SYMMETRIES AND MODULI SPACES OF THE SELF-DUAL YANG-MILLS EQUATIONS

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ABSTRACT. We review the construction of infinite-dimensional symmetry algebras of the self-dual Yang-Mills equations on $\mathbb{R}^4$ and the ADHM description of the moduli space of instantons on S^4. We report on recent work describing the action of the corresponding symmetry (pseudo)-group on the instanton moduli spaces.

1. INTRODUCTION

In Hamiltonian mechanics, one has a well defined notion of an integrable system. Such a system is defined by a $2n$-dimensional symplectic manifold $(X, \boldsymbol{\omega})$ with the dynamics being determined by a function $H : X \to \mathbb{R}$. The system is completely integrable in the sense of Liouville if there exist n functions preserved by the flow generated by the Hamiltonian that have vanishing Poisson brackets with one another and that are functionally independent on X, perhaps minus a set of measure zero, S. From a more global perspective, this implies that there is an action of an n-dimensional Abelian Lie group on $X \setminus S$ that, in the case that this group is compact, corresponds to a Hamiltonian torus action. Thus the space $X \setminus S$ is foliated by n-dimensional tori.

When one considers systems of partial differential equations, it is more difficult to define a suitable notion of integrability. One approach has been to study systems that admit infinite-dimensional symmetry algebras. In order to make contact with the finite-dimensional theory, however, one would like to know whether there is a corresponding group action on the space of solutions of the system and, if so, what the corresponding orbit structure is. As an example for particular types of harmonic maps, the action of the corresponding symmetry group (the *dressing action*) on the space of solutions is well understood (see [Gu] and references therein).

A set of equations that is of particular interest from the point of view of integrable systems theory is the self-dual Yang-Mills equations, which is a system of partial differential equations defined on an arbitrary oriented four-manifold, X. If the four-manifold is self-dual then the self-dual Yang-Mills equations are generally considered to be integrable in the sense that there is a twistor-theoretic method of constructing solutions which is analogous to the inverse-scattering methods

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ubiquitous in standard integrable systems theory [AHS, WW, MW]. Most known integrable systems can be derived as symmetry reductions of the self-dual Yang-Mills equations on a self-dual manifold for particular choices of gauge group and manifold X [MW]. It is therefore important to know whether there is a corresponding group action on the space of solutions to the self-dual Yang-Mills equations corresponding to the above symmetry algebra and, if so, what the orbits of this group action are.

The purpose of this article is twofold. Firstly to review the relevant literature and results on the various aspects of the self-dual Yang-Mills equations that this problem entails, and secondly to briefly present some preliminary results on the action of non-local symmetries on instanton moduli spaces (for full details see [Gr1]). The problem of understanding the orbit structure lies at the interface between local considerations (i.e. symmetry algebras of differential equations) and global ones (moduli spaces of instanton solutions). One of our aims at this point is give a unified notation appropriate to both points of view. For simplicity we restrict ourselves to the self-dual Yang-Mills equations on $\mathbb{R}^4$ and take the gauge group to be SU(2). In this case one can construct explicitly an infinite-dimensional algebra of non-local symmetries of the self-dual Yang-Mills equations [Ch, D]. Since for standard integrable systems (e.g. the KdV equation) it is necessary to restrict oneself to a suitable class of solutions in order for inverse-scattering methods to work (e.g. periodic solutions or solutions with rapid asymptotic fall-off), we restrict ourselves to instanton solutions of the self-dual Yang-Mills equations on $\mathbb{R}^4$ (i.e. solutions with curvature that is L^2). By conformal invariance and a theorem of Uhlenbeck [U], this means we may equivalently consider the instanton problem on S^4. In this case we have a full description of the moduli space of solutions of the self-dual Yang-Mills equations given by the ADHM construction [ADHM, At1]. Our main result (see Section 4) is that the tangent space to each instanton moduli space is generated by non-local symmetries of the form given in [Ch, D]. As such, since the instanton moduli spaces are connected [Do], the corresponding symmetry pseudo-group acts transitively on them.

We end by discussing some further lines of research that we are pursuing.

2. The self-dual Yang-Mills equations

Let $(X, \mathbf{g})$ be a connected, oriented, Riemannian four-manifold. Since X is oriented, we have a volume form $\boldsymbol{\nu} \in \Omega^4(X)$, and thus we may define a Hodge $*$ operation

$$* : \Omega^p(X) \to \Omega^{4-p}(X), \qquad p = 0, \ldots, 4,$$

by the relation

$$\boldsymbol{\alpha} \wedge *\boldsymbol{\beta} = \langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle \, \boldsymbol{\nu}, \qquad \forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \Omega^p(X).$$

When restricted to $\Omega^2(X)$ on a Riemannian four-manifold, the map $*$ has two important properties:

- It is conformally invariant;

- $\left(*|_{\Omega^2(X)}\right)^2 = \mathrm{Id}|_{\Omega^2(X)}$. We therefore have a direct sum decomposition

$$\Omega^2(X) = \Omega^{2+}(X) \oplus \Omega^{2-}(X) \tag{$*$}$$

of the space of two-forms into self-dual and anti-self-dual two-forms.

Let $\pi : E \to X$ be a vector bundle over X with structure group G. A connection on E may be represented by a $\mathfrak{g}$-valued one-form $\mathbf{A} \in \Omega^1(X, \mathfrak{g})$, with curvature $\mathbf{F_A} \in \Omega^2(X, \mathfrak{g})$. Using the decomposition of $\Omega^2(X)$ we may therefore write

$$\mathbf{F_A} = \mathbf{F}^+_\mathbf{A} + \mathbf{F}^-_\mathbf{A},$$

where $\mathbf{F}^\pm_\mathbf{A} \in \Omega^{2\pm}(X, \mathfrak{g})$ are the self-dual and anti-self-dual parts of the curvature.

Definition 2.1. A connection on a vector bundle $\pi : E \to X$ is a solution of the *self-dual Yang-Mills equations* if its curvature obeys the condition

$$*\mathbf{F_A} = \mathbf{F_A}.$$

(Equivalently $\mathbf{F}^-_\mathbf{A} = 0$.)

If we take E to be TX, the tangent bundle of X, with the Levi-Civita connection then the curvature of the connection is the Riemann tensor, $\mathbf{R}$. Viewing this as an element of $S^2\left(\Omega^2(X)\right)$ we may decompose it in block diagonal form relative to the decomposition (*) to yield

$$\mathbf{R} = \begin{pmatrix} W^+ + \frac{s}{12} & \mathrm{Ric}_0 \\ \mathrm{Ric}_0 & W^- + \frac{s}{12} \end{pmatrix}$$

where $W^\pm$ denote the self-dual and anti-self-dual parts of the Weyl tensor, Ric_0 denotes the trace-free part of the Ricci tensor, and s denotes the scalar curvature of the metric $\mathbf{g}$.

Definition 2.2. A Riemannian manifold $(X, \mathbf{g})$ is *self-dual* if the anti-self-dual part of the Weyl tensor vanishes:

$$W^- = 0.$$

From now on we fix X to be S^4 with its standard (self-dual) conformal structure. For simplicity we take $G = \mathrm{SU}(2) \cong \mathrm{Sp}(1)$, although it is straightforward to generalise our results to other groups.

2.1. **Instanton numbers and index theorem results.**

Lemma 2.3. [BPST, AHS] *Let $\pi : E \to S^4$ be a rank-2 Hermitian complex vector bundle associated to a principal $\mathrm{SU}(2)$ bundle $P \to S^4$. Such bundles are characterised by the second Chern class*

$$c_2(E) := -\frac{1}{8\pi^2}\int_X \|\mathbf{F_A}\|^2\, d\mathbf{vol}_X,$$

where $\mathbf{A}$ *is any connection on* E *and* $\mathbf{F_A} \in \Omega^2(S^4, \mathrm{End}(E))$ *is its curvature. If* $\int_X \|\mathbf{F_A}\|^2 \, d\mathbf{vol}_X < \infty$*, then* $p_2(E) = -k$ *where* k *is an integer, referred to as the* instanton number*. If the connection* $\mathbf{A}$ *is a self-dual, then* $k \geq 0$*, with equality if and only if the connection is flat.*

There is a natural action of the group of gauge transformations (i.e. maps $S^4 \to \mathrm{SU}(2)$) on the space of connections. We therefore define the moduli space of k-instanton solutions:

$$\mathcal{M}_k := \frac{\{(E, \mathbf{A}) : \mathbf{F_A} = *\mathbf{F_A}, c_2(E) = -k\}}{\text{Gauge transformations}}.$$

Theorem 2.4. [AHS] $\mathcal{M}_k$ *is a manifold (possibly with singularities corresponding to reducible connections) of dimension* $8k - 3$*.*

2.2. **The ADHM construction.** The ADHM construction [ADHM, At1] allows us to construct the $(8k-3)$-parameter family of solutions of the self-dual Yang-Mills equations on S^4 using quaternionic linear algebra. Viewing S^4 as $\mathbb{H}P^1$ then a point $p \in S^4$ corresponds to a quaternionic line $\Sigma \subset \mathbb{H}^2$. Choosing a complex structure on $\mathbb{H}^2$, we can identify it with $\mathbb{C}^4$, with Σ corresponding to a complex surface. Since, as a subspace of $\mathbb{H}^2$, Σ is invariant under right multiplication by the unit quaternion j, this implies that as a subspace of $\mathbb{C}^4$ it is invariant under the corresponding anti-linear anti-involution

$$\sigma : \mathbb{C}^4 \to \mathbb{C}^4 : (z_1, z_2, z_3, z_4) \mapsto (-\overline{z_2}, \overline{z_1}, -\overline{z_4}, \overline{z_3}).$$

Under the natural projection $\mathbb{C}^4 \to \mathbb{C}P^3$ the surface Σ projects to a rational curve which we denote by $\sigma(p) \cong \mathbb{C}P^1 \subset \mathbb{C}P^3$. Curves that arise in this way are called *real lines* in $\mathbb{C}P^3$. The involution on $\mathbb{C}P^3$ induced by σ leaves the real lines invariant, and acts as the anti-podal map on each real $\mathbb{C}P^1$.

Theorem 2.5. [Wa] *There is a bijective correspondence between a). Solutions of the self-dual Yang-Mills equations on* S^4 *and b). holomorphic vector bundles on* $\mathbb{C}P^3$ *that are (holomorphically) trivial when restricted to each real line. We refer to this correspondence as the* Ward correspondence*.*

Remark. The holomorphic bundle over $\mathbb{C}P^3$ will carry additional structures depending on the particular group G that we are considering. For the case $G = \mathrm{SU}(2)$, we require that the determinant bundle $\det E$ is trivial and that E admits a positive real form.

The ADHM construction uses methods from algebraic geometry to construct holomorphic vector bundles on $\mathbb{C}P^3$ and therefore, via the Ward correspondence, self-dual Yang-Mills connections on S^4. For each $z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4$, we define a linear map

$$A(z) : W \to V,$$

where W, V are complex vector spaces of dimension $k, 2k+2$ respectively, which is of the form

$$A(z) = \sum_{i=1}^{4} z_i A_i.$$

The space W is assumed to admit an anti-linear involution $\sigma_W : W \to W$. The space V is assumed to have a non-degenerate, skew-symmetric form $(\cdot,\cdot)$ and an anti-linear anti-involution $\sigma_V : V \to V$ that is compatible with the form in the sense that

$$(\sigma_V u, \sigma_V v) = \overline{(u, v)}, \qquad \forall\, u, v \in V.$$

We then require that the map $A(z)$ satisfies the compatibility condition that

$$\sigma_V\left(A(z)w\right) = A(\sigma(z))\sigma_W(w), \qquad \forall\, w \in W \tag{2.1}$$

and impose the following conditions:

- For all $z \in \mathbb{C}^4$, the space $U_z := A(z)(W) \subset V$ is of dimension k;
- For all $z \in \mathbb{C}^4$, U_z is isotropic with respect to $(\cdot,\cdot)$ i.e. $U_z \subseteq U_z^{\perp}$, where $^{\perp}$ denotes the complement with respect to the form $(\cdot,\cdot)$.

If we then define the quotient $E_z := U_z^{\perp}/U_z$, then the dimension and isotropy constraints on U_z imply that the collection of E_z defines a holomorphic, rank-2 complex vector bundle $E \to \mathbb{C}P^3$ with structure group $\mathrm{SL}(2,\mathbb{C})$. The reality condition (2.1) then imply that the bundle is trivial on restriction to any real line and that the structure group reduces from $\mathrm{SL}(2,\mathbb{C})$ to $\mathrm{SU}(2)$.

The power of the ADHM construction is that all k-instanton solutions of the self-dual Yang-Mills equations arise in this fashion.

2.3. **Patching matrix description.** Given a finite point $\mathbf{x} := (x^1, x^2, x^3, x^4) \in \mathbb{R}^4 \subset S^4$, then we may view

$$x := x^1 + ix^2 + jx^3 + kx^4 = u + jv \in \mathbb{H}$$

as an affine coordinate on $\mathbb{R}^4 \cong \mathbb{H} \subset \mathbb{H}P^1$, where

$$u := x^1 + ix^2, \qquad v := x^3 - ix^4.$$

As mentioned above, the point $\mathbf{x}$ therefore defines a quaternionic line $\Sigma(\mathbf{x}) \subset \mathbb{H}^2$ and thence a real line $\sigma(\mathbf{x}) \cong \mathbb{C}P^1 \subset \mathbb{C}P^3$ which takes the form

$$\sigma(\mathbf{x}) = \Big\{ [z_1, z_2, z_1\, u - z_2\, \overline{v}, z_1\, v + z_2\, \overline{u}] \,:\, (z_1, z_2) \in \mathbb{C}^2 \setminus \{(0,0)\} \Big\}.$$

On a fixed real line $\sigma(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^4$, we introduce the affine coordinate $z = z_2/z_1 \in \mathbb{C}P^1$. The image of point $\infty \in S^4$ is the real line

$$l_\infty := \big\{ [0, 0, z_3, z_4] \,:\, (z_3, z_4) \in \mathbb{C}^2 \setminus \{(0,0)\} \big\}.$$

Given an instanton solution on S^4, we may consider the restriction of the solution to $\mathbb{R}^4 \subset S^4$. (Uhlenbeck's theorem [U] implies that there is no loss of information in doing so.) We may therefore consider the corresponding restriction of the bundle $E|_{\mathbb{C}P^3\setminus l_\infty}$ which, for convenience, we denote by $E \to \mathbb{C}P^3 \setminus l_\infty$. We may split the region $\mathbb{C}P^3 \setminus l_\infty$ into coordinate regions

$$\mathcal{S}_0 := \left\{((u,v),z) \in \mathbb{C}^2 \times \mathbb{C}P^1 : z \neq \infty\right\} = \mathbb{C}^2 \times U_0,$$
$$\mathcal{S}_\infty := \left\{((u,v),z) \in \mathbb{C}^2 \times \mathbb{C}P^1 : z \neq 0\right\} = \mathbb{C}^2 \times U_\infty,$$

where

$$U_0 := \left\{[z_1, z_2] \in \mathbb{C}P^1 : z_1 \neq 0\right\}, \qquad U_\infty := \left\{[z_1, z_2] \in \mathbb{C}P^1 : z_2 \neq 0\right\}.$$

Since $\mathcal{S}_0, \mathcal{S}_\infty \cong \mathbb{C}^3$, a result of Grauert (see e.g. [OSS]) implies that the bundle E restricted to either of these regions is trivial. Therefore the bundle E is characterised by the transition functions $G : \mathcal{S}_0 \cap \mathcal{S}_\infty \to \mathrm{SL}(2,\mathbb{C})$.

We may construct the map G directly from the ADHM data. With respect to bases on V, W, $A(z)$ can be viewed as a $(2k+2) \times k$ matrix with complex coefficients. The columns of $A(z)$ then define a set of k vectors $v_1(z), \dots, v_k(z) \in \mathbb{C}^{2k+2}$ that span the space U_z. These vectors obey the reality condition

$$\sigma_V\left(v_i(z)\right) = v_i(\sigma(z)), \qquad i = 1, \dots, k.$$

Since U_z is isotropic with respect to the symplectic form $(\cdot,\cdot)$, we deduce that

$$(v_i(z), v_j(z)) = 0.$$

We now restrict to a real line $\sigma(\mathbf{x}) = \{\mathbf{x}\} \times (U_0 \cup U_\infty) \subset \mathbb{C}P^3 \setminus l_\infty$. On the subset $\{\mathbf{x}\} \times U_0$ the annihilator $U_z^\perp$ is spanned by $\{v_i(z)\}$ along with two vectors $\{e_A(z) : A = 1,2\}$ that span $U_z^\perp/U_z$. We therefore have

$$(v_i(z), e_A(z)) = 0,$$

and may, without loss of generality, assume that

$$(e_1(z), e_2(z)) = -\left(e_2(z), e_1(z)\right) = 1. \tag{2.2}$$

We may also define a basis $\{f_A(z) : A = 1,2\}$ for $U_z^\perp/U_z$ on the region $\{\mathbf{x}\} \times U_\infty$ by the relations

$$f_1(z) := -\sigma_V\left(e_2\left(\sigma(z)\right)\right), \qquad f_2(z) := \sigma_V\left(e_1\left(\sigma(z)\right)\right).$$

This basis has the property that

$$(f_1(z), f_2(z)) = 1, \qquad (v_i(z), f_A(z)) = 0.$$

Given that $\{e_A(z)\}$ and $\{f_A(z)\}$ are both bases for $U_z^\perp/U_z$ for $z \in U_0 \cap U_\infty$, there exist functions ${G_A}^B(z), {\lambda_A}^i(z)$ with the property that

$$f_A(z) = {G_A}^B(z) e_B(z) + {\lambda_A}^i(z) v_i(z). \tag{2.3}$$

The matrix G is then the transition function of our bundle E. The definition of the vectors $\{f_A\}$ in terms of the $\{e_A\}$ implies that

$$G^* = G,$$

$$\det G = 1,$$

where $G^*(z) := \overline{{}^tG(\sigma(z))}$.

Finally note that since the bundle E is holomorphic over $\mathbb{C}P^3 \setminus l_\infty$ the transition functions are holomorphic with respect to the natural complex structure on $\mathbb{C}P^3$ restricted to $\mathbb{C}P^3 \setminus l_\infty$. In terms of the coordinates introduced above this implies that

$$G = G\left(u - z\overline{v}, v + z\overline{u}, z\right).$$

3. Symmetries of the self-dual Yang-Mills equations

In terms of the complex coordinates u, v introduced on $\mathbb{C}^2 \cong \mathbb{R}^4$ above, the standard flat metric on $\mathbb{R}^4$ takes the form

$$\mathbf{g} = \frac{1}{2}\left(du \otimes d\overline{u} + d\overline{u} \otimes du + dv \otimes d\overline{v} + d\overline{v} \otimes dv\right).$$

The corresponding volume form is then

$$\epsilon = dt \wedge dx \wedge dy \wedge dz = \frac{1}{4} du \wedge d\overline{u} \wedge dv \wedge d\overline{v}.$$

In terms of these coordinates, the self-dual Yang-Mills equations for a connection, $\mathbf{A}$, correspond to the following conditions on the components of the curvature tensor, $\mathbf{F_A}$, of the connection:

$$F_{uv} = 0; \tag{3.1a}$$

$$F_{u\overline{u}} + F_{v\overline{v}} = 0; \tag{3.1b}$$

$$F_{\overline{uv}} = 0. \tag{3.1c}$$

Introducing the vector fields

$$\mathbf{X}(z) := \partial_{\overline{v}} + z\partial_u, \qquad \mathbf{Y}(z) := \partial_{\overline{u}} - z\partial_v,$$

which depend on an arbitrary parameter $z \in \mathbb{C} \cup \infty = \mathbb{C}P^1$, then the self-dual Yang-Mills equations (3.1) are equivalent to the condition that

$$\mathbf{F_A}\left(\mathbf{X}(z), \mathbf{Y}(z)\right) = 0, \qquad \forall z \in \mathbb{C}P^1.$$

3.1. **Non-local symmetries of the self-dual Yang-Mills equations.** An important property of the self-dual Yang-Mills equations is that they are the compability condition for the following overdetermined equations [Ch, BZ, T, Cr]

$$(\partial_{\overline{v}} + z\partial_u)\, \psi(x,z) = -\left(A_{\overline{v}} + zA_u\right) \psi(x,z), \tag{3.2a}$$

$$(\partial_{\overline{u}} - z\partial_v)\, \psi(x,z) = -\left(A_{\overline{u}} - zA_v\right) \psi(x,z), \tag{3.2b}$$

for a map $\psi : \mathbb{R}^4 \times U \to \mathrm{SL}(2,\mathbb{C})$, where $U \subset \mathbb{C}P^1$ is a suitable domain. In particular we may find a solution $\psi_0 : \mathbb{R}^4 \times U_0 \to \mathrm{SL}(2,\mathbb{C})$ that is analytic in z for $z \neq \infty$. Given such a solution we may then construct a solution $\psi_\infty(x,z) : \mathbb{R}^4 \times U_\infty \to \mathrm{SL}(2,\mathbb{C})$ that is analytic in z for $z \neq 0$ by taking

$$\psi_\infty(x,z) = \left(\psi_0(x,\sigma(z))^{-1}\right)^\dagger,$$

where

$$\sigma(z) := -\frac{1}{\overline{z}}.$$

Note that σ may be viewed as the anti-podal map on $\mathbb{C}P^1$ viewed in terms of affine coordinates.

Equations (3.2) for ψ_0 and ψ_∞ imply that we may write the components of the connection in the form

$$A_u = -\left(\partial_u \psi_\infty(\infty)\right) \psi_\infty(\infty)^{-1}, \qquad A_v = -\left(\partial_v \psi_\infty(\infty)\right) \psi_\infty(\infty)^{-1}, \tag{3.3}$$

$$A_{\overline{u}} = -\left(\partial_{\overline{u}} \psi_0(0)\right) \psi_0(0)^{-1}, \qquad A_{\overline{v}} = -\left(\partial_{\overline{v}} \psi_0(0)\right) \psi_0(0)^{-1}. \tag{3.4}$$

If we define $J := \psi_\infty(\infty)^{-1} \cdot \psi_0(0)$ then the remaining part of the self-dual Yang-Mills equations imply that J obeys the Yang-Pohlmeyer equation

$$\partial_u \left(J_{\overline{u}} J^{-1}\right) + \partial_v \left(J_{\overline{v}} J^{-1}\right) = 0. \tag{3.5}$$

It is known that the only local symmetries of the self-dual Yang-Mills equations on flat $\mathbb{R}^4$ are gauge transformations and those generated by the action of the conformal group. On the other hand, there exists a non-trivial family of non-local symmetries of the self-dual Yang-Mills equations [Ch, D]. Let $J(t)$ denote a 1-parameter family of solutions of (3.5) that is assumed to depend smoothly on the parameter $t \in I$, where I is a open subinterval of the real line that contains the origin. Taking the derivative of (3.5) with respect to t we find that we require

$$\partial_u \left(J \partial_{\overline{u}} \left(J^{-1} \frac{\partial J}{\partial t} \right) J^{-1} \right) + \partial_v \left(J \partial_{\overline{v}} \left(J^{-1} \frac{\partial J}{\partial t} \right) J^{-1} \right) = 0. \tag{3.6}$$

The construction of [Ch, D] proceeds as follows. Let $\chi(z)$ be a solution of the system

$$\left[\left(\partial_{\overline{v}} - J_{\overline{v}} J^{-1}\right) + z\partial_u\right] \chi(x,z) = 0,$$

$$\left[\left(\partial_{\overline{u}} - J_{\overline{u}} J^{-1}\right) - z\partial_v\right] \chi(x,z) = 0.$$

If we then define

$$\frac{\partial J}{\partial t} = \chi(x,z)T(x,z)\chi(x,z)^{-1} \cdot J, \tag{3.7}$$

where the function T obeys the relations

$$(\partial_{\overline{v}} + z\partial_u)\,T = (\partial_{\overline{u}} - z\partial_v)\,T = 0, \tag{3.8}$$

then $\partial J/\partial t$ is a solution of the linearisation (3.6). Without loss of generality, we may take

$$\chi(x,z) = \psi_\infty(\infty)^{-1} \cdot \psi_\infty(z),$$

which is analytic for $z \neq 0$, and has the property that $\chi(x,\infty) = \mathrm{Id}$. If we expand the right-hand-side of (3.7) as a Laurent series in z then the coefficients in the expansion define a (generally infinite) family of solutions of the linearisation equations for J. Moving from one coefficient in this expansion to the next defines a map between solutions of the linearisation equations which, in integrable systems terminology, defines the recursion operator of the self-dual Yang-Mills equations and its related integrable hierarchy [MW].

Such symmetries of the self-dual Yang-Mills equations with gauge group $G = \mathrm{GL}(n,\mathbb{C})$ have been studied within the Sato/Segal-Wilson approach to integrable systems by Takasaki [T], and thus have the interpretation of an infinite-dimensional family of projective transformations on an infinite-dimensional Grassmannian. It is not clear, however, how to implement this approach for $G = \mathrm{SU}(n)$, as the transformations generally do not preserve the unitarity of field J. Takasaki's approach has, however, been used to study dressing actions on harmonic maps, where the reality conditions are more straightforward [AJS-A].

In order to maintain unitarity of J, we will consider the symmetry

$$\begin{aligned}\frac{\partial J}{\partial t} &= \chi(x,z)T(x,z)\chi(x,z)^{-1} \cdot J + J \cdot \left(\chi(x,z)^{-1}\right)^\dagger T(x,z)^\dagger \chi(x,z)^\dagger \\ &= \psi_\infty(\infty)^{-1}\left[\psi_\infty(z)T(x,z)\psi_\infty(z)^{-1} + \psi_0(\sigma(z))T(x,z)^\dagger\psi_0(\sigma(z))^{-1}\right]\psi_0(0),\end{aligned} \tag{3.9}$$

where $T(x,z)$ obeys the condition (3.8).

If we now consider the symmetry (3.9) with $T(x,z)$ a constant element of $\mathfrak{g}\otimes\mathbb{C}$, then the algebra of such symmetries is isomorphic to the Kac-Moody algebra of $\mathfrak{g}\otimes\mathbb{C}$. The natural question is whether there is a corresponding group action on the space of solutions. A partial solution to this problem was given by Crane [Cr], who showed that taking T to be constant, one could define an action of the (analytic) loop group of $G_{\mathbb{C}}$ (i.e. the group of analytic maps from S^1 to $G_{\mathbb{C}}$) on the space of solutions of the self-dual Yang-Mills equations. As Crane showed by example, however, this action does not preserve the instanton condition that the curvature of the connection be L^2. More generally, for a holomorphic bundle on $\mathbb{C}P^3 \setminus l_\infty$ defined by the patching matrix $G(u - z\overline{v}, v + z\overline{u}, z)$, a general $T(u - z\overline{v}, v + z\overline{u}, z)$ generates a transformation of the form

$$G(u - z\overline{v}, v + z\overline{u}, z) \mapsto g(u - z\overline{v}, v + z\overline{u}, z) \cdot G(u - z\overline{v}, v + z\overline{u}, z) \cdot g^*(u - z\overline{v}, v + z\overline{u}, z),$$

where

$$g^*(u - z\overline{v}, v + z\overline{u}, z) := g\left(u + \frac{\overline{v}}{\overline{z}}, v - \frac{\overline{u}}{\overline{z}}, -\frac{1}{\overline{z}}\right)^{\dagger}.$$

The function g is an arbitrary holomorphic functions of its arguments. If g is analytic for $z \neq \infty$, this transformation is simply a holomorphic change of basis on the bundle over $\mathbb{C}P^3$, and leaves the self-dual Yang-Mills field unchanged. However, if g has poles at finite values of z, it will have a non-trivial effect on the connection and generates a distinct solution of the self-dual Yang-Mills equations. This action has been given a cohomological description by Park (see [Pa] and references therein), which has been further investigated by Popov and Ivanova (see [Po, Iv] and references therein).

4. One-parameter families of ADHM data

We now consider a one-parameter family of ADHM data $A(t, z) : W \to V$, where $t \in I$ is a parameter with values in an open subset $I \subseteq \mathbb{R}$ that contains the origin. We assume that $A(t, z)$ is a smooth, continuous function of t. Such a one-parameter family of data defines a one-parameter family of holomorphic vector bundles $E(t) \to \mathbb{C}P^3$ and hence a one-parameter family of instanton solutions of the self-dual Yang-Mills equations on S^4. We now wish to investigate how the elements of the explicit constructions of the previous sections depend on $A(t, z)$.

The image $A(t, z)\,(W)$ is now spanned by the vectors $\{v_i(t, z)\}$. On a fixed real line the vector space $U_z^{\perp}/U_z$ is spanned by $\{e_A(t, z)\}$ for each $z \in U_0$, which we assume are normalised such that (2.2) is satisfied for each $t \in I$. Constructing the vectors $\{f_A(t, z)\}$, we then define the patching matrix $G(t, z)$ and the functions ${\lambda_A}^i(t, z)$ as in equation (2.3).

A short calculation (see [Gr1]) implies that the t-derivative of the patching matrix obeys the relation

$$\frac{\partial}{\partial t} G(t, z) = d(t, z)G(t, z) + G(t, z)d(t, z)^*, \tag{4.1}$$

where $d(t, z)$ is a matrix with components

$${d_A}^B(t, z) = \sum_{C=1}^{2} \epsilon^{BC} \left(\frac{\partial}{\partial t} f_A(t, z), f_C(t, z) - \sum_{i=1}^{k} {\lambda_C}^i(t, z) v_i(t, z) \right),$$

where $\epsilon^{BC} = -\epsilon^{CB}$ and $\epsilon^{12} = 1$ are the components of the volume form on the bundle and we have defined $d^*(t, z) = d(t, \sigma(z))^{\dagger}$.

At the formal level, it follows from equation (4.1) that

$$G(t, z) = \alpha(t, z)\, G(0, z)\, \left(\alpha(t, z)\right)^*, \tag{4.2}$$

where $\alpha(t, z)$ satisfies the first order ordinary differential equation

$$\dot{\alpha}(t, z) = d(t, z)\alpha(t, z), \qquad \alpha(0 : z) = \mathrm{Id}.$$

The form of $G(t,z)$ given in equation (4.2) is then precisely of the form given in [Cr]. More precisely, the differential equation (4.1) satisfied by $G(t,z)$ is a flow generated by symmetries of the form (3.9). Therefore given any smooth path in $\gamma : I \to \mathcal{M}_k$, where I is an open sub-interval of $\mathbb{R}$, then the tangent vector to γ at any point is the fundamental vector field corresponding to a symmetry of the form given in [Ch, D].

At the global level the transformations generated by (3.9) are generally only locally defined and therefore form a pseudo-group (or groupoid) rather than a Lie group. Since the moduli spaces $\mathcal{M}_k$ are connected [Do], the above calculations imply that this pseudo-group acts transitively on each $\mathcal{M}_k$. Since the moduli spaces are finite-dimensional, the tangent space at each point will be generated by a finite-dimensional sub-algebra of the algebra of symmetries (3.9) at each point. An implicit description of this sub-algebra may be derived from the ADHM data via equation (4.1). Whether this finite-dimensional sub-algebra gives rise to a transitive action of a finite-dimensional group on each moduli space $\mathcal{M}_k$, and whether the transitive action of a pseudo-group on the $\mathcal{M}_k$ leads to any information concerning the global structure of the moduli spaces is currently under investigation.

5. Open problems

Outside of clarifying some of the technical issues surrounding our results, we mention some natural ways in which the current work could be extended. A reformulation of the ADHM construction due to Atiyah [At2] and Donaldson [Do] shows that there is a $1-1$ correspondence between (framed) instantons solutions of instanton number k and based holomorphic maps $f : \mathbb{C}P^1 \to \Omega G$ of degree k. Given that such maps are in many ways analogous to harmonic maps from $\mathbb{C}P^1 \to \mathbb{C}P^N$, it seems likely that the dressing action on instanton moduli spaces may be most clearly understood within this formalism.

Although the generalisation from $G = \mathrm{SU}(2)$ to arbitrary G is essentially trivial, a natural question is whether the results we have found hold for other four-manifolds. In the case where the base manifold is $\mathbb{C}P^2$ with $G = \mathrm{SU}(2)$ we can analyse the one-instanton moduli space, $\mathcal{M}_1(\mathbb{C}P^2)$. This moduli space is isomorphic to a cone on $\mathbb{C}P^2$, with the vertex of the cone corresponding to the unique, homogeneous, reducible $\mathrm{SU}(2)$ connection on the bundle $L \oplus L^{-1}$. Preliminary investigations [Gr2] indicate that in this case the corresponding symmetry group action has two, distinct orbits, the first consisting of the irreducible connections and the second consisting of the reducible connection which is a fixed point of the group action. Given that on a general four-manifold the space of reducible connections is central to Donaldson's theory of four-manifolds [DK], this might suggest an interesting correspondence between topological field theory and group actions in integrable systems.

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