VALUE DISTRIBUTION AND SPECTRAL THEORY OF SCHRÖDINGER OPERATORS WITH L^2-SPARSE POTENTIALS

S.V. BREIMESSER, JAMES D.E. GRANT, AND D.B. PEARSON

Abstract. We apply the methods of value distribution theory to the spectral asymptotics of Schrödinger operators with L^2-sparse potentials.

1. Introduction

A real valued, locally integrable function V, defined on the half-line $0 \leq x < \infty$, is said to be a L^2-sparse potential if, given arbitrary $\delta, N > 0$, there exists a subinterval (a, b) of $[0, \infty)$ such that $b - a = N$ and $\int_a^b (V(x))^2 \, dx < \delta$. In other words, if V is L^2-sparse then one can find arbitrarily long intervals on which the L^2 norm of V is arbitrarily small. Given an L^2-sparse potential, we can define a Schrödinger operator $T = -\frac{d^2}{dx^2} + V(x)$ acting in $L^2(0, \infty)$ and subject to Dirichlet boundary condition at $x = 0$. By considering an appropriate sequence of approximate eigenfunctions (see for example [G], Theorem 22) one may verify that the Weyl spectrum of T contains the whole of $\mathbb{R}^+$. It follows that we have the limit point case at infinity, so that T can be uniquely defined as a self-adjoint operator, subject to the single boundary condition at $x = 0$.

Any L^2-sparse potential is a sum $V_1 + V_2$, where V_1 is a sparse potential and $V_2 \in L^2(0, \infty)$; here a potential V is said to be sparse if arbitrarily long intervals exist on which V is identically zero. There is a considerable literature on sparse potentials and their perturbations, in particular establishing conditions for the existence of absolutely continuous and singular continuous spectra. For recent results in this field, see [KLS, R, SS] and references therein.

Spectral theory for the Schrödinger operator T can be closely linked to the theory of value distribution for real-valued functions, and in particular value distribution for functions which are defined as boundary values of Herglotz functions. (A Herglotz or Nevanlinna function is a function of a complex variable, analytic in the upper half-plane with positive imaginary part.)

For a measurable function $F_+ : \mathbb{R} \to \mathbb{R}$, the value distribution may be described by means of a map $\mathcal{M} : (A, S) \mapsto \mathcal{M}(A, S) \in \mathbb{R} \cup \{\infty\}$, called the value distribution function of F_+, and defined for Borel subsets A, S of $\mathbb{R}$ by

$$\mathcal{M}(A, S) = \left| A \cap F_+^{-1}(S) \right|. \tag{1}$$

Here $\left| \cdot \right|$ stands for Lebesgue measure. Thus $\mathcal{M}(A, S)$ is the Lebesgue measure of the set of $\lambda \in A$ for which $F_+(\lambda) \in S$. In the particular case that F_+ is the almost everywhere boundary value of a Herglotz function, i.e.

$$F_+(\lambda) = \lim_{d \to 0^+} F(\lambda + id), \qquad \text{(almost all } \lambda \in \mathbb{R}),$$

2000 *Mathematics Subject Classification.* 47E05, 34L05, 82Q10.

Key words and phrases. Herglotz functions, Schrödinger operator, value distribution, spectral theory, sparse potentials.

Published in *Journal of Computational and Applied Mathematics* **148**, 307–322 (2002).

we can write (see [BP1])

$$\mathcal{M}\,(A,S)=\lim_{d\,\to\,0^{+}}\frac{1}{\pi}\int_{A}\theta\,(F(\lambda+id),S)\,d\lambda, \tag{2}$$

where $\theta(z,S)$ denotes the angle subtended at a point z by the Borel subset S of the real line. (For $\lambda\in\mathbb{R}$, we define $\theta(\lambda,S)$ to be $\pi\chi_S(\lambda)$, where χ_S is the characteristic function of the set S.) In fact, given A and S with $|A|<\infty$, the limit in (2) will exist for *any* Herglotz function F (whether or not F has real boundary values a.e.) and may be used to define the value distribution function $\mathcal{M}$ associated with an arbitrary Herglotz function. In general $\mathcal{M}$ may not describe the value distribution of any single real-valued function $F_+(\lambda)$, but there will always be sequences $\{F^{(n)}\}$ of real-valued functions for which $\mathcal{M}$ describes the limiting value distribution.

Value distribution for boundary values of Herglotz functions is also closely connected with the geometric properties of the upper half-plane, regarded as a hyperbolic space [BP1, BP2]. Given two points $z_1,z_2\in\mathbb{C}^+$, we define a measure of separation

$$\gamma\,(z_1,z_2)=\frac{|z_1-z_2|}{\sqrt{\operatorname{Im}z_1}\sqrt{\operatorname{Im}z_2}}, \tag{3}$$

which is related to the hyperbolic distance $D(z_1,z_2)$ by the equation

$$\gamma\,(z_1,z_2)=2\sinh\left(\frac{1}{2}D(z_1,z_2)\right).$$

The relevance of hyperbolic distance to estimates of value distribution comes from the fact that if F_1 and F_2 are two Herglotz functions satisfying the estimate

$$\gamma\,(F_1(z),F_2(z))<\epsilon,$$

for all z such that $\operatorname{Im}z=d$ and $\operatorname{Re}z\in A$, then the value distribution $\mathcal{M}_2(A,S)$ associated with F_2 is a good approximation to the value distribution $\mathcal{M}_1(A,S)$ associated with F_1, in the sense that

$$\Big|\mathcal{M}_1(A,S)-\mathcal{M}_2(A,S)\Big|\le\epsilon\,|A|+2E_A(d). \tag{4}$$

Here $E_A(d)$ is an error estimate which is an increasing function of d, and which converges to zero in the limit $d\to 0$, for fixed Borel set A. For details of this and related results, see [BP1, BP2]. Estimates such as (4) imply that if $\{F^{(n)}\}_{n=1,2,3,\ldots}$ is a sequence of Herglotz functions converging uniformly to $F(z)$ for z lying in any fixed compact subset of $\mathbb{C}^+$, then the value distribution associated with $F^{(n)}$ will converge in the limit $n\to\infty$ to the value distribution associated with F.

The main purpose of this paper is to apply the above analysis to the spectral asymptotics of the Schrödinger operator with L^2-sparse potential, as described by solutions $f(x,\lambda)$ of the Schrödinger equation at real spectral parameter λ. Herglotz functions of particular interest in this context are the Weyl m-function $m(z)$ for the operator $-\frac{d^2}{dx^2}+V$ in $L^2\,(0,\infty)$ with Dirichlet boundary condition at $x=0$, and the Weyl m-function $m^N(z)$ for $-\frac{d^2}{dx^2}+V$ regarded as operating in $L^2\,(N,\infty)$ for some fixed $N>0$, with Dirichlet boundary condition at $x=N$. Estimates of both of these m-functions may be carried out, for z in some fixed compact subset of $\mathbb{C}^+$, in terms of the logarithmic derivative $f'(x,z)/f(x,z)$, for asymptotically large x, of particular solutions $f(\cdot,z)$ of the Schrödinger equation at *complex* spectral parameter z. The main general results of the paper are presented in Theorems 1 and 2. Theorem 1 provides an estimate of the large x

asymptotics of f'/f, based on an L^2 bound for the potential across a finite interval. Theorem 2 is an analysis of asymptotic value distribution in the case of L^2-sparse potential, linking this to the asymptotics of m^N.

Finally, we indicate some consequences of the analysis for spectral theory of L^2-sparse potentials, implying in particular the absence of absolutely continuous spectrum at negative λ.

2. Asymptotics of v'/v

We consider the differential expression $\tau = -\frac{d^2}{dx^2} + V(x)$ on the half-line $0 \leq x < \infty$, where the potential function V is assumed to be real valued and integrable over any finite subinterval of $[0, \infty)$. Assume limit-point case at infinity, implying that a self-adjoint operator $T = -\frac{d^2}{dx^2} + V(x)$ can be defined, acting in $L^2(0, \infty)$ and subject to a Dirichlet boundary condition at $x = 0$.

The Weyl m-function $m(z; V)$ may be defined in terms of solutions $f(\cdot, z)$ of the Schrödinger equation at complex spectral parameter z, namely

$$-\frac{d^2}{dx^2} f(x, z) + V(x) f(x, z) = z f(x, z) \qquad (\operatorname{Im} z > 0, 0 \leq x < \infty). \tag{5}$$

First define two solutions $u(x, z), v(x, z)$ of (5), subject respectively to initial conditions

$$u(0, z) = 1, \qquad u'(0, z) = 0, \tag{6a}$$

$$v(0, z) = 0, \qquad v'(0, z) = 1, \tag{6b}$$

where prime denotes differentiation with respect to x. (Solutions of (5) and (6) with z replaced by a real spectral parameter λ will be denoted by $u(x, \lambda), v(x, \lambda)$ respectively and, for fixed x, are the boundary values of $u(x, z)$, $v(x, z)$ as z approaches the real axis.)

Then (in the limit-point case at infinity) we define $m(z; V)$ uniquely by the condition that

$$u(\cdot, z) + m(z; V)\, v(\cdot, z) \in L^2(0, \infty).$$

An alternative characterisation of the Weyl function is that if $f(\cdot, z)$ is any (non-trivial) $L^2(0, \infty)$ solution of (5), then

$$m(z; V) = \frac{f'(0, z)}{f(0, z)}. \tag{7}$$

It follows from the limit point/limit circle theory [CL] that $m(z; V)$ is an analytic function of z for $\operatorname{Im} z > 0$. In addition $\operatorname{Im} m(z; V) > 0$ for $\operatorname{Im} z > 0$, so that $m(z; V)$ is a Herglotz function (analytic in the upper half-plane with positive imaginary part).

Given any $N > 0$, we can also define the Dirichlet m-function $m^N(z; V)$ for the truncated problem on the interval $N \leq x < \infty$, and an analysis of the large N asymptotics of m^N will play an important role in this paper. Here we are strongly motivated by the recent results of Deift and Killip [DK] for L^2 potentials.

Since, according to equation (7), the m-function is dependent on the logarithmic derivative of a solution $f(\cdot, z)$ of equation (5), a first step in our analysis will be to carry out a comparison between logarithmic derivatives of solutions of equation (5) as the potential is varied. We begin with the logarithmic derivative of the solution $v(\cdot, z)$ subject to the initial conditions (6b). Here it is $-v'/v$ rather than v'/v that is a Herglotz function for $x > 0$. The following elementary estimate provides a bound for the γ-separation of the logarithmic derivative as the potential is varied.

Lemma 1. *Let $v(x,z), \tilde{v}(x,z)$ be solutions of equation* (5) *with potentials $V(x)$, $\tilde{V}(x)$ respectively, and subject to initial conditions*

$$v(0,z) = \tilde{v}(0,z) = 0, \qquad v'(0,z) = \tilde{v}'(0,z) = 1.$$

Then, for any $x > 0$,

$$\gamma\left(-\frac{v'(x,z)}{v(x,z)}, -\frac{\tilde{v}'(x,z)}{\tilde{v}(x,z)}\right) \le (\operatorname{Im} z)^{-1} \frac{\left(\int_0^x \left(V(t) - \tilde{V}(t)\right)^2 |\tilde{v}(t,z)|^2 \, dt\right)^{1/2}}{\left(\int_0^x |\tilde{v}(t,z)|^2 \, dt\right)^{1/2}} \tag{8}$$

Proof. Abbreviating the notation for simplicity, we have

$$\gamma\left(-\frac{v'}{v}, -\frac{\tilde{v}'}{\tilde{v}}\right) = \frac{\left|\frac{v'}{v} - \frac{\tilde{v}'}{\tilde{v}}\right|}{\sqrt{\operatorname{Im}\left(-\frac{v'}{v}\right) \operatorname{Im}\left(-\frac{\tilde{v}'}{\tilde{v}}\right)}}, \tag{9}$$

where

$$\left|\frac{v'}{v} - \frac{\tilde{v}'}{\tilde{v}}\right| = \frac{|\tilde{v}v' - v\tilde{v}'|}{|v\tilde{v}|}. \tag{10}$$

Using the Schrödinger equation $-v'' + Vv = zv$, and similarly for $\tilde{v}$, we have

$$\frac{d}{dx}\left(\tilde{v}v' - v\tilde{v}'\right) = \tilde{v}v'' - v\tilde{v}'' = \left(V - \tilde{V}\right) v\tilde{v},$$

which, with the initial conditions, gives

$$\tilde{v}v' - v\tilde{v}' = \int_0^x \left(V(t) - \tilde{V}(t)\right) v(t)\tilde{v}(t)\, dt.$$

We also have

$$\operatorname{Im}\left(-\frac{v'}{v}\right) = \frac{1}{2i}\left(\frac{\overline{v}'}{\overline{v}} - \frac{v'}{v}\right) = \frac{1}{2i|v|^2}\left(v\overline{v}' - \overline{v}v'\right),$$

which again on considering $\frac{d}{dx}\left(v\overline{v}' - \overline{v}v'\right)$ gives

$$\operatorname{Im}\left(-\frac{v'}{v}\right) = \frac{\operatorname{Im} z}{|v|^2}\int_0^x |v(t)|^2 \, dt,$$

with a similar equation for $\tilde{v}$. Using (9) and (10), and substituting for $\operatorname{Im}(-v'/v)$, $\operatorname{Im}(-\tilde{v}'/\tilde{v})$ and $(\tilde{v}v' - v\tilde{v}')$ results in the bound

$$\gamma\left(-\frac{v'}{v}, -\frac{\tilde{v}'}{\tilde{v}}\right) = \frac{\left|\int_0^x \left(V(t) - \tilde{V}(t)\right) v(t)\tilde{v}(t)\, dt\right|}{(\operatorname{Im} z)\left(\int_0^x |v(t)|^2 \, dt \int_0^x |\tilde{v}(t)|^2 \, dt\right)^{1/2}},$$

from which (8) follows on applying Schwarz's inequality to the integral in the numerator. □

If both potentials $V, \tilde{V}$ are bounded, we can use the result of Lemma 1 to derive simple bounds for the separation γ between the two logarithmic derivatives. For example we have, from (8), for any $L > 0$,

$$\gamma\left(-\frac{v'}{v}, -\frac{\tilde{v}'}{\tilde{v}}\right)\Big|_{x=L} \le \frac{1}{\operatorname{Im} z} \sup_{t\in[0,L]} \left|V(t) - \tilde{V}(t)\right|.$$

In particular, we see that any *uniformly* convergent sequence V_n of potentials will result in a corresponding sequence $-v_n'/v_n$ which will converge uniformly in γ-separation (and hence also uniformly in the hyperbolic metric).

We turn now to the case of a potential subject to an L^2-type condition, for which we take in the first instance the comparison potential to be $\tilde{V}(x) = 0$. Let $v(x,z)$ be defined as before to be the solution of equation (5) with potential $V(x)$ and subject to $v(0,z)=0, v'(0,z)=1$, and let $v_0(x,z)$ satisfy the equation

$$-\frac{d^2 v_0(x,z)}{dx^2} = z v_0(x,z)$$

with the same initial conditions. Again we take $\operatorname{Im} z > 0$, and write $\sqrt{z} = a + ib$ with a, b real and $a, b > 0$. An explicit expression for v_0 is then

$$v_0(x,z) = \left(2i\sqrt{z}\right)^{-1}\left(\mathrm{e}^{ix\sqrt{z}} - \mathrm{e}^{-ix\sqrt{z}}\right) = \left(2\left(b - ia\right)\right)^{-1}\left(\mathrm{e}^{-iax}\mathrm{e}^{bx} - \mathrm{e}^{iax}\mathrm{e}^{-bx}\right),$$

so that

$$|v_0(x,z)|^2 = \left(2\left(a^2+b^2\right)\right)^{-1}\left(\cosh 2bx - \cos 2ax\right),$$

and, from (8), we have

$$\gamma\left(-\frac{v'}{v}, -\frac{v_0'}{v_0}\right)\Big|_{x=L} \le \frac{\left(\int_0^L V(t)^2 \left(\cosh 2bt - \cos 2at\right)\, dt\right)^{1/2}}{\operatorname{Im} z \left(\int_0^L \left(\cosh 2bt - \cos 2at\right)\, dt\right)^{1/2}}. \tag{11}$$

Here the integral in the numerator may be written

$$\begin{aligned}
-\int_0^L \left\{\left(\cosh 2bt - \cos 2at\right)\frac{d}{dt}\int_t^L V(s)^2\, ds\right\} dt &= \int_0^L \left\{\left(2b\sinh 2bt + 2a\sin 2at\right)\int_t^L V(s)^2\, ds\right\} dt \\
&\le \int_0^L \left\{\left(2b\sinh 2bt + 2a\right)\int_0^L V(s)^2\, ds\right\} dt \\
&= \left(2aL + \cosh 2bL - 1\right)\int_0^L V(s)^2\, ds \qquad (12)
\end{aligned}$$

To complete the estimate of (11), we need a lower bound for the denominator integral, which comes to

$$\frac{\sinh 2bL}{2b} - \frac{\sin 2aL}{2a}.$$

We shall make the assumption $L \ge 1/\sqrt{|z|}$. Such a condition, with $\sqrt{z} = a + ib$, implies that either $L \ge 1/\left(\sqrt{2}a\right)$ or $L \ge 1/\left(\sqrt{2}b\right)$. (If $L < 1/\left(\sqrt{2}a\right)$ and $L < 1/\left(\sqrt{2}b\right)$ then $|z| = a^2 + b^2 < \frac{1}{2L^2} + \frac{1}{2L^2} = \frac{1}{L^2}$, which contradicts the assumption.)

We consider the two possibilities in turn:

Case 1 : $L \ge \frac{1}{\sqrt{2}a}$

From the bound $\sinh x/x > 1$ for $x > 0$, we have

$$\frac{\sinh 2bL}{2b} > L,$$

whereas

$$\left|\frac{\sin 2aL}{2a}\right| \le \frac{1}{2a} \le \frac{L}{\sqrt{2}},$$

so that

$$\left|\frac{\sin 2aL}{2a}\right| < \frac{1}{\sqrt{2}}\frac{\sinh 2bL}{2b},$$

and it follows that

$$\frac{\sinh 2bL}{2b} - \frac{\sin 2aL}{2a} > \left(1 - \frac{1}{\sqrt{2}}\right)\frac{\sinh 2bL}{2b}. \tag{13}$$

Case 2 : $L \ge \frac{1}{\sqrt{2}b}$

Since the function $\sinh x/x$ is increasing for $x \geq 0$, we then have

$$\frac{\sinh 2bL}{2b} \geq \frac{L \sinh \sqrt{2}}{\sqrt{2}},$$

whereas

$$\left|\frac{\sin 2aL}{2a}\right| < L.$$

Hence in this case we find

$$\left|\frac{\sin 2aL}{2a}\right| < \frac{\sqrt{2}}{\sinh \sqrt{2}} \left(\frac{\sinh 2bL}{2b}\right),$$

so that

$$\frac{\sinh 2bL}{2b} - \frac{\sin 2aL}{2a} > \left(1 - \frac{\sqrt{2}}{\sinh \sqrt{2}}\right) \frac{\sinh 2bL}{2b}. \tag{14}$$

Noting that $\sinh \sqrt{2} < 2$, we see that the bound (14) holds both in case 1 and in case 2.

Using (12) and (14) as upper and lower bounds for the numerator and denominator respectively of (11), we have, now, for $L \geq 1/\sqrt{|z|}$, the estimate

$$\begin{aligned}
\gamma\left(-\frac{v'(L,z)}{v(L,z)}, -\frac{v_0'(L,z)}{v_0(L,z)}\right) &\leq \frac{1}{\operatorname{Im} z} \left(\frac{(2aL + \cosh 2bL - 1)\int_0^L V(s)^2\, ds}{\left(1 - \frac{\sqrt{2}}{\sinh \sqrt{2}}\right) \frac{\sinh 2bL}{2b}}\right)^{1/2} \\
&= \frac{1}{\operatorname{Im} z} \left(1 - \frac{\sqrt{2}}{\sinh \sqrt{2}}\right)^{-1/2} \left(2a \left(\frac{2bL}{\sinh 2bL}\right) + 2b \left(\frac{\cosh 2bL - 1}{\sinh 2bL}\right)\right)^{1/2} \\
&\qquad \times \left(\int_0^L V(s)^2\, ds\right)^{1/2}
\end{aligned}$$

Noting that

$$\frac{2bL}{\sinh 2bL} < 1$$

and that

$$\frac{\cosh 2bL - 1}{\sinh 2bL} = \tanh bL < 1,$$

we can use the estimate $(a+b)^{1/2} \leq \left(2\left(a^2+b^2\right)\right)^{1/4} = (2|z|)^{1/4}$ to obtain the following result.

Lemma 2. *Define $v(x,z)$ as in Lemma 1, and let $v_0(x,z)$ be the corresponding solution of* (5) *with zero potential. Then, for any $L \geq 1/\sqrt{|z|}$, we have the bound*

$$\gamma\left(-\frac{v'(L,z)}{v(L,z)}, -\frac{v_0'(L,z)}{v_0(L,z)}\right) \leq \frac{C|z|^{1/4}}{\operatorname{Im} z} \left(\int_0^L V(s)^2\, ds\right)^{1/2}, \tag{15}$$

where C is a positive constant. (In fact we can take $C = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sinh\sqrt{2}}\right)^{-1/2}$ in which case $C < 3.3$.)

Notice that Lemma 2 provides a simple bound for the hyperbolic distance between $-v'/v$ and $-v_0'/v_0$ at $x = L$, in terms of the L^2 norm of the potential V across the interval $[0, L]$.

Since, as is easily verified, we have

$$\lim_{L \to \infty} -\frac{v_0'(L,z)}{v_0(L,z)} = i\sqrt{z},$$

we can make a comparison, for large L, of $-v_0'/v_0$ with its asymptotic limit, leading to the following result.

Lemma 3. *With $v_0(x,z)$ defined as in Lemma 2, for any $L \geq 1/\sqrt{|z|}$ we have the bound*

$$\gamma\left(-\frac{v_0'(L,z)}{v_0(L,z)}, i\sqrt{z}\right) \leq C' \frac{\left(1+\left(\frac{b}{a}\right)^2\right)^{1/2}}{\left(e^{4bL}-1\right)^{1/2}}, \tag{16}$$

where C' is a positive constant. (In fact we can take $C' = 2^{1/4}C$, where C is the constant defined in Lemma 2, in which case $C' < 3.9$.)

Proof. Explicitly, we have

$$-\frac{v_0'}{v_0} = \frac{(ia-b)\left(e^{-iax}e^{bx} + e^{iax}e^{-bx}\right)}{e^{-iax}e^{bx} - e^{iax}e^{-bx}},$$

and multiplying numerator and denominator by the complex conjugate of the denominator gives

$$\operatorname{Im}\left(-\frac{v_0'}{v_0}\right) = \frac{2a\sinh 2bx - 2b\sin 2ax}{\left|e^{-iax}e^{bx} - e^{iax}e^{-bx}\right|^2},$$

Moreover,

$$\left|-\frac{v_0'}{v_0} - i\sqrt{z}\right| = \left|-\frac{v_0'}{v_0} + b - ia\right| = \frac{2\sqrt{a^2+b^2}e^{-bx}}{\left|e^{-iax}e^{bx} - e^{iax}e^{-bx}\right|}.$$

Putting these results together we find, at $x = L$,

$$\gamma\left(-\frac{v_0'}{v_0}, i\sqrt{z}\right)\Big|_{x=L} = \left(\frac{2\left(a^2+b^2\right)}{a}\right)^{1/2} \frac{e^{-bL}}{(a\sinh 2bL - b\sin 2aL)^{1/2}}.$$

Substituting in the denominator the lower bound obtained previously in (14) and simplifying, we arrive at (16). ☐

In using (16) to make precise estimates of the convergence to $i\sqrt{z}$ of $-v_0'/v_0$, it is useful to note the inequalities:

(i) if $\operatorname{Re} z \geq 0$ then $b/a \leq 1$;
(ii) $\left(1+\left(\frac{b}{a}\right)^2\right) \leq 4\left(1+\left(\frac{\operatorname{Re} z}{\operatorname{Im} z}\right)^2\right)$;
(iii) $b > \frac{\operatorname{Im} z}{2\sqrt{|z|}}$.

These inequalities imply, in particular, that $-v_0'/v_0$ converges uniformly in hyperbolic norm to $i\sqrt{z}$, for z in any fixed compact subset of the upper half-plane.

3. Estimates of u and v for L^1-bounded potentials

We consider solutions $u(x,z), v(x,z)$ of equation (5) on a fixed interval $0 \leq x \leq N$, subject to initial conditions (6) at $x = 0$. We compare these solutions with the corresponding solutions $u_0(x,z), v_0(x,z)$ with zero potential, and subject to the same initial conditions as for u and v.

Lemma 4. *Let K be a fixed compact subset of $\mathbb{C}^+$, and let $N > 0$ be fixed. Then, given any $\epsilon > 0$, there exists $\delta_0 > 0$ (δ_0 depending on ϵ, N and K) such that, for any potential function V satisfying*

$$\int_0^N |V(t)|\, dt < \delta_0,$$

we have, for all $z \in K$ and for all $x \in [0,N]$,

$$|u(x,z) - u_0(x,z)| < \epsilon, \qquad |v(x,z) - v_0(x,z)| < \epsilon.$$

Proof. The proof is a standard perturbation argument using the Gronwall inequality.

Let M be the 2×2 transfer matrix given by

$$M = M(x,z) = \begin{pmatrix} u & v \\ u' & v' \end{pmatrix},$$

and let

$$M_0 = \begin{pmatrix} u_0 & v_0 \\ u_0' & v_0' \end{pmatrix}.$$

Then

$$\frac{dM}{dx} = \begin{pmatrix} 0 & 1 \\ V-z & 0 \end{pmatrix} M, \qquad \frac{dM_0}{dx} = \begin{pmatrix} 0 & 1 \\ -z & 0 \end{pmatrix} M_0,$$

and we have

$$\frac{d}{dx}\left(M_0^{-1}M\right) = VA\left(M_0^{-1}M\right),$$

where

$$A = A(x,z) = (-v_0, u_0)^T (u_0, v_0)$$

and $V = V(x)$. Hence

$$\left(M_0^{-1}M\right)(x) = I + \int_0^x V(t)A(t)\left(M_0^{-1}M\right)(t)\,dt,$$

where I is the 2×2 identity matrix and, for notational convenience, we have suppressed the dependence on z. If $\|A\|$ denotes operator norm of the matrix A in the two-dimensional space l_2, we have, for $x \geq 0$,

$$\left\|\left(M_0^{-1}M\right)(x) - I\right\| \leq \int_0^x |V(t)|\,\|A(t)\|\,dt + \int_0^x |V(t)|\,\|A(t)\|\,\left\|\left(M_0^{-1}M\right)(t) - I\right\|\,dt.$$

An application of the Gronwall inequality now leads to the bound, valid for all $x \in [0,N]$,

$$\begin{aligned} \|M(x) - M_0(x)\| &\leq \|M_0(x)\|\,\left\|\left(M_0^{-1}M\right)(x) - I\right\| \\ &\leq \|M_0(x)\|\left\{\exp\left(\int_0^N |V(t)|\,\|A(t)\|\,dt\right) - 1\right\}. \end{aligned} \tag{17}$$

Noting that

$$\|M_0(x)\| \leq \left(|u_0|^2 + |v_0|^2 + |u_0'|^2 + |v_0'|^2\right)^{1/2},$$

and

$$\|A\| = |u_0|^2 + |v_0|^2,$$

we see that both $\|M_0(x,z)\|$ and $\|A(t,z)\|$ are bounded for $x,t \in [0,N]$ and $z \in K$.

The result of the Lemma now follows from (17) and the observation that

$$|u - u_0| \leq \|M - M_0\|, \qquad |v - v_0| \leq \|M - M_0\|.$$

□

The following Corollary is a straightforward consequence of the Lemma.

Corollary 1. *Let K be a fixed compact subset of $\mathbb{C}^+$, and let $N > 0$ be fixed. Define u, v, u_0, v_0 as in Lemma 4. Then given any $\epsilon > 0$, there exists $\delta_0 > 0$ (δ_0 depending on ϵ, N and K) such that, for all potential functions V satisfying $\int_0^N |V(t)|\, dt < \delta_0$, we have, for all $z \in K$,*

$$\left| \int_0^N \operatorname{Im}\left(\overline{u}(t,z)v(t,z)\right) dt - \int_0^N \operatorname{Im}\left(\overline{u_0}(t,z)v_0(t,z)\right) dt \right| < \epsilon. \tag{18}$$

4. Estimate of $-f'/f$ for potentials subject to an L^2-type condition

We can now state an estimate of convergence of $-f'/f$ to $i\sqrt{z}$ based on an L^2-type condition on the potential.

Theorem 1. *Let $f(x,z)$ be any solution for $x \in [0,\infty)$ of the Schrödinger equation (5) at complex spectral parameter z ($\operatorname{Im} z > 0$) which satisfies the condition*

$$\operatorname{Im}\left(-\frac{f'(0,z)}{f(0,z)}\right) > 0.$$

Let K be any fixed compact subset of $\mathbb{C}^+$.

Then, given any $\epsilon > 0$, there exist $\delta, N > 0$ (δ, N depending on ϵ and K) such that, for all $L \geq N$ and for all potential functions V satisfying the L^2 bound

$$\int_0^L |V(t)|^2\, dt < \delta,$$

the estimate

$$\gamma\left(-\frac{f'(L,z)}{f(L,z)}, i\sqrt{z}\right) < \epsilon \tag{19}$$

holds for all $z \in K$.

Proof. In using the γ measure of separation to carry out the estimate (19), it should be noted that, unlike the hyperbolic metric which is a function of γ, the separation $\gamma(z_1, z_2)$ between two points $z_1, z_2 \in \mathbb{C}^+$ does not satisfy the triangle inequality. However, the following result can be useful as a substitute for the triangle inequality:

If $z_1, z_2, z_3 \in \mathbb{C}^+$ and it is given that

$$\gamma(z_1,z_2) < \alpha, \qquad \gamma(z_2,z_3) < \beta, \qquad \text{with} \qquad 0 < \alpha, \beta \leq 2,$$

then it follows that $\gamma(z_1,z_3) < \sqrt{2}\,(\alpha+\beta)$. (To verify this result, note that if $0 < \alpha, \beta \leq 2$ and

$$\gamma(z_1,z_2) = 2\sinh\left(\frac{D(z_1,z_2)}{2}\right) < \alpha, \qquad \gamma(z_2,z_3) = 2\sinh\left(\frac{D(z_2,z_3)}{2}\right) < \beta,$$

then

$$\begin{aligned}
\gamma(z_1,z_3) &= 2\sinh\left(\frac{D(z_1,z_3)}{2}\right) \\
&\leq 2\sinh\left(\frac{D(z_1,z_2)}{2} + \frac{D(z_2,z_3)}{2}\right) \\
&= 2\sinh\left(\frac{D(z_1,z_2)}{2}\right)\cosh\left(\frac{D(z_2,z_3)}{2}\right) + 2\sinh\left(\frac{D(z_2,z_3)}{2}\right)\cosh\left(\frac{D(z_1,z_2)}{2}\right) \\
&\leq \alpha\sqrt{1+\frac{\beta^2}{4}} + \beta\sqrt{1+\frac{\alpha^2}{4}} \\
&\leq (\alpha+\beta)\sqrt{2}
\end{aligned}$$

as required.) As a simple consequence of this result, the three inequalities $\gamma(z_1, z_2) < \frac{\epsilon}{6}, \gamma(z_2, z_3) < \frac{\epsilon}{6}, \gamma(z_3, z_4) < \frac{\epsilon}{6}$, with $0 < \epsilon < 1$, together imply that $\gamma(z_1, z_4) < \epsilon$.

If, then, we define u, v, u_0, v_0 as in the proofs of the previous Lemmas, it will be sufficient, to verify (19), to show that if $z \in K$ then we have the three inequalities, at $x = L$,

$$\gamma\left(-\frac{f'}{f}, -\frac{v'}{v}\right) < \frac{\epsilon}{6}, \qquad \gamma\left(-\frac{v'}{v}, -\frac{v_0'}{v_0}\right) < \frac{\epsilon}{6}, \qquad \gamma\left(-\frac{v_0'}{v_0}, i\sqrt{z}\right) < \frac{\epsilon}{6}. \tag{20}$$

We begin by fixing the value of N. Given $\epsilon > 0$ and a compact subset K of $\mathbb{C}^+$, we take $N = N(\epsilon, K)$ to satisfy, for all $z \in K$, the three inequalities

$$\int_0^N \operatorname{Im}\left(\overline{u_0} v_0\right) dt > \frac{12}{\epsilon \operatorname{Im} z}, \tag{21a}$$

$$\frac{C'\left(1 + \left(\frac{b}{a}\right)^2\right)^{1/2}}{\left(\mathrm{e}^{4bN} - 1\right)^{1/2}} < \frac{\epsilon}{6}, \tag{21b}$$

$$N > \frac{1}{\sqrt{|z|}}. \tag{21c}$$

That N may be chosen to satisfy the first of these inequalities for $z \in K$ follows from the fact that $\int_0^\infty \operatorname{Im}\left(\overline{u_0} v_0\right) dt = \infty$ and that, for fixed N, the integral $\int_0^N \operatorname{Im}\left(\overline{u_0} v_0\right) dt$ depends continuously on z for $\operatorname{Im} z > 0$. In the second inequality we have $\sqrt{z} = a + ib$, where both b and b/a are bounded for $z \in K$; the constant C' is defined in the proof of Lemma 3. Note also that $1/\sqrt{|z|}$ is bounded for $z \in K$ in the third inequality.

From the Corollary to Lemma 4 we know that, for $z \in K$, the integral $\int_0^N \operatorname{Im}\left(\overline{u} v\right) dt$ is close to $\int_0^N \operatorname{Im}\left(\overline{u_0} v_0\right) dt$ provided that $\int_0^N |V(t)|\, dt$ is sufficiently small. In particular, the inequality (21a) implies that there exists $\delta_0 = \delta_0(\epsilon, K) > 0$ such that, for all $z \in K$, we have

$$\int_0^N |V(t)|\, dt < \delta_0 \Rightarrow \int_0^N \operatorname{Im}\left(\overline{u} v\right) dt > \frac{6}{\epsilon \operatorname{Im} z}. \tag{22}$$

Having fixed the values of N and δ_0, now define $\delta = \delta(\epsilon, K)$ to satisfy the two inequalities

(i) $N\delta < \delta_0^2$;
(ii) $\frac{C|z|^{1/4}}{\operatorname{Im} z}\sqrt{\delta} < \frac{\epsilon}{6}$ for all $z \in K$.

Here the constant C has been defined in the statement of Lemma 2. Now suppose that $L \geq N$ and $\int_0^L |V(t)|^2\, dt < \delta$. By the Schwarz inequality we then have

$$\int_0^N |V(t)|\, dt \leq \left(N \int_0^N |V(t)|^2\, dt\right)^{1/2} < (\delta N)^{1/2} < \delta_0,$$

by inequality (i). Hence, (22) implies that

$$\int_0^N \operatorname{Im}\left(\overline{u} v\right) dt > \frac{6}{\epsilon \operatorname{Im} z}.$$

By Lemma 3 of [BP1] (see also Lemma 2 of [BP2]) we have, for any solution f of (5) satisfying $\operatorname{Im}\left(-f'(0,z)/f(0,z)\right) > 0$,

$$\gamma\left(-\frac{f'}{f}, -\frac{v'}{v}\right)\Bigg|_{x=L} \leq \frac{1}{\operatorname{Im} z \int_0^L \operatorname{Im}\left(\overline{u} v\right) dt} < \frac{\epsilon}{6}.$$

Thus we have derived the first inequality in (20). The second inequality in (20) follows from (15) and (ii) above, using $\int_0^L |V(t)|^2\,dt < \delta$. We can also use Lemma 3 with the inequality (21b) to complete the proof of (20), which also completes the proof of the Theorem. □

We now explore some consequences of Theorem 1 in the case of L^2-sparse potentials. Let V be an L^2-sparse potential. Then a sequence of subintervals $\{(a_k, b_k)\}$ $(k = 1, 2, 3, \dots)$ of $\mathbb{R}^+$ can be found such that, with $L_k = b_k - a_k$,

$$\lim_{k\to\infty} L_k = \infty \qquad \text{and} \qquad \lim_{k\to\infty} \int_{a_k}^{b_k} (V(t))^2\,dt = 0.$$

Given a fixed, bounded, measurable subset A of $\mathbb{R}$, having closure $\overline{A}$, and given any $\epsilon > 0$, we first of all find $d > 0$ (d depending on ϵ and A) such that $E_A(d) < \epsilon\,|A|/2$. Here $E_A(\cdot)$ is the error estimate on the right hand side of (4), and from (4) we deduce that

$$|\mathcal{M}_1(A, S) - \mathcal{M}_2(A, S)| < 2\epsilon\,|A|\,, \tag{23}$$

provided $\gamma\,(F_1(z), F_2(z)) < \epsilon$ for all $z \in K$, where K is the compact subset of $\mathbb{C}^+$ defined by the conditions $\operatorname{Im} z = d, \operatorname{Re} z \in \overline{A}$.

Now use Theorem 1 to define δ and N such that, for all $L \geq N$ and for all potentials V satisfying the bound $\int_0^L |V(t)|^2\,dt < \delta$ we have

$$\gamma\left(-\frac{f'(L,z)}{f(L,z)}, i\sqrt{z}\right) < \epsilon. \tag{24}$$

Here $f(\cdot, z)$ is a solution of the Schrödinger equation (5) for which

$$\operatorname{Im}\left(-\frac{f'(0,z)}{f(0,z)}\right) > 0.$$

We take k sufficiently large (say $k > k_0$) so that $L_k \geq N$ and such that the bound $\int_{a_k}^{b_k} |V(t)|^2\,dt < \delta$ is satisfied by our sparse potential V.

We can now apply (24) with $L = L_k$, where f is a suitably chosen solution of the Schrödinger equation (5), but with potential modified by an appropriate change of x-coordinate. There are two separate cases to be considered:

Firstly, define $f(x, z) = v(x + a_k, z)$ (for $0 \leq x \leq L_k = b_k - a_k$). Then, for $x \in [0, L_k]$, $f(\cdot, z)$ satisfies the Schrödinger equation (5) with potential $V(x + a_k)$. Moreover, we have

$$\int_0^{L_k} (V(t + a_k))^2\,dt = \int_{a_k}^{b_k} (V(t))^2\,dt < \delta.$$

Hence (24) is satisfied in this case, and we have

$$\gamma\left(-\frac{v'(b_k,z)}{v(b_k,z)}, i\sqrt{z}\right) < \epsilon.$$

From (23) we now deduce that the respective value distributions for the Herglotz functions $-v'(b_k, z)/v(b_k, z)$ and $i\sqrt{z}$ differ by at most $2\epsilon\,|A|$, for all $k > k_0$.

Secondly, let $F(\cdot, z)$ be a (non-trivial) solution in $L^2\,(0, \infty)$ of the Schrödinger equation (5), with sparse potential V. The m-function $m^{a_k}(z)$ for the Schrödinger operator $-\frac{d^2}{dx^2} + V$ acting in $L^2\,(a_k, \infty)$ is then given by

$$m^{a_k}(z) = \frac{F'(a_k, z)}{F(a_k, z)}.$$

We can now define $f(\cdot, z)$ by

$$f(x,z) = F(b_k - x, z) \qquad (0 \le x \le L_k)$$

so that $f(\cdot, z)$ satisfies the Schrödinger equation with potential $V(b_k - x)$. Since $F'(b_k, z)/F(b_k, z)$ has positive imaginary part, we also have $\mathrm{Im}\,(-f'(0,z)/f(0,z)) > 0$. In this case, an application of (19) with $L = L_k$ results in the estimate

$$\gamma\left(m^{a_k}(z), i\sqrt{z}\right) < \epsilon,$$

and it follows as before that the respective value distributions for the Herglotz functions m^{a_k} and $i\sqrt{z}$ differ by at most $2\epsilon\,|A|$, for all $k > k_0$.

The following Theorem summarises the situation regarding asymptotic value distribution in the case of L^2-sparse potentials[1]. The Theorem implies in particular, for the special case of L^2 potentials, that the value distribution of $v'(N,\lambda)/v(N,\lambda)$ approaches an asymptotic limit as $N \to \infty$.

Theorem 2. *Let $v(\cdot,\lambda)$ be the solution of the Schrödinger equation at real spectral parameter λ, subject to initial conditions $v(0,\lambda) = 0, v'(0,\lambda) = 1$, in the case of an L^2-sparse potential V.*

Let $\{(a_k, b_k)\}$ be a sequence of subintervals of $\mathbb{R}^+$, for which $\lim_{k\to\infty}(b_k - a_k) = \infty$ and $\lim_{k\to\infty}\int_{a_k}^{b_k}|V(t)|^2\,dt = 0$.

Then for Borel subsets A, S of $\mathbb{R}$, with $|A| < \infty$, we have

$$\lim_{k\to\infty}\frac{1}{\pi}\int_A \theta\left(m_+^{a_k}(\lambda), S\right)d\lambda = \frac{1}{\pi}\int_A \theta\left(i\sqrt{\lambda}, S\right)d\lambda,$$

$$\lim_{k\to\infty}\left|\left\{\lambda \in A : \frac{v'(b_k,\lambda)}{v(b_k,\lambda)} \in S\right\}\right| = \frac{1}{\pi}\int_A \theta\left(i\sqrt{\lambda}, -S\right)d\lambda.$$

The conclusion of the Theorem, which applies in the first instance in the case that A is bounded and of finite measure, may be extended to the more general case in which A is not necessarily bounded. (Let A have finite measure. Given $\epsilon > 0$, fix N sufficiently large that the complement of $[-N,N]\cap A$ has measure less than ϵ. Denoting by A_N this truncated set, the theorem may be applied first of all to A_N, which is bounded. Since the integrals to be estimated are then within ϵ of the corresponding integrals for the set A, the more general conclusion follows on letting ϵ approach zero.)

5. Spectral analysis

Here we present some consequences of Theorem 2 for the spectral theory of Schrödinger operators with L^2-sparse potentials. The first result implies that absolutely continuous spectrum can occur only for $\lambda > 0$.

Corollary 2. *Suppose V is L^2-sparse. Then the support of the a.c. measure μ_{ac} of $T = -\frac{d^2}{dx^2} + V$ is contained in $\mathbb{R}^+$.*

Proof. Suppose the contrary. Then if μ_{ac} is the a.c. part of the spectral measure, we can find a subset A of $\mathbb{R}^-$ having finite Lebesgue measure for which $\mu_{ac}(A) > 0$. Then $|A| > 0$, and we may also suppose that A is a subset of an essential support of μ_{ac}.

[1]We are indebted to A. Pushnitski for pointing out the close connection between estimates of m-functions at complex z and asymptotic resolvent estimates in the case of potentials with an L^2 condition.

Now define intervals (a_k, b_k) as in Theorem 2, and set $N_k = (a_k + b_k)/2$. Then N_k may be regarded either as the left hand endpoint of an interval (N_k, b_k), or as the right hand endpoint of an interval (a_k, N_k). An application of Theorem 2 then implies that

$$\lim_{k\to\infty} \frac{1}{\pi} \int_A \theta\left(m_+^{N_k}(\lambda), S\right) d\lambda = \frac{1}{\pi} \int_A \theta\left(i\sqrt{\lambda}, S\right) d\lambda, \tag{25}$$

whereas

$$\lim_{k\to\infty} \left|\left\{\lambda \in A : \frac{v'(N_k,\lambda)}{v(N_k,\lambda)} \in S\right\}\right| = \frac{1}{\pi} \int_A \theta\left(i\sqrt{\lambda}, -S\right) d\lambda. \tag{26}$$

Since A is a subset of an essential support of μ_{ac}, we also have

$$\lim_{k\to\infty} \left[\left|\left\{\lambda \in A : \frac{v'(N_k,\lambda)}{v(N_k,\lambda)} \in S\right\}\right| - \frac{1}{\pi} \int_A \theta\left(m_+^{N_k}(\lambda), S\right) d\lambda\right] = 0. \tag{27}$$

(For a proof of this result, which holds for any sequence N_k with $N_k \to \infty$, and for arbitrary locally L^1 potentials, see [BP1].) Equations (25), (26) and (27) now imply that

$$\int_A \theta\left(i\sqrt{\lambda}, S\right) d\lambda = \int_A \theta\left(i\sqrt{\lambda}, -S\right) d\lambda. \tag{28}$$

However $i\sqrt{\lambda} \in \mathbb{R}^-$ for $\lambda \in A$, and taking $S = \mathbb{R}^-$ we see that the left-hand-side of (28) is strictly positive, whereas the right-hand-side is zero.

Hence we have a contradiction, and the Corollary is proved. □

There are interesting applications of Corollary 2 to L^2 perturbations of slowly oscillating potentials such as $\cos\sqrt{x}$. For example, if $V(x) = \cos\sqrt{x} + V_0$ with $V_0 \in L^2(\mathbb{R}^+)$, then $V(x) - 1$ is an L^2-sparse potential, and it follows from Corollary 2 that $T = -\frac{d^2}{dx^2} + V$ has no a.c. measure for $\lambda < 1$. (In fact, $[-1, 1]$ is contained in the singular spectrum of T; for related results on spectral theory with slowly oscillating potentials see [S].)

We can also consider various perturbations of L^2-sparse potentials. A typical result is the following:

Corollary 3. *Let V be a L^2-sparse potential. Define intervals $\{(a_k, b_k)\}$, with $N_k = (a_k + b_k)/2$, as in the proof of Corollary 2. Then the Schrödinger operator*

$$-\frac{d^2}{dx^2} + V(x) + \sum_{k=1}^{\infty} \delta(x - N_k)$$

has purely singular spectral measure.

Proof. The proof follows from Theorem 2, using similar arguments to those applied in [BP1, BP2] to the special case in which V is a sparse rather than L^2-sparse potential. □

Acknowledgements. This work has been partially supported by the EPSRC.

References

[BP1] Breimesser S.V., Pearson D.B.: Asymptotic value distribution for solutions of the Schrödinger equation, *Mathematical Physics, Analysis and Geometry* **3**, 385–403 (2000).

[BP2] Breimesser S.V., Pearson D.B.: Geometrical aspects of spectral theory and value distribution for Herglotz functions. Preprint 2001.

[CL] Coddington E.A., Levinson N.: *Theory of Ordinary Differential Equations* (McGraw-Hill, New York, 1955).

[DK] Deift R., Killip R.: On the absolutely continuous spectrum of one-dimensional Schroedinger operators with square summable potentials, *Communications in Mathematical Physics* **203**, 341–347 (1999).

[G] Glazman I.M.: *Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators* (Israel program for scientific translations, Jerusalem, 1965).

[KLS] Kiselev A., Last Y., Simon B.: Modified Prüfer and EFGP transforms and the spectral analysis of one-dimensional Schrödinger operators, *Communications in Mathematical Physics* **194**, 1–45 (1998).

[R] Remling C.: A probabilistic approach to one-dimensional Schrödinger operators with sparse potentials, *Communications in Mathematical Physics* **185**, 313–323 (1997).

[SS] Simon B., Stolz G.: Operators with singular continuous spectrum, V. Sparse potentials, *Proceedings of the American Mathematical Society* **124**, 2073–2080 (1996).

[S] Stolz G.: Spectral theory for slowly oscillating potentials II. Schrödinger operators, *Mathematische Nachrichten* **183**, 275–294 (1997).

Department of Mathematics, University of Hull, Hull HU6 7RX, U.K.
E-mail address: `s.v.breimesser@hull.ac.uk`
E-mail address: `j.d.grant@hull.ac.uk`
E-mail address: `d.b.pearson@hull.ac.uk`