A polyhedral study of the diameter constrained minimum spanning tree problem*

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Abstract

This paper provides a study of integer linear programming formulations for the diameter constrained spanning tree problem (DMSTP) in the natural space of edge design variables. After presenting a straightforward model based on the well known jump inequalities a new stronger family of *circular-jump inequalities* is introduced. These inequalities are further generalized in two ways: either by increasing the number of partitions defining a jump, or by combining jumps with cutsets. Most of the proposed new inequalities are shown to define facets of the DMSTP polyhedron under very mild conditions. Currently best known lower bounds for the DMSTP are obtained from an extended formulation on a layered graph using the concept of central nodes/edges. The new families of inequalities are shown not to be implied by this layered graph formulation.

Keywords: Integer programming, Diameter constrained trees, Facet defining inequalities

1 Introduction

Given a graph $G = (V, E)$, with node set $V = \{1, 2, \dots, n\}$ and edge set $E = \{1, 2, \dots, m\}$, a spanning tree of G, denoted by $T = (V, E_t)$ is a connected subgraph of G without cycles. The diameter of the graph G is the length of the longest shortest path between any two nodes in G (in this paper we consider the length of a path to be its number of edges). The diameter constrained minimum spanning tree problem (DMSTP) is defined as follows: Given a graph G with edge costs $c_e \geq 0$, for all $e \in E$, and a diameter limit D, the goal is to identify a minimum cost spanning tree of G whose diameter does not exceed D.

The question on how to provide a *strong formulation* for the DMSTP in the *natural space* of edge variables (using only one variable associated to each edge) remained open for some time. Up to now, only *extended* formulations for the DMSTP are considered in the literature, see, e.g., Gouveia and Magnanti [8], Gouveia et al. [9, 10, 11]. The formulations leading to the most successful methods are based on graph concepts related with centers in trees and sophisticated reformulations tailored for network design problems with length constraints (see [11]). Despite providing very tight linear programming (LP) bounds, these formulations involve a large number of variables and are therefore not (directly) suitable to be used for solving the DMSTP on very large graphs.

For some time it was not known how to describe inequalities that guarantee the length of the paths using only edge variables. This might explain why DMSTP formulations defined in the space of the edge variables have not been considered in the past DMSTP literature. The work by Dahl [2] has made a significant contribution in this area by proposing the so-called jump constraints to model constrained

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shortest paths. Jump constraints have been later on also considered in the context of the hop-constrained trees ([4, 5]), and they can even be adapted in a straightforward way to model the DMSTP. However, this leads to a rather weak integer programming (ILP) model (as demonstrated below).

The main question addressed by this article is how to use the structure of spanning trees to derive non standard generalizations of jump constraints leading to ILP models with tighter LP relaxations. In general, no non-trivial generalizations of jump inequalities have been studied in the literature. Past research has focused on showing how to adapt the jump inequalities for more general problems (e.g., network design with survivability- and hop-constraints and providing polyhedral results, see, e.g., [1, 6, 13, 14]).

Scientific Contribution. The main contribution of our work is the introduction of a first ILP model for the DMSTP in the natural space of edge variables. The model uses a new family of circular-jump inequalities which imply the jump inequalities and are specific for modeling diameter constraints in spanning trees. These basic inequalities are further generalized in two ways: by increasing the number of partitions within a jump, or by combining jumps with cutsets. We show that most of the proposed inequalities define facets of the DMSTP polyhedron under very mild conditions. We observe that these generalization techniques can also be applied to the original jump inequalities and can be of further relevance for many network design problems dealing with length constraints (e.g., hop-constrained (survivable) network design problems).

Currently best known lower bounds for the DMSTP are obtained from an extended formulation on a layered graphs using the concept of central nodes/edges. The new families of inequalities are shown not be implied contained in the projection of the LP-relaxation of the layered graph polytope into the space of edge variables.

In order to contextualize our study we start by introducing a generic formulation (1)–(5) for the problem, considering undirected edge design variables $x_e \in \{0, 1\}$, for all $e \in E$, which indicate whether edge e is used in the solution.

$$\min \sum_{e \in E} c_e x_e \tag{1}$$

$$x(E(S)) \leq |S| - 1 \quad \forall S \subset V, |S| \geq 2 \tag{2}$$

$$x(E) = |V| - 1 \tag{3}$$

$$\vec{x} \in \mathcal{F} \tag{4}$$

$$\vec{x} \in \{0, 1\}^{|E|} \tag{5}$$

In this formulation $x(M)$, $M \subseteq E$, stands for $\sum_{e \in M} x_e$, and $E(S)$ are the edges with both endpoints in $S \subset V$. Constraints (2) are the subtour elimination constraints, that together with equation (3) guarantee that the obtained solution is a spanning tree. Let $E(\mathbf{x})$ denote the subgraph of G induced by the edges $e \in E$, such that $x_e = 1$, and let $P_{E(\vec{x})}(u, v)$ denote the shortest path in this subgraph between $u, v \in V$. The set $\mathcal{F}$ is defined as $\mathcal{F} = \{\vec{x} \in \{0, 1\}^{|E|} \mid \text{ for each } u, v \in V, u \neq v, \exists P_{E(\vec{x})}(u, v) : |P_{E(\vec{x})}(u, v)| \leq D\}$. Hence, $\mathcal{F}$ is the set of incidence vectors such that the induced subgraph contains a feasible path between any two nodes $u \in V$ and $v \in V$, i.e., a path of length at most D.

Let $\mathcal{P}$ be the convex hull of all DMSTP feasible solutions, i.e.,

$$\mathcal{P} = \text{conv}\{\mathbf{x} \in \{0, 1\}^{|E|} \mid \mathbf{x} \text{ satisfies } (2) - (4)\}.$$

Outline of the Article. In the remainder of this section we establish the dimension of the DMSTP polytope. In Section 2, we study jump-like inequalities based on partitions of the node set V. Besides revisiting the jump inequalities introduced by Dahl we propose new circular-jump inequalities and two generalizations that can be used for describing $\mathcal{F}$ in the space of $\mathbf{x}$-variables. In Section 3, cut-jump inequalities based on defining a jump inequality on a node subset $P \subset V$ are studied. In Section 4 we study packing-type inequalities that provide an alternative way to describe $\mathcal{F}$ in the natural space of design variables. Section 5 shows that the inequalities introduced in this article are not implied by the theoretically strongest DMSTP model from the literature and discusses some computational issues.

Finally, conclusions are drawn in Section 6 where we also discuss open questions that can be addressed in future research.

Throughout this paper we assume that G is a complete graph. We first analyze the dimension of the DMST polytope $\dim(\mathcal{P})$ given by the following theorems whose proofs are left for Appendix 1.

Theorem 1. *For $D = 2$, $\dim(\mathcal{P}) = n - 1$.*

Theorem 2. *For $D \geq 3$, $\dim(\mathcal{P}) = m - 1$.*

2 (Generalized) Jump Inequalities

2.1 Jump Inequalities

Jump inequalities were originally proposed by Dahl [2] in the context of the undirected hop-constrained shortest path problem. The so-called (s, t, D)-jump inequalities, used to model the constrained paths are defined as follows: the set V is partitioned into $D+2$ disjoint, nonempty subsets $(J_0, J_1, \ldots, J_{D+1})$, $V = \bigcup_{i=0}^{D+1} J_i$, $J_i \cap J_j = \emptyset$, $0 \leq i < j \leq D+1$, such that $s \in J_0$ and $t \in S_{D+1}$. Dahl has shown that $x(J) \geq 1$ for any such partition prevent paths with more than D edges, where $J = \bigcup_{0 \leq i < j-1 \leq D} [J_i, J_j] \subset E$ and $[J_i, J_j] = \{\{u, v\} \in E \mid u \in J_i, v \in J_j\}$.

The explanation why these inequalities prevent paths with more than D edges, is as follows (see, e.g., Dahl [2]): Suppose that there exists a solution such that it contains no edges of a given jump J. Since this solution must be connected, there must exist a path starting at node in subset J_0, passing through nodes from all subsets from J_1 to J_D and ending at a node in subset J_{D+1}. This path has length $D + 1$ and thus the solution cannot be feasible.

The jump inequalities can be adapted in a straightforward way to model constrained shortest paths in the context of the DMSTP as well. Let $(J_0, J_1, \ldots, J_{D+1})$ be a non-trivial partition of the node set V such that all sets are nonempty as above. Then $J = \bigcup_{i<j-1} [J_i, J_j] \subset E$, where $[J_i, J_j] = \{\{u, v\} \in E \mid u \in J_i, v \in J_j\}$, is a *jump*. Let $\mathcal{J}_u$ be the set of all possible jumps on G (regarding all possible partitions into $D + 2$ nonempty subsets), then the jump constraints for the DMSTP are defined as follows:

$$x(J) \geq 1 \qquad \forall J \in \mathcal{J}_u \tag{J}$$

We obtain a valid formulation for the DMSTP by using these inequalities in place of $\mathcal{F}$ in the generic description given above. Note that it is sufficient to consider only those jumps where $|J_0| = |J_{D+1}| = 1$ and that those jumps dominate the remaining ones. For the undirected shortest path problem with at most D edges, jump inequalities are facet defining (see, Dahl [2]). As we shall show in the next section, this is not the case for the DMSTP.

2.2 Circular-Jump Inequalities

Let $(J_0, J_1, \ldots, J_{D+1})$ be a non-trivial partition of V as described above. Then, $CJ = \bigcup_{0 \leq i < j-1 \leq D} [J_i, J_j] \setminus [J_0, J_{D+1}]$ is a *circular jump.* Note that the difference between a circular jump CJ and a jump J is that the former one does not contain the edges connecting the first set with the last one. Consequently, the circular jumps obtained from e.g., $(J_0, J_1, \ldots, J_{D+1})$, $(J_0, J_{D+1}, J_D, \ldots, J_1)$ and $(J_D, J_{D-1}, J_0, \ldots, J_{D+1})$ are all the same. Thus, while in a jump we have ordered subsets with a given first subset and a given last subset, in a circular jump there is no first set neither last set and thus, the designation. Figure 1 illustrates the set of edges of a circular jump for a graph with $n \geq 6$ and $D = 4$. Let $\mathcal{J}$ be the set of all circular jumps, then the corresponding circular-jump constraints are defined as follows:

$$x(CJ) \geq 1 \qquad \forall CJ \in \mathcal{J} \tag{cJ}$$

Note, that based on the previous observation when comparing a jump inequality with a circular-jump inequality, it is clear that each jump inequality (J) is dominated by a circular-jump inequality (cJ) obtained from the same partition. Note also that a circular-jump inequality, dominates several jump inequalities (by choosing any subset of the circular jump to be the first set of the jump inequality and keeping the same order for the remaining subsets).

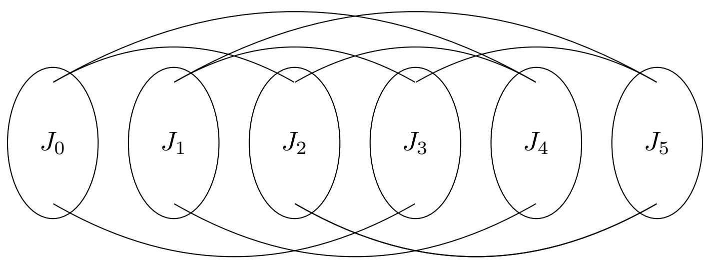

Figure 1: Circular jump CJ for a given partition $(J_0, J_1, \dots, J_5)$. $D = 4$.

Theorem 3. *Circular-jump inequalities* (cJ) *are valid for the DMST polytope* $\mathcal{P}$*.*

Proof. Circular jumps obviously forbid paths with more than D edges. To show that these inequalities do not cut off feasible solutions, let us assume the opposite, i.e., there exists a feasible solution T that violates one (cJ) inequality which is specified by a partition $(\tilde{J}_0, \tilde{J}_1, \dots, \tilde{J}_{D+1})$. Since we have seen that jump inequalities are valid, this (cJ) inequality is violated because one edge linking a node, say i_0, in the first partition with a node, say i_{D+1}, in the last partition is in the solution. However, since the solution must be connected, there exists k, $0 < k < D + 1$ such that there is a path from node i_0 to a node i_k, passing though all subsets $\tilde{J}_0$ to $\tilde{J}_k$ and a path from node i_{D+1} to a node i_{k+1}, passing through subsets $\tilde{J}_{D+1}, \tilde{J}_D, \tilde{J}_{D-1}, ..., \tilde{J}_{k+1}$. Since no jump edges belong to this solution, the length of this path is $D + 1$ and hence the solution T is infeasible, which is a contradiction. □ □

Hence, we obtain another valid formulation for the DMSTP by using the inequalities (cJ) in place of $\mathcal{F}$ in the generic description given above. As noted before, in contrast to the jump inequalities (J), the validity of these inequalities follows from the fact that the underlying solution is a spanning tree and they are not valid for related problems with hop constraints.

Clearly, the LP bound of this formulation is not worse than the LP bound provided by the formulation based on jump inequalities (J) which consequently cannot define facets of $\mathcal{P}$. The results provided below show that this is not the case for circular jumps (cJ).

Before proving that circular jumps (cJ) are facet defining, we observe that the DMSTP can be solved in polynomial time for $D = 2, 3$. In case of $D = 2$, all feasible solutions are stars, and the following result shows that (cJ) are facets of the underlying polytope:

Theorem 4. *For $D = 2$, circular-jump inequalities define facets of $\mathcal{P}$ if and only if at most one set of the partition (J_0, J_1, J_2, J_3) contains more than one node.*

Proof. See Appendix 2. □

In the rest of the paper we will concentrate on the more general case, when $D \geq 3$. The two following results are proved in Appendix 2.

Theorem 5. *For $D = 3$, circular-jump inequalities* (cJ) *define facets of $\mathcal{P}$ if two consecutive sets of the partition contain exactly one node.*

Proof. See Appendix 2. □

Theorem 6. *For $4 \leq D \leq n - 2$, circular-jump inequalities* (cJ) *define facets of $\mathcal{P}$.*

Proof. See Appendix 2. □

Before moving to the next sections where we show how to generalize these inequalities, we observe that from a given infeasible solution we may derive several violated (cJ) inequalities. Consider a spanning tree solution with at least one path being too long and for the moment let us assume that the length of this path is $D + 1$. The $D + 2$ nodes of this path determine the "seeds" of the $D + 2$ subsets in the partition defining a circular jump. There are two intuitive strategies for assigning the remaining nodes to the subsets of the partition that have been called path and layered approach in the context of classical jump constraints in the literature [4]. One strategy is to assign all nodes of a subtree rooted at a seed node to the subset seeded but that node. The other is two pick one extreme node of the path as a root of a tree that is directed away from that node. Then, the distance of each node to the root will define its partition subset. Typically, the longest path of an infeasible solution will be much longer than $D+1$. In Sections 2.3 and 2.4 we propose two more general classes of circular-jump inequalities that consider more than $D+1$ subsets (and thus allow to assign nodes of infeasible paths with length strictly greater than $D+1$ to different subsets). One generalization is based on ideas proposed by Dahl and Gouveia [3] for generalizing the jump constraints. The other is new and it is worth to investigate whether a similar generalization of jump constraints can be used in practice for other network design problems. In Section 3 we introduce different generalizations which allow us to consider the three classes of (generalized) circular-jump inequalities on a subset of nodes together with an additional set containing all remaining nodes.

2.3 Generalized-Circular-Jump Inequalities

Consider an infeasible solution with a path of length equal to $D+k$, where k is an arbitrary integer number such that $2 \leq k \leq D-1$. Besides the need to assign nodes not in the path to the subsets of the partition (as above), we also need to decide what to do with the extra nodes in the infeasible path. Essentially, we need to "contract" the infeasible path. Although these contractions may lead to violated circular-jump inequalities, there may be too many of them to generate. Also it is not clear which contractions are the best ones to select.

In this section we describe one way of generalizing the circular-jump inequalities for partitions with more than $D+2$ subsets. These generalized inequalities are slightly more complicated than the original ones, the right hand side depends on the number of partitions and for an inequality with the right hand side equal to k ($k>1$), the coefficients of the edge variables go up to k.

For a given $k \in \mathbf{N}$, $1 < k < D$, k-generalized circular jumps can be defined as follows. Let $(J_0, J_1, \ldots, J_{D+k})$ be a non-trivial partition of the node set V, i.e., $V = \bigcup_{i=0}^{D+k} J_i$, $J_i \cap J_j = \emptyset$, $0 \leq i < j \leq D+k$, such that all sets are nonempty, i.e., $J_i \neq \emptyset$, $0 \leq i \leq D+k$.

For $\ell \in \{1, \ldots, k-1\}$, let

$$C_\ell = \bigcup_{i=0}^{D+k} [J_i, J_{(i+\ell+1) \bmod (D+2)}]$$

Sets C_ℓ are subsets of edges that jump over *exactly* ℓ partitions, for each $\ell \in \{1, \ldots, k-1\}$. Finally, let CJ be the circular jump as defined above (derived from the partition $(J_0, J_1, \ldots, J_{D+k})$), and let

$$C_k = CJ \setminus \bigcup_{\ell'=1}^{k-1} C_{\ell'}$$

be the set of edges that jump over *at least* k partitions.

Then $GCJ = \bigcup_{\ell=1}^{k} C_\ell$ is a *k-jump*. Let $\mathcal{J}^k$ be a family of all possible k-jumps for a given k. For $1 < k < D$, the associated *generalized-circular-k-jump inequality* is given as:

$$\sum_{\ell=1}^{k} \ell \cdot x(C_\ell) \geq k \qquad \forall GCJ = \bigcup_{\ell=1}^{k} C_\ell \in \mathcal{J}^k \tag{gcJ}$$

Clearly, for $k=1$, we obtain the standard circular-jump inequality (cJ). To give some intuition on these inequalities, consider such an inequality with $D+3$ partitions (i.e., with $k=2$). The inequality states that any feasible solution needs to use at least one edge that jumps over two subsets or at least two edges that jump over exactly one partition. In the general case (with right hand side equal to any $2 \leq k \leq D-1$) any feasible solution has to use at least k edges that jump over exactly one partition (i.e., k edges from C_1), or at least one edge that jumps over at least k partitions (i.e., an edge from C_k), or a combination of edges from $\cup_{\ell=1}^{k} C_\ell$ so that in total at least k partitions are jumped.

Figure 2 illustrates this inequality for $D=4$ and $k=2$: edges shown above the sets jump over two partitions each (and the associated coefficients in (gcJ) are two), and the remaining jump edges skip a

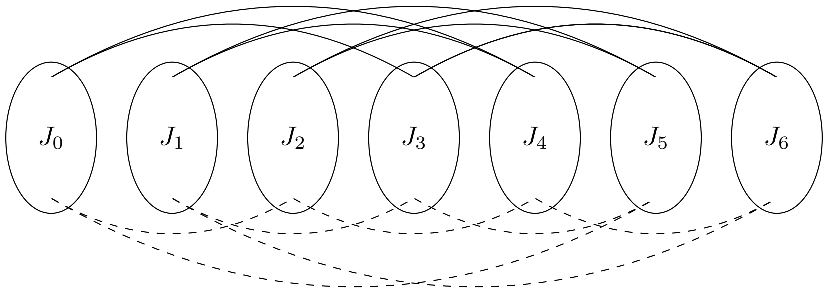

Figure 2: Illustration of a generalized circular jump GCJ for a given partition $(J_0, J_1, \ldots, J_6)$. $D = 4$. Solid edges above the partitions jump over two partitions (and thus the corresponding variables have coefficients equal to two) while dashed edges below the partitions skip a single partition (and the corresponding variables have coefficients of one).

single partition. The corresponding (gcJ) inequality is given as:

$$
\begin{array}{llll}
x(J_0,J_2)+2x(J_0,J_3) & +2x(J_0,J_4)+x(J_0,J_5) & & \\
\qquad\quad +x(J_1,J_3) & +2x(J_1,J_4)+2x(J_1,J_5) & +x(J_1,J_6) & \\
 & \quad +x(J_2,J_4)+2x(J_2,J_5) & +2x(J_2,J_6) & \\
 & \qquad\qquad +x(J_3,J_5) & +2x(J_3,J_6) & \\
 & & +x(J_4,J_6) \geq 2. &
\end{array}
$$

To see that the later inequality is valid, we will use a Chvátal-Gomory rounding argument. Consider (cJ) inequalities defined for partitions $(J_0 \cup J_1, , J_2, \ldots, J_6)$, $(J_0, J_1 \cup J_2, J_3, \ldots, J_6), \ldots, (J_0, J_1, J_2, \ldots, J_5 \cup J_6)$, and $(J_1, J_2, \ldots, J_6 \cup J_0)$. By summing up these seven inequalities, and dividing the obtained inequality by six and rounding up the right-hand side to the nearest integer, we obtain the desired result. In the general case, the validity argument also follows from the Chvátal-Gomory rounding, as shown in the proof of the following result:

Theorem 7. *Generalized-circular-jump inequalities* (gcJ) *are valid for the DMST polytope* $\mathcal{P}$.

Proof. We will prove the theorem by induction using the fact that inequalities (gcJ) are the standard circular-jump inequalities when $k = 1$ as the induction starting step. Assuming that the inequalities are valid for $k-1$, we will use a Chvátal-Gomory rounding argument to prove their validity for k. Let $(J_0, J_1, \ldots J_{D+k})$ be the partition associated to the k-(gcJ). To derive the corresponding inequality, we consider $D+k+1$ inequalities of type $k-1$-(gcJ), each one associated with a partition set defined by using the previous $D+k+1$ partitions in the following way:

$P_1 = \{J_0 \cup J_1, J_2, \ldots J_{D+k}\}$

$P_2 = \{J_0, J_1 \cup J_2, J_3, \ldots J_{D+k}\}$

...

$P_{D+k} = \{J_0, \ldots, J_{D+k-2}, J_{D+k-1} \cup J_{D+k}\}$

$P_{D+k+1} = \{J_1, \ldots, J_{D+k-1}, J_{D+k} \cup J_0\}$

Consider an edge $e \in C_\ell$. As will be shown in the following, the coefficient obtained from summing over all $D+k+1$ inequalities corresponding to partitions $P_1, \ldots, P_{D+k+1}$ is given by

$$\xi_\ell = \begin{cases} \ell(D+k)-1 & \text{if } \ell < k \\ (k-1)(D+k+1) & \text{if } \ell \geq k \end{cases}.$$

In the following, without loss of generality (due to circularity) we assume that $e = \{u, v\}$ with $u \in J_0$ and $v \in J_{\ell+1}$.

For $\ell < k$ we observe that the coefficient associated to edge e is equal to $\ell - 1$ for the inequalities corresponding to the $\ell + 1$ partitions $P_1, \dots, P_{\ell+1}$ and equal to ℓ for the inequalities associated to the remaining $D + k - \ell$ partitions $P_{\ell+2}, \dots, P_{D+k+1}$. Overall, we obtain $(\ell - 1)(\ell + 1) + \ell \cdot (D + k - \ell) = \ell(D + k) - 1$. For $\ell \geq k$, the coefficient associated to edge e is equal to $k - 1$ for the inequalities corresponding to all $D + k + 1$ partition, also giving the claimed overall value of $(k - 1)(D + k + 1)$.

Since the smallest coefficient $\xi_1 = D + k - 1$ we obtain

$$\left\lceil \frac{\xi_\ell}{\xi_1 + 1} \right\rceil = \begin{cases} \ell \text{ if } \ell < k \\ k \text{ otherwise} \end{cases}$$

Finally, since the obtained right hand side is $(D + k + 1)(k - 1)$ we obtain inequality (gcJ) by dividing through $\xi_1 + 1$ and rounding up all coefficients on the left and right hand side, respectively. □

Theorem 8. *For* $4 \leq D \leq n - k - 1$ *and* $2 \leq k \leq D - 2$*, generalized-circular-jump inequalities* (gcJ) *define facets of* $\mathcal{P}$*.*

Proof. See Appendix 3. □

2.4 Stretched-Circular-Jump Inequalities

In this section, we introduce a different class of valid inequalities that also generalize circular jumps, by considering more than $D + 2$ partitions. In contrast to the inequalities described in the previous section, all coefficients associated to jump edges will be equal to one in the new proposed constraints. Let $p \in \mathbb{N}$ ($p \geq D + 2$) and let $(J_0, J_1, \dots, J_{p-1})$ be a non-trivial partition of the node set V, i.e., $V = \bigcup_{i=0}^{p-1} J_i$, $J_i \cap J_j = \emptyset$, $0 \leq i < j \leq p - 1$, and $J_i \neq \emptyset$, $0 \leq i \leq p - 1$. Then, the set of edges $SJ = \bigcup_{0 \leq i < j-1 \leq p-2} [J_i, J_j] \setminus [J_0, J_{p-1}]$ skipping at least one partition is a *stretched circular jump* (or just *stretched jump*). Note, that a stretched circular jump corresponds to a circular jump if $p = D + 2$ and it is a straightforward generalization with more partitions when $p > D + 2$, thus the designation.

In order to motivate the new inequalities, we first observe that if we consider partitions with up to $2D$ node subsets, there exist feasible solutions that use only one edge from the associated stretched jump. If $p = 2D + 1$, then two edges from the associated SJ are needed. Furthermore, we observe that for each additional D partitions, one additional edge from the SJ is needed, and we conclude that for $p \geq l \cdot D + 1$ partitions ($l \in N$, $l \geq 2$), at least l jump edges will be used (the proof of this and other much stronger properties is provided in the proofs of Lemmas 1 and 2). Thus, if $\mathcal{S}$ denotes the set of all possible stretched circular jumps on G for a given p, we obtain the following constraints (wscJ)

$$x(SJ) \geq \left\lfloor \frac{p-1}{D} \right\rfloor \qquad \forall SJ \in \mathcal{S} \qquad \text{(wscJ)}$$

to which we will refer to as *weak stretched-circular-jump constraints.* We do not formally prove the validity of (wscJ) since their stronger variant is proposed below, with a less intuitive right hand side value.

Next, we introduce a stronger variant to which we will refer to as *stretched-circular-jump constraints* (scJ). These inequalities are given as follows:

$$x(SJ) \geq \begin{cases} \left\lfloor \frac{p-3}{D-1} \right\rfloor & \text{if } D \text{ even} \\ \max\left\{1, \left\lfloor \frac{p-5}{D-2} \right\rfloor\right\} & \text{if } D \text{ odd} \end{cases} \qquad \forall SJ \in \mathcal{S} \qquad \text{(scJ)}$$

It can be easily shown that (scJ) dominate their weaker counterparts (wscJ), since for $p \geq D + 2$, the right hand side of inequalities (scJ) is at least as large as the one of inequalities (wscJ) and strictly larger in many cases. Also note that a different interpretation of constraints (scJ) (for some special cases) will be given in Section 4.

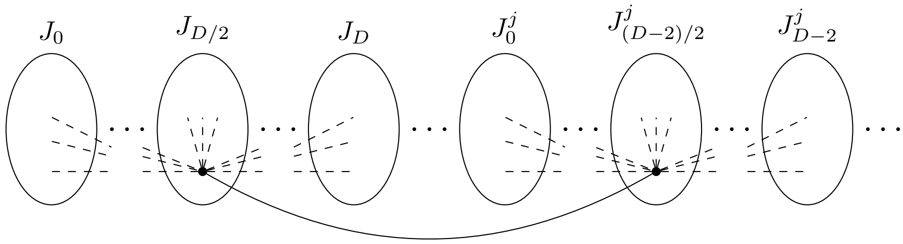

Figure 3: Sketch of an exemplary solution for D even that connects the nodes of $D+1+l(D-1)$ partitions by using l jump edges.

To see that these inequalities are valid, we first prove tight upper bounds on the maximum number of partitions that can be connected using at most a given number of jump edges. Let $(J_0, J_1, \dots, J_{p-1})$ be a non-trivial partition of the node set V into $p \geq D+2$ nonempty subsets as described above, and let SJ be the stretched circular jump associated to it.

Lemma 1. *Let D be an even number. Then, the maximum number of partitions p in a stretched jump SJ such that there exists a spanning tree of diameter D using at most $l \geq 1$ jump edges from SJ is $p = D + 1 + l \cdot (D-1)$.*

Proof. We first show that the upper bound we claim is tight, i.e., given a partition containing $D+1+l\cdot(D-1)$ subsets, it is indeed possible to create a feasible solution using l edges from SJ. To see this, consider a partition of V into $D+1+l\cdot(D-1)$ subsets such that the subsets are ordered so that the first group consists of $D+1$ subsets $(J_0,\dots,J_D)$ which is then followed by l groups of $D-1$ subsets $(J_0^j,\dots,J_{D-2}^j)$, for $1 \leq j \leq l$ (see Figure 3). In each group, there is a *middle subset*, i.e., $J_{D/2}$ for the first group, and $J^j_{(D-2)/2}$ for the remaining ones. To construct a feasible solution, we first construct spanning subtrees in each group without using jump edges (with diameter D and $D-2$, respectively) as follows: in each middle set a node is chosen as the center node and all other nodes from the same group are connected to it with paths of length at most $D/2$ for the first group and at most $(D-2)/2$ for the remaining l groups (see Figure 3). In the last step, for each $1 \leq j \leq l$ the center node of the middle set $J^j_{(D-2)/2}$ is connected to the center node from $J_{D/2}$. That way we obtain a spanning tree using exactly l edges from the stretched jump, with the diameter equal D. Finally, we observe that one cannot connect any group of at least $D+2$ subsets without using jump edges (cf. validity of standard circular jumps). Thus, for a given l, the only possibility to exceed $p = D+1+l\cdot(D-1)$ subsets is to have a part of the solution in which two groups, one with $D+1$ subsets, the other with D subsets, are connected by one jump edge or in which three groups with D subsets are connected by two jump edges. It is, however, easy to verify that both options are impossible without violating the given diameter bound. ☐ ☐

Lemma 2. *Let D be an odd number. Then, the maximum number of partitions p in a stretched jump SJ such that there exists a spanning tree of diameter D using at most $l \geq 1$ jump edges from SJ is $p = 2D + (l-1)\cdot(D-2)$.*

Proof. Along the lines of the proof of Lemma 2, we first observe that the upper bound is tight. Given a partition containing $2D+(l-1)\cdot(D-2)$ subsets, we create a feasible solution using l jump edges. To this end, the subsets of the partition are divided into two groups each consisting of D subsets, $(J_0^0,\dots,J_{D-1}^0)$ and $(J_0^1,\dots,J_{D-1}^1)$, and then followed by $l-1$ groups of $D-2$ subsets: $(J_0^j,\dots,J_{D-3}^j)$, for $2 \leq j \leq l$ (see Figure 4). Similarly, each group contains a middle set, and a node is chosen from each middle set as the center of the group. A spanning subtree in each group is created around the corresponding center node, without using jump edges. In the last step, center node from $J^0_{(d-1)/2}$ is connected to center nodes from $J^1_{(D-1)/2}$ and $J^j_{(D-3)/2}$, for all $2 \leq j \leq l$. As one cannot connect more than $D+2$ subsets without using jump edges, there are only two chances to exceed $p = 2D+(l-1)\cdot(D-2)$. Either there is a part of the solution in which two groups, one with $D+1$ subsets, the other with D subsets, are connected by one jump edge or three groups with D, D, and $D-1$ subsets, respectively, are connected by two jump

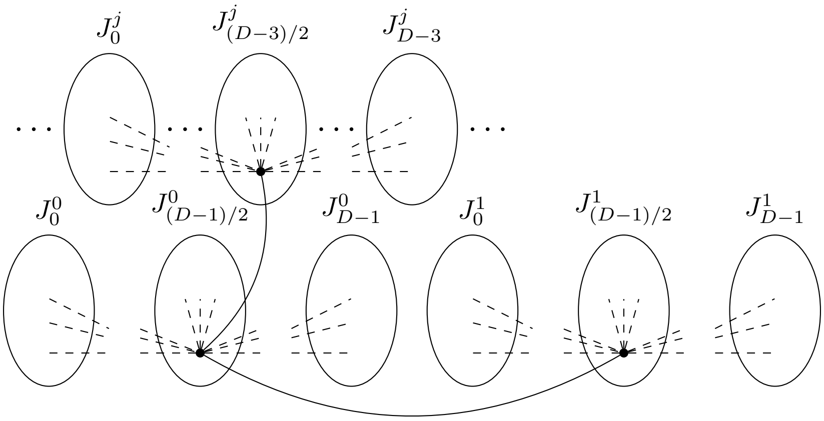

Figure 4: Sketch of an exemplary solution for D odd that connects the nodes of $2D + (l-1)(D-2)$ partitions by using l jump edges.

edges. It is, however, easy to verify that both options are impossible without violating the diameter bound. □ □

Using Lemmas 1 and 2, it is easy to prove Theorem 9 stating the validity of inequalities (scJ).

Theorem 9. *Stretched-circular-jump inequalities* (scJ) *are valid for the DMST polytope* $\mathcal{P}$.

Proof. We first consider the case when D is even and observe that from Lemma 1 if $p \geq D+1+(l-1)(D-1)+1 = (D+2)+(l-1)(D-1)$ we need l (or more) edges from the stretched jump SJ. Rewriting the last inequality, we obtain $l \leq \frac{p-3}{D-1}$ and thus, $\lfloor \frac{p-3}{D-1} \rfloor$ is the largest natural number satisfying this condition.
Similarly, if D is odd we first observe that $\lfloor \frac{p-5}{D-2} \rfloor \leq 1$ if $p < 2D+1$ in which case inequality (scJ) is obviously valid. For $p \geq 2D+1$, from Lemma 2, we have that l (or more) jump edges are necessary if the number of partitions is at least $2D+(l-2)(D-2)+1$, i.e., if $p \geq 2D+1+(l-2)(D-2)$. Again, we obtain that $\lfloor \frac{p-5}{D-2} \rfloor$ is the largest natural number satisfying this condition which concludes the proof. □ □

Since different numbers of partitions may lead to the same right hand side in the corresponding inequalities (scJ) it is natural to check whether some of them dominate others. Indeed, as shown in Theorem 10, for each right hand side value the inequalities that are derived for partitions with a minimum number of subsets (generating this particular right hand side value) dominate the others.

Theorem 10. *Let SJ and SJ' be stretched jumps obtained from non-trivial partitions of the set V into $p+1$ and p partitions, respectively. Let furthermore, $x(SJ) \geq \alpha_0$ and $x(SJ') \geq \alpha_1$ be the stretched jump inequalities corresponding to SJ and SJ'. Then $x(SJ') \geq \alpha_1$ dominates $x(SJ) \geq \alpha_0$ if $\alpha_1 = \alpha_0$.*

Proof. Let $(J_0, J_1, \ldots, J_p)$ be the partition corresponding to stretched circular jump SJ and let SJ' be the stretched circular jump obtained from merging the last two partitions, i.e. obtained from partition $(J_0, J_1, \ldots, J_{p-1} \cup J_p)$. Since $SJ \subset SJ'$ ($SJ \setminus SJ' = [J_0, J_{p-1}] \cup [J_{p-2}, J_p]$) and since $\alpha_0 = \alpha_1$ by assumption, the result follows. □ □

3 (Generalized) Circular-Jump Inequalities on Node Subsets

In this section, we derive new families of inequalities by associating nodes of an infeasible path of length $> D$ to a jump subset, and combining it with the remaining nodes using a cutset.

Theorem 11. *Let (P, S) be a partition of the set of nodes V such that $|P| \geq D+2$ and $S \neq \emptyset$ and let $(J_0, J_1, \ldots, J_{D+1})$ be a non-trivial partition of set P into $D+2$ non-empty subsets. Let, furthermore CJ*

be the set of edges defining a circular jump on P induced by this partition and let $\mathcal{J}(P)$ be the set of all circular jumps on P. Then, the following cut circular-jump inequalities

$$x(CJ)+x(P,S)+x(E(S)) \geq 1+|S| \quad \forall P \subset V, |P| \geq D+2, S=V \setminus P, CJ \in \mathcal{J}(P) \tag{ccJ}$$

are valid for the DMST polytope $\mathcal{P}$.

Proof. To see the validity of these constraints assume that $x(E(S)) = k$ ($k \leq |S|-1$). In this case, the set S is composed of $|S|-k$ components. Each of these components needs to be connected to P, that is $x(P,S) \geq |S|-k$. If $x(P,S) > |S|-k$ then the inequality is obviously valid. If $x(P,S) = |S|-k$, then $x(CJ) \geq 1$ must be satisfied (otherwise we will have an infeasible path in the partition defined by P). □ □

Theorem 12. *For $4 \leq D \leq n-3$, cut circular-jump inequalities* (ccJ) *define facets of $\mathcal{P}$.*

Proof. See Appendix 4. □

A similar reasoning can be used to combine generalized-circular-jump inequalities with a cutset. Let $k \geq 2$ and let (P,S) be a partition of the set of nodes V such that $|P| \geq D+k+1$ and $S \neq \emptyset$, and $2 \leq k \leq D-1$. Let $(J_0, J_1, \ldots, J_{D+k})$ be a non-trivial partition of set P into $D+k+1$ non-empty subsetss, $GCJ = \cup_{\ell=1}^{k} \mathcal{C}_\ell$ be the set of edges defining a generalized circular k-jump on P induced by this partition (cf. Section 2.3), and $\mathcal{J}^k(P)$ be the set of all generalized circular k-jumps on P.

Consider the following inequalities.

$$\sum_{\ell=1}^{k} \ell \cdot x(\mathcal{C}_\ell)+kx(P,S)+kx(E(S)) \geq k \cdot (|S|+1) \quad \forall P \subset V, |P| \geq D+2, S=V \setminus P, GCJ \in \mathcal{J}^k(P) \tag{cgcJ}$$

We denote these inequalities by *cut generalized-circular-jump inequalities* .

Theorem 13. *The inequalities* (cgcJ) *are valid for the DMST polytope $\mathcal{P}$.*

Proof. The theorem is shown by induction on k and using a Chvátal-Gomory analogous to the one used in the proof of Theorem 7. We observe that for $k=1$, constraints (cgcJ) are the standard cut circular-jump inequalities (ccJ) which are valid due to Theorem 11. To derive their validity for k, assuming the validity for $k-1$, we keep the partition of V into S and P fixed, and consider the $D+k+1$ partitions of P (as in the proof of Theorem 7). We recall, that by summing up and dividing through $D+k$, we obtain the appropriate coefficients for all edges $e \in C_\ell$, $1 \leq \ell \leq k$. It is easy to see that the coefficients ξ_e of edges $e \in [P:S] \cup E(S)$ obtained from summing all $D+k+1$ partitions are $\xi_e = (D+k+1)\cdot(k-1)$ and thus dividing through $D+k$ and rounding up yields coefficients k as claimed in the theorem. Similarly, the obtained right hand side is $\left\lceil \frac{(D+k+1)\cdot(k-1)\cdot(|S|+1)}{D+k} \right\rceil = k \cdot (|S|+1)$. □ □

Theorem 14. *For $4 \leq D \leq n-k-2$ and $2 \leq k \leq D-3$, cut generalized-circular-jump inequalities* (cgcJ) *define facets of $\mathcal{P}$.*

Proof. See Appendix 4. □

Finally, we also show that stretched-circular-jumps can be combined with a cutset as well.

Theorem 15. *Let (P,S) be a partition of the set of nodes V such that $|P| \geq D+2$ and $S \neq \emptyset$ and let $(J_0, J_1, \ldots, J_{p-1})$, $p \geq D+2$, be a non-trivial partition of set P into p subsets. Let, furthermore SJ be the set of edges defining a stretched circular jump on P induced by this partition and let $\mathcal{S}(P)$ be the set of all stretched circular jumps on P. Then, the following* cut stretched-circular-jump inequalities

$$x(SJ)+x(P,S)+x(E(S)) \geq \begin{cases} |S| + \lfloor \frac{p-3}{D-1} \rfloor & \textit{if } D \textit{ even} \\ |S| + \max\left\{1, \lfloor \frac{p-5}{D-2} \rfloor\right\} & \textit{if } D \textit{ odd} \end{cases} \quad \forall SJ \in \mathcal{S}(P) \tag{cscJ}$$

are valid for the DMST polytope $\mathcal{P}$.

Proof. To prove the theorem, we indirectly assume that there exists a feasible DMST solution $G_0 = (V_0, E_0)$ that violates inequalities (cscJ) and such that

$$x(SJ(P)) + x(P,S) + x(E(S)) = \alpha_0. \tag{6}$$

Thereby, α_0 is strictly less than the right hand side given in (cscJ). We will show how to construct a counterexample where set S contains one node less. Consequently applying this procedure for $|S|$ steps, a counterexample to a stretched-circular-jump inequality is obtained which yields a contradiction. In the following, let $v \in S$ be an arbitrary node and let $\deg_{G_0}(v)$ denote its degree in G_0.

$\deg_{G_0}(v) = 1$: Consider the graph $G_1 = (V_1, E_1)$ obtained from G_0 after removing v and the single edge incident to it. Clearly G_1 is a feasible DMST solution. Now consider the partition $(P, S \setminus \{v\})$ of V_1 and observe that from equation (6) we also obtain that $x(SJ(P)) + x(P, S \setminus \{v\}) + x(E(S \setminus \{v\})) = \alpha_0 - 1$. Thus, if $S \setminus \{v\} \neq \emptyset$, the partition $(P, S \setminus \{v\})$ violates a cut stretched-circular-jump inequality (scJ) on graph G_1.

$\deg_{G_0}(v) = t$, $t > 1$: Let $\{v_1, v_2, \ldots, v_t\} \subset V_0 \setminus \{v\}$ be the set of nodes adjacent to v in G_0. Consider the graph $G_2 = (V_0 \setminus \{v\}, E_2)$ where $E_2 = (E_0 \cup \{\{v_1, v_i\} \mid 2 \leq i \leq t\}) \setminus \{\{v, v_i\} \mid 1 \leq i \leq t\}$, i.e., the graph obtained from G_0 by removing node v and its incident edges, and connect all nodes previously adjacent to v by a star (choosing any of of them to be the center). Since G_0 is a feasible DMST solution, G_2 also is a feasible solution (it is a tree, since G_0 is a tree and its diameter cannot be larger than the diameter of G_0). Since the coefficients of all removed edges $\{v, v_i\}$, $1 \leq i \leq t$ in (cscJ) are equal to one and the coefficients of the $t-1$ newly added edges are at most one, we obtain $x(SJ(P)) + x(P, S \setminus \{v\}) + x(E(S \setminus \{v\})) \leq \alpha_0 - 1$. As above if $S \setminus \{v\} \neq \emptyset$, the partition $(P, S \setminus \{v\})$ violates a cut stretched-circular-jump inequality.

We note that after repeating above steps $|S|$ times, we obtain a counterexample of a stretched-circular-jump inequality which contradicts our assumption since validity of constraints (scJ) is shown in Theorem 9. □ □

4 (Rounded) Circuit Packing Inequalities

For many network design problems there are two equivalent ways of expressing natural space formulations. Either by using so-called cut inequalities that in some sense guarantee connectivity of the solutions or by using so-called packing inequalities stating that feasible solutions cannot have more than a certain number of edges in given subsets. One example of this is given by formulations for the spanning tree problem (see, e.g., [15]) and Steiner tree problem (see, e.g., [7]). Several routing problems can also be modeled in these two ways (e.g., the well known TSP). Usually, the two approaches are shown to be equivalent (or more, precisely, to lead to formulations with equivalent LP relaxations) since one model can be transformed into the other by using equalities that are included in the models. Up to now, the inequalities that have been discussed for modeling "path constraints" in the DMSTP are cut-type' inequalities. In this section, we briefly discuss a set of packing-type inequalities. We will show that using these *circuit packing* inequalities (7) in place of $\mathcal{F}$ yields a valid DMSTP formulation and that they are related, via equality (3), to some of the inequalities that have been discussed before. For simplicity we give this description for complete graphs.

$$x(C) \leq |C| - \frac{|C|}{D} \qquad \text{for all cycles } C \subset E, D+2 \leq |C| \leq |V| \tag{7}$$

These inequalities state that for any cycle $C \subset E$ with at least $D+2$ edges, the number of solution edges taken in any feasible solution is bounded from above by $|C| - \frac{|C|}{D}$. By rounding arguments, these inequalities can be further strengthened into the *rounded circuit packing* inequalities:

$$x(C) \leq |C| - \left\lceil \frac{|C|}{D} \right\rceil \qquad \text{for all cycles } C \subset E, D+2 \leq |C| \leq |V| \tag{8}$$

A similar set of inequalities is presented in for the directed hop-constrained minimum spanning tree problem.

Theorem 16. *Replacing* $\mathbf{x} \in \mathcal{F}$ *in the generic formulation* (2)–(5) *by* (7) *or* (8) *gives a valid model for the DMSTP.*

Proof. We will only show that any infeasible solution is cut off by the given inequalities. The validity of constraints follows from the discussion below which shows that these constraints are dominated by other valid inequalities studied in this paper.

Assume that there exists an integer solution feasible to generic model with constraints (7), such that, without loss of generality, it contains a path of length $D+1$. Let V_C be the set of nodes of this path. Obviously $|V_C| = D+2$. By concatenating the first and the last node of the path, we obtain a cycle C (this is possible, since G is complete), and for this cycle, we notice that (7) is violated. On the left hand side, we have $D+1$, on the right hand side, we have a value which is strictly less than $D+1$, which is a contradiction. □ □

The following theorem shows that rounded circuit packing inequalities of length $D+2$ are equivalent to some of the inequalities studied in the previous section.

Theorem 17. *Let* C *be a cycle in* G *of length* $D+2$ *such that* $C = \{\{i, i+1 \mod (D+2), 0 \leq i \leq D+1\}$. *Then, a rounded circuit packing inequality induced by* C *corresponds to:*

1. *a circular-jump constraint* (cJ), *if* $|V| = D+2$.
2. *a cut circular-jump constraint* (ccJ), *if* $|V| > D+2$.

Proof.

1. Let $V = \{0, \dots, D+1\}$ and let CJ be the circular jump corresponding to the partition $(\{0\}, \{1\}, \dots, \{D+1\})$ in which each set is a single node from V following the ordering of nodes from C. By subtracting the rounded circuit packing inequality implied by C from $x(E) = |V| - 1$, we obtain a circular-jump constraint $x(E) - x(C) = x(CJ) \geq 1$.
2. Let CJ be the circular jump corresponding to the partition $(\{0\}, \{1\}, \dots, \{D+1\})$ of node set $P = \{0, 1, \dots, D+1\}$, and $S = V \setminus P$. We observe that using the same transformation as above we obtain a cut circular-jump constraint $x(E) - x(C) = x(CJ) + x(E(S)) + x(P, S) \geq 1 + |S|$. □ □

The following theorem shows that rounded circuit packing inequalities induced by cycles of length greater than $D+2$ are dominated by some of the inequalities studied in the previous section.

Theorem 18. *Let* $P = \{0, \dots, D+k-1\}$, $S = V \setminus P$ *and let* $C = \{\{i, i+1 \mod (D+k)\}, 0 \leq i \leq D+k-1\}$ *be a cycle of length* $D+k$ $(k > 2)$ *in* P. *A rounded circuit packing inequality induced by the cycle* C *is dominated by:*

1. *a stretched-circular-jump constraint* (scJ) *if* $|V| = D+k$.
2. *a cut stretched-circular-jump constraint* (cscJ) *if* $|V| > D+2$.

Proof. We only show the second claim and observe that the first follows if $S = \emptyset$.

As before, we subtract the rounded circuit packing inequality implied by C from $x(E) = |V| - 1$, to obtain $x(E) - x(C) = x(SJ) + x(P:S) + x(E(S)) \geq |S| + \lceil \frac{D+k}{D} \rceil - 1 = |S| + \lceil \frac{k}{D} \rceil$. It only remains to show that the right hand side of inequalities (cscJ) is at least as large as this value, i.e., that for any $k > 2$ we have $\lceil \frac{k}{D} \rceil \leq \left\lfloor \frac{D+k-3}{D-1} \right\rfloor$ if D is even and $\lceil \frac{k}{D} \rceil \leq \max\{1, \left\lfloor \frac{D+k-5}{D-2} \right\rfloor\}$ if D is odd. If $2 < k \leq D$, these claims obviously hold. Else, if $k \geq D+1$, let $D+k = l \cdot D + f$ with $l, f \in \mathbb{N}$, $l \geq 2$, $0 \leq f \leq D-1$, and $l + f \geq 3$.

If D is even and using the assumption $f + l \geq 3$, the result follows since

$$\left\lceil \frac{k}{D} \right\rceil = \left\lceil \frac{(l-1) \cdot D + f}{D} \right\rceil \leq l \leq \left\lfloor l + \frac{f + l - 3}{D-1} \right\rfloor = \left\lfloor \frac{D+k-3}{D-1} \right\rfloor.$$

For D odd, we observe that $l \geq 2$ and $l + f \geq 3$ implies $2l + f \geq 5$ and the result thus follows from

$$\left\lceil \frac{k}{D} \right\rceil \leq l \leq \left\lfloor l + \frac{f + 2l - 5}{D-2} \right\rfloor = \left\lfloor \frac{D+k-5}{D-2} \right\rfloor.$$

□ □

Due to this equivalence we do not discuss more families of packing-type inequalities. We observe, however, that equivalent sets of packing-type inequalities can be obtained from the cut-type inequalities discussed before by using the same relations as used in the proofs of Theorems 17 and 18.

5 Comparison to Layered Graph Approach

As noted before, the most efficient method for solving the DMSTP is still the one given in Gouveia et al. [11]. The underlying formulation models hop-constraints with layered graphs and makes use of the property of tree centers. For simplicity, here we discuss only the case for D even. Essentially Gouveia et al. [11] used the fact that the DMSTP can be modeled as a special hop-constrained minimum spanning tree problem, where the root of the tree is appropriately selected and showed that the DMSTP can be reformulated as a Steiner arborescence problem with few additional constraints on an appropriately defined layered graph which implicitly ensures that the diameter of a solution is at most D. For the case of D even, this graph $G_{\mathrm{L}} = (V_{\mathrm{L}}, A_{\mathrm{L}})$ is defined as $V_{\mathrm{L}} = \{r\} \cup \{i_h \mid i \in V, 0 \le h \le D/2\}$ and $A_{\mathrm{L}} = \{(i_h, j_h+1) \mid \{i,j\} \in E,\ 0 \le h \le D/2-1\} \cup \{(r, i_0) \mid i \in V\}$. Using arc decision variables $a_{ij} \in \{0,1\}$, $\forall (i,j) \in A = \{(i,j) \mid \{i,j\} \in E\}$, and layered arc design variables $X^h_{ij} \in \{0,1\}$, $\forall (i_{h-1}, j_h) \in A_{\mathrm{L}}$, the layered-graph formulation for the DMSTP is given by (9)–(14) where $X[M]$, $M \subseteq A_{\mathrm{L}}$, stands for $\sum_{(i_{h-1},j_h)\in A_{\mathrm{L}}} X^h_{ij}$.

$$\min \sum_{(i,j)\in A} c_{ij} a_{ij} \tag{9}$$

$$\text{s.t. } X[\delta^-(W)] \ge 1 \qquad \forall W \subset V_{\mathrm{L}} \setminus \{r\}, \exists i \in V : \{i_h \mid 0 \le h \le D/2\} \subseteq W \tag{10}$$

$$X[\delta^+(r)] = 1 \tag{11}$$

$$\sum_{h=0}^{D/2} X[\delta^-(i_h)] = 1 \qquad \forall i \in V \tag{12}$$

$$a_{ij} = \sum_{h=1}^{D/2-1} X^h_{ij} \qquad \forall (i,j) \in A \tag{13}$$

$$\mathbf{X} \ge \mathbf{0},\ \mathbf{a} \in \{0,1\}^{|A|} \tag{14}$$

This model has been shown to theoretically dominate all previously proposed ones. A branch-and-cut algorithm developed from this model is the current state-of-the-art for solving DMSTP instances to proven optimality. Although the linear programming relaxation of (9)–(14) is integral for most of the instances considered in [11], we show in the following that it does not imply most of the inequalities introduced in this article.

Let $v_{LP}(LG)$ denote the value of the LP-relaxation of the layered graph model, and let $v_{LP}(CJ)$ be the value of the LP-relaxation of the model defined by the tree constraints (2) and (3) and the circular-jump inequalities (cJ). The following result shows that the lower bounds of the layered graph model can be as worse as 1/2 of the bounds obtained by our model involving only circular-jump inequalities.

Theorem 19. *There exists DMSTP instances such that*

$$\frac{v_{LP}(LG)}{v_{LP}(CJ)} \le \frac{1}{2}.$$

Proof. Consider an input graph given in Figure 5(a) with edge costs given as follows $c_{i,i+1} = \epsilon$, for all $0 \le i \le 7$, $c_{04} = M$ and $c_e = \infty$ for all the remaining edges (G is complete). Let us assume $D = 6$. Optimal LP-solution of the layered graph model is given in Figure 6(a), its value is $v_{LP}(LG) = 6.5\epsilon + M/2$. Projected back in the space of $\mathbf{x}$ variables, this fractional solution is shown in Figure 5(a). This solution violates a circular-jump inequality (cJ) obtained from the partition $P = \{\{0\}, \{1\}, \ldots, \{7\}\}$, since $x(P) = 0.5 < 1$. The optimal LP-solution of the ILP model derived from the circular-jump inequalities is $v_{LP}(CJ) = 6\epsilon + M$. Letting $M \mapsto \infty$ and $\epsilon \mapsto 0$, we observe that

$$\frac{v_{LP}(LG)}{v_{LP}(CJ)} \mapsto \frac{1}{2}.$$

□

□

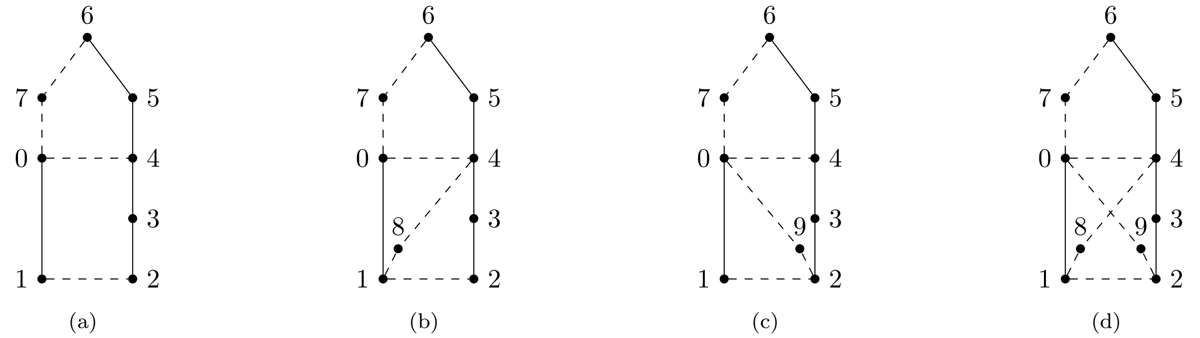

Figure 5: Optimal LP-solutions of the layered graph model for $D = 6$ (projected back into the space of $\mathbf{x}$ variables) that do violate (a) circular jump constraints, (b) cut-circular jump constraints, (c) generalized circular jump constraints, and (d) cut generalized circular jump constraints; dashed edges indicate LP values of 0.5, solid edges of 1.

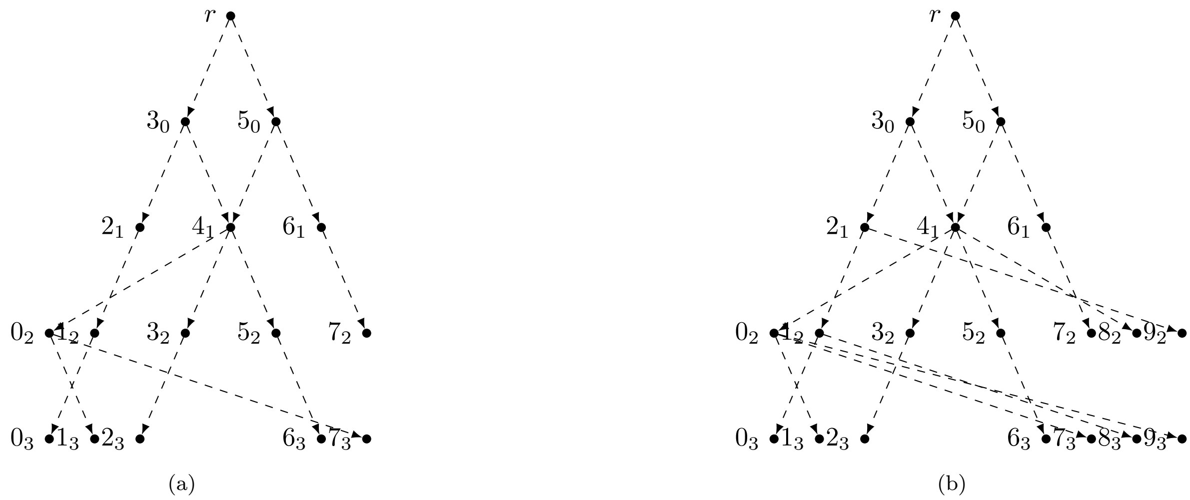

Figure 6: (a) Optimal LP-solution of the layered graph model for the instance given in Figure 5(a). (b) Optimal LP-solution of the layered graph model for the instance given in Figure 5(d). After removing certain nodes and their incident arcs the remaining two solutions are obtained (5(b) 9_2, 9_3; 5(c) 8_2, 8_3).

We furthermore observe that each of the solutions given in Figure 5 violates one of the new inequalities proposed in this article:

- Figure 5(b): Violates a cut circular-jump inequality (ccJ) obtained from the partition $P = \{\{0\}, \{1\}, \ldots, \{7\}\}$ and $S = \{8\}$, since $x(CJ) + x(S, P) + x(E(S)) = 1.5 < 2$.
- Figure 5(c): Violates a generalized-circular-jump inequality (gcJ) obtained for $k = 2$ from the partition $P = \{\{0\}, \{1\}, \ldots, \{7\}, \{9\}\}$, since $\sum_{\ell=1}^{2} \ell \cdot x(C_\ell) = 1.5 < 2$ (edge $\{2, 9\} \in C_1$ and edge $\{0, 4\} \in C_2$).
- Figure 5(d): Violates a cut generalized-circular-jump inequality (cgcJ) obtained for $k = 2$ from the partition $P = \{\{0\}, \{1\}, \ldots, \{7\}, \{9\}\}$ and $S = \{8\}$, since $\sum_{\ell=1}^{2} \ell \cdot x(C_\ell) + 2x(P, S) + 2x(E(S)) = 2.5 < 4$ (edge $\{2, 9\} \in C_1$, edge $\{0, 4\} \in C_2$, edges $\{1, 8\}$ and $\{4, 8\}$ in $[P, S]$.

Let $\text{proj}_x(LG)$ be the polytope of the LP-relaxation of the layered graph model into the space of $\mathbf{x}$ variables. We conclude that the polytope of the LP-relaxation of the new formulation is not fully contained in $\text{proj}_x(LG)$.

Computational Observations. While the new inequalities contribute to strengthening the obtained LP bounds, they are quite hard to separate due to their "partitioning like" structure. More precisely, to separate the new inequalities studied in this paper, we need to solve NP-hard partitioning problems with different side constraints. Our experiments on some instances from the literature showed that exact separation on graphs with 20 nodes already takes quite a long amount of time. Contrary to the examples provided in Figure 5, for the family of randomly generated Euclidean complete graphs from [11], the resulting LP gaps are still relatively large. The LP bounds of the layered graph approach on the same set of instances are very small (or even integer), which indicates that there are other practically relevant families of valid inequalities included in the layered graph center based model, that are not contained in our formulation.

We have implemented a branch-and-cut approach based on the proposed natural formulation with exact separation of new inequalities (performed by solving the underlying ILP models, similar to the one proposed by Gruber and Raidl [12]). We observed that for complete graphs with 30 or more nodes, the branch-and-cut approach is typically not able to close the remaining LP gaps even when given a relatively long time limit. Since partitioning problems are even more difficult to solve than the DMSTP, we expect that both heuristic (rather than exact) separation of new inequalities as well as the identification of further valid inequalities are needed to significantly improve the performance of our branch-and-cut.

Thus, further studies are needed to make the natural space formulation competitive in terms of computations. These studies include a) the identification of additional valid inequalities projected from the layered graph center based model, and b) the development of heuristic separation routines and a branch-and-cut algorithm tailored to tackle very large scale instances that cannot be solved by the layered graph approach.

6 Conclusions

In this article, we performed the first study of natural space formulations for the diameter constrained minimum spanning tree problem (DMSTP) which use variables associated to undirected edges only. We proposed different classes of inequalities generalizing the concept of jump inequalities [2], which have been used in the literature for other problems with length constraints. For most of the new inequalities, we showed that they define facets of the DMST polyhedron under very mild conditions. Validity of *circular-jump* type inequalities relies on the fact that the underlying problem is a spanning tree. In contrast, the concepts of *stretched-jump inequalities* (i.e., increasing the number of sets of the partition without increasing the coefficients associated with the individual variables) and *cut-jump inequalities* (i.e., jumps on subsets of nodes) are worth to be analyzed for other network design problems with length constraints as well. We also show that most of the new inequalities are not implied by current state-of-the-art ILP model based on a layered graph reformulation [11].

Several future research issues are raised with this work: i) The identification of additional valid inequalities in the natural space and the analysis of which of them are specifically relevant to significantly reduce the resulting LP gaps. ii) The development of efficient (heuristic) separation routines for separating them in order to make a corresponding branch-and-cut approach applicable to very large scale instances which cannot be treated by the layered graph approach. iii) Derivation of a compact (layered graph?) reformulation that implies the inequalities studied in this work. In particular we refer to a model implying the circular-jump constraints. Recall that a compact model that contains standard s-t-jump inequalities can be easily derived [5, 11]. An appropriate mixed integer linear programming formulation which contains jump-constraints for the DMSTP is obtained by the intersection of hop-constrained spanning trees, each with a root on a different node and such that distance from any node to the root node is at most D (we skip this part here but refer the reader to Dahl et al. [5] and Gouveia et al. [11] to see that our statement is valid). The main point is that we can show that the strongest such model does not imply the circular-jump constraints, and our attempts to provide a "similar" model implying the circular-jump constraints have failed. iv) A better understanding of the projection of layered graph models to the natural space of design variables (which applies to any problem that is modeled in this way, not only the DMSTP).

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Appendix 1: Dimension Proofs

Proof of Theorem 1

To show that $\dim(\mathcal{P}) = n - 1$ if $D = 2$, we first recall that the set of feasible solutions to the DMSTP with $D = 2$ is given by the n "stars" $S^i = (V, E_i)$ with $E_i = \{\{i, j\} \mid \forall j \in V \setminus \{i\}\}$, $\forall i \in V$. By looking at the matrix in which each of the incidence vectors represents its row, we easily observe that this matrix has a full rank, which is n. □

Proof of Theorem 2

To show for $D \geq 3$, $\dim(\mathcal{P}) = m - 1$ holds, we first note that the dimension of the DMSTP polytope cannot be larger than $m-1$ since all feasible solutions satisfy equation (3). To show that the dimension is at least $m-1$ we now construct a set $\mathcal{T}$ of m solutions whose edge-incidence vectors are affinely independent. We will start with the empty set, and insert the incidence vectors in the order described below, that guarantees that each newly added vector is affinely independent with respect to the current set $\mathcal{T}$.

For $i, j \in V$, let $S^k_{i,j} = (V, E_k \setminus \{k,i\} \cup \{i,j\}$, $E_k = \{\{k,i\} \mid \forall i \in V \setminus \{k\}\}$ be the graph obtained from the star $S^k = (V, E_k)$ after attaching node i to node j instead to node k. Note that each graph $S^k_{i,j}$ is a feasible solution since its diameter is precisely three.

We first insert the star S^1 into $\mathcal{T}$. We then add additional $n-2$ affinely independent solutions to $\mathcal{T}$ as follows: node 2 is detached from 1 and attached to j, for all $j = 3, \ldots, n$. So far, these solutions create a linearly independent set of points. At the end of this step, we add S^2 to $\mathcal{T}$. To see that this final step guarantees affine independence of points, we need to show that the points S^2 together with $S^1_{2,j}$, $j = 3, \ldots, n$ and S^1 are affinely independent. This is illustrated in Table 1 that shows the submatrix associated with the non-zero columns represented by these solutions. By subtracting the first row from all the remaining rows, we easily observe that such obtained matrix has a full rank.

In the next step, additional $n-2$ incidence vectors are added to $\mathcal{T}$ by deleting the edge $\{1,3\}$ from S^1 and attaching the node 3 to node j, for all $j = 4, \ldots, n$. At the end of this step, S^3 is added to $\mathcal{T}$. By looking at the submatrix defined by columns $x_{12}, x_{13}, \ldots, x_{1n}$ and $x_{34}, x_{35}, \ldots, x_{3n}$, and using the same arguments as above, we can show that these points are affinely independent.

This procedure is repeated for all nodes $i = 4, \ldots, n-1$. That way, we create $(n-1)n/2 + 1$ affinely independent points, which concludes the proof. □

Table 1: A submatrix showing the affine independence of points S^1, S^2 and $S^1_{2,j}$, for $j = 3, \ldots, n$.

	x_{12}	x_{13}	$\ldots$	x_{1n}	x_{23}	x_{24}	$\ldots$	x_{2n}
S^1	1	1	$\ldots$	1	0	0	$\ldots$	0
S^1_{23}	0	1	$\ldots$	1	1	0	$\ldots$	0
S^1_{24}	0	1	$\ldots$	1	0	1	$\ldots$	0
	$\ldots$				$\ldots$			
S^1_{2n}	0	1	$\ldots$	1	0	0	$\ldots$	1
S^2	1	0	$\ldots$	0	1	1	$\ldots$	1

Appendix 2: Facet Proofs for Circular-Jump Inequalities

Given a circular jump determined by a partition $(J_0, \ldots, J_{D+1})$, to simplify the notation and avoid case distinction, we refer to partitions J_i of a given jump for any $i \in \mathbb{Z}$ implicitly assuming an appropriate modulo calculation $i \mod (D+2)$ whenever $i < 0$ or $i > D+1$. In general, for a partition $(J_0, \ldots, J_p)$, when we state J_i, we refer to $J_{i \mod (p+1)}$. Furthermore, for $v_i \in J_i$, $0 \leq i \leq D+1$, we introduce the following notation:

- $I(v_i) = \{\{v_i, j\} \mid j \in J_i, j \neq v_i\}$, $v_i \in J_i$, i.e., $I(v_i)$ is the edge set of a star containing all nodes of J_i with center v_i (*"in-edges"*)
- $B(v_i) = \{\{j, v_i\} \mid j \in J_{i-1}\}$, $v_i \in J_i$, i.e., $B(v_i)$ is a set of edges connecting all nodes from J_{i-i} to one particular node v_i from J_i. (*"back-edges"*)
- $F(v_i) = \{\{v_i, j\} \mid j \in J_{i+1}\}$, $v_i \in J_i$, i.e., $F(v_i)$ is a set of edges connecting all nodes from J_{i+1} to one particular node v_i from J_i (*"forward-edges"*)

Similarly, let $I(J_i) = \bigcup_{v_i \in J_i} I(v_i)$ be all inner edges of J_i, for $0 \leq i \leq D+1$, and notice that edges between two consecutive partitions J_i and J_{i+1} can be represented as $[J_i, J_{i+1}] = B(J_{i+1}) = F(J_i)$ where $B(J_i) = \bigcup_{v_i \in J_i} B(v_i)$ and $F(J_i) = \bigcup_{v_i \in J_i} F(v_i)$.

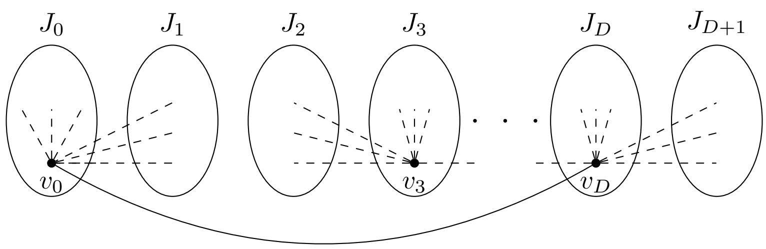

Figure 7: Solution $T^{0,D}$.

Let $v_i \in J_i$ be arbitrarily chosen nodes, $0 \le i \le D+1$, and CJ a given circular jump determined by a partition $(J_0, \ldots, J_{D+1})$. Let $T^{p,q}$ denote a feasible solution using exactly one jump edge $\{v_p, v_q\} \in CJ$ where $q - p \ge 2$, $p, q \in \{0, \ldots, D+1\}$. For example, Figure 7 illustrates a solution $T^{0,D}$ that contains edges from $I(v_0) \cup F(v_0) \cup B(v_3) \cup F(v_D)$, a path $\{v_3, v_4\}, \ldots, \{v_{D-1}, v_D\}$ and sets $I(v_i)$, i.e., stars centered at each v_i, $i = 3, \ldots, D$.

Proof of Theorem 4

First note that since for $D = 2$ we have $\dim(\mathcal{P}) = n - 1$, and each feasible solution is a star, the face F corresponding to a circular-jump inequality must contain the incidence vectors of all but one feasible solutions to be a facet. We now show that each inequality satisfying the above condition and such that all but one sets in the partition are singletons is facet defining. Without loss of generality assume that $(J_0, J_1, J_2, J_3) = (\{1\}, \{2\}, \{3\}, \{4, \ldots, n\})$. Then, the corresponding circular-jump inequality is given by

$$x(CJ) = x_{13} + \sum_{i=4}^{n} x_{2i} \ge 1 \tag{15}$$

We observe that $\sum_{\{2,i\} \in CJ} x_{2i} = n - 3$, while for a given node $i \in V \setminus \{2\}$, the number of its adjacent edges from CJ is exactly one. Hence, each star centered at a node $i \ne 2$ is a feasible DMSTP solution that satisfies (15) with equality. Thus, we have $\dim(\mathcal{P}) = n - 1$ affinely independent points.

To show the converse, assume that at least two sets of partition (J_0, J_1, J_2, J_3) contain two or more nodes. Without loss of generality (due to circularity), we assume that J_3 is one of them and J_i (for some $i \in \{0, 1, 2\}$) is the second one. Then, for a node $j \in J_1$, $\sum_{\{i,j\} \in CJ} x_{ij} \ge 2$ holds and thus all points corresponding to graphs with center node $j \in J_1$ cannot be contained in F. Thus, for CJ to be a facet $|J_1| = 1$ must hold. Thus, either J_0 or J_2 contains at least two nodes and at least one additional node v with $\sum_{\{i,v\} \in CJ} x_{iv} \ge 2$ exists, which for the same reasons cannot lie on F. Thus, we conclude that at most $n - 2$ affinely independent points can lie on the face F corresponding to a circular-jump inequality obtained from a partition with at most two singleton sets if $D = 2$ and thus such an inequality cannot be facet defining. $\square$

Proof of Theorem 5

We now show that for $D = 3$, circular jumps define facets of $\mathcal{P}$ if two consecutive sets of the partition contain exactly one node. Without loss of generality, we assume that $J_0 = \{v_0\}$ and $J_4 = \{v_4\}$ are the two consecutive partitions that contain only a single node. Let $\mathcal{H}(CJ) = \{\mathbf{x} \in \mathcal{P} \mid x(CJ) = 1\}$ and consider a facet defining inequality of the form $\vec{\alpha}x(CJ) + \vec{\beta}x(E \setminus CJ) \ge \xi$. We show that if all points in $\mathcal{H}(CJ)$ satisfy

$$\vec{\alpha}x(CJ) + \vec{\beta}x(E \setminus CJ) = \xi \tag{16}$$

then (16) is a positive multiple of (cJ).

Our proof consists of the following four steps:

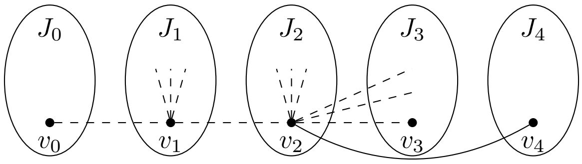

Figure 8: Solution $T^{2,4}$ with edge set $E(T^{2,4}) = \{\{v_0, v_1\}, \{v_1, v_2\}, \{v_2, v_4\}\} \cup I(v_1) \cup I(v_2) \cup F(v_2)$.

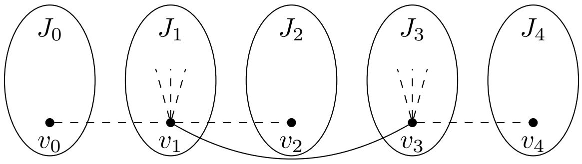

Figure 9: Solution $T^{1,3}$ with edge set $E(T^{1,3}) = \{\{v_0, v_1\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_3, v_4\}\} \cup I(v_1) \cup I(v_3)$.

- Step 1: $\beta_{u,v} = 0$, for all $\{u, v\} \in I(J_1) \cup I(J_2) \cup [J_1, J_2]$
- Step 2: $\beta_{u,v} = 0$, for all $\{u, v\} \in [J_2, J_3] \cup I(J_3)$
- Step 3: $\beta_{u,v} = 0$, for all $\{u, v\} \in [J_0, J_1] \cup [J_3, J_4] \cup [J_0, J_4]$
- Step 4: $\alpha_{u,v} = \zeta$, for all $\{u, v\} \in CJ$.

Step 1: Since $\dim(\mathcal{P}) = m-1$ we can fix one coefficient from (16). Let $\beta_{v_1,v_2} = 0$. If $|J_1| = |J_2| = 1$ the claim holds since $\{v_1, v_2\} = I(J_1) \cup I(J_2) \cup [J_1, J_2]$.
Thus, assume $|J_1| > 1$ or $|J_2| > 1$. If $|J_2| > 1$, we consider the solutions $T^{2,4}$ shown in Figure 8 and $T_u^{2,4}$, $u \in J_2 \setminus \{v_2\}$, obtained from $T^{2,4}$ by replacing edge $\{v_2, u\}$ by $\{v_1, u\}$. By plugging in the characteristic vectors of $T^{2,4}$ and $T_u^{2,4}$ into (16) and subtracting them from each other we obtain $\beta_{v_2,u} = \beta_{v_1,u}$, $\forall u \in J_2 \setminus \{v_2\}$. If $|J_1| > 1$ we obtain $\beta_{v_1,w} = \beta_{v_2,w}$, $\forall w \in J_1 \setminus \{v_1\}$ in a similar way by using solutions $T_w^{2,4}$, $w \in J_1$ that are created from $T^{2,4}$ by replacing edge $\{v_1, w\}$ with $\{v_2, w\}$.

These steps are repeated starting with initial solutions where different centers $v_1' \in J_1$, $v_1' \neq v_1$, and $v_2' \in J_2$, $v_2' \neq v_2$, are chosen (if they exist, i.e., if the corresponding set has at least two nodes). Since, by assumption, either $|J_1| > 1$ or $|J_2| > 1$, by using the fact that $\beta_{v_1,v_2} = 0$, we obtain $\beta_{u,v} = 0$, for all $\{u, v\} \in I(J_1) \cup I(J_2) \cup [J_1, J_2]$.

Step 2: We first observe that $\beta_{u,v} = \rho$, for all $\{u, v\} \in [J_2, J_3] \cup I(J_3)$, can be shown using analogous arguments as in Step 1 (initially starting with a solution where the roles of J_0 and J_4 as well as J_1 and J_3 are interchanged). Thus, if $|J_2| > 1$, $\beta_{u,v} = 0$, for all $\{u, v\} \in [J_2, J_3] \cup I(J_3)$ follows. Else, we obtain this result after additionally showing that $\beta_{v_1,v_2} = \beta_{v_2,v_3}$. The latter follows from plugging in the characteristic vectors of $T^{1,3}$ (see Figure 9) and the solution obtained from $T^{1,3}$ by replacing edge $\{v_1, v_2\}$ by $\{v_2, v_3\}$ into equation (16) and subtracting them from each other.

Step 3: To see that $\beta_{uv} = 0$, for all $\{u, v\} \in [J_0, J_1]$, we consider the solution $T^{0,2}$ given in Figure 10 and observe that another feasible solution can be constructed from $T^{0,2}$ by replacing edge $\{v_2, u\}$ by $\{v_0, u\}$ for all $u \in J_1$. Since $J_0 = \{v_0\}$ the result follows (from Step 1). Starting from a solution $T^{2,4}$ with jump edge $\{v_2, v_4\}$, $\beta_{uv} = 0$, for all $\{u, v\} \in [J_3, J_4]$, follows by analogous arguments. Finally, $\beta_{v_0,v_4} = 0$ (i.e., $\beta_{uv} = 0$, for all $\{u, v\} \in [J_0, J_4]$), is obtained by plugging in the characteristic vectors of $T^{1,4}$ (see Figure 11) and the solution obtained from $T^{1,4}$ by replacing edge $\{v_0, v_1\}$ by $\{v_0, v_4\}$ (both solutions are in $\mathcal{H}(CJ)$ since $\{v_0, v_4\}$ is not a jump edge).

Step 4: From the previous steps, we conclude that $\beta_{u,v} = 0$, for all $\{u, v\} \in E \setminus CJ$. It remains to show that the coefficients of all jump edges $\{u, v\} \in CJ$ are identical, i.e., $\alpha_{u,v} = \zeta$, $\forall \{u, v\} \in CJ$. Let $\alpha_{v_0,v_2} = \zeta$. Plugging in the characteristic vectors of $T^{0,2}$ and $T^{1,3}$ into (16) and subtracting them from each other, it follows that $\alpha_{v_1,v_3} = \zeta$. Replacing $T^{1,3}$ by $T^{2,4}$ or $T^{1,4}$, respectively, yields $\alpha_{v_2,v_4} = \zeta$ and $\alpha_{v_1,v_4} = \zeta$. By constructing a solution using jump edge $\{v_0, v_3\}$ (which is easy) we also obtain

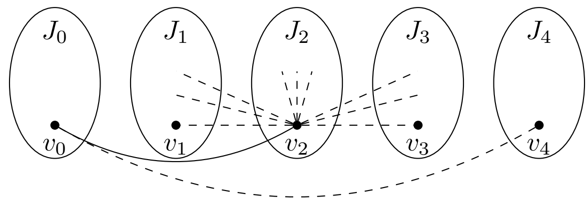

Figure 10: Solution $T^{0,2}$ with edge set $E(T^{0,2}) = \{\{v_0, v_2\}, \{v_0, v_4\}\} \cup B(v_2) \cup I(v_2) \cup F(v_2)$.

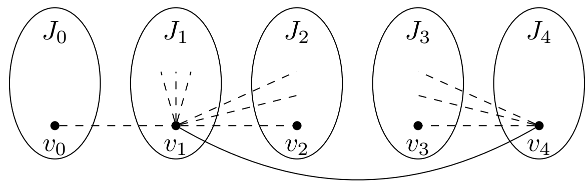

Figure 11: Solution $T^{1,4}$ with edge set $E(T^{1,4}) = \{\{v_0, v_1\}, \{v_1, v_4\}\} \cup I(v_1) \cup F(v_1) \cup B(v_4)$.

$\alpha_{v_0,v_3} = \zeta$. Finally, by varying the chosen center nodes in the different partitions, and repeating these steps we obtain $\alpha_{uv} = \zeta$, for all $\{u, v\} \in CJ$, which concludes the proof. $\square$

Proof of Theorem 6

To see that circular-jump inequalities define facets of $\mathcal{P}$ for $D \geq 4$, let $\mathcal{H}(CJ) = \{\mathbf{x} \in \mathcal{P} \mid x(CJ) = 1\}$. Consider a facet defining inequality of the form $\vec{\alpha}x(CJ) + \vec{\beta}x(E \setminus CJ) \geq \xi$. We will show that if all points in $\mathcal{H}(CJ)$ satisfy

$$\vec{\alpha}x(CJ) + \vec{\beta}x(E \setminus CJ) = \xi \tag{17}$$

then (17) is a positive multiple of (cJ).

Our proof follows the following steps:

- Step 1: $\beta_{u,v} = \nu_i$, for all $\{u, v\} \in I(J_i)$, $0 \leq i \leq D + 1$
- Step 2: $\beta_{u,v} = \nu_{i-1,i}$, for all $\{u, v\} \in [J_{i-1}, J_i]$, $0 \leq i \leq D + 1$
- Step 3: $\nu_{i-1,i} = \nu_i$ if $|J_i| > 1$, $0 \leq i \leq D + 1$
- Step 4: $\nu_{i-1,i} = \nu_{i-1}$ if $|J_{i-1}| > 1$, $0 \leq i \leq D + 1$
- Step 5: $\nu_{i-2,i-1} = \nu_{i-1,i}$, $0 \leq i \leq D + 1$
- Step 6: $\alpha_{u,v} = \mu$, for all $\{u, v\} \in CJ$

From Steps 1-5, we obtain that $\beta_{u,v} = \nu$, for all $\{u, v\} \in E \setminus CJ$. Since, $\dim(\mathcal{P}) = m - 1$ we can choose the coefficient of one variable. After choosing $\beta_{u,v} = 0$ for an arbitrary $e = \{u, v\} \in E \setminus CJ$ we can conclude that $\nu = 0$ and the theorem follows since $\mathcal{H}(CJ)$ contains all points satisfying $\mu x(CJ) = \mu$ with $x(CJ) = 1$.

Step 1: We show that $\beta_{u,v} = \nu_i$, for all $\{u, v\} \in I(J_i)$, $0 \leq i \leq D + 1$, whenever such edges exist, i.e., if $|J_i| > 1$. We note, that if $|I(J_i)| = 1$, this claim obviously holds. Thus, let $|I(J_i)| > 1$ (that is $|J_i| \geq 3$) and consider the solution $T^{i-4,i}$ given in Figure 12 which is feasible and in $F(CJ)$ for any $0 \leq i \leq D + 1$. Now, let $k, l \in J_i$, $k \neq l$, be two arbitrary nodes from $J_i \setminus \{v_i\}$, and let $T^{i-4,i}_{k,l}$ be the solution in $\mathcal{H}(CJ)$ obtained from $T^{i-4,i}$ by replacing the edge $\{v_i, l\}$ by $\{k, l\}$. By plugging in the characteristic vectors of $T^{i-4,i}$ and $T^{i-4,i}_{k,l}$ into (17) and subtracting them from each other, we obtain $\beta_{v_i,l} = \beta_{k,l}$, for all $k, l \in J_i \setminus \{v_i\}$, $k \neq l$.

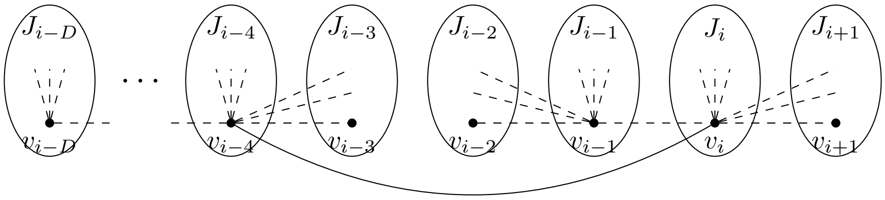

Figure 12: Solution $T^{i-4,i}$ with edge set $E(T^{i-4,i}) = \bigcup_{j=i-D}^{i-4}(I(v_j) \cup \{\{v_j, v_{j+1}\}\}) \cup F(v_{i-4}) \cup \{\{v_{i-4}, v_i\}, \{v_{i-1}, v_i\}\} \cup B(v_{i-1}) \cup I(v_{i-1}) \cup I(v_i) \cup F(v_i)$.

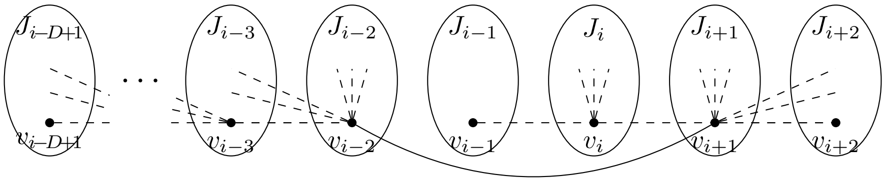

Figure 13: Solution $T^{i-2,i+1}$ with edge set $E(T^{i-2,i+1}) = \bigcup_{j=i-D+2}^{i-2} B(v_j) \cup I(v_{i-2}) \cup \{\{v_{i-2}, v_{i+1}\}, \{v_{i-1}, v_i\}, \{v_i, v_{i+1}\}\} \cup I(v_i) \cup I(v_{i+1}) \cup F(v_{i+1})$.

Now consider solution $\hat{T}^{i-4,i}$ in $\mathcal{H}(CJ)$ in which a node $v_i' \in J_i$ different from v_i is chosen as center of J_i. Repeating the previous steps using $\hat{T}^{i-4,i}$ we obtain $\beta_{v_i',l} = \beta_{k,l}$, for all $k, l \in J_i \setminus \{v_i'\}$, $k \neq l$. Since the edge $\{v_i, v_i'\}$ is contained in both results obtained, we conclude that $\beta_{u,v} = \nu_i$, for all $\{u, v\} \in I(J_i)$.

Step 2: To show that $\beta_{u,v} = \nu_{i-1,i}$, for all $\{u, v\} \in [J_{i-1}, J_i]$, $0 \leq i \leq D + 1$, we again consider solution $T^{i-4,i}$ given in Figure 12 and solution $\hat{T}^{i-4,i}$ with a node $v_i' \neq v_i$ chosen as the center of J_i. By plugging in the corresponding characteristic vectors into (17) and subtracting them from each other we obtain $\beta_{v_{i-1},v_i} = \beta_{v_{i-1},v_i'}$, for all $v_i' \in J_i \setminus \{v_i\}$, by using the result obtained in Step 1. Furthermore, by changing the center in partition J_{i-1} to $l \in J_{i-1} \setminus \{v_{i-1}\}$ and by using the same arguments we also obtain $\beta_{v_{i-1},v_i} = \beta_{l,v_i}$, for all $l \in J_{i-1} \setminus \{v_{i-1}\}$. Repeating these steps for all possible pairs of central nodes from J_{i-1} and J_i, we conclude that $\beta_{u,v} = \nu_{i-1,i}$, for all $\{u, v\} \in [J_{i-1}, J_i]$, $0 \leq i \leq D + 1$.

Step 3: To see that $\nu_{i-1,i} = \nu_i$ if $|J_i| > 1$, we again consider the solution $T^{i-4,i}$ and observe that after replacing edge $\{v_i, k\}$ by $\{v_{i-1}, k\}$ for an arbitrary node $k \in J_i \setminus \{v_i\}$ (which exists by assumption), we obtain another solution in $\mathcal{H}(CJ)$. Plugging in the characteristic vectors in (17) and subtracting them from each other we obtain $\nu_i = \beta_{v_i,k} = \beta_{v_{i-1},k} = \nu_{i-1,i}$.

Step 4: Similar to the previous step we show that $\nu_{i-1,i} = \nu_{i-1}$ if $|J_{i-1}| > 1$ by considering the solution $T^{i-4,i}$ and another solution in $\mathcal{H}(CJ)$ obtained by replacing edge $\{v_{i-1}, k\}$ by $\{v_i, k\}$ for an arbitrary node $k \in J_{i-1} \setminus \{v_{i-1}\}$. Plugging in the characteristic vectors in (17) and subtracting them from each other we obtain $\nu_{i-1} = \beta_{v_{i-1},k} = \beta_{v_i,k} = \nu_{i-1,i}$.

Step 5: We first observe that if $|J_{i-1}| > 1$ the claim holds since the results of the previous steps imply that $\nu_{i-2,i-1} = \nu_{i-1} = \nu_{i-1,i}$. For $|J_{i-1}| = 1$ we consider the solution $T^{i-2,i+1}$ given in Figure 13 which is feasible and in $\mathcal{H}(CJ)$ for any $D \geq 4$ and $0 \leq i \leq D + 1$. Analogously to the previous steps, we can construct another solution in $\mathcal{H}(CJ)$ by replacing edge $\{v_{i-1}, v_i\}$ by edge $\{v_{i-2}, v_{i-1}\}$ to obtain $\nu_{i-2,i-1} = \beta_{v_{i-2},v_{i-1}} = \beta_{v_{i-1},v_i} = \nu_{i-1,i}$.

Step 6: It finally remains to show that the coefficients of all jump edges are identical, i.e., $\alpha_{uv} = \mu$, for all $\{u, v\} \in CJ$. To show this, consider the set of solutions $\tilde{T}^{i,i+l}$ (see Figure 14) showing that a solution in $\mathcal{H}(CJ)$ exists using a jump arc between J_i and J_{i+l} for each relevant values of i and l, i.e., $0 \leq i \leq D + 1$, $2 \leq l \leq \lceil \frac{D+1}{2} \rceil$. Note that all cases with $l > \lceil \frac{D+1}{2} \rceil$ are included since they yield a jump over less than $\lceil \frac{D+1}{2} \rceil$ partitions due to circularity. By systematically comparing any two of these solutions for all possible values of i and l and by doing this for all possible center nodes of partitions J_i and J_{i+l}, respectively, we obtain the desired result.

□

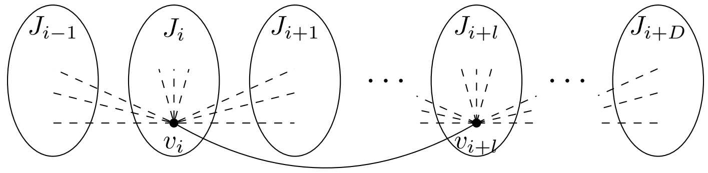

Figure 14: Solution $\tilde{T}^{i,i+l}$, $2 \leq l \leq \lceil \frac{D+1}{2} \rceil$, with edge set $E(\tilde{T}^{i,i+l}) = \{\{v_i, v_{i+l}\} \cup B(v_i) \cup I(v_i) \cup F(v_i) \cup \bigcup_{j=i+3}^{i+l} B(v_j) \cup I(v_{i+l}) \cup \bigcup_{j=i+l}^{i+D-1} F(v_j)$.

Appendix 3: Facet Proofs for Generalized-Circular-Jump Inequalities

Proof of Theorem 8

To see that generalized-circular-jump constraints define facets of $\mathcal{P}$ if $D \geq 4$ and $2 \leq k \leq D-2$, let $J = (J_0, J_1, \ldots, J_{D+k})$ be a non-trivial partition of V, let subsets $C_\ell \subset E$ be defined as above and let $\mathcal{H}(GCJ) = \{\mathbf{x} \in \mathcal{P} \mid \sum_{\ell=1}^{k} \ell \cdot x(C_\ell) = k\}$ be the set of feasible points that satisfy this particular generalized circular jump with equality. Consider a facet defining inequality of the form $\vec{\alpha}x(GCJ) + \vec{\beta}x(E \setminus GCJ) \geq \xi$ that contains the face $\mathcal{H}(GCJ)$. We will show that if all points in $\mathcal{H}(GCJ)$ satisfy

$$\vec{\alpha}x(GCJ) + \vec{\beta}x(E \setminus GCJ) = \xi \tag{18}$$

then (18) is a positive multiple of (gcJ).

By T^i_{k*1}, we denote a feasible solution that contains a chain of k consecutive jump edges from C_1, starting from the partition J_i (see Figure 15). Similarly, $T^i_{1*\ell,(k-\ell)*1}$ will denote a feasible solution with a chain of jump edges starting at J_i such that the first jump edge is from C_ℓ (i.e., it jumps over ℓ subsets), and it is followed by $k-\ell$ jump edges from C_1, see Figure 16.

Our proof follows the following steps:

- Step 1: $\beta_{u,v} = \nu$, for all $\{u,v\} \in E \setminus GCJ$
- Step 2: $\alpha_{u,v} = \mu$, for all $\{u,v\} \in C_1$
- Step 3: $\alpha_{u,v} = \ell \cdot \mu$, for all $\{u,v\} \in C_\ell$, $2 \leq \ell \leq k$.

After choosing $\beta_{u,v} = 0$ for an arbitrary $e = \{u,v\} \in E \setminus GCJ$ the result follows.

Step 1: One can show that the coefficients of all edges $e \in E \setminus GCJ$ are the same by the same technique used in the proof of Theorem 6 (by using slightly adapted solutions each using a single jump edge from set C_k). We therefore skip this part of the proof.

Step 2: To show that $\alpha_{u,v} = \mu$, for all $\{u,v\} \in C_1$, we first observe that for any $D \geq 4$, any $0 \leq \ell \leq D+k$ and $2 \leq k \leq D-2$, we can construct a feasible solution T^i_{k*1} using a sequence of k jump edges from C_1 starting from a node in partition i and which are adjacent to each other, see Figure 15. By plugging in the incidence vectors of T^i_{k*1} and T^{i+2}_{k*1} into equation (18) and subtracting them from each other we obtain $\alpha_{v_i,v_{i+2}} = \alpha_{v_{i+2k},v_{i+2(k+1)}}$. Systematical repetition with solutions using different central nodes in partitions J_i, $J_{i+2}, \ldots, J_{i+2k}$, and $J_{i+2(k+1)}$ yields $\alpha_{uv} = \alpha_{st}$, $u \in J_i$, $v \in J_{i+2}$, $s \in J_{i+2k}$, $t \in J_{i+2(k+1)}$, $0 \leq i \leq D+k$. If $D+k$ is even (and thus the number of partitions is odd), we obtain $\alpha_{uv} = \mu$, for all $\{u,v\} \in C_1$ by performing the previous steps for any $i \in \{0, 1, \ldots, D+k\}$. If $D+k$ is odd, after considering each $0 \leq i \leq D+k$, we obtain $\alpha_{uv} = \mu_1$, for all $\{u,v\} \in C_1$ and $u \in J_i$ with $i \mod 2 = 0$ and $\alpha_{uv} = \mu_2$, for all $\{u,v\} \in C_1$ and $u \in J_i$ with $i \mod 2 = 1$. By inserting the characteristic vectors of T^1_k and T^2_k into (18) and subtracting the results from each other we obtain $k\mu_1 = k\mu_2$ and hence, $\mu_1 = \mu_2 = \mu$ follows.

Step 3: We first consider jump edges $\{u,v\} \in C_\ell$ for $1 \leq \ell < k$, i.e., those that skip precisely ℓ partitions. Consider a feasible solution $T^i_{1*\ell,(k-\ell)*1}$ from $\mathcal{H}(GCJ)$ using one jump edge from J_ℓ skipping exactly ℓ partitions ($2 \leq \ell < k$) and $k-\ell$ jump edges from C_1 skipping one partition (see Figure 16).

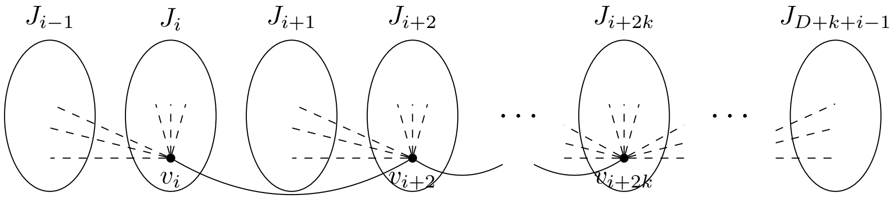

Figure 15: Solution T^i_{k*1} with edge set $E(T^i_{k*1}) = \bigcup_{j=0}^{k-1}(\{\{v_{i+2j}, v_{i+2j+2}\}\} \cup I(v_{i+2j}) \cup B(v_{i+2j})) \cup \bigcup_{j=i+2k}^{D+k+i-2} F(v_j)$.

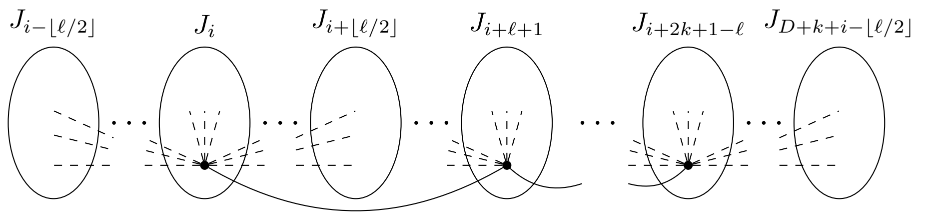

Figure 16: Solution $T^i_{1*\ell,(k-\ell)*1}$ with edge set $E(T^i_{1*\ell,(k-\ell)*1}) = \{\{v_i, v_{i+\ell+1}\}\} \cup I(v_i) \cup \bigcup_{j=0}^{k-\ell}(I(v_{i+\ell+2j+1}) \cup B(v_{i+\ell+2j}) \cup \bigcup_{j=i-\lfloor \ell/2 \rfloor+1}^{i} B(v_j) \cup \bigcup_{j=i+\lfloor \ell/2 \rfloor+2}^{i+\ell+1} B(v_j) \cup \bigcup_{j=i}^{i+\lfloor \ell/2 \rfloor-1} F(v_j) \cup \bigcup_{j=i+2k+1-\ell}^{D+k+i-\lfloor \ell/2 \rfloor-1} F(v_j)$.

Inserting the characteristic vectors of $T^i_{1*\ell,(k-\ell)*1}$ and the previously considered solution T^i_{k*1} using k edges from C_1 into (18), subtracting the results from each other, and systematically repeating these steps for all jump edges from C_ℓ we obtain $\alpha_{uv} = \ell \cdot \mu$, for all $\{u,v\} \in C_\ell$, $1 \le \ell < k$.

We also observe that the same argument suffices to show that $\alpha_{uv} = k \cdot \mu$ for all edges $\{u,v\} \in C_k$ that skip precisely k partitions. To see that this equation also holds for those edges skipping more than k partitions, we observe that for each $\{u,v\} \in C_k$, we can create a solution in $\mathcal{H}(GCJ)$, see Figure 17. This exemplary solution is denoted by $T^i_{1*\ell}$, where, due to circularity, it is sufficient to consider values of ℓ such that $k \le \ell \le \lceil \frac{D+k}{2} \rceil$. We observe that $T^i_{1*\ell}$ is a feasible solution, that contains two paths of length $\lfloor \ell/2 \rfloor$ left and right from v_i and similarly, a path of the same length left of $v_{i+\ell+1}$ and a path of length at most $\lfloor \ell/2 \rfloor$ on the right. Thus, the longest path in such constructed solution does not exceed D. Finally, by plugging in the characteristic vectors of $T^i_{1*\ell}$ and a solution using k edges from C_1 (e.g., T^i_{k*1}) into (18) and subtracting the results from each other we obtain the desired result.

□

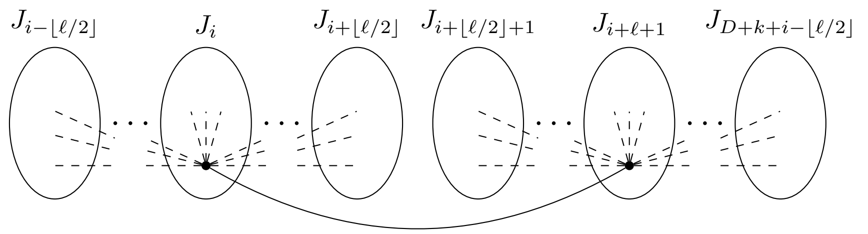

Figure 17: Solution $T^i_{1*\ell}$ with edge set $E(T^i_{1*\ell}) = \{\{v_i, v_{i+\ell+1}\}\} \cup I(v_i) \cup I(v_{i+\ell+1}) \cup \bigcup_{j=i-\lfloor \ell/2 \rfloor+1}^{i} B(v_j) \cup \bigcup_{j=i+\lfloor \ell/2 \rfloor+2}^{i+\ell+1} B(v_j) \cup \bigcup_{j=i}^{i+\lfloor \ell/2 \rfloor-1} F(v_j) \cup \bigcup_{j=i+\ell+1}^{D+k+i-\lfloor \ell/2 \rfloor-1} F(v_j)$. For $D \ge 4$, $k < D-1$, and $k \le \ell \le \lceil (D+k)/2 \rceil$, the longest path of $Ti_{1*\ell,(k-\ell)*1}$ has no more than D edges.

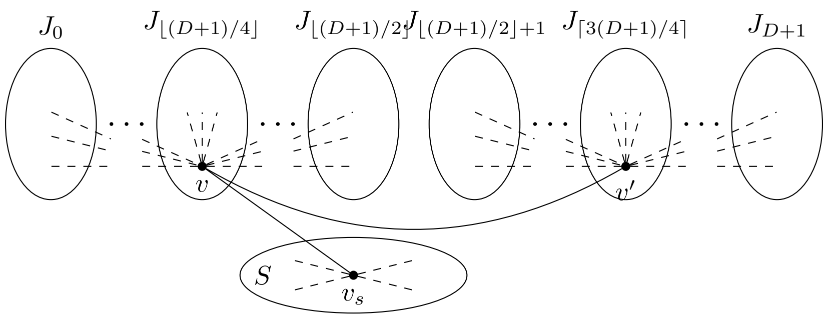

Figure 18: A feasible solution for $D \geq 4$ with edge set $\{\{v, v_s\}, \{v, v'\}\} \cup I(v_{\lfloor (D+1)/4 \rfloor}) \cup I(v_{\lceil 3(D+1)/4 \rceil}) \cup I(v_s) \cup \bigcup_{j=1}^{\lfloor (D+1)/4 \rfloor} B(v_j) \cup \bigcup_{j=\lfloor (D+1)/2 \rfloor+2}^{\lceil 3(D+1)/4 \rceil} B(v_j) \cup \bigcup_{j=\lfloor (D+1)/4 \rfloor}^{\lfloor (D+1)/2 \rfloor-1} F(v_j) \cup \bigcup_{k=\lceil 3(D+1)/4 \rceil}^{D} F(v_j)$. We set $v = v_{\lfloor (D+1)/4 \rfloor}$ and $v' = v_{\lceil 3(D+1)/4 \rceil}$.

Appendix 4: Facet Proofs for Cut Circular-Jump and Cut Generalized-Circular-Jump Inequalities

Proof of Theorem 12

To show that cut circular-jump inequalities (ccJ) define facets of $\mathcal{P}$ for $4 \leq D \leq n-3$, let (P, S) determine a partition of V and let CJ be the edges of the circular jump defined on P and $\mathcal{H}(CJ, S) = \{\mathbf{x} \in \mathcal{P} \mid x(CJ) + x(P, S) + x(E(S)) = 1 + |S|\}$. Consider a facet defining inequality of the form $\vec{\alpha}x(CJ) + \vec{\beta}x(E(P) \setminus CJ) + \gamma x(P, S) + \delta x(E(S)) \geq \xi$. We will show that if all points in $\mathcal{H}(CJ, S)$ satisfy

$$\vec{\alpha}x(CJ) + \vec{\beta}x(E(P) \setminus CJ) + \gamma x(P, S) + \delta x(E(S)) = \xi \tag{19}$$

then (19) is a positive multiple of (ccJ).

Our proof follows the following steps:

- Step 1: $\gamma_{u,v} = \mu'$, for all $\{u, v\} \in [P, S]$ and $\delta_{u,v} = \mu'$, for all $\{u, v\} \in E(S)$
- Step 2: $\alpha_{u,v} = \mu$, for all $\{u, v\} \in CJ$ and $\beta_{u,v} = \nu$, for all $\{u, v\} \in E(P) \setminus CJ$
- Step 3: $\mu = \mu'$

By choosing an arbitrary coefficient $\beta_{uv} = 0$, $\{u, v\} \in E(P) \setminus CJ$, the theorem follows from the results of Steps 1-3.

Step 1: We will first show that $\gamma_{v_i,u} = \delta_{v_s,u}$ for $v_i \in J_i$, $v_s \in S$, and for all $u \in S \setminus \{v_s\}$. Thus, consider solution $\hat{T}^{i-4,i}$ obtained from $T^{i-4,i}$ (see Figure 12) by choosing an arbitrary center $v_s \in S$ and adding edges $\{v_i, v_s\}$ and $\{v_s, u\}$, for all $u \in S \setminus \{v_s\}$. Clearly, $\hat{T}^{i-4,i}$ is feasible if $D \geq 4$ and contained in $\mathcal{H}(CJ, S)$. Another solution from $\mathcal{H}(CJ, S)$ is constructed from $\hat{T}^{i-4,i}$ by replacing edge $\{v_s, u\}$ by $\{v_i, u\}$ for some $u \in S \setminus \{v_s\}$. Plugging in the characteristic vectors into equation (19) and subtracting the result from each other, we obtain $\gamma_{v_i,u} = \delta_{v_s,u}$ for $v_i \in J_i$, $v_s \in S$, and for all $u \in S \setminus \{v_s\}$. Furthermore, repeating the same procedure using each $v \in S$ as initial center of S (i.e., taking the role of v_s), we obtain $\gamma_{u,v} = \mu'$, for all $\{u, v\} \in E(S)$, and by varying the center of J_i we also obtain $\gamma_{u,v} = \mu'$, for all $u \in J_i$, for all $v \in S$. Finally, repetition for $i = 0, 1, \ldots D+1$, yields $\gamma_{u,v} = \mu'$, for all $\{u, v\} \in [P, S]$ and $\delta_{u,v} = \mu'$, for all $\{u, v\} \in E(S)$.

Step 2: We observe that for any solution T' using exactly one jump edge $\{u, v\}$, at least one of its incident nodes, say u, will not be a leaf. Thus, we can create a solution T'' in $\mathcal{H}(CJ, S)$ by simply attaching all nodes from S to node u, i.e., by adding the edge set $\{\{u, k\} \mid k \in S\}$. Thus, by repeating the proof of Theorem 6 and correspondingly augment all used solutions together with the result from Step 1 we obtain $\alpha_{u,v} = \mu$, for all $\{u, v\} \in CJ$ and $\beta_{u,v} = \nu$, for all $\{u, v\} \in E(P) \setminus CJ$.

Step 3: It remains to show that the coefficients of jump edges $\{u, v\} \in CJ$ and coefficients of edges from $E \setminus E(P)$ coincide, i.e., that $\mu = \mu'$. Observe that the solution given in Figure 18 is feasible for $D \geq 4$

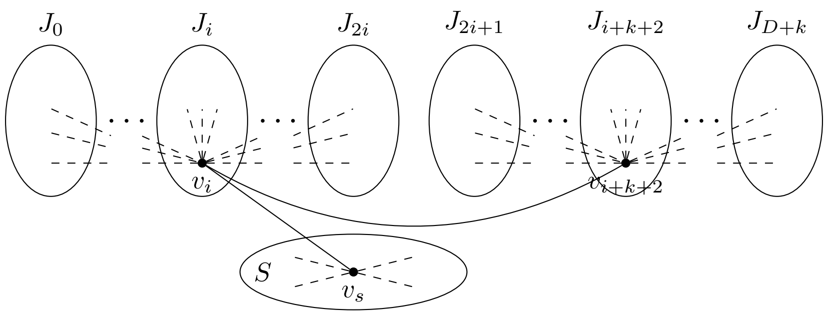

Figure 19: A feasible solution for $D \geq 4$, $2 \leq k \leq D-2$, $\lceil \ell/2 \rceil \leq i \leq \lfloor D/2 \rfloor$, with edge set $\{\{v_i, v_s\}, \{v_i, v_{i+k+2}\}\} \cup I(v_s) \cup I(v_i) \cup I(v_{i+k+2}) \cup \bigcup_{j=1}^{i} B(v_j) \cup \bigcup_{j=2i+2}^{i+2k+2} B(v_j) \cup \bigcup_{j=i}^{2i-1} F(v_j) \cup \bigcup_{j=i+k+2}^{D+k-1} F(v_j)$.

and in $\mathcal{H}(CJ, S)$, furthermore, another feasible solution in $\mathcal{H}(CJ, S)$ can be created by replacing the jump edge $\{v, v'\}$ by $\{v_s, v'\}$. Plugging in the incidence vectors of these two solutions into equation (19) and subtracting the results from each other we obtain $\alpha_{v,v'} = \gamma_{v_s,v'}$ and thus (using the results of the previous steps) we have $\mu = \mu'$. □

Proof of Theorem 14

To show that cut generalized-circular-jump inequalities (cgcJ) define facets of $\mathcal{P}$ for $4 \leq D \leq n-3$, $2 \leq k \leq D-3$, let (P, S) be a non-trivial partition of the set of nodes V such that $P = (J_0, J_1, \ldots, J_{D+k})$, let subsets $J_\ell \subset E$ be defined as above and let $\mathcal{H}(GCJ, S) = \{\mathbf{x} \in \mathcal{P} \mid \sum_{\ell=1}^{k} \ell \cdot x(C_\ell) + k \cdot x(P, S) + k \cdot x(E(S)) = k \cdot (|S| + 1)\}$. Consider a facet defining inequality of the form $\vec{\alpha}x(GCJ) + \vec{\beta}x(E(P) \setminus J) + \gamma x(P, S) + \delta x(E(S)) \geq \xi$. We will show that if all points in $\mathcal{H}(GCJ, S)$ satisfy

$$\vec{\alpha}x(GCJ) + \vec{\beta}x(E(P) \setminus GCJ) + \gamma x(P, S) + \delta x(E(S)) = \xi \tag{20}$$

then (20) is a positive multiple of (cgcJ).

Our proof follows the following steps:

- Step 1: $\gamma_{u,v} = \mu'$, for all $\{u, v\} \in [P, S]$ and $\delta_{u,v} = \mu'$, for all $\{u, v\} \in E(S)$
- Step 2: $\beta_{u,v} = \nu$, for all $\{u, v\} \in E(P) \setminus GCJ$, $\alpha_{u,v} = \mu$, for all $\{u, v\} \in C_1$, $\alpha_{u,v} = \ell \cdot \mu$, for all $\{u, v\} \in C_\ell$, $1 \leq \ell \leq k$
- Step 3: $\mu' = k \cdot \mu$

After choosing an arbitrary coefficient $\beta_{uv} = 0$, $\{u, v\} \in E(P) \setminus GCJ$, the theorem follows from the results of Steps 1-3.

Step 1: One can show that the coefficients of edges from the cut $[P, S]$ as well as those of edges from $E(S)$ are identical using a similar technique than in Step 1 of the proof of Theorem 12. By considering solutions with exactly one jump edge from set C_k, all steps are quite similar and we therefore skip the details.

Step 2: Using the result of Step 1, these results can be shown by repeating the necessary steps from the proof of Theorem 8 while directly connecting all nodes from S to a non-leaf node from P incident to at least one jump edge. Since it is easy to see that such a node always exists we skip the details.

Step 3: To show that $\mu' = k \cdot \mu$ consider the solution in $\mathcal{H}(GCJ, S)$ given in Figure 19 and observe that $\{v_i, v_{i+k+2}\} \in J_k$ if $k \leq D-2$ and that for $2 \leq k \leq D-3$ (which is true by assumption of the theorem), another feasible solution from $\mathcal{H}(GCJ, S)$ is obtained by replacing jump edge $\{v_i, v_{i+k+2}\}$ by edge $\{v_s, v_{i+k+2}\}$. Plugging in the incidence vectors corresponding to these two solutions into (20) and subtracting the result from each other, we obtain that $\mu' = \gamma_{v_s,v_{i+k+2}} = \alpha_{v_i,v_{i+k+2}} = k \cdot \mu$.

□