

LSDIFFM AND THE EINSTEIN EQUATIONS

JAMES D.E. GRANT

ABSTRACT. We give a formulation of the vacuum Einstein equations in terms of a set of volume-preserving vector fields on a four-manifold \mathcal{M} . These vectors satisfy a set of equations which are a generalisation of the Yang-Mills equations for a constant connection on flat spacetime.

It is known (Mason and Newman 1989) that the equations which describe self-dual Ricci-flat metrics can be derived from the self-dual Yang-Mills equations in flat space, with a specific choice of gauge group. In particular, consider the self-dual Yang-Mills equations on flat space, (\mathbb{M}, η) ,

$$F_{ij} = \frac{1}{2} \epsilon_{ij}{}^{kl} F_{kl}, \quad (1)$$

where F_{ij} is the curvature of an algebra valued connection A_i . If we now impose that the connection A_i is constant on \mathbb{M} , then equations (1) become a set of algebraic conditions on the connection

$$[A_i, A_j] = \frac{1}{2} \epsilon_{ij}{}^{kl} [A_k, A_l].$$

Mason and Newman showed (Mason and Newman 1989) that if we take the connection A to take values in $Lsdiff\mathcal{M}$, the volume-preserving diffeomorphisms of an auxiliary four-manifold \mathcal{M} , and write $A_i = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$, then the (contravariant) metric $\mathbf{g} = \eta^{ij} \mathbf{e}_i \otimes \mathbf{e}_j$ defines, up to a known conformal factor, a self-dual metric on the manifold \mathcal{M} . If we consider the case where the connection A satisfies the full Yang-Mills equations, we are led to the equations

$$\eta^{ij} [\mathbf{e}_i, [\mathbf{e}_j, \mathbf{e}_k]] = 0, \quad (2)$$

which, if we allow the connection to have torsion, correspond to Einstein-Cartan theory (Mason and Newman 1989).

The question we will consider here is whether it is possible to find a similar formulation of the full vacuum Einstein equations. We begin with four vectors, \mathbf{V}_i , and an internal metric η_{ij} , with inverse η^{ij} such that the (contravariant) metric

$$\mathbf{g} = \eta^{ij} \mathbf{V}_i \otimes \mathbf{V}_j \quad (3)$$

satisfies the vacuum Einstein equations. (Letters i, j, \dots will denote internal indices, which will be raised and lowered using η^{ij} and η_{ij} . We will consider complex metrics, and will not discuss reality conditions.) Defining the structure functions of the vectors \mathbf{V}_i by

$$[\mathbf{V}_i, \mathbf{V}_j] = \tilde{C}_{ij}{}^k \mathbf{V}_k,$$

then, by an internal rotation, we impose the condition that

$$\tilde{C}_{ij}{}^j = -\mathbf{V}_i(\log f),$$

for some function f . (This is possible for *all* metrics for some function f . There is no integrability condition.) We define a conformally related set of vectors $\mathbf{e}_i = f\mathbf{V}_i$ with

$$[\mathbf{e}_i, \mathbf{e}_j] = C_{ij}{}^k \mathbf{e}_k,$$

where

$$C_{ij}{}^k = f\tilde{C}_{ij}{}^k + \mathbf{e}_i(\log f)\delta_j^k - \mathbf{e}_j(\log f)\delta_i^k.$$

(The reason for considering this conformal transformation will become clear later.) We now note that

$$C_{ij}{}^j \equiv \Gamma_i = 2\mathbf{e}_i(\log f). \quad (4)$$

This means that the \mathbf{e}_i are volume preserving (Mason and Newman 1989). In particular, there exists a volume form $\epsilon \in \Lambda^4(\mathcal{M})$ such that

$$\mathcal{L}_{\mathbf{e}_i}\epsilon = 0,$$

where \mathcal{L} denotes Lie derivative, and such that

$$\epsilon(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4) = f^2. \quad (5)$$

(Taking the Lie derivative of equation (5) along \mathbf{e}_i then gives us equation (4).) Expressing the Ricci tensor of the metric (3) in terms of the structure functions and Ricci tensor of the vectors \mathbf{e}_i , we find that

$$\begin{aligned} R(\mathbf{V})_{ij} = \frac{1}{f^2} [& R(\mathbf{e})_{ij} + \eta_{ij} (\Gamma^k \mathbf{e}_k(\log f) - \eta^{kl} \mathbf{e}_k \mathbf{e}_l(\log f) - 2\eta^{kl} \mathbf{e}_k(\log f) \mathbf{e}_l(\log f)) \\ & + 2\mathbf{e}_i(\log f) \mathbf{e}_j(\log f) + 2C_{(ij)}^k \mathbf{e}_k(\log f) - 2\mathbf{e}_{(i} \mathbf{e}_{j)}(\log f)]. \end{aligned} \quad (6)$$

Using condition (4), the equations $R(\mathbf{V})_{ij} = 0$ can be reduced to the form

$$[\mathbf{e}_k, [\mathbf{e}^k, \mathbf{e}_{(i)}]] \cdot \mathbf{e}_j = \eta_{ij} {}^+ \Gamma_k - \Gamma^k + 2 {}^+ C_{(i}{}^{kl} - C_{j)kl} - 2 {}^+ \Gamma_{(i} - \Gamma_{j)}, \quad (7)$$

where we have defined

$$\begin{aligned} {}^\pm C_{ij}{}^k &= \frac{1}{2} [C_{ij}{}^k \pm \frac{1}{2} \epsilon_{ij}{}^{lm} C_{lm}{}^k], \\ {}^\pm \Gamma_i &= {}^\pm C_{ij}{}^j, \end{aligned}$$

and

$$\mathbf{e}_i \cdot \mathbf{e}_j \equiv \eta_{ij}.$$

Taking the trace of equation (7), we find the condition for the metric (3) to be scalar flat,

$$[\mathbf{e}_i, [\mathbf{e}^i, \mathbf{e}_j]] \cdot \mathbf{e}^j = 2 {}^+ \Gamma_i - \Gamma^i.$$

We thus have:

Theorem *Given a linearly-independent set of vector fields $\{\mathbf{e}_i\}$ which obey equations (7), and a volume-form ϵ such that $\mathcal{L}_{\mathbf{e}_i}\epsilon = 0$, then the set of vector fields $\mathbf{V}_i = f^{-1}\mathbf{e}_i$ define*

a vacuum metric, where f is defined by equation (5). Conversely, for all vacuum Einstein metrics, there exists a set of vectors, $\{\mathbf{e}_i\}$, (unique up to a restricted set of internal rotations) with the above properties.

This theorem constitutes a generalisation of the result of Mason and Newman (Mason and Newman 1989). They showed that if the commutator is self-dual, or anti-self-dual (in the sense that ${}^-C_{ij}{}^k = 0$ or ${}^+C_{ij}{}^k = 0$, respectively) then the corresponding metric (3) is Ricci-flat with self-dual, or anti-self-dual Weyl tensor, respectively. In our case, if the commutator is self-dual or anti-self-dual, then equations (7) are satisfied identically due to the Jacobi identity. In particular, the right-hand side of equations (7) is purely an interaction term between the self-dual and anti-self-dual parts of the commutator: there is no contribution from purely self-dual/anti-self-dual fields. We therefore seem to have discovered a formalism where the full gravitational field can be looked on in terms of an interaction between its self-dual and anti-self-dual constituent parts.

The initial motivation for this work was to see if it was possible to give a flat space ambi-twistor construction for the vacuum Einstein equations, in a spirit similar to the construction for the full Yang-Mills equations (Witten 1978, Green *et al* 1978). The Einstein equations, as written in (7), can be looked on as arising from a generalisation of the Yang-Mills equations, since the term on the left-hand side is a generalisation of the Yang-Mills operator in flat space for a constant connection with values in $Lsdiff\mathcal{M}$ (c.f. equation (2)). (Note that if we had not performed the conformal transformation detailed above, we would be led to an expression which also involved terms of the form $[\mathbf{e}_{(i}, [\mathbf{e}_j), \mathbf{e}_k]] \cdot \mathbf{e}^k$, so the connection with Yang-Mills theory would be lost.) It is therefore possible that an ambi-twistor construction for the vacuum Einstein equations can be found, via equations (7), using similar ideas to those in (Witten 1978, Green *et al* 1978). This construction would be based upon the extension of an $sdiff\mathcal{M}$ bundle over flat ambi-twistor space \mathbb{A} , to a bundle over $\mathbb{P}_3 \times \mathbb{P}_3$, with the extra condition that the connection on the diagonal subspace is constant. However, if we were to simply follow (Witten 1978, Green *et al* 1978) and consider the first obstruction to extending a given $sdiff\mathcal{M}$ bundle over \mathbb{A} to a bundle over $\mathbb{P}_3 \times \mathbb{P}_3$, this would lead to a constant $Lsdiff\mathcal{M}$ -valued connection which would satisfy the full Yang-Mills equations on \mathbb{M} , as opposed to equation (7). We must split up the four-dimensional vectors \mathbf{e}_i as a sum of two sets of four eight-dimensional vectors, so that $\mathbf{e}_i = \mathbf{f}_i + \mathbf{g}_i$ on $\mathcal{M} \times \mathcal{M}$. We then wish that equations (7) should be satisfied identically on the diagonal subsurface which defines our original manifold. There are two main possibilities we can consider:

- We impose that the commutator of the \mathbf{f}_i is self-dual, and the commutator of the \mathbf{g}_i is anti-self-dual in the sense that

$$[\mathbf{f}_i, \mathbf{f}_j] = \frac{1}{2}\epsilon_{ij}{}^{kl} [\mathbf{f}_k, \mathbf{f}_l], \quad [\mathbf{g}_i, \mathbf{g}_j] = -\frac{1}{2}\epsilon_{ij}{}^{kl} [\mathbf{g}_k, \mathbf{g}_l].$$

In order to satisfy equations (7) on the diagonal subsurface (Witten 1978, Green *et al* 1978), however, we would require $[\mathbf{f}_i, \mathbf{g}_j] \neq 0$. The value of this commutator can easily be derived from equations (7).

- We require that $[\mathbf{f}_i, \mathbf{g}_j] = 0$, but impose that the vectors \mathbf{f}_i are only gauge equivalent, under some internal rotation, to a set of volume-preserving vectors with self-dual commutator. Similarly, the vectors \mathbf{g}_i are gauge equivalent to a set of volume-preserving vectors with anti-self-dual commutator.

The second alternative seems more tempting, since it maintains the independence of the self-dual and anti-self-dual parts of the field. In both the above cases, it remains to be seen to what extent the resulting equations can be simplified, or possibly solved, by a judicious choice of internal gauges of the two sets of vectors. It seems, however, that any construction on flat ambi-twistor space for the vacuum Einstein equations will be considerably more involved than that for the Yang-Mills equations. This would, however, be an alternative to the deformed ambi-twistor space approach to the vacuum Einstein equations (Isenberg and Yasskin 1982, LeBrun 1985, Yasskin 1986, Baston and Mason 1987, Lebrun 1990).

An alternative procedure (Chakravarty *et al* 1991, Grant 1993) would be to choose a suitable coordinate representation of the vectors \mathbf{e}_i which would lead to a set of differential equations for various potentials. These equations would be the generalisation of the Heavenly equations for half-flat metrics (Plebański 1975). It should also be possible to insert the formalism of Kozameh and Newman (Iyer *et al* 1992) into our equations, which would give a closed form for (possibly some variant of) the Light Cone Cut Equation. The formalism used here suggests it may be worth investigating a modified version of the usual spin-coefficient formalism (Newman and Penrose 1962), where we replace components of the spin-connection of a tetrad by the commutator components of our volume-preserving vectors fields as the basic objects.

In addition, it may be possible to simplify the equations of conformal gravity by means of a well chosen conformal gauge. This would be a set of fourth order equations (the vanishing of the Bach tensor (Kozameh *et al* 1985)) for the relevant vector fields, so it is unlikely that there will be any obvious relation with the Yang-Mills equations, although our equations (7) would, of course, be a sub-system of these equations. These, along with the vanishing of the Dighton-Eastwood tensor, seem to be the set of equations that arise naturally in the deformation of ambi-twistor spaces (Baston and Mason 1987, Lebrun 1990). Alternatively, from equation (6), we can construct the equations for certain types of non-Ricci-flat metrics, e.g. Einstein metrics with constant non-zero scalar curvature. It appears, however, that the most natural description of such metrics is not in terms of volume-preserving vectors (Grant 1995).

It should perhaps be pointed out that in order to make full contact between our equation (7) and the Yang-Mills equations, we would have to evaluate the skew-symmetric part of the operator $[\mathbf{e}_k, [\mathbf{e}^k, \mathbf{e}_i]] \cdot \mathbf{e}_j$ which appears in equation (7). Unfortunately, the skew-symmetric part of this operator is not related to the Einstein equations (this is precisely the reason it does not appear in equation (7)). The only equations which can be written down for the skew-symmetric part, without imposing extra constraints on the frame, are identities. Therefore, one can obtain an expression for the Yang-Mills operator, but only at the expense of introducing a large amount of redundant information. For example, one

can show that

$$[\mathbf{e}_k, [\mathbf{e}^k, \mathbf{e}_{[i]}]] \cdot \mathbf{e}_j = 2 \mathbf{e}_k (-C^k_{[ij]}) + {}^{-}C_{ijk} {}^{-}\Gamma^k - 2 {}^{-}C_{[i|kl]} + C_j{}^{kl}.$$

This equation is an identity, derived using the volume preserving condition, and cannot be simplified using the Einstein equations. Adding this equation to equation (7) and rearranging, we obtain

$$\begin{aligned} [\mathbf{e}_k, [\mathbf{e}^k, \mathbf{e}_i]] &= {}^{+}\Gamma_k {}^{-}\Gamma^k \mathbf{e}_i + 2 {}^{+}C_i{}^{kl} - C_{jkl} \mathbf{e}^j - 2 {}^{+}\Gamma_{(i} - \Gamma_{j)} \mathbf{e}^j \\ &\quad + 2 \mathbf{e}_k (-C^k_{[ij]}) \mathbf{e}^j + {}^{-}C_{ijk} {}^{-}\Gamma^k \mathbf{e}^j. \end{aligned} \quad (8)$$

The left hand side of this equation is, in our context, the Yang-Mills operator. However, the content of equation (8) is exactly the same as that of equation (7). Moreover, the right hand side of the equation is no longer a pure interaction term, which is one of the main points of interest of equation (7).

Finally, a set of divergenceless vectors appear naturally in the recent spin- $\frac{3}{2}$ work on the vacuum Einstein equations (Penrose 1994). It would be interesting to know if these vectors are related to our ones.

These issues will be considered in more detail elsewhere.

The author would like to thank to R Capovilla, H A Chamblin, L E Morales, J F Plebański and K P Tod for suggestions. This work was begun while the author was in Pittsburgh, supported under NSF grant PHY 89-04035. It was continued in Mexico, with funding from the Centro de Investigacion y de Estudios Avanzados del I.P.N. and CONACYT. The final version was produced in the University of Newcastle, under the Sir Wilfred Hall fellowship.

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DEPARTMENT OF PHYSICS, UNIVERSITY OF NEWCASTLE, NEWCASTLE UPON TYNE, NE1 7RU,
UNITED KINGDOM

DEPARTAMENTO DE FÍSICA, CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS DEL I. P. N.,
APARTADO POSTAL 14-740, 07000 MÉXICO D.F., MÉXICO.

DEPARTMENT OF PHYSICS AND ASTRONOMY, UNIVERSITY OF PITTSBURGH, PITTSBURGH, PA 15260,
U.S.A.