the relationship between the complex structures and the vectors $\mathbf{V}_i$ given in Section 2 and the fact that ξ is a Killing vector, we see that we require $\mathcal{L}_\xi \mathbf{V}_i = 0$.

If ∂_x is a tri-holomorphic Killing vector, this means that $\partial_x \mathbf{X} = \partial_x \mathbf{W} = 0$, where $\mathbf{X}$ and $\mathbf{W}$ are as in (2.13) and (2.14). This means h is of the form $a(t,y,z) + x\, b(y,z)$ for some functions a and b. In terms of functions $A = h_t$ and $B = h_x$ this means that $A = A(t,y,z)$, $B = B(y,z)$, so (3.2) is automatically satisfied. If we take $A(t=0) = a_1(y,z)$ and $B(t=0) = \phi(y,z)$, it is straightforward to show that the solution to (3.1) is then

$$A(t,y,z) = \exp\{ t (\phi_y \partial_z - \phi_z \partial_y) \} a_1(y,z), \qquad B(y,z) = \phi(y,z). \tag{5.1}$$

For given functions ϕ and a_1 it is straightforward to do the exponentiation [16], giving A explicitly. Using the exponentiated form of (4.1) and (4.2), we could now use these solutions to generate new solutions which had some restricted x-dependent initial data as well.

We can also consider metrics with a tri-holomorphic Killing vector ∂_z. This means we require $\partial_z \mathbf{V}_i = 0$. In this case, we take $h = -tz + g(t,x,y)$. We then recover the result [24, 22] that g must satisfy the three dimensional Laplace equation $g_{tt} + g_{xy} = 0$. The general solution to this is known [23], and can be written in terms of two arbitrary functions $a_0(x,y)$ and $a_1(x,y)$. An almost identical reduction occurs if we take ∂_y as a tri-holomorphic Killing vector. Again, using the symmetries (4.1) and (4.2), we can generate infinite dimensional families of new solutions, that in general have no Killing vectors.

We note in passing that the solution corresponding to the multi-centre Eguchi-Hansen metric [24] is

$$A = -z + \alpha \sum_{i=1}^{s} \operatorname{arcsinh} \left(\frac{(t-t_i)}{2\sqrt{(x-x_i)(y-y_i)}} \right), \tag{5.2}$$

$$B = -\frac{\alpha}{2} \sum_{i=1}^{s} \frac{\sqrt{(t-t_i)^2 + 4(x-x_i)(y-y_i)}}{(x-x_i)}, \tag{5.3}$$

where α is a constant. This is the only metric with a tri-holomorphic Killing vector that has a non-singular real (Euclidean) section [25].

[3] I would like to thank Professor C.J Isham for pointing this out to me.

It would be interesting now to study the case of a holomorphic Killing vector ζ. In terms of the complex structures this is characterised by

$$\mathcal{L}_\zeta \mathbf{J}^1 = 0, \qquad \mathcal{L}_\zeta \mathbf{J}^2 = \mathbf{J}^3, \qquad \mathcal{L}_\zeta \mathbf{J}^3 = -\mathbf{J}^2. \tag{5.4}$$

In terms of the vectors $\mathbf{V}_i$ this means that

$$\mathcal{L}_\zeta \mathbf{V}_1 = 0, \qquad \mathcal{L}_\zeta \mathbf{V}_2 = \mathbf{V}_3, \qquad \mathcal{L}_\zeta \mathbf{V}_3 = -\mathbf{V}_2. \tag{5.5}$$

It should be possible to relate this to the known results on such metrics [26, 27]. It should also be noted that since we have an initial value formulation of the self-duality problem, we can systematically generate multi-centre generalisations of a given metric. For example if a solution has initial data $a_0(x,y,z), a_1(x,y,z)$, then the multi-centre generalisation will have initial data of the form

$$\begin{aligned} a_0'(x,y,z) &= \sum_{i=1}^{s} a_0(x-x_i, y-y_i, z-z_i), \\ a_1'(x,y,z) &= \sum_{i=1}^{s} a_1(x-x_i, y-y_i, z-z_i), \end{aligned}$$

for any points $\{(x_i, y_i, z_i) : i = 1, \ldots, s\}$. It would therefore be of interest to study the Atiyah-Hitchin metric [28] in this formalism, since it would give a systematic way of generating a multi-Atiyah-Hitchin solution which, though known to exist, has not been constructed explicitly.

6. Conclusion

We have shown how, at least formally, to construct the general complex metric with self-dual Riemann tensor. We have also studied the symmetry algebra of the system, which turns out to be a generalised version of $W_\infty \oplus W_\infty$. This leads to an infinite dimensional family of conserved quantities with non-vanishing Poisson brackets. This should be of interest if we were to try and quantise the system [29]. Finally, although we have managed to characterise metrics with a tri-holomorphic Killing vector, it remains to relate the holomorphic Killing vector case to known results.

It should be emphasised that all the considerations here have been inherently *local* in nature, and we have imposed no sorts of boundary conditions on our solutions. If we were to look for metrics that are well defined globally, this would lead us to problems in sheaf cohomology [30], which appear to be best tackled using the twistor formalism [4]. However, the moduli space of conformally self-dual metrics on a given manifold is now reasonably well understood [31, 32].

The techniques developed here have now been used by Plebański et al to reduce the holomorphic Killing vector equation and some special cases of the hyper-heavenly equation to equations of Cauchy-Kowalesky form, and to construct their formal solutions [33].

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Appendix A. The First Heavenly Equation

We now show how our formalism is related to Plebański's [7]. Starting with (3.1) and (3.2), instead of looking on A as a function of t, x, y and z we take A as a coordinate and look on $f \equiv t$ and $g \equiv B$ as functions of $p \equiv A$, $q \equiv x$, $r \equiv y$, $s \equiv z$. This transformation is well defined as long as $A_t \neq 0$. Inverting (3.1) and (3.2) gives

$$f_q \;=\; -\, g_p, \tag{A.1}$$

$$f_r\, g_s \;-\; f_s\, g_r \;=\; 1. \tag{A.2}$$

From (A.1) we deduce that there exists a function $\Omega(p, q, r, s)$ such that $f \;=\; -\,\Omega_p$, $g \;=\; \Omega_q$. Equation (A.2) then means that Ω must satisfy $\Omega_{ps}\,\Omega_{qr} \;-\; \Omega_{pr}\,\Omega_{qs} \;=\; 1$. Carrying out the same transformation on the line element (2.21), we find it becomes $ds^2 \;=\; \Omega_{pr}\, dp\, dr \;+\; \Omega_{ps}\, dp\, ds \;+\; \Omega_{qr}\, dq\, dr \;+\; \Omega_{qs}\, dq\, ds$. Thus we have recovered the Plebański formalism[4]. It would be interesting to see if a similar transformation can be used to turn the problem of conformally self-dual metrics with non-zero cosmological constant into an initial value problem.

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