NULL INJECTIVITY ESTIMATE
UNDER AN UPPER CURVATURE BOUND

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ABSTRACT. We establish a uniform estimate for the injectivity radius of the past null cone of a point in a general Lorentzian manifold foliated by spacelike hypersurfaces and satisfying an upper curvature bound. Precisely, our main assumptions are, on one hand, upper bounds on the null curvature of the spacetime and the lapse function of the foliation, and sup-norm bounds on the deformation tensors of the foliation. Our proof is inspired by techniques from Riemannian geometry, and it should be noted that we impose no restriction on the size of the curvature or deformation tensors, and allow for metrics that are "far" from the Minkowski one. The relevance of our estimate is illustrated with a class of plane-symmetric spacetimes which satisfy our assumptions but admit no uniform lower bound on the curvature not even in the L^2 norm. The geometric conditions identified here provide a uniform control of the geometry of spacetimes, and should be useful in the context of general relativity.

1. INTRODUCTION

In this paper, we consider time-oriented, Lorentzian manifolds satisfying certain geometric bounds and, by suitably adapting techniques from Riemannian geometry, we derive geometric estimates about null cones, that is, the boundary of the past of a point in the manifold. Our main purpose is to investigate the role of a one-sided bound on the curvature, as opposed to integral bounds and, specifically, to establish a uniform lower bound on the injectivity radius of null cones.

In the recent work [15], Klainerman and Rodnianski derived such an estimate for the null injectivity radius of a four-dimensional Ricci-flat Lorentzian manifolds in terms of the L^2 norm of the curvature tensor on spacelike hypersurfaces and additional geometric quantities. Null cones play a central role in the (harmonic) analysis of nonlinear wave equations and having a good control of null cones allows one, for instance, to construct parametrices and tackle the initial value problem, as explained in [15, 16]. Recall also that Chen and LeFloch [9, 17] covered Lorentzian manifolds whose Riemann curvature is bounded above and below. Imposing solely an upper curvature bound, as we do in this paper, raises new conceptual and technical difficulties.

Our aim, throughout, is to identify minimal conditions required to obtain an injectivity estimate, without imposing the Einstein equations. This is important if one wants to cover large classes of spacetimes which need not be vacuum and, additionally, it is important even in the vacuum to identify optimal conditions.

We provide here a technique of proof that uses only a one-sided curvature bound and relies entirely on differential geometric arguments. In particular, we avoid assumptions concerning the existence of coordinate systems in which the metric would be close to the flat metric. We can state the assumptions required for a null injectivity estimate directly in terms of geometrical data, especially an upper bound on the null curvature and a bound on deformation tensors and lapse function, that place our results more in line with standard injectivity radius estimates in Riemannian geometry, such as those of Cheeger [6], Heintze and Karcher [14], and Cheeger, Gromov, and Taylor [8].

Indeed, the present paper builds on an extensive literature in both the Riemannian and the Lorentzian settings, and we especially gained insights from the papers by Ehrlich and co-authors [3, 4, 5, 10]. Recall that sectional curvature bounds in the context of Lorentzian geometry were

studied by Andersson and Howard [2] and allowed them to derive various comparison and rigidity theorems. More recently, Alexander and Bishop [1] have derived triangle comparison theorems for semi-Riemannian manifolds satisfying sectional curvature bounds.

The results in the present paper are relevant to, and provide a set-up for analyzing, the long-time behavior of solutions to Einstein's field equations of general relativity. Our presentation is directly applicable to identify all geometric information required *before* imposing the Einstein equations.

This paper is organized as follows. In Section 2, we begin with some terminology and state the main result established in this paper; cf. Theorem 2.2. In Section 3, we derive a lower bound on the null conjugacy radius of a point, by analyzing Jacobi fields along null geodesics and using, first, an affine parameter and, next, the time parameter of the foliation. Our main estimate about the injectivity radius is proven in Sections 4 and 5. Finally, in Section 6, we exhibit a class of spacetimes satisfying all the bounds assumed in our main theorem but no lower curvature bound.

2. Terminology and main result

2.1. **Lorentzian manifolds endowed with a foliation.** Let (M, g) be a time-oriented, $(n+1)$-dimensional, Lorentzian manifold (without boundary). We write $\langle X, Y \rangle = \langle X, Y \rangle_g := g(X, Y)$ for the scalar product of two vectors X, Y with respect to the metric g.

Fix a point $p \in M$ and suppose there exists a subset, say $M_I \subseteq M$, containing p that is globally hyperbolic, and foliated by the level hypersurfaces of a time function $t: M_I \to I$, which is monotonically increasing towards the future and is onto an interval $I \subseteq \mathbb{R}$. Moreover, denote the level sets of this function by $\mathcal{H}_t$, for $t \in I$, and assume the normalization $I = [-1, 0]$ with $p \in \mathcal{H}_0$.

From the exterior derivative of the function t, we define the **lapse function** of the foliation, $n: M_I \to (0, +\infty)$ by the relation

$$n := (-g(dt, dt))^{-1/2},$$

where $g(dt, dt)$ denotes the $(2, 0)$-version of the metric acting on the one-form dt. (Notationally, we do not distinguish between the $(0, 2)$- and $(2, 0)$-versions of the metric, denoting both by g.) We define the future-directed, unit normal vector field T on M_I to be the unique vector field on M_I determined by the relation

$$g(T, \cdot) = -n dt.$$

Since M_I is geodesically complete and globally hyperbolic, a theorem of Geroch [11] implies that the hypersurfaces $\mathcal{H}_t$ are necessarily diffeomorphic. More specifically in our situation, we may define a diffeomorphism

$$\varphi_t: \mathcal{H}_0 \to \mathcal{H}_t, \qquad t < 0,$$

whose inverse is defined by transporting a point $q \in \mathcal{H}_t$ along the integral curve of T through q to its point of intersection with $\mathcal{H}_0$. The manifold $\mathcal{H}_0$ inherits a one-parameter family of Riemannian metrics $g_t := \varphi_t^*\big(g|_{\mathcal{H}_t}\big)$, $t \in I$, where $g|_{\mathcal{H}_t}$ is the induced Riemannian metric on the hypersurface $\mathcal{H}_t$.

To state our other geometrical assumptions, we require some additional objects associated with the foliation. Firstly, given $p \in M_I$ and the normal vector field T associated with the foliation, we define the *reference Riemannian metric* on M_I by

$$g_T = g + 2g(T, \cdot) \otimes g(T, \cdot).$$

The metric g_T can be used to define inner products and norms on tensor bundles on M_I, which we denote by $\langle \cdot, \cdot \rangle_T$ and $|\cdot|_T$, respectively. An additional geometrical object characterizing the foliation is the **deformation tensor**, which is an element of the space of all symmetric $(0, 2)$-tensor fields on M_I, defined by $\boldsymbol{\pi} := \mathscr{L}_T g$ where $\mathscr{L}$ denotes the Lie derivative operator.

Given arbitrary local coordinates, $\{x^i\}$, on a subset $U \subseteq \mathcal{H}_0$, we may define a transported coordinate system (t, x^i) on the set $I \times U \subseteq M_I$ by translating points in $\mathcal{H}_0$ along the integral curves of the vector field T. In terms of these local coordinates, the Lorentzian metric takes the form

$$g = -n^2\, dt^2 + g_{ij} dx^i dx^j, \tag{2.1}$$

where n is the lapse function and $g_t(x) := \varphi_t^* \big(g_{ij}(t, \varphi_t(x)) dx^i dx^j \big)$, for $x \in \mathcal{H}_0$ gives the induced metric on $\mathcal{H}_0$ in local coordinates. In terms of the transported coordinate system, we have

$$T = \frac{1}{n} \partial_t,$$

and the reference Riemannian metric takes the local form

$$g_T = n^2 dt^2 + g_{ij} dx^i dx^j.$$

Finally, the deformation tensor, $\boldsymbol{\pi}$ has components

$$\boldsymbol{\pi}_{tt} = 0, \qquad \boldsymbol{\pi}_{ti} = \partial_i n, \qquad \boldsymbol{\pi}_{ij} = \frac{1}{n} \partial_t g_{ij} \tag{2.2}$$

with respect to the transported coordinate system.

2.2. **Geometric bounds.** On the initial slice, we assume a lower bound $\iota_0 > 0$ on the **initial injectivity radius** corresponding to the manifold $(\mathcal{H}_0, g_0)$:

Condition (ι_0): $\qquad \mathrm{Inj}(p, \mathcal{H}_0, g_0) \geq \iota_0. \tag{2.3}$

Recall that, for instance in the case where the manifold $\mathcal{H}_0$ is closed (i.e. compact without boundary), a result of Cheeger [6] (see also [14]) provides an estimate for the injectivity radius of $\mathcal{H}_0$ in terms of the diameter, volume, and upper and lower bounds on the sectional curvature of $(\mathcal{H}_0, g_0)$. If $\mathcal{H}_0$ is non-compact, then a result of Cheeger, Gromov, and Taylor [8] provides an injectivity radius estimate under an upper bound on the sectional curvature of g_0 and a lower bound on the volume of metric balls at p.

One assumption we make throughout is that there exists a positive constant $\overline{K}_n$, referred to as the **upper lapse constant**, with the property:

Condition $(\overline{K}_n)$: $\quad n \leq \overline{K}_n \qquad$ in the set $M_I. \tag{2.4}$

We do not require a lower bound on the lapse, a fact that may be of interest in applications; see, for instance, [19].

Next, considering the **past null cone** $\mathcal{N}^-(p)$ from p and, for $t \in I$, we introduce the intersection of this cone with the slice $\mathcal{H}_t$:

$$\mathcal{S}_t := \mathcal{N}^-(p) \cap \mathcal{H}_t.$$

Considering the null cone $N^-(p) \subset T_p M$ in the tangent space at p, we denote the corresponding subset of this cone by

$$\Sigma_t \subset N^-(p) \subset T_p M$$

and their union by

$$N_t^-(p) := \bigcup_{t \leq s \leq 0} \Sigma_s.$$

Note that, for sufficiently small values of t, the sets Σ_t define a foliation of $N_t^-(p)$. Finally, we define the image of the set $\mathcal{S}_t$ in the slice $\mathcal{H}_0$ by

$$S_t := \varphi_t^{-1}(\mathcal{S}_t) \subset \mathcal{H}_0.$$

For sufficiently small $t < 0$, the set S_t is topologically a sphere of dimension $(n-1)$, and the set

$$\mathcal{N}_t^-(p) := \bigcup_{t \leq s \leq 0} S_t \subset \mathcal{H}_0$$

is (topologically) a closed n-dimensional ball.

In addition, we denote by $\mathcal{F}_t^-(p)$ the family of all (restrictions of) t-parametrized radial geodesics $\gamma \colon [0,1] \to \mathcal{N}_t^-(p)$ that originate at p from a null tangent vector in $N_t^-(p)$. We also introduce the bundle $T\mathcal{F}_t^-(p)$ of all tangent vectors to geodesics in $\mathcal{F}_t^-(p)$.

Given a causal vector X, we denote by $Rm_X \colon \{X\}^\perp \to \{X\}^\perp$ the *curvature operator*, regarded as a linear map defined by

$$\langle Rm_X(Y), Y \rangle_g := \langle Rm(X, Y)Y, X \rangle_g, \qquad Y \in \{X\}^\perp.$$

Our main assumption is that the curvature operator is bounded above along null geodesics and, specifically, we impose that there exists a real constant $\overline{K}_{Rm}$, called the **upper null curvature constant of** p, with the property:

Condition $(\overline{K}_{Rm})$:
$$\begin{aligned} \langle Rm_X(Y), Y \rangle_g &\leq \overline{K}_{Rm} \langle Y, Y \rangle_g, \\ X \in T\mathcal{F}_t^-(p), \quad Y &\in \{X\}^\perp. \end{aligned} \tag{2.5}$$

In addition, there exists a constant $K_{\boldsymbol{\pi}} \geq 0$, the **first deformation constant,** such that $\boldsymbol{\pi}$ is bounded with respect to the metric g_T:

Condition $(K_{\boldsymbol{\pi}})$:
$$\begin{aligned} |\boldsymbol{\pi}(V,V)| &\leq K_{\boldsymbol{\pi}}\langle V,V \rangle_T, \\ V \in T_q M, \quad q &\in M_I. \end{aligned} \tag{2.6}$$

The final bound that we require relates to the properties of the null geodesics when projected to the manifold $\mathcal{H}_0$. Given a past-directed null geodesic γ parametrized by the parameter t, we let $x(t) := \varphi_t^{-1}(\gamma(t)) \in \mathcal{H}_0$. We refer to $x(t)$ as the *projection* of the null geodesic γ to $\mathcal{H}_0$. Let ∂_r denote the radial vector field with respect to the metric g_0 from the point p in $\mathcal{H}_0$. Letting ∇ denote the Levi-Civita connection for the metric g_0, our final geometrical bound is about the **second deformation tensor,** $\boldsymbol{\omega}$, defined as the difference between the Levi-Civita connections of the metrics g_t and g_0:

$$\boldsymbol{\omega}(X,Y) := \nabla_X^{g_t} Y - \nabla_X Y, \qquad X, Y \text{ vector fields on } \mathcal{H}_0. \tag{2.7}$$

We assume that there exists a constant $K_{\boldsymbol{\omega}}$, referred to as the **second deformation constant,** with the property:

Condition $(K_{\boldsymbol{\omega}})$:
$$\begin{aligned} |\boldsymbol{\omega}(V,V)| &\leq K_{\boldsymbol{\omega}}\langle V,V \rangle_T, \\ V \text{ spacelike or null, in the set } M_I. \end{aligned} \tag{2.8}$$

We emphasize that there is no assumption that our constants are *small*, merely that the geometrical quantities mentioned above are uniformly bounded. In particular, we have not assumed that our metric is in any sense *close* to the (flat) Minkowski metric, or that the curvature of the metric g is small. For instance, if the metric, and its connection, are changing rapidly along the t-foliation (as in the "bump" example mentioned in Example 5.1, below), the values of the constants would be large.

2.3. **Null injectivity estimate.** Our aim is to estimate the null injectivity radius at p. As noted above, the sets Σ_t, for $t \in I$ define a foliation of the corresponding subset of $N^-(p)$.

Definition 2.1. *Given a point p and a local foliation, the **past null injectivity radius** of p (with respect to the foliation), denoted by*

$$Null\ Inj^-(p),$$

is the supremum of all values $|t|$ such that the exponential map $\exp_p$ is a global diffeomorphism from the pointed null cone $N_t^-(p) \setminus \{0\}$ in $T_p M$ to its image in the manifold.

Theorem 2.2 (Null injectivity estimate). *Fix some positive constants ι_0, $\overline{K}_n$, $\overline{K}_{Rm}$, $K_{\boldsymbol{\pi}}$, and $K_{\boldsymbol{\omega}}$. Let (M, g, p) be a time-oriented, pointed Lorentzian manifold of dimension $(n+1)$ such that, along some foliation defined in a domain M_I, the conditions (ι_0), (K_n), $(\overline{K}_{Rm})$, $(K_{\boldsymbol{\pi}})$, and $(K_{\boldsymbol{\omega}})$ are satisfied. Then there exists a real $\iota > 0$, depending only on the dimension n and the constants above, with the property that the past null injectivity radius of p is bounded below by ι, that is,*

$$Null\ Inj^-(p) \geq \iota.$$

In earlier works, injectivity radius estimates were established under an L^2 curvature bound (Klainerman and Rodnianski [15]) or under a sup-norm bound on the curvature (Chen and LeFloch [9]). The above theorem encompasses spacetimes not covered in these works and for which *no uniform lower bound* on the curvature is available.

Note that $(\overline{K}_{Rm})$ is the only condition involving second-order derivatives of the metric, while the remaining conditions involve zero- or first-order derivatives. Most importantly, in Theorem 2.2, we do not assume a lower bound curvature nor on the lapse function.

To establish the above theorem, we must derive an estimate for the largest value $|t|$, denoted by $|t_1|$, such that for all $|t| < |t_1|$, the set $N_t^-(p) \setminus \{0\}$ is globally diffeomorphic to its image via the exponential map. It follows from [5] that the null exponential map, $\exp_p| N_{t_1}^-(p)$, breaks down as a global diffeomorphism if and only one (or both) of the following possibilities occur:

- There exists a point $q \in \mathcal{H}_{t_1}$ that is conjugate to p along a null geodesic from p to q.
- There exists $q \in \mathcal{H}_{t_1}$ such that there exist distinct null geodesics from p that intersect at the point q.

This situation is quite similar to the situation on a complete Riemannian manifold, where a result of Whitehead states that a geodesic, γ, from a point p ceases to be minimizing at a point q if and only if either q is conjugate to p along γ and/or there exists a distinct geodesic from p to q of the same length as γ.

Let us recall that in Riemannian geometry, there are therefore two key ingredients involved in proving injectivity radius estimates (see, for instance, [6]). Firstly, one derives an estimate that gives a lower bound on the conjugacy radius, i.e. the distance that one must travel along a radial geodesic from a point before one encounters a conjugate point. Such an estimate is usually found by a Rauch comparison argument, and requires an upper bound on the sectional curvature along the geodesics.

The second ingredient, required for an injectivity radius estimate, is a lower bound on the length of the shortest geodesic loop through a point in the manifold (or the shortest closed geodesic, in the case that the manifold is compact). Such an estimate generally requires different geometrical conditions. For example, Cheeger's lower bound on the length of the shortest closed geodesic on a compact manifold [6] (cf. also [14]) requires a lower bound on the sectional curvature and volume of the manifold and an upper bound on the diameter.

We tackle the problem of determining an estimate for the null injectivity radius on a Lorentzian manifold in a similar way.

In the next section, we consider conjugate points along null geodesics. A lower bound on the null conjugacy radius for affinely-parametrized null geodesics is obtained under upper bound on the curvature along null geodesics (following here [12]). We then translate this result in terms of the t-foliation. It is at this point that our assumed bounds on the lapse and second fundamental form of the t-foliation are required.

The second issue, that of intersecting null geodesics, is treated in Sections 4 and 5 in which the strategy of proof is presented.

3. NULL CONJUGACY RADIUS

3.1. **Estimate based on the foliation parameter.**

Definition 3.1. *Given a point p and a local foliation, the **past null conjugacy radius** of p (with respect to the t-foliation) is denoted by*

$$Null\ Conj^-(p),$$

and is the supremum of all values $|t|$ for which the map $\exp_p \colon N_t^-(p) \to \mathcal{N}_t^-(p)$ is a local diffeomorphism onto its image.

Our main result in this section is as follows.

Proposition 3.2 (Conjugacy radius estimate based on the foliation parameter)**.** *Consider the null cone $\mathcal{N}^-(p)$ from a point p and assume that the conditions $(\overline{K}_{Rm})$, $(\overline{K}_n)$, and $(K_{\boldsymbol{\pi}})$ on the curvature operator, lapse, and deformation tensor, respectively (i.e. (2.5), (2.4), and (2.6)) are satisfied. Then, if $\overline{K}_{Rm} \leq 0$, then no null geodesics from p have conjugate points. If $\overline{K}_{Rm} > 0$, there exists a real $\iota = \iota(\overline{K}_{Rm}, \overline{K}_n, K_{\boldsymbol{\pi}}) > 0$ such that no null geodesics from p have conjugate points for t larger than ι, i.e.*

$$Null\ Conj^-(p, T_p) \geq \iota := \begin{cases} \frac{1}{\overline{K}_n K_{\boldsymbol{\pi}}} \log\left(\frac{\sqrt{\overline{K}_{Rm}}}{\sqrt{\overline{K}_{Rm} + \pi K_{\boldsymbol{\pi}}}} \right), & K_{\boldsymbol{\pi}} > 0, \\ -\frac{1}{\overline{K}_n} \frac{\pi}{\sqrt{\overline{K}_{Rm}}}, & K_{\boldsymbol{\pi}} = 0. \end{cases}$$

To prove this result, we must derive from the curvature bound a corresponding bound on the curvature along affinely-parametrized geodesics. We may then apply a result of Harris [12] to estimate the corresponding conjugacy radius along affinely-parametrized geodesics, as explained in Section 3.2, below. Finally, in Section 3.3, we then translate this bound into an estimate for the conjugacy radius with respect to the t-foliation.

3.2. **Estimate based on the affine parameter.** We begin with the following result.

Proposition 3.3 (Conjugacy radius estimate based on the affine parameter). *Let γ be an affinely-parametrized, past directed null geodesic from the point p. Let K be a constant such that, along the geodesic γ, the curvature operator with respect to $\gamma' := \frac{d\gamma}{ds}$ is bounded above by K, i.e.*

$$Rm\left(\frac{d\gamma}{ds}, X, \frac{d\gamma}{ds}, X\right) \leq K\langle X, X\rangle, \qquad X \in \left\{\frac{d\gamma}{ds}\right\}^\perp.$$

If $K \leq 0$, then the point $\gamma(s)$ is not conjugate to p along γ for $s > 0$. If $K > 0$, then the point $\gamma(s)$ is not conjugate to p along γ for $s < \frac{\pi}{\sqrt{K}}$.

In order to prove this proposition, we recall that conjugate points are determined by the differential of the exponential map. This map is non-degenerate at $V \in T_p M$ precisely when all non-trivial Jacobi fields, Y, along the geodesic γ_V with $Y(0) = 0$ are non-vanishing at the point $\exp_p V$. The Jacobi equation is essentially a system of second-order differential equations, and the behavior of its solutions may be controlled by comparison with solutions of model differential equations.

Let γ be a past-oriented, affinely-parametrized, null geodesic from p. We consider an arbitrary Jacobi field Y along γ, satisfying, by definition, the Jacobi equation

$$Y''(s) + Rm\big(Y(s), \gamma'(s)\big)\gamma'(s) = 0,$$
$$Y(0) = 0, \qquad Y'(0) \neq 0,$$

where $Y' := \nabla_{\gamma'} Y$, etc. It follows directly from the Jacobi equation and the condition that $Y(0) = 0$ that

$$\langle \gamma'(s), Y(s)\rangle = \langle \gamma'(0), Y'(0)\rangle s.$$

Therefore, if $\langle \gamma'(0), Y'(0)\rangle \neq 0$, then $\langle \gamma'(s), Y(s)\rangle \neq 0$ for $s > 0$ and hence $Y(s) \neq$ for $s > 0$. Since such a Jacobi field cannot give rise to a conjugate point along γ, we may, without loss of generality from the point of view of detecting conjugate points, restrict our attention to Jacobi fields that are orthogonal to γ'.

Since γ is a null geodesic, the condition that $Y \perp \gamma'$ implies that Y may have a component parallel to γ'. As noted in [3], a Jacobi field parallel to γ leads to the index form along γ having a degeneracy since $I[Y, V] = 0$, for all $V \in V_\perp(c)$. As such, the link between non-definiteness of the index form and the existence of conjugate points is lost. There are two distinct, but essentially equivalent, ways to dealing with this issue:

   a). Uhlenbeck [21], Beem and Ehrlich [3] and Hawking and Ellis [13] consider equivalence classes of Jacobi fields where $Y_1 \sim Y_2$ if $Y_1 - Y_2$ is a multiple of γ'.
   b). Harris [12] impose that Jacobi fields along γ are "nowhere tangential" to γ.

For our purposes, it is more convenient to follow the second approach. Recall that Harris defines a (perpendicular) Jacobi field Y along a null geodesic γ to be *nowhere tangential* to γ if, for any s such that $Y(s) \neq 0$, then $Y(s)$ is not proportional to $\gamma'(s)$. He defines a Jacobi field to be *purely tangential* if the proportionality condition $Y(s) \propto \gamma'(s)$ holds for all s. It is then straightforward to prove, from the uniqueness theorems for second-order ordinary differential equations, that up to the first conjugate point along γ, a perpendicular Jacobi field is either purely tangential or nowhere tangential. The Jacobi equation implies that any purely tangential Jacobi field is of the form $Y(s) = As\gamma'(s)$, where A is a constant, and hence will be non-zero for $s > 0$. As such, purely tangential Jacobi fields do not give rise to conjugate points along the geodesic γ. We are therefore

finally lead to restrict ourselves to non-tangential, perpendicular Jacobi fields along γ. Such a Jacobi field may be written in the form

$$Y(s) = \alpha(s)\gamma'(s) + X(s),$$

where $X(s)$ is a space-like vector field along γ that is orthogonal to γ'. We then note that

$$\langle Y(s), Y(s)\rangle = \langle X(s), X(s)\rangle \geq 0,$$

with equality if and only if $Y(s) = 0$ (since Y is assumed nowhere tangential). As such, there is an induced Euclidean inner product induced on the space of Jacobi fields under consideration. We therefore write

$$|Y(s)| := \sqrt{\langle Y(s), Y(s)\rangle}.$$

Motivated by the study of Jacobi fields on Riemannian manifolds with constant curvature, it is natural to introduce for each $K \in \mathbb{R}$ the real-valued function

$$\Phi_K(t) := \begin{cases} |K|^{-1/2}\,\sinh(|K|^{1/2}\,t), & K < 0, \\ t, & K = 0, \\ K^{-1/2}\,\sin(K^{1/2}\,t), & K > 0, \end{cases} \tag{3.1}$$

which we define for all $t \geq 0$ if $K \leq 0$, and for $t \in [0, \pi\,K^{-1/2}]$ if $K > 0$. Observe that Φ_K thus defined is non-negative on its domain of definition, and satisfies

$$\ddot{\Phi}_K + K\,\Phi_K = 0, \qquad \Phi_K(0) = 0, \quad \dot{\Phi}_K(0) = 1. \tag{3.2}$$

The following lemma provides one with an estimate of the length of Y. Note that Proposition 3.3 follows immediately from this lemma whose proof is adapted from arguments in Riemannian geometry. A (more general) version of this result, with a different proof, appears in Harris [12].

Lemma 3.4 (Jacobi field estimate). *Let γ be a past-directed, affinely-parametrized, null geodesic with $\gamma(0) = p$. Let Y be a nowhere tangential Jacobi field along γ with $Y(0) = 0$ and $Y(s) \perp \gamma'(s)$. Let $K \in \mathbb{R}$ be defined as in Proposition 3.3. Then, the Jacobi field Y satisfies*

$$|Y(s)| \geq |Y'(0)|\,\Phi_K(s), \qquad s > 0. \tag{3.3}$$

In particular, if $K \leq 0$ then for any $s \in [0, 1]$ the point $\gamma(s)$ is not conjugate to p along γ. If $K > 0$ then $\gamma(s)$ is not conjugate to p along γ for any $s < \pi/\sqrt{K}$.

Proof. We first check that

$$\lim_{s \to 0} \frac{d}{ds}|Y(s)| = |Y'(0)|.$$

Namely, with $\varphi(s) := \big|Y(s)\big|$ we can write

$$\begin{aligned} \lim_{s \to 0}(\phi'(s))^2 &= \lim_{s \to 0} \frac{\langle Y(s), Y'(s)\rangle^2}{\langle Y(s), Y(s)\rangle} \\ &= \lim_{s \to 0} \frac{2\,\langle Y(s), Y'(s)\rangle}{2\,\langle Y(s), Y'(s)\rangle}\Big(\langle Y'(s), Y'(s)\rangle^2 - Rm(Y(s), \gamma'(s), Y(s), \gamma'(s))\Big) \\ &= |Y'(0)|^2, \end{aligned}$$

where we have used L'Hôpital's rule and Jacobi field equation.

In addition, we have

$$\frac{1}{2}\frac{d}{ds}|Y(s)|^2 = g(Y(s), Y'(s)),$$

and an application of the Cauchy-Schwarz inequality yields the Kato-type inequality

$$\left|\frac{d}{ds}|Y(s)|\right| \leq |Y'(s)| \tag{3.4}$$

for all s such that $Y(s) \neq 0$.

We now calculate

$$\frac{1}{2}\frac{d^2}{ds^2}\big|Y(s)\big|^2 = \frac{d}{ds}g(Y(s),Y'(s))$$
$$= \big|Y'(s)\big|^2 + g(Y(s),Y''(s)),$$

thus

$$\frac{1}{2}\frac{d^2}{ds^2}\big|Y(s)\big|^2 = \big|Y'(s)\big|^2 - Rm\big(Y(s),\gamma'(s),Y(s),\gamma'(s)\big).$$

Therefore, imposing our curvature assumption, we deduce that

$$\frac{1}{2}\frac{d^2}{ds^2}\big|Y(s)\big|^2 \geq \big|Y'(s)\big|^2 - K\big|Y(s)\big|^2.$$

As above, let $\varphi(s) := \big|Y(s)\big|$, which has the properties that $\varphi(0) = 0$, $\phi'(0) = |Y'(0)|$, which is non-vanishing (and positive) since $Y'(0) \neq 0$. Applying (3.4), we then deduce that

$$\varphi''(s) + K\,\varphi(s) \geq 0$$

at all points where $\varphi(s) \neq 0$. Using the function Φ defined in equation (3.1), we now define the function

$$\psi(s) := \phi'(0)\Phi_K(s),$$

which satisfies

$$\psi''(s) + K\,\psi(s) = 0$$

with initial conditions $\psi(0) = \varphi(0) = 0$, $\psi'(0) = \phi'(0)$. We then deduce that

$$\frac{d}{ds}\left(\phi'(s)\psi(s) - \varphi(s)\psi'(s)\right) \geq 0, \qquad s > 0,$$

so $\phi'(s)\psi(s) - \varphi(s)\psi'(s)$ is non-decreasing for $s > 0$. The initial conditions therefore imply that $\phi'(s)\psi(s) - \varphi(s)\psi'(s) \geq 0$ for $s > 0$. Therefore, we have

$$\frac{d}{ds}\left(\frac{\varphi(s)}{\psi(s)}\right) \geq 0,$$

so the ratio $\varphi(s)/\psi(s)$ is non-decreasing. An application of L'Hôpital's rule implies that the ratio $\varphi(s)/\psi(s)$ tends to 1 at the origin, giving

$$\varphi(s) \geq \psi(s), \qquad t \geq 0.$$

Rewriting this inequality in terms of the Jacobi field Y and the functions Φ, we arrive at the required inequality (3.3). In particular, we note that the first zero of φ cannot occur prior to the first zero of ψ, $\hfill\square$

3.3. **Derivation of the main conjugate radius estimate.** To derive the conjugacy radius estimate with respect to the t-foliation, we require a more detailed analysis of the geodesic equation. Let $\gamma \colon [0,a] \to M$ be an affinely-parametrized, past-directed, null geodesic emanating from the point p. In terms of the local transported coordinate description (2.1), writing the components of the geodesic in the form $s \mapsto (t(s), x^i(s))$, the equations for affinely parametrized geodesics with respect to the metric g take the form

$$\frac{d^2t}{ds^2} + \frac{n_t}{n}\left(\frac{dt}{ds}\right)^2 + 2\frac{n_i}{n}\left(\frac{dt}{ds}\right)\left(\frac{dx^i}{ds}\right) + \frac{1}{2n^2}\left(\partial_t g_{ij}\right)\frac{dx^i}{ds}\frac{dx^j}{ds} = 0, \qquad (3.5a)$$

$$\frac{d^2x^i}{ds^2} + ng^{ij}n_j\left(\frac{dt}{ds}\right)^2 + g^{ij}\left(\partial_t g_{jk}\right)\left(\frac{dt}{ds}\right)\left(\frac{dx^k}{ds}\right) + \Gamma^i{}_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0, \qquad (3.5b)$$

where the coefficients $\Gamma^i{}_{jk}$ are Christoffel symbols. We wish to consider such null geodesics parametrized by the foliation parameter t. As such, we view a null geodesic as a map $I \to M_I \cong I \times \mathcal{H}_0; t \mapsto (t, x(t))$, where $x(t) \in \mathcal{H}_0$ is the spatial projection of the geodesic. In terms of the local coordinates above, we now consider the affine parameter, s, and the components x^i

of the geodesic as functions of t. Re-arranging the above equations, we find that they imply that $s = s(t)$ and $x^i = x^i(t)$ satisfy the equations

$$\frac{\ddot{s}}{\dot{s}} = \frac{n_t}{n} + 2\frac{n_i}{n}\dot{x}^i + \frac{1}{2n^2}\left(\partial_t g_{ij}\right)\dot{x}^i\dot{x}^j, \tag{3.6a}$$

$$\ddot{x}^i + ng^{ij}n_j + g^{ij}\left(\partial_t g_{jk}\right)\dot{x}^k + \Gamma^i{}_{jk}\dot{x}^j\dot{x}^k = \frac{\ddot{s}}{\dot{s}}\dot{x}^i, \tag{3.6b}$$

where a dot $\dot{}$ denotes $\frac{d}{dt}$, and a dash $'$ denotes $\frac{d}{ds}$.

We wish to consider geodesics, parametrized by t, that are, in addition, null. From (2.1), we deduce that such geodesics have the additional property that

$$n(t, x(t))^2 = g_{ij}(t, x(t))\frac{dx^i(t)}{dt}\frac{dx^j(t)}{dt}.$$

Lemma 3.5. *Along past-directed geodesics from p, one has*

$$\frac{d}{dt}\log\left(\frac{1}{n}\left|\frac{ds}{dt}\right|\right) = \frac{1}{2n}\boldsymbol{\pi}\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right) \tag{3.7}$$

and, using the affine parameter,

$$\frac{d}{ds}\left(n\frac{dt}{ds}\right) = -\frac{1}{2}\boldsymbol{\pi}\left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds}\right). \tag{3.8}$$

Proof. The result is local, so we may carry out the calculations in the adapted coordinate system described above. The future-directed, unit normal T takes the form $\frac{1}{n}\partial_t$, whereas the t-parametrized tangent vector takes the form

$$\dot{\gamma} = \frac{d\gamma}{dt} = \partial_t + \frac{dx^i}{dt}\partial_{x^i},$$

and we therefore have

$$\langle T, \dot{\gamma}\rangle = -n.$$

If s is the affine parameter along the geodesic γ, and we denote $\frac{d}{ds}$ by $'$, then

$$\gamma' = \frac{d\gamma}{ds} = \frac{d\gamma}{dt}\left(\frac{ds}{dt}\right)^{-1} = \frac{\dot{\gamma}}{\dot{s}}.$$

We therefore have

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \frac{\ddot{s}}{\dot{s}}\dot{\gamma},$$

and thus

$$\boldsymbol{\pi}(\dot{\gamma}, \dot{\gamma}) = (\mathscr{L}_T g)(\dot{\gamma}, \dot{\gamma}) = 2\langle\nabla_{\dot{\gamma}}T, \dot{\gamma}\rangle = 2\left(\nabla_{\dot{\gamma}}\langle T, \dot{\gamma}\rangle - \langle T, \nabla_{\dot{\gamma}}\dot{\gamma}\rangle\right)$$

$$= 2\left(-\nabla_{\dot{\gamma}}n - \frac{\ddot{s}}{\dot{s}}\langle T, \dot{\gamma}\rangle\right) = 2\left(-\frac{dn}{dt} + n\frac{\ddot{s}}{\dot{s}}\right)$$

$$= 2n\frac{d}{dt}\log\left(\frac{1}{n}\left|\frac{ds}{dt}\right|\right).$$

(Note that, since we are considering past-directed null geodesics, we have $\frac{ds}{dt} < 0$.)

For the second result, we simply observe that

$$\boldsymbol{\pi}(\gamma', \gamma') = 2\langle\nabla_{\gamma'}T, \gamma'\rangle = 2\nabla_{\gamma'}\langle T, \gamma'\rangle = 2\frac{d}{ds}\left(-n\frac{dt}{ds}\right).$$

$\square$

The above lemma allows us to translate between the behavior of geodesics in terms of the affine parameter s, and the time parameter t. We will also require the following observation.

Lemma 3.6. *After a suitable normalization of the affine parameter, one has*

$$\left.\frac{ds}{dt}\right|_{t=0} = -n(p).$$

Proof. Given a past-directed, null vector $L \in T_p M$ such that $g(T, L) = +1$, then s is the affine parameter along the geodesic $\gamma_L : [0, s_L] \to M$ uniquely determined by the condition that $\gamma_L(0) = p$, $\dot{\gamma}_L(0) = L$. Therefore, from the definition of T, we obtain

$$n(p) \left. \frac{dt}{ds} \right|_{t=0} = \langle n(p)dt, \dot{\gamma}_L(0) \rangle = -\langle T(p), L \rangle = -1.$$

$\square$

Lemma 3.7. *Under the conditions $(\overline{K}_n)$ and $(K_{\boldsymbol{\pi}})$ imposed along past-directed, null geodesics from p, one has*

$$- e^{\overline{K}_n K_{\boldsymbol{\pi}} t} \le \frac{1}{n} \frac{ds}{dt} \le -e^{-\overline{K}_n K_{\boldsymbol{\pi}} t} \tag{3.9}$$

and, in terms of the affine parameter,

$$-1 - K_{\boldsymbol{\pi}} s \le \frac{1}{n} \frac{ds}{dt} \le -1 + K_{\boldsymbol{\pi}} s. \tag{3.10}$$

The affine parameter and t-parameter along a past-directed null geodesic satisfy the inequality

$$s(t) \le \frac{1}{K_{\boldsymbol{\pi}}} \left(1 - e^{K_{\boldsymbol{\pi}} \overline{K}_n t} \right). \tag{3.11}$$

Proof. From condition (2.6), we find that

$$\left| \boldsymbol{\pi} \left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right) \right| \le K_{\boldsymbol{\pi}} \left| \frac{d\gamma}{dt} \right|_T^2.$$

Moreover, we have

$$\left| \frac{d\gamma}{dt} \right|_T^2 = n^2 + |\dot{x}(t)|_{g_t}^2 = 2n^2,$$

since the geodesic γ is assumed null and, therefore, from (3.7)

$$\left| \frac{d}{dt} \log \left(\frac{1}{n} \left| \frac{ds}{dt} \right| \right) \right| \le K_{\boldsymbol{\pi}} n \le K_{\boldsymbol{\pi}} \overline{K}_n.$$

Integrating this inequality from t to 0 and recalling that $t < 0$, we find

$$-K_{\boldsymbol{\pi}} \overline{K}_n t \le -\log \left(\frac{1}{n} \left| \frac{ds}{dt} \right| \right) \le K_{\boldsymbol{\pi}} \overline{K}_n t,$$

and re-arranging gives (3.9). The first inequality in (3.9) then yields

$$\frac{ds}{dt} \ge -n e^{\overline{K}_n K_{\boldsymbol{\pi}} t} \ge -\overline{K}_n e^{\overline{K}_n K_{\boldsymbol{\pi}} t}.$$

Integrating this inequality from t to 0 then gives (3.11).

In order to deduce the inequality (3.10), we note that

$$\left| \boldsymbol{\pi} \left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds} \right) \right| \le K_{\boldsymbol{\pi}} \left| \frac{d\gamma}{ds} \right|_T^2 = 2 K_{\boldsymbol{\pi}} n^2 \left(\frac{dt}{ds} \right)^2.$$

Therefore, from (3.7), we deduce that

$$\left| \frac{d}{ds} \left(n \frac{dt}{ds} \right) \right| \le K_{\boldsymbol{\pi}} \left(n \frac{dt}{ds} \right)^2$$

and, hence

$$\left| \frac{d}{ds} \left(n \frac{dt}{ds} \right)^{-1} \right| \le K_{\boldsymbol{\pi}}.$$

Integrating, with the boundary condition (3.6), gives the inequalities (3.10). $\square$

Proof of Proposition 3.2. Let γ be a past-directed, null geodesic from the point p parametrised by the foliation parameter t. Our bound (2.5) on the curvature operator implies that, for all $Y \perp \dot{\gamma}$, we have

$$R\left(\frac{d\gamma}{dt}, Y, \frac{d\gamma}{dt}, Y\right) \leq \overline{K}_{Rm} n^2 \langle Y, Y \rangle.$$

Changing to affine-parametrisation of the geodesic γ, we therefore find that

$$R\left(\frac{d\gamma}{ds}, Y, \frac{d\gamma}{ds}, Y\right) \leq \overline{K}_{Rm} \left(n\frac{dt}{ds}\right)^2 \langle Y, Y \rangle.$$

From the second inequality in (3.10), we therefore deduce that

$$R\left(\frac{d\gamma}{ds}, Y, \frac{d\gamma}{ds}, Y\right) \leq \frac{\overline{K}_{Rm}}{(-1 + K_{\boldsymbol{\pi}} s)^2} \langle Y, Y \rangle.$$

For the moment, we assume that $K_{\boldsymbol{\pi}} > 0$. Let $s_0 = s_0(K_{\boldsymbol{\pi}}) := \frac{1}{K_{\boldsymbol{\pi}}}$, and assume that $0 \leq s \leq s_1$ for fixed $s_1 < s_0$. For $s \in [0, s_1]$, the curvature operator for the affinely parametrised geodesic γ satisfies

$$Rm_{\frac{d\gamma}{ds}} \leq \frac{\overline{K}_{Rm}}{(-1 + K_{\boldsymbol{\pi}} s_1)^2}.$$

If follows, from Proposition 3.3, that for $\overline{K}_{Rm} \leq 0$, the geodesic γ will contain no conjugate points for $s \leq s_1$. If $\overline{K}_{Rm} > 0$, there will be no conjugate points for $s^2 < \pi^2 / K(\overline{K}_{Rm}, K_{\boldsymbol{\pi}}, s_1)$, where

$$K(\overline{K}_{Rm}, K_{\boldsymbol{\pi}}, s_1) := \frac{\overline{K}_{Rm}}{(-1 + K_{\boldsymbol{\pi}} s_1)^2}.$$

If $\pi^2 / K(\overline{K}_{Rm}, K_{\boldsymbol{\pi}}, s_1) > (s_1)^2$, then no conjugate points occur for $s < s_1$. We may therefore repeat our conjugate point estimate with a larger value of s_1. Alternatively, if $\pi^2 / K(\overline{K}_{Rm}, K_{\boldsymbol{\pi}}, s_1) < (s_1)^2$, then a conjugate point occurs for some $s < s_1$. In this case, we should repeat our conjugate point calculations with a smaller value of s_1. The optimal estimate is therefore achieved if we solve for s_1 such that $\pi^2 / K(\overline{K}_{Rm}, K_{\boldsymbol{\pi}}, s_1) = (s_1)^2$. This yields the estimate

$$s_1 = \frac{\pi}{\sqrt{\overline{K}_{Rm}}} \left(1 + K_{\boldsymbol{\pi}} \frac{\pi}{\sqrt{\overline{K}_{Rm}}}\right)^{-1}. \tag{3.12}$$

Note that $s_1 \leq s_0$ since $K_{\boldsymbol{\pi}} > 0$.

As such, our estimates show that a past-directed, affinely-parametrised null geodesic from p will encounter no conjugate points for $s < s_1$, with $s_1 := s_1(\overline{K}_{Rm}, K_{\boldsymbol{\pi}})$ defined as in equation (3.12). We must now translate this condition for the geodesic parametrised by the foliation parameter t. Any conjugate point must occur for a value of s greater than or equal to s_1. It follows from (3.11) that this occurs at a value of t, t_1, such that

$$s_1 \leq \frac{1}{K_{\boldsymbol{\pi}}} \left(1 - e^{K_{\boldsymbol{\pi}} \overline{K}_n t_1}\right).$$

Re-arranging this expression yields the estimate stated in the proposition.

Finally, if $K_{\boldsymbol{\pi}} = 0$, then $Rm_{\frac{d\gamma}{ds}} \leq \overline{K}_{Rm}$. Again, if $\overline{K}_{Rm} \leq 0$, there are no conjugate points and if $\overline{K}_{Rm} > 0$ there are no conjugate points prior to $s_0 = \pi / \sqrt{\overline{K}_{Rm}}$. Applying the inequality (3.11) in the limiting case $K_{\boldsymbol{\pi}} \to 0$ then yields

$$t_1 \leq -\frac{1}{\overline{K}_n} \frac{\pi}{\sqrt{\overline{K}_{Rm}}},$$

which completes the proof of the proposition. $\qquad\square$

4. Geodesic intersections far from the vertex

4.1. **Strategy of proof.** We now concentrate on the case where past-directed null geodesics from p reintersect at a point $q \in \mathcal{H}_{t_0}$ for some $t_0 \in I$ with $t_0 < 0$. We assume that the values of t that we consider are sufficiently small that there are no null conjugate points, and that the null exponential map is therefore a local diffeomorphism. The breakdown of the null exponential map as a global diffeomorphism at t_0 implies that we have distinct null geodesics from p, γ_1 and γ_2 (which we take to be parametrized by the parameter t) such that $\gamma_1(t_0) = \gamma_2(t_0) =: q \in \mathcal{H}_{t_0}$, and that this phenomenon does not happen for any $t > t_0$. By construction, the tangent vectors $\dot{\gamma}_1(0), \dot{\gamma}_2(0) \in T_p M$ are distinct null vectors at p. Following [15, Lemma 3.1], we prove the following result which we observe to be valid for arbitrary metrics.

Lemma 4.1 (Projection of intersecting null geodesics)**.** *The spatial projections with respect to the t-foliation of the null tangent vectors $\dot{\gamma}_1(t_0), \dot{\gamma}_2(t_0) \in T_q M$ are opposite.*

Proof. We first translate this information into our picture on the manifold $\mathcal{H}_0$. The intersection of null geodesics from p on the hypersurface $\mathcal{H}_{t_0}$ implies that the sphere S_{t_0} has a self-intersection at the point $q_0 := \varphi_{t_0}^{-1}(\gamma_1(t_0)) = \varphi_{t_0}^{-1}(\gamma_2(t_0))$. The definition of t_0 implies that the spheres S_t, for $t_0 < t < 0$ have no self-intersection. We consider the projections

$$x_1 := \varphi_t^{-1} \circ \gamma_1 \colon [t_0, 0] \to \mathcal{H}_0, \qquad x_2 := \varphi_t^{-1} \circ \gamma_2 \colon [t_0, 0] \to \mathcal{H}_0.$$

We wish to prove that $\dot{x}_1(t_0) \propto -\dot{x}_2(t_0)$, where the constant of proportionality is positive. Since we have no conjugate points and the spheres S_t do not self-intersect for $t_0 < t$, the S_t are embedded spheres in $\mathcal{H}_0$ while S_{t_0} is immersed but not embedded. As is geometrically clear (and technically follows from a transversality argument [15]) the self-intersection of the sphere S_{t_0} must be tangential. Since the normal vectors (with respect to the metric g_t on $\mathcal{H}_0$) to S_{t_0} at the point of intersection are the tangent vectors $\dot{x}_1(t_0), \dot{x}_2(t_0)$, it follows that these vectors must be proportional. The constant of proportionality cannot be positive, since uniqueness of solutions of the geodesic equations would then imply that $\gamma_1 = \gamma_2$. Moreover, since x_1, x_2 are projections of null geodesics, it follows that $\dot{x}_1$ and $\dot{x}_2$ must be non-vanishing. Hence the constant of proportionality cannot be zero, and therefore must be negative. $\square$

We wish to study the minimal value of t, denoted by t_0, for which the sphere S_t self-intersects. Recall that $\mathrm{inj}(g_0, \mathcal{H}_0, p) =: r_0$ denotes the injectivity radius at the point $p \in \mathcal{H}_0$ with respect to the metric g_0. Then there are two possibilities:

- **Geodesic intersections far from** p**.** A point in S_t leaves the ball $B_{g_0}(p, r_0)$ at or before time t_0, i.e. there exists a null geodesic from p with the property that its projection $\Gamma = \Gamma(t)$ satisfies $d_{g_0}(p, \Gamma(t)) \geq r_0$ for some $t \leq t_0$;
- **Geodesic intersections near** p**.** S_t self-intersects before any point in S_t reaches distance r_0 from p.

We will study the first possibility in the next subsection and, in this case, Proposition 4.2, below, gives a lower bound on t_0. The second possibility is more involved, and is the subject of Section 5.

4.2. **Geodesic intersections far from** p**.** We first note that we may recover the second fundamental form of the hypersurface $\mathcal{H}_t$, $\mathbf{k}_t$, from the spatial projection of the deformation tensor $\boldsymbol{\pi}$. In particular, if X is a vector field on $\mathcal{H}_t$ (and, therefore, $X \perp T$), then we have

$$\boldsymbol{\pi}(X, X) = (\mathscr{L}_T g)(X, X) = 2\langle \nabla_X T, X \rangle =: 2\mathbf{k}_t(X, X). \tag{4.1}$$

It therefore follows that if the deformation tensor satisfies condition $(K_{\boldsymbol{\pi}})$, then we have a corresponding bound on the second fundamental form:

$$|\mathbf{k}_t(X, X)| = \frac{1}{2}|\boldsymbol{\pi}(X, X)| \leq \frac{1}{2}K_{\boldsymbol{\pi}}g_T(X, X) = \frac{1}{2}K_{\boldsymbol{\pi}}g_t(X, X). \tag{4.2}$$

The following is the main result of this subsection.

Proposition 4.2. *Suppose that the foliation satisfy the conditions (ι_0), $(\overline{K}_n)$, and $(K_{\boldsymbol{\pi}})$. If S_t does not intersect the cut locus of $p \in \mathcal{H}_0$, that is, $S_t \cap \mathrm{Cut}_{g_0}(p) \neq \emptyset$ then one has*

$$|t| \geq \begin{cases} \frac{2}{\overline{K}_n K_{\boldsymbol{\pi}}} \log\left(1 + \frac{1}{2}K_{\boldsymbol{\pi}}\iota_0\right), & K_{\boldsymbol{\pi}} \neq 0, \\ \frac{\iota_0}{\overline{K}_n}, & K_{\boldsymbol{\pi}} = 0. \end{cases}$$

Hence, Proposition 4.2 provides us with a lower bound on the value of t for which the sphere S_t leaves $B_{g_0}(p, \iota_0)$. To establish this proposition, we require some estimates for length of the spatial projection of a null geodesic.

Lemma 4.3. *For $-1 \leq s \leq t \leq 0$, one has*

$$g_t \leq e^{\overline{K}_n K_{\boldsymbol{\pi}}(t-s)} g_s. \tag{4.3}$$

Proof. We first note that, in the transported coordinate system, we have

$$\mathbf{k}_t := \frac{1}{2n}\partial_t g_t.$$

If X is a smooth vector field on $\mathcal{H}_0$ (independent of t), then we have

$$\partial_t\left(g_t(X,X)\right) = 2n(t)\mathbf{k}_t(X,X)$$
$$\leq n(t)K_{\boldsymbol{\pi}}g_t(X,X) \leq \overline{K}_n K_{\boldsymbol{\pi}}g_t(X,X).$$

Integrating from s to t, we therefore have

$$g_t(X,X) \leq e^{\overline{K}_n K_{\boldsymbol{\pi}}(t-s)} g_s(X,X).$$

This inequality holds for all X, so we deduce the inequality (4.3). $\qquad\square$

Given a null geodesics $\gamma\colon [-1,0] \to M$, we wish to consider the length of its spatial projection Γ with respect to the metric g_0.

Lemma 4.4. *For $-1 \leq t \leq 0$ one has*

$$L_{g_0}[\Gamma|_{[0,t]}] \leq \begin{cases} \frac{2}{K_{\boldsymbol{\pi}}}\left(-1 + e^{-\frac{1}{2}\overline{K}_n K_{\boldsymbol{\pi}} t}\right), & K_{\boldsymbol{\pi}} > 0, \\ \overline{K}_n|t|, & K_{\boldsymbol{\pi}} = 0. \end{cases} \tag{4.4}$$

Proof. It is sufficient to treat the case $K_{\boldsymbol{\pi}} > 0$, and we then have

$$L_{g_0}[\Gamma|_{[0,t]}] = \int_t^0 |\dot{\mathbf{x}}(u)|_{g_0}\, du \leq \int_t^0 e^{-\frac{1}{2}\overline{K}_n K_{\boldsymbol{\pi}} u} |\dot{\mathbf{x}}(u)|_{g_u}\, du$$
$$= \int_0^t e^{-\frac{1}{2}\overline{K}_n K_{\boldsymbol{\pi}} u}\, n(u, x(u))\, du \leq \overline{K}_n \int_t^0 e^{-\frac{1}{2}\overline{K}_n K_{\boldsymbol{\pi}} u}\, du$$
$$= \frac{2}{K_{\boldsymbol{\pi}}}\left(-1 + e^{-\frac{1}{2}\overline{K}_n K_{\boldsymbol{\pi}} t}\right),$$

as required. $\qquad\square$

Corollary 4.5. *Given any null geodesic and its projection*

$$\gamma\colon [-1,0] \to M_I, \qquad \Gamma\colon [-1,0] \to \mathcal{H}_0,$$

then, for $-1 \leq t \leq 0$, one has

$$d_{g_0}(p, \Gamma(t)) \leq \begin{cases} \overline{K}_n|t|, & K_{\boldsymbol{\pi}} = 0, \\ \frac{2}{K_{\boldsymbol{\pi}}}\left(-1 + e^{-\frac{1}{2}\overline{K}_n K_{\boldsymbol{\pi}} t}\right), & K_{\boldsymbol{\pi}} \neq 0. \end{cases} \tag{4.5}$$

Remark 4.6. It follows from the above results that, if $\gamma_1, \gamma_2\colon [t_0, 0] \to M_I$ are null geodesics that intersect at p and again in the surface $\mathcal{H}_{t_0}$ for some t_0, then the lengths of their spatial projections $\Gamma_1, \Gamma_2\colon [0, t_0] \to \mathcal{H}_0$ satisfy (4.4). The closed loop at p defined by concatenating these curves, which we denote by Γ, therefore satisfies the inequality

$$L_{g_{t_0}}[\Gamma] \leq \frac{4}{K_{\boldsymbol{\pi}}}\left(-1 + e^{-\frac{1}{2}\overline{K}_n K_{\boldsymbol{\pi}} t}\right).$$

Proof of Proposition 4.2. If $S_t \cap \mathrm{Cut}_{g_0}(p) \neq \emptyset$ then there exists a null geodesic γ such that its projection, Γ, satisfies

$$i_0 \leq d_{g_0}(p, \Gamma(t)),$$

and the result then follows from Corollary 4.5.   □

5. Geodesic intersections near the vertex

5.1. **Tangent space calculations.** We now consider the case where there exists $t_0 < 0$ such that the spheres S_t have no self-intersections for $t_0 < t < 0$, the sphere S_{t_0} has self-intersections, and the spheres $S_t, 0 \leq t \leq t_0$ are contained in the ball of center p and radius ι_0.

Since we remain within the injectivity radius (with respect to the metric g_0) at p, we may use the exponential map (with respect to g_0) to define spheres

$$\Sigma_t := \left(\exp_p^{g_0}\right)^{-1}(S_t) \subset T_p \mathcal{H}_0.$$

Since $\exp_p$ is a global diffeomorphism from $B(0, \iota_0)$ to $B(p, \iota_0)$, a sphere S_t will have self-intersections if and only if the sphere Σ_t has self-intersections.

In their approach to the null injectivity radius problem [15], where the metric g is shown to be ϵ-close to the Minkowski metric in a particular local coordinate system, Klainerman and Rodnianski argue that the intersection of null geodesics that we are considering cannot occur within the local coordinate chart. Intuitively, it seems clear that, in order for the light cone to become so distorted that it self-intersects in the required fashion, we would require a significant amount of curvature in our manifold, and hence the metric cannot be assumed globally close to the Minkowski metric. The following example shows, however, the phenomenon that we are considering *cannot* be ruled out, in general.

Example 5.1 (Growing bump metric). *Let $\mathcal{H} = \mathbb{R}^2$, with g_0 the (flat) Euclidean metric. Therefore the injectivity radius of the initial slice is $+\infty$, so any intersections of the S_t that happen will occur before they intersect the cut locus of p with respect to g_0. We evolve the metric for $t > 0$ so that it gains a bump, of height t say, with g_t being the induced metric from flat $\mathbb{R}^3$. If we pick a point away from the bump, then the geodesic balls will start out round but, once they hit the bump (which is growing with t), they will bend around the bump. The geodesic balls will then intersect at the back of the bump for sufficiently large t. If we project to the surface $\mathcal{H}_0$, then the arrangement of the spheres S_t is a family of nested spheres developing a self-intersection.*

Note that, although we have assumed $\mathcal{H}_0$ to be $\mathbb{R}^2$, it is clear that this argument may be suitably localised in order to make $\mathcal{H}_0$ compact, and to any dimension greater than or equal to 2.

One of the notable features of this example is that the metric g_t is evolving with t, so the spatial geometry is undergoing significant change.

Having established that a self-intersection of a sphere S_t can occur, the main result of this section is that we can find an explicit lower bound on the corresponding value of t. We first must find a condition that is necessary for the intersection of null geodesics, for which we can then develop an estimate. Although the criterion that we will use is not optimal, it does fulfil this requirement.

Let $\gamma \colon [0, a] \to M$ be a past-directed null geodesic with $\gamma(0) = p$, with $\gamma(s) = (t(s), x(s))$, where $x(s) \in \mathcal{H}_0$ (for $s \in [0, a]$) is its projection. The equations that t and x must satisfy in an arbitrary transported local coordinate system are given in equations (3.5).

Generically, for sufficiently negative values of t, there may exist values of t for which distinct null geodesics from p intersect on the hypersurface $\mathcal{H}_t$. Let $t_* < 0$ denote the largest (i.e. least negative) value of t for which there exist distinct null geodesics from p, γ_1 and γ_2, such that $\gamma_1(t_*) = \gamma_2(t_*) \in \mathcal{H}_{t_*}$. We denote this point of intersection by q. Lemma 4.1 shows that the spatial projection with respect to the t-foliation of the tangent vectors $\dot\gamma_1(t_*)$ and $\dot\gamma_2(t_*)$ at q are opposite. Denoting the projections of these geodesics to $\mathcal{H}_0$ by x_1 and x_2, we therefore deduce that $\dot{x}_1(t_*) \propto -\dot{x}_2(t_*)$, where the constant of proportionality is positive.

Let ∂_r denote the unit radial vector field with respect to the metric g_0 on $\mathcal{H}_0$. (Our notation in this regard follows [20].) This is a smooth vector field on $\mathcal{H} \setminus (\{p\} \cup \mathrm{Cut}(p))$ and, hence, is smooth on $B_{g_0}(p, i_0) \setminus \{p\}$.

We then have the following preliminary result.

Lemma 5.2. *One of the inner products* $\langle \dot{x}_1(t_*), \partial_r \rangle_{g_0}, \langle \dot{x}_2(t_*), \partial_r \rangle_{g_0}$ *is non-negative.*

Proof. From the fact that $\dot{x}_1(t_*) \propto -\dot{x}_2(t_*)$, we have

$$\langle \dot{x}_1(t_*), \partial_r \rangle_{g_0} \propto -\langle \dot{x}_2(t_*), \partial_r \rangle_{g_0},$$

where the constant of proportionality is positive. □

We may therefore derive an upper bound on the possible value of t_* by finding an upper bound on the value of t for which the spatial projection of null geodesics satisfies $\langle \dot{x}(t), \partial_r \rangle_{g_0} \geq 0$. For calculational simplicity, what we will work with affinely-parametrized geodesics. Since $\frac{ds}{dt} < 0$ for past-directed null geodesics, what we will derive is a lower bound for the first value of s, s_*, for which the tangent vector $\frac{d\gamma}{ds}$ satisfies

$$\left\langle \frac{dx}{ds}(s), \partial_r \right\rangle_{g_0} \leq 0. \tag{5.1}$$

We will then turn this estimate into an estimate for the corresponding value of t using the same method as we used for the conjugate point estimate.

Therefore, let $x \colon [0, a] \to \mathcal{H}_0$ be the projection to $\mathcal{H}_0$ of an affinely-parametrized null geodesic, with affine parameter s. Therefore $x(0) = p \in \mathcal{H}_0$ and, by assumption, $x(s)$ remains within the ball of center p and radius $i_0 := \iota_0$, i.e. $x(s) \in B_{g_0}(p, i_0)$ for $s \in [0, a]$. As such, there exists a unique, affinely-parametrized, radial geodesic (with respect to g_0) $\gamma_s \colon [0, 1] \to \mathcal{H}_0$, from p to $x(s)$. This geodesic has the property that

$$d_{g_0}(p, x(s)) = L_{g_0}[\gamma_s] = \int_0^1 \sqrt{g_0 \left(\frac{d\gamma_s(u)}{du}, \frac{d\gamma_s(u)}{du} \right)} \, du.$$

Let ∇ denote the Levi-Civita connection defined by the metric g_0 and, as before, let $'$ denote $\frac{d}{ds}$. We then have the following result.

Lemma 5.3. *One has*

$$\frac{d}{ds} d_{g_0}(p, x(s)) = \left\langle \partial_r, \frac{dx}{ds}(s) \right\rangle_{g_0},$$

$$\frac{d^2}{ds^2} d(p, x(s)) \geq \langle \partial_r, \nabla_{x'(s)} x'(s) \rangle_{g_0},$$

and also

$$\left. \frac{d}{ds} d_{g_0}(p, x(s)) \right|_{s=0} = 1.$$

Proof. The first equality follows directly from the first variation of arc-length formula. Given a geodesic γ in a manifold and an orthogonal vector field, W, along γ, then the index form is given by (see, e.g., [7])

$$I[W] := \int_0^1 \left(|\nabla_{\gamma'} W|^2 - \langle \gamma', \mathbf{R}(W, \gamma') W \rangle \right) \, ds$$

Recall that the index form along a geodesic is positive-definite prior to the first conjugate point along the geodesic. We define the variation through geodesics $\alpha \colon [0, 1] \times [0, a] \to \mathcal{H}_0$ with $(u, s) \to \gamma_s(u)$, and the tangent vectors $T := \alpha_*(\partial_u), V := \alpha_*(\partial_s)$. The second variation of arc-length formula then states that

$$\frac{d^2}{ds^2} d(p, x(s)) = \left[\frac{1}{|T|} \langle T, \nabla_V V \rangle_{g_0} \right]_{u=0}^{u=1} + \frac{1}{|T|} I[V^\perp],$$

where $V^\perp$ denotes the part of the vector field V along γ_s that is orthogonal (with respect to g_0) to T.

We now apply this formula to our situation. Since $\gamma_s(0) = p$, for each s, we deduce that $V(u = 0) = 0$. $V^\perp$ is therefore a vector field along γ_s that vanishes at p, and is equal to $x'(s)$ at the point $x(s)$. A standard index argument (cf. again [7]) implies that $I[V^\perp] \geq I[\mathbf{J}]$, where $\mathbf{J}$

is the Jacobi vector field along γ_s such that $\mathbf{J}(p) = 0$ and $\mathbf{J}(x(s)) = V^{\perp}(x(s))$. By assumption, there are no points conjugate to p along the geodesics γ_s (since we are working within $B_{g_0}(p, i_0)$). Therefore the index form is positive definite along the geodesic γ_s, so $I[\mathbf{J}] > 0$. We therefore have

$$\frac{d^2}{ds^2} d(p, x(s)) \geq \left\langle \partial_r, \nabla_{x'(s)} x'(s) \right\rangle_{g_0},$$

as required.

Finally, note that $x'(0)$ is an outward-directed, radial vector at the point p. Therefore, as $s \to 0$, the inner product $\left\langle \partial_r, \frac{dx}{ds}(s) \right\rangle_{g_0}$ converges to $\left| \frac{dx}{ds}(0) \right|_{g_0}$. We then have

$$\left| \frac{dx}{ds} \right|_{g_0} \bigg|_{s=0} = \frac{dt}{ds} \bigg|_{s=0} \left| \frac{dx}{dt} \right|_{g_0} \bigg|_{t=0} = \left| \frac{ds}{dt} \right|^{-1} \bigg|_{t=0} n(p) = 1,$$

by Lemma 3.6.   □

We now impose, for some constant $\underline{K}_{acc.}$, the following **radial acceleration condition:**

$$(\underline{K}_{acc.}) \qquad \left\langle \partial_r, \nabla_{x'(s)} x'(s) \right\rangle_{g_0} \geq \underline{K}_{acc.} \tag{5.2}$$

along null geodesics x'.

Lemma 5.4. *Under the assumed conditions, any pair of past-directed, null geodesics from p will not intersect for $0 < s < s_*$, where*

$$s_* = s_*(\underline{K}_{acc.}) = \begin{cases} -\frac{1}{\underline{K}_{acc.}}, & \underline{K}_{acc.} < 0, \\ +\infty, & \underline{K}_{acc.} \geq 0. \end{cases}$$

Proof. From Lemma 5.3 we deduce that

$$\frac{d}{ds} d_{g_0}(p, x(s)) \geq 1 + \underline{K}_{acc.} s.$$

Therefore, $\frac{d}{ds} d_{g_0}(p, x(s)) > 0$ for $s < s_*$, where s_* is as stated in the lemma. Lemma 5.2 and the following discussion then complete the proof.   □

We are in a position to establish the main result of this section.

Proposition 5.5. *Under the conditions $(\overline{K}_n)$, $(K_{\boldsymbol{\pi}})$, $(\underline{K}_{acc.})$, any pair of past-directed, null geodesics from p do not intersect for $t_0 < t < 0$, where*

$$t_0 = t_0(\overline{K}_n, K_\pi, \underline{K}_{acc.}) := \frac{1}{\overline{K}_n K_\pi} \log \left(1 + \frac{K_\pi}{\underline{K}_{acc.}} \right).$$

Proof of Proposition 5.5. It is now sufficient to apply the same technique as employed in the proof of Proposition 3.2 to translate the affine parameter estimate in Lemma 5.4 to an estimate for the corresponding value of t.   □

5.2. **The radial acceleration condition.** We now discuss the nature of the condition (5.2), where we have assumed a lower bound on the radial acceleration. In the transported coordinate system, the $\nabla_{x'(s)} x'(s)$ term takes the form

$$\left(\nabla_{x'(s)} x'(s) \right)^i$$
$$= \frac{d^2}{ds^2} x^i(s) + \Gamma^i_{jk}(0, x(s)) \frac{dx^j}{ds} \frac{dx^k}{ds}$$
$$= \left(\Gamma^i_{jk}(0, x(s)) - \Gamma^i_{jk}(t(s), x(s)) \right) \frac{dx^j}{ds} \frac{dx^k}{ds} - n \frac{dt}{ds} g^{ij} \left(\boldsymbol{\pi}_{jt} \frac{dt}{ds} + \boldsymbol{\pi}_{jk} \frac{dx^k}{ds} \right),$$

where we have used the fact that $x(s)$ is the projection of a null geodesic in the manifold M. Recall, from Section 2.2, that we have defined the $(1, 2)$ tensor field $\boldsymbol{\omega}$ on $\mathcal{H}_0$ as the difference between the Levi-Civita connections of the metrics g_t and g_0. It then follows that

$$\left\langle \partial_r, \nabla_{x'(s)} x'(s) \right\rangle_{g_0} = -\boldsymbol{\alpha}(t) \left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds} \right),$$

where
$$\boldsymbol{\alpha}(t)\,(V,V)$$
$$:= \langle \partial_r, n\,\mathrm{grad}_{g_t}\,n \rangle_{g_0} \left(V^t\right)^2 + 2nV^t \langle \partial_r, \mathbf{k}(V^\perp) \rangle_{g_0} + \langle \partial_r, \boldsymbol{\omega}(V^\perp, V^\perp) \rangle_{g_0}. \tag{5.3}$$

Our radial acceleration condition is therefore equivalent to a bound of the form
$$-\boldsymbol{\alpha}(t)\,(\gamma', \gamma') \geq \underline{K}_{acc.}$$

along the null geodesics γ. It is clear, from the explicit form of $\boldsymbol{\alpha}$ given in (5.3), that such a bound follows from our assumptions $(K_{\boldsymbol{\pi}})$ and $(K_{\boldsymbol{\omega}})$.

The tensor $\boldsymbol{\omega}$ itself may be estimated by calculating the difference between the Christoffel symbols of the metric g_t and g_0.

$$\boldsymbol{\omega}^i{}_{jk}(t, x(t)) = \Gamma^i{}_{jk}(t, x(t)) - \Gamma^i{}_{jk}(0, x(t)) = \int_0^t \left(\partial_1 \Gamma^i{}_{jk}\right)(u, x(t))\,du.$$

The derivative in the integrand may be written in the form

$$\partial_t \Gamma^i{}_{jk} = R_{tj}{}^i{}_k + \frac{1}{2}\nabla_j \left(n\boldsymbol{\pi}^i{}_k\right) + \frac{1}{2}\left(\boldsymbol{\pi}_{tk}\boldsymbol{\pi}^i{}_j - \boldsymbol{\pi}_t{}^i \boldsymbol{\pi}_{jk}\right).$$

As such, the $\boldsymbol{\omega}$ term could be bounded by assuming additional bounds on the $R_{ti}{}^j{}_k$ parts of the curvature, on the spatial derivative of the $n\boldsymbol{\pi}_{ij}$ (i.e. the spatial derivative of $\partial_t g_{ij}$) and additional bounds on the deformation tensor.

It is clear that the radial acceleration condition, or a condition of a similar type, is required here. The crux of our argument is that the projection of the null geodesics in M to the hypersurface $\mathcal{H}_0$ are not geodesics on $\mathcal{H}_0$. In order to estimate the deformation of the spheres, S_t, we need to control how much these projections deviate from geodesics with respect to g_0. The radial acceleration (or equivalently, the form $\boldsymbol{\alpha}$) is the most direct way of measuring this deviation.

6. A CLASS OF SPACETIMES WITH CURVATURE BOUNDED ABOVE

The conditions assumed in our main theorem in this paper are satisfied by a large class of spacetimes.

Proposition 6.1 (Family of spacetimes with curvature unbounded below)**.** *Fix some positive constants ι_0, $\overline{K}_n$, $\overline{K}_{Rm}$, $K_{\boldsymbol{\pi}}$, and $K_{\boldsymbol{\omega}}$. There exists a family of spacetimes satisfying all of the assumptions in Theorem 2.2, but whose curvature operator is not uniformly bounded below in terms of the given constants, even in the L^2 norm.*

Proof. We search for the desired spacetimes in the class of plane wave solutions. The standard four-dimensional plane wave metric takes the form

$$g = -2\,dudv + 2H(u, x, y)du^2 + dx^2 + dy^2,$$

where u, v, x, y are local coordinates. With respect to the null coframe

$$e^1 = du, \qquad e^2 = dv - H(u, x, y)du, \qquad e^3 = dx, \qquad e^4 = dy,$$

in terms of which

$$g = -e^1 \otimes e^2 - e^2 \otimes e^1 + e^3 \otimes e^3 + e^4 \otimes e^4,$$

we find that the non-vanishing components of the curvature tensor are

$$R_{1313} = -H_{xx}, \qquad R_{1314} = -H_{xy}, \qquad R_{1414} = -H_{yy}. \tag{6.1}$$

If we wish to consider the curvature operator, then we must consider the curvature quantity $R(X, Y, X, Y)$, where X is a null vector and Y is orthogonal to X. Letting $\Sigma^{ij} := X^i Y^j - X^j Y^i$ then, from the explicit formula for the curvature components above, we find that

$$R(X, Y, X, Y) = -H_{xx}\left(\Sigma^{13}\right)^2 - 2H_{xy}\Sigma^{13}\Sigma^{14} - H_{yy}\left(\Sigma^{14}\right)^2. \tag{6.2}$$

In particular, if we impose that the Hessian (in the variables x, y) of the function H is positive semi-definite,

$$H_{xx} \geq 0, \qquad H_{xx}H_{yy} - H_{xy}^2 \geq 0, \tag{6.3}$$

then we deduce that the curvature operator Rm_X is non-positive

$$Rm_X \leq 0.$$

The additional geometrical conditions required for our theorems require only the first derivatives of the metric. As such, these conditions can be satisfied by choosing H to have bounded first derivatives.

It is clear, however, from (6.2) that we can make the curvature arbitrarily negative by letting (for example) H_{xx} and H_{yy} become arbitrarily large and positive.

A special case of this construction occurs if we take the function H to be of the special form $H(x, y, u) = a(x, u) + b(y, u)$. The Hessian condition (6.3) is satisfied if $a_{xx} \geq 0$ and $b_{yy} \geq 0$, i.e. the functions a and b are convex in x and y, respectively. We may choose a and b to have small first derivatives (in order to satisfy the conditions of Theorem 2.2), but such that there exist points at which a_{xx} and b_{yy} are large and positive. In particular, we may consider a sequence of such metrics where the limiting functions a, b are convex but not C^2 so that the curvature of the spacetime approaches a distribution containing a Dirac-mass singularity, say on the hypersurface $x = x_0$ for some x_0. Solutions with distributional curvature [18] could be used here to handle the limiting spacetimes with limited regularity obtained in the limit.   □

Acknowledgments

References

[1] S.B. Alexander and R. L. Bishop, *Lorentz and semi-Riemannian spaces with Alexandrov curvature bounds*, Comm. Anal. Geom. 16 (2008), 251–282.

[2] L. Andersson and R. Howard, *Comparison and rigidity theorems in semi-Riemannian geometry,* Comm. Anal. Geom. 6 (1998), 819–877.

[3] J.K. Beem and P.E. Ehrlich, *A Morse index theorem for null geodesics*, Duke Math. J. 46 (1979), 561–569.

[4] J.K. Beem and P.E. Ehrlich, *Cut points, conjugate points and Lorentzian comparison theorems*, Math. Proc. Cambridge Philos. Soc. 86 (1979), 365–384.

[5] J.K. Beem, P.E. Ehrlich, and K.L. Easley, *Global Lorentzian Geometry*, Monographs Textbooks in Pure Appl. Math., Vol 202, Marcel Dekker Inc., New York, 1996.

[6] J. Cheeger, *Finiteness theorems for Riemannian manifolds*, Amer. J. Math. 92 (1970), 61–74.

[7] J. Cheeger and D.G. Ebin, *Comparison Theorems in Riemannian Geometry*, North-Holland, Amsterdam, Vol. 9, 1975.

[8] J. Cheeger, M. Gromov, and M. Taylor, *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds*, J. Differential Geom. 17 (1982), 15–53.

[9] B.-L. Chen and P.G. LeFloch, *Injectivity radius of Lorentzian manifolds*, Comm. Math. Phys. 278 (2008), 679–713.

[10] P.E. Ehrlich and M. Sanchez, *Some semi-Riemannian volume comparison theorems,* Tohoku Math. J. 52 (2000), 331–348.

[11] R. P. Geroch, *Topology in general relativity*, J. Math. Phys. 8 (1967), 782–786.

[12] S.G. Harris, *A triangle comparison theorem for Lorentz manifolds,* Indiana Univ. Math. J. 31 (1982), 289–308.

[13] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Space-time*, Cambridge Monographs on Math. Phys., Vol 1, Cambridge Univ. Press, London, 1973.

[14] E. Heintze and H. Karcher, *A general comparison theorem with applications to volume estimates for submanifolds,* Ann. Sci. Ecole Norm. Sup. 11 (1978), 451–470.

[15] S. Klainerman and I. Rodnianski, *On the radius of injectivity of null hypersurfaces*, J. Amer. Math. Soc. 21 (2008), 775–795.

[16] S. Klainerman and I. Rodnianski, *On the breakdown criterion in general relativity*, J. Amer. Math. Soc. 23 (2010), 345–382.

[17] P.G. LeFloch, *Injectivity radius and optimal regularity for Lorentzian manifolds with bounded curvature*, Actes Sémin. Théor. Spectr. Géom. 26 (2007), 77–90.

[18] P.G. LeFloch and C. Mardare, *Definition and weak stability of spacetimes with distributional curvature*, Port. Math. 64 (2007), 535–573.

[19] E. Malec and N. Ó Murchadha, *Constant mean curvature slices in the extended Schwarzschild solution and the collapse of the lapse*, Phys. Rev. D (3) 68 (2003), no. 12, 124019, 16.

[20] P. Petersen, *Riemannian Geometry*, Graduate Texts in Math. Vol. 171, Springer, New York, 2006.

[21] K. Uhlenbeck, *A Morse theory for geodesics on a Lorentz manifold*, Topology 14 (1975), 69–90.

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