INJECTIVITY RADIUS ESTIMATES I

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1. Injectivity radius estimates for Riemannian metrics

Preliminaries. Let $(M, \mathbf{g})$ be a smooth Riemannian manifold[1]. Given a piece-wise smooth curve $c : [a, b] \to M$, we define its length

$$L[c] := \int_a^b |\dot{c}(t)| dt,$$

where

$$|\dot{c}(t)| := \sqrt{\mathbf{g}(\dot{c}(t), \dot{c}(t))}.$$

From this, we define the distance function such that, for $p, q \in M$, we have

$$d_{\mathbf{g}}(p, q) := \inf \left\{ L[c] \;\middle|\; c : [a, b] \to M, c \text{ piece-wise smooth}, c(a) = p, c(b) = q \right\}.$$

A curve $\gamma : [0, \ell] \to M$ is a *geodesic* if it satisfies the equation

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0,$$

where ∇ denotes the Levi-Civita connection corresponding to the Riemannian metric $\mathbf{g}$. The standard existence and uniqueness theorem for ordinary differential equations implies that, given $p \in M$ and $\mathbf{v} \in T_p M$ then there exists a unique geodesic $\gamma_{\mathbf{v}}$ in M with the properties that

$$\gamma_{\mathbf{v}}(0) = p, \qquad \dot{\gamma}_{\mathbf{v}}(0) = \mathbf{v}.$$

All manifolds that we will consider will be assumed complete, in the sense that M with the metric $d_{\mathbf{g}}$ is a complete metric space. According to the Hopf–Rinow theorem, this is equivalent to the condition that $(M, \mathbf{g})$ be geodesically complete i.e. $\gamma_{\mathbf{v}}(t)$ is well-defined for all $t \in \mathbb{R}$. Given $p \in M$, we define the *exponential map* at p as the map

$$\exp_p : T_p M \to M, \quad \mathbf{v} \mapsto \gamma_{\mathbf{v}}(1).$$

Injectivity radius. Given $p \in M$ and $\mathbf{v} \in T_p M$, for sufficiently small $t > 0$ the geodesic $\gamma_{\mathbf{v}}$ will be a minimising curve between the points p and $\gamma(t)$, in the sense that

$$d_{\mathbf{g}}(p, \gamma(t)) = L[\gamma|\,[0, t]].$$

Generically, however, there exists $t_0 > 0$ such that γ is not minimising for $t > t_0$, in the sense that

$$d_{\mathbf{g}}(p, \gamma(t)) < L[\gamma|\,[0, t]], \qquad \text{for } t > t_0.$$

On a complete Riemannian manifold, there are only two ways that a geodesic becomes non-minimising in this way[2]:

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[1]So M is a smooth manifold and $\mathbf{g}$ defines an inner product on each tangent space $T_p M$, for each $p \in M$, and this inner product varies smoothly with p.

[2]This result is due to Whitehead.

1). there exist two distinct geodesics $\gamma, \widetilde{\gamma}$ from p to $q := \gamma(t_0) = \widetilde{\gamma}(\widetilde{t}_0)$ of the same length: $L[\gamma|[0,t_0]] = L[\widetilde{\gamma}|[0,\widetilde{t}_0]$.
2). The point $\gamma(t_0)$ is *conjugate* to p along the geodesic γ.

To define the concept of a conjugate point, let $\gamma : [0,\ell] \to M$ be a geodesic parametrised proportional to arc-length (i.e. $|\dot{\gamma}| = 1$). A vector field along a geodesic γ, $\mathbf{J}$, is a *Jacobi field* along γ if it satisfies the *Jacobi equation*

$$\nabla_{\dot{\gamma}}^2 \mathbf{J} + \mathbf{R}(\mathbf{J}, \dot{\gamma})\dot{\gamma} = 0.$$

Without any loss of generality (in the context that we are interested in), we will assume that Jacobi fields are orthogonal to the tangent vector to γ i.e. $\mathbf{g}(\mathbf{J}, \dot{\gamma}) = 0$.

Definition 1.1. A point $q \in M$ is *conjugate* to $p \in M$ along a geodesic $\gamma : [0,1] \to M$ with $\gamma(0) = p, \gamma(1) = q$ if there exists a non-zero Jacobi field, $\mathbf{J}$, along γ with the property that $\mathbf{J}(0) = \mathbf{J}(1) = 0$.

Remark 1.2. The condition that the point $\gamma_{\mathbf{v}}(1)$ is conjugate to p along $\gamma_{\mathbf{v}}$ may be shown to be equivalent to the condition that the derivative of the exponential map at $\mathbf{v}$ (i.e. $D_{\mathbf{v}} : T_{\mathbf{v}}(T_pM) \to T_{\exp_p \mathbf{v}}M$) is singular. Therefore the exponential map does not define a local diffeomorphism from a neighbourhood of $\mathbf{v}$ to a neighbourhood of $\exp_p \mathbf{v}$. Away from conjugate points, the exponential map is a local diffeomorphism.

In order to define the injectivity radius, we consider the ball $B(\mathbf{0}, r) := \{\mathbf{v} \in T_pM : \mathbf{g}_p(\mathbf{v},\mathbf{v}) < r\} \subseteq T_pM$. For sufficiently small $r > 0$, the exponential map maps this set diffeomorphically onto $B(p,r) := \{q \in M : d_{\mathbf{g}}(p,q) < r\}$. Once the geodesics from p become non-minimising, however, then this map will either be non-injective (if there exist distinct geodesics between p and a point q) or not a local diffeomorphism (if there are conjugate points along the geodesics.) under the exponential map.

Definition 1.3. The *injectivity radius of* $(M, \mathbf{g})$ *at* p, $\mathrm{inj}(p)$, is the supremum of values of r such that the exponential map defines a global diffeomorphism from $B(\mathbf{0}, r)$ onto its image in M.

Remark 1.4. The injectivity radius essentially measures the size of the largest ball around p for which radial geodesics behave like geodesics in $\mathbb{R}^n$. Equivalently, it defines the largest ball on which geodesic normal coordinates around p may be used.

Rauch comparison theorem. In order to estimate the injectivity radius of M at p, we need to understand conjugate points along geodesics from p and geodesic loops through p[3].

In this section, we show how the conjugate points may be controlled under the assumption that the sectional curvature of $(M, \mathbf{g})$ is bounded above. Given $p \in M$ and a two-dimensional plane $\boldsymbol{\pi} \subseteq T_pM$ that is spanned by two vectors $\mathbf{X}, \mathbf{Y} \in T_pM$, then the sectional curvature of the plane $\boldsymbol{\pi}$, $K_M(\boldsymbol{\pi})$ is defined as

$$K_M(\boldsymbol{\pi}) \equiv K_M(\mathbf{X} \wedge \mathbf{Y}) \equiv \frac{\mathbf{R}(\mathbf{X}, \mathbf{Y}, \mathbf{X}, \mathbf{Y})}{\mathbf{g}(\mathbf{X}, \mathbf{X})\mathbf{g}(\mathbf{Y}, \mathbf{Y}) - \mathbf{g}(\mathbf{X}, \mathbf{Y})^2}.$$

[3] An argument due to Klingenberg implies that if the injectivity radius at p is recognised by two distinct geodesics from p to a point q, then these geodesics must form a loop that is smooth at q, but not necessarily smooth at p.

One can check that this definition depends only on the plane $\boldsymbol{\pi}$, and not on $\mathbf{X}, \mathbf{Y}$. For example, if we choose $\mathbf{X}, \mathbf{Y}$ such that $|\mathbf{X}| = |\mathbf{Y}| = 1$ and $\mathbf{g}(\mathbf{X}, \mathbf{Y}) = 0$, then

$$K_M(\mathbf{X} \wedge \mathbf{Y}) = \mathbf{R}(\mathbf{X}, \mathbf{Y}, \mathbf{X}, \mathbf{Y}),$$

so K_M may be identified with the components of the curvature with respect to an orthonormal frame.

Let $K \in \mathbb{R}$. We define the functions

$$sn_K(r) := \begin{cases} \frac{1}{\sqrt{K}} \sin\left(\sqrt{K} r\right) & K > 0, \qquad r \in [0, \pi/\sqrt{K}] \\ r & K = 0, \qquad r \in [0, \infty) \\ \frac{1}{\sqrt{|K|}} \sinh\left(\sqrt{|K|} r\right) & K < 0, \qquad r \in [0, \infty). \end{cases}$$

We then have the following result

Rauch comparison theorem. *Let $(M, \mathbf{g})$ be a Riemannian manifold such that $K_M \le K$ for some $K \in \mathbb{R}$. Let $p \in M$ and γ be a geodesic in M with $\gamma(0) = p$, $|\dot\gamma| = 1$. Let $\mathbf{J}$ be a Jacobi vector field along γ such that $\mathbf{J}(0) = 0$ and $\mathbf{J} \perp \dot\gamma$. Then*

$$|\mathbf{J}(t)| \ge \left(\frac{d}{dt}|\mathbf{J}(t)|\right)\Big|_{t=0} sn_K(t), \tag{1.1}$$

for $t > 0$ (up to the first zero of the right-hand-side in the case $K > 0$).

Corollary 1.5. *Let $(M, \mathbf{g})$ be a Riemannian manifold such that $K_M \le K$ for some $K \in \mathbb{R}$.*

1). *If $K \le 0$, then geodesics in M contain no conjugate points.*
2). *If $K > 0$, then, given any $p \in M$ and a geodesic γ with $\gamma(0) = p$ and $|\dot\gamma| = 1$, then $\gamma(t)$ will not be conjugate to p along γ for any $t < \pi/\sqrt{K}$.*

Remark 1.6. There exist much sharper versions of the Rauch theorem (see, e.g., [4]). It is a powerful tool for controlling conjugate points, and is used in the proof of the sphere theorems (Rauch, Berger, Klingenberg, …), the Toponogov triangle comparison theorem [3, 4] and the Soul theorem [11].

Moral: Given an upper bound on the sectional curvature of $(M, \mathbf{g})$, we can find a lower bound on the distance along any geodesic parametrised by arc-length from a point $p \in M$ before we meet conjugate points.

Geodesic loops. Consider the flat two-dimensional cylinder $M := \mathbb{R} \times S^1$, where the circumference of the S^1 factor is L. Given any point, p, in this manifold, the length of the shortest geodesic loop through p is then L (i.e. take the closed geodesic that goes around the S^1 factor). Since L is completely undetermined by the curvature of the metric (the curvature is zero, for any value of L), it follows that we cannot expect to get a direct estimate for the shortest geodesic loop through p in terms of the curvature of the metric.

On the other hand, if we consider the ball centre 0, radius $r > 0$ in $M_0 := \mathbb{R}^2$, then its area equals πr^2. If we consider the ball radius r in the above cylinder then, for $r >> L$, its area is approximately $2rL$. (The region can be identified, approximately, with a rectangle of sides L and $2r$.) In particular, the rate of growth of these volumes is different for $\mathbb{R}^2$ and $\mathbb{R}^1 \times S^1$.

In particular, consider the following ratio:

$$\frac{\mathrm{Vol}\,(B_M(p,r))}{\mathrm{Vol}\,(B_{M_0}(0,2r))} \sim \frac{2rL}{\pi(2r)^2} = \frac{L}{2\pi r}.$$

Note that this ratio goes to zero like $1/r$ as $r \to \infty$, and that the constant of proportionality is (up to a factor of 2π) the length L i.e. the length of the shortest geodesic loop through the point p.

This example suggests that we may be able to find estimates for the shortest geodesic loop through a point p and hence, when combined with the Rauch results from above, for the injectivity radius at p in terms of volumes of geodesic balls. This is the approach followed by Cheeger, Gromov and Taylor [6] who derived, for example, the following result:

Theorem 1.7. *Let $(M, \mathbf{g})$ be a complete Riemannian manifold such that $K_M \le K$ for constants $K \in \mathbb{R}$. Let $0 < r < \frac{\pi}{4\sqrt{K}}$ if $K > 0$ and $r \in (0, \infty)$ if $K \le 0$. Then the injectivity radius at p satisfies*

$$\mathrm{inj}(p) \ge r\frac{\mathrm{Vol}\,(B_M(p,r))}{\mathrm{Vol}\,(B_M(p,r)) + \mathrm{Vol}_{T_pM}(B_{T_pM}(\mathbf{0},2r))},$$

where $\mathrm{Vol}_{T_pM}(B_{T_pM}(\mathbf{0},2r))$ denotes the volume of the ball radius $2r$ in T_pM, where both the volume and the distance function are defined using the metric $\mathbf{g}^\star := \exp_p^\star \mathbf{g}$ i.e. the pull-back of the metric $\mathbf{g}$ to T_pM via the exponential map[4].

Plausibility argument: We can get the rough idea of the proof of this result from our above example with $\mathbb{R}^2$ and the cylinder. In this case, the constant curvature space is $\mathbb{R}^2$ with the flat metric, and $(M, \mathbf{g})$ is $S^1 \times \mathbb{R}^1$ with its flat metric. Since there are no conjugate points in this case (since the metric is flat, so $K = 0$) then, given $p \in M$, the exponential map defines a local diffeomorphism $\exp_p : T_pM \to M$. Noting that $T_pM \cong \mathbb{R}^2$, the exponential map may be looked on as "wrapping up" the y axis in $\mathbb{R}^2$ with period L in order to construct the cylinder.

The exponential map maps the set $B(\mathbf{0}, r) \subseteq T_pM$ to the set $B(p, r)$. If $r >> L$ then a point $q \in B(p, r)$ will, generally, have several inverse-images in $B(\mathbf{0}, r)$. In particular, the point p will have inverse images $\widetilde{t}_{-m}, \dots, \widetilde{t}_0, \dots, \widetilde{t}_m \in B(\mathbf{0}, r)$ which we take to be of the form $t_k = k\mathbf{L}$, where $\mathbf{L}$ is a vector in the y direction of length L.

If we now consider a point $q \in B(p, r)$, then it will have at least $2m+1$ inverse images in the set $\Omega := \cup_{i=-m}^{n} B(\widetilde{t}_i, r)$. Since $\widetilde{t}_i \in B(\mathbf{0}, r)$, it follows that $\Omega \subseteq B(\mathbf{0}, 2r)$. Hence each point $q \in B(p, r)$ has at least $(2m + 1)$ inverse images in the set $B(\mathbf{0}, 2r)$. Therefore

$$(2m + 1)\mathrm{Vol}\,(B(p, r)) \le \mathrm{Vol}\,(B(\mathbf{0}, 2r)). \tag{1.2}$$

In particular,

$$m \le \frac{1}{2}\left(-1 + \frac{\mathrm{Vol}\,(B(\mathbf{0}, 2r))}{\mathrm{Vol}\,(B(p, r))}\right). \tag{1.3}$$

We now note that, from the definition of m, that we have $mL < r$ and $(m + 1)L \ge r$, then we have

$$m \ge \frac{r}{L} - 1.$$

Combining the last two inequalities and rearranging gives

$$L \ge 2r\left(1 + \frac{\mathrm{Vol}\,(B(\mathbf{0}, 2r))}{\mathrm{Vol}\,(B(p, r))}\right)^{-1}.$$

[4]Note that since $2r < \pi/\sqrt{K}$, the exponential map is non-singular on $B(\mathbf{0}, 2r)$, so $\mathbf{g}^\star$ is well-defined.

In this case, we have no conjugate points on geodesics, so the injectivity radius at p is greater than or equal to half the length of the shortest geodesic loop through p. We therefore have

$$\mathrm{inj}(p) \geq r\left(1 + \frac{\mathrm{Vol}\,(B(\mathbf{0}, 2r))}{\mathrm{Vol}\,(B(p, r))}\right)^{-1},$$

which is of the form in the Cheeger–Gromov–Taylor theorem.

Remark 1.8. As is clear from the construction, the above estimate is not particularly sharp. For our cylinder example, it tells us that

$$\mathrm{inj}(p) \geq r\left(1 + \frac{2\pi r}{L}\right)^{-1},$$

for all $r > 0$. The best estimate comes by letting $r \to \infty$, in which case we have

$$\mathrm{inj}(p) \geq \frac{L}{2\pi},$$

whereas the actual value of $\mathrm{inj}(p) = L/2$. The important point, however, is that we have an explicit lower bound for the injectivity radius, rather than a particularly accurate one.

Remark 1.9. The length of the shortest homotopically non-trivial closed geodesic on a manifold is called the *systole* of the manifold. The relationship between the systole and the volume of the manifold for *essential* manifolds is the topic of a famous/notorious paper by Gromov [8].

Note that in going from (1.2) to (1.3), we have divided by $\mathrm{Vol}\,(B(p, r))$. It is therefore important in this argument that there exists a $p \in M$ and $r > 0$ such that $\mathrm{Vol}\,(B(p, r)) \neq 0$. In particular, one of the main advances that Cheeger, Gromov and Taylor made in their paper was to prove the following:

Theorem 1.10. *Let $(M, \mathbf{g})$ be a complete Riemannian manifold such that*

1). *There exists a constant K such that $0 \leq K < \infty$ with $|K_M| \leq K$;*
2). *There exists a point $p \in M$ and a constant $v_0 > 0$ such that*

$$\mathrm{Vol}_{\,g}(B_{\mathbf{g}}(p, 1)) \geq v_0.$$

Then there exists a positive constant $i_1 = i_1(K, v_0, n)$ such that

$$\mathrm{inj}(p) \geq i_1.$$

Remark 1.11. The volume condition turns out to be crucial. In particular, for example, there exists a one-parameter family of metrics on S^3 (the Berger sphere metrics) $\{\mathbf{g}_\epsilon\}_{\epsilon \in (0,1]}$ with the property that for each point $p \in S^3$, the injectivity radius $\mathrm{inj}_{\mathbf{g}_\epsilon}(p)$ converges to zero as $\epsilon \to 0$ uniformly on S^3. However, the sectional curvatures of these metrics lies in the range $[\epsilon^2, 4 - 3\epsilon^2]$. As such, curvature bounds alone are not enough to expect the injectivity radius to be bounded away from zero. In the case of the Berger spheres, volumes converge to zero like ϵ as $\epsilon \to 0$. This phenonemenon of a sequence of Riemannian manifolds collapsing to a manifold of lower dimension with the curvature remaining bounded is often referred to as "Cheeger–Gromov collapse" (see, e.g., [5]).

2. Lorentzian metrics

In work with P.G. LeFloch (Paris VI, CNRS), I am trying to adapt some of the above techniques to Lorentzian metrics rather than Riemannian metrics. This work is (so far, at least) partly based on his previous paper with Chen [7]. Since this work is still in its early stages, I will simply summarise the main results that we have so far.

Let $(M, \mathbf{g})$ be a geodesically complete, smooth Lorentzian manifold. In order to do analysis on M, we need to define a Riemannian metric. This we do by letting $p \in M$ and fixing a future-directed, unit time-like vector, $\mathbf{T}_p \in T_pM$ at p. We then parallel transport $\mathbf{T}_p$ along all the geodesics that emanate from p (i.e. transport along geodesics γ such that $\nabla_{\dot\gamma}\mathbf{T} = 0$). This defines a unique smooth, future-directed, unit time-like vector field $\mathbf{T} \in \mathfrak{X}(\Omega)$, where Ω is any neighbourhood of p that does not intersect the cut-locus of p.

Using the Lorentzian metric, $\mathbf{g}$, and the vector field $\mathbf{T}$, we may construct a Riemannian metric on the set Ω defined by

$$\mathbf{g_T}(\mathbf{X}, \mathbf{Y}) := \mathbf{g}(\mathbf{X}, \mathbf{Y}) + 2\mathbf{g}(\mathbf{T}, \mathbf{X})\mathbf{g}(\mathbf{T}, \mathbf{Y}), \qquad \mathbf{X}, \mathbf{Y} \in \mathfrak{X}(\Omega),$$

where $\mathfrak{X}(\Omega)$ denotes the set of smooth vector fields on Ω. This metric is used to define a norm on tensor fields defined on Ω, which we denote by $\left| \cdot \right|_{\mathbf{g_T}}$.

It is then straightforward to check that

$$\nabla_{\dot\gamma}\mathbf{g_T} = 0,$$

so $\mathbf{g_T}$ is parallel along radial geodesics from p.

We wish to determine to what extent the exponential map, $\exp_p$, is a local diffeomorphism. We define the open ball

$$B_r^{\mathbf{T}}(0) := \left\{ \mathbf{v} \in T_pM \;\middle|\; \mathbf{g}_{\mathbf{T},p}(\mathbf{v}, \mathbf{v}) < r \right\} \subseteq T_pM.$$

We then define the *conjugacy radius at p* to be the supremum of the values of r for which the map $\exp_p\big| B_r^{\mathbf{T}}(0)$ is a local diffeomorphism onto its image. The *injectivity radius at p* to be the supremum of the values of r for which the map $\exp_p\big| B_r^{\mathbf{T}}(0)$ is a global diffeomorphism onto its image. To determine the conjugacy radius, we again consider Jacobi vector fields along the geodesics from p.

Proposition 2.1. *Let $\gamma : [0, \ell] \to \Omega$ be a geodesic with respect to $\mathbf{g}$ normalised such that $\left|\dot\gamma(0)\right|_{\mathbf{g_T}} = 1$ and $\mathbf{J}$ a Jacobi vector field along γ with $\mathbf{J}(0) = 0$. Assume that the curvature of the metric $\mathbf{g}$ satisfies condition*

$$\sup_{\Omega} \left|\mathbf{R}\right|_{\mathbf{g_T}} \leq K_2, \tag{2.1}$$

for some $K_2 < \infty$. Then $\mathbf{J}(t) \neq 0$ for $t \in (0, \pi/\sqrt{K_2})$.

Corollary 2.2. *The conjugacy radius at p obeys*

$$\mathbf{conj}(p) \geq \frac{\pi}{\sqrt{K_2}}.$$

Remark 2.3. Note that the estimate is insensitive to the causal nature (i.e. time-like, space-like or null) of the geodesics. This is in contradistinction to the previous Lorentzian Rauch-type theorems given in [9] and [2, 1].

Outlook. At the moment, we are working on adapting the Cheeger–Gromov–Taylor homotopy arguments for geodesic loops to Lorentzian manifolds. This should then give injectivity radius estimates along the lines of those quoted above in the Riemannian case.

Additional topics that we are intending to address include:

- The adaption of Berger's version of the Rauch comparison theorem for totally geodesic submanifolds to the Lorentzian case, and derive estimates for the distance from submanifolds to focal points under L^∞ curvature bounds.
- Whether, rather than working with bounds on the curvature (i.e. second derivatives of the metric), one can get injectivity radius estimates from bounds on first derivatives on the metric (e.g. the extrinsic curvature of sub-manifolds). This should lead us in a similar direction to recent work of Klainerman and Rodnianski on the initial value problem for Ricci-flat Lorentzian manifolds [10].

Proofs

Proof of Rauch comparison theorem.

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}\left(|\mathbf{J}(t)|^2\right) &= \frac{1}{2}\nabla_{\dot\gamma}\nabla_{\dot\gamma}\left(\mathbf{g}(\mathbf{J}(t),\mathbf{J}(t))\right) = \nabla_{\dot\gamma}\left(\mathbf{g}(\mathbf{J}(t),\dot{\mathbf{J}}(t))\right)\\ &= \mathbf{g}(\dot{\mathbf{J}}(t),\dot{\mathbf{J}}(t)) + \mathbf{g}(\mathbf{J},\ddot{\mathbf{J}})\\ &= |\dot{\mathbf{J}}|^2 + \mathbf{g}(\mathbf{J}, -\mathbf{R}(\mathbf{J},\dot\gamma)\dot\gamma)\\ &= |\dot{\mathbf{J}}|^2 - \mathbf{R}(\mathbf{J},\dot\gamma,\mathbf{J},\dot\gamma)\\ &= |\dot{\mathbf{J}}|^2 - K_M(\mathbf{J}\wedge\dot\gamma)\left(|\mathbf{J}|^2|\dot\gamma|^2 - \mathbf{g}(\mathbf{J},\dot\gamma)^2\right)\\ &= |\dot{\mathbf{J}}|^2 - K_M(\mathbf{J}\wedge\dot\gamma)|\mathbf{J}|^2\\ &\geq |\dot{\mathbf{J}}|^2 - K|\mathbf{J}|^2,\end{aligned}$$

where we have used the notation $\dot{\mathbf{J}} := \nabla_{\dot\gamma}\mathbf{J}$, $\ddot{\mathbf{J}} := \nabla_{\dot\gamma}\nabla_{\dot\gamma}\mathbf{J}$, etc.In addition, we have

$$|\mathbf{J}|\left|\frac{d}{dt}|\mathbf{J}|\right| = \frac{1}{2}\left|\frac{d}{dt}|\mathbf{J}|^2\right| = \frac{1}{2}|\nabla_{\dot\gamma}\mathbf{g}(\mathbf{J},\mathbf{J})| = |\mathbf{g}(\mathbf{J},\dot{\mathbf{J}})| \leq |\mathbf{J}||\dot{\mathbf{J}}|.$$

Therefore, we have the Kato inequality

$$\left|\frac{d}{dt}|\mathbf{J}|\right| \leq |\dot{\mathbf{J}}| \qquad \text{for } \mathbf{J}\neq 0.$$

Substituting this into the equation above gives

$$\frac{1}{2}\frac{d^2}{dt^2} \geq \left(\frac{d}{dt}|\mathbf{J}|\right)^2 - K|\mathbf{J}|^2.$$

Letting $\varphi(t) := |\mathbf{J}(t)|$ then this inequality implies that

$$\ddot\varphi(t) + K\varphi(t) \geq 0 \qquad \text{for } \varphi \neq 0.$$

The boundary conditions on $\mathbf{J}$ imply that $\varphi(0) = 0$ and $\dot\varphi(0) \geq 0$. We then define

$$\psi(t) := \dot\varphi(0) sn_K(t),$$

which is the quantity that appears on the right-hand-side of (1.1), and note that

$$\ddot\psi(t) + K\psi(t) = 0$$

and satisfies $\psi(0) = \varphi(0)$, $\dot\psi(0) = \dot\varphi(0)$. Note, in addition, that $\psi(t) > 0$ for all $t > 0$ if $K \leq 0$ and for $t \in (0, \pi/\sqrt{K})$ if $K > 0$.

Letting $f(t) := \ddot\varphi(t) + K\varphi(t) \geq 0$, we then calculate

$$\frac{d}{dt}\left(\varphi\dot\psi - \dot\varphi\psi\right) = -f\psi \leq 0,$$

for all $t > 0$ if $K \leq 0$ and for $t \in (0, \pi/\sqrt{K})$ if $K > 0$. Therefore $\varphi\dot\psi - \dot\varphi\psi$ is non-increasing. Since it vanishes as $t \to 0+$, we deduce that

$$\varphi(t)\dot\psi(t) - \dot\varphi(t)\psi(t) \leq 0.$$

Therefore

$$\frac{d}{dt}\left(\frac{\psi}{\varphi}\right) \leq 0,$$

so ψ/φ is non-increasing. Since this ratio converges to 1 as $t \to 0+$, we deduce that

$$\frac{\psi(t)}{\varphi(t)} \leq 1.$$

Rearranging gives (1.1). $\square$

Proof of Proposition 2.1. This is a more complicated version of the Riemannian proof given above. We first calculate

$$\begin{aligned}
\frac{1}{2}\frac{d^2}{dt^2}\big|\mathbf{J}(t)\big|^2_{\mathbf{g_T}} &= \frac{1}{2}\frac{d^2}{dt^2}\left(\mathbf{g}(\mathbf{J}(t),\mathbf{J}(t)) + 2\mathbf{g}(\mathbf{T},\mathbf{J}(t))^2\right)\\
&= \frac{d}{dt}\Big(\mathbf{g}(\mathbf{J}(t),\dot{\mathbf{J}}(t)) + 2\mathbf{g}(\mathbf{T},\mathbf{J}(t))\mathbf{g}(\mathbf{T},\dot{\mathbf{J}}(t))\Big)\\
&= \mathbf{g}(\dot{\mathbf{J}}(t),\dot{\mathbf{J}}(t)) + \mathbf{g}(\mathbf{J}(t),\ddot{\mathbf{J}}(t)) + 2\mathbf{g}(\mathbf{T},\dot{\mathbf{J}}(t))\mathbf{g}(\mathbf{T},\dot{\mathbf{J}}(t)) + 2\mathbf{g}(\mathbf{T},\mathbf{J}(t))\mathbf{g}(\mathbf{T},\ddot{\mathbf{J}}(t))\\
&= \big|\dot{\mathbf{J}}(t)\big|^2_{\mathbf{g_T}} - \mathbf{R}(\mathbf{J}(t),\dot\gamma(t),\mathbf{J}(t),\dot\gamma(t)) - 2\mathbf{g}(\mathbf{T},\mathbf{J}(t))\mathbf{R}(\mathbf{J}(t),\dot\gamma(t),\mathbf{T},\dot\gamma(t))\\
&= \big|\dot{\mathbf{J}}(t)\big|^2_{\mathbf{g_T}} - \mathbf{R}(\mathbf{J}(t),\dot\gamma(t),\mathbf{J}(t) + 2\mathbf{g}(\mathbf{T},\mathbf{J}(t))\mathbf{T},\dot\gamma(t))\\
&\geq \big|\dot{\mathbf{J}}(t)\big|^2_{\mathbf{g_T}} - K_2\big|\mathbf{J}(t)\big|_{\mathbf{g_T}}\big|\dot\gamma(t)\big|^2_{\mathbf{g_T}}\big|\mathbf{J}(t) + 2\mathbf{g}(\mathbf{T},\mathbf{J}(t))\mathbf{T}\big|_{\mathbf{g_T}}\\
&\geq \big|\dot{\mathbf{J}}(t)\big|^2_{\mathbf{g_T}} - K_2\big|\mathbf{J}(t)\big|^2_{\mathbf{g_T}}.
\end{aligned} \tag{.2}$$

Again, using the Kato inequality, we have

$$\frac{1}{2}\frac{d^2}{dt^2}\big|\mathbf{J}(t)\big|^2_{\mathbf{g_T}} \geq \left(\frac{d}{dt}\big|\mathbf{J}(t)\big|_{\mathbf{g_T}}\right)^2 - K_2\big|\mathbf{J}(t)\big|^2_{\mathbf{g_T}}. \tag{.3}$$

For simplicity of notation, let

$$\varphi(t) := \big|\mathbf{J}(t)\big|_{\mathbf{g_T}}.$$

Note that $\varphi(0) = 0$, $\dot\varphi(0) > 0$ and $\varphi(t) \geq 0$. Substituting into (.3), we find that φ satisfies the inequality

$$\ddot\varphi(t) + K_2\varphi(t) \geq 0$$

at all points where $\varphi(t) \neq 0$. We now define the function

$$\psi(t) := \begin{cases} \frac{\dot\varphi(0)}{\sqrt{K_2}}\sin\sqrt{K_2}t & K_2 > 0,\\ \dot\varphi(0)t & K_2 = 0,\end{cases}$$

which satisfies the differential equation

$$\ddot\psi(t) + K_2\psi(t) = 0$$

with initial conditions $\psi(0) = \varphi(0) = 0$, $\dot\psi(0) = \dot\varphi(0)$. A standard Sturm comparison argument then shows that the smallest $t_0 > 0$ for which $\varphi(t_0) = 0$ is greater than or equal to the first $t > 0$ for which $\psi(t) = 0$. Since the latter is equal to $\pi/\sqrt{K_2}$, we deduce that $t_0 \geq \pi/\sqrt{K_2}$. Therefore $\mathbf{J}(t) \neq 0$ for $t \in (0, \pi/\sqrt{K_2})$, as required. $\square$

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