Modelling the Hop Constrained Connected Facility Location Problem on Layered Graphs

Ivana Ljubić [1] and Stefan Gollowitzer [2]

Department of Statistics and Decision Support Systems
Faculty of Business, Economics, and Statistics
University of Vienna, Austria

Abstract

Gouveia et al. [3] show how to model the Hop Constrained Minimum Spanning tree problem as Steiner tree problem on a layered graph. Following their ideas, we provide three possibilities to model the Hop Constrained (HC) Connected Facility Location problem (ConFL) as ConFL on layered graphs. We show that on all three layered graphs the respective LP relaxations of two cut based models are of the same quality. In our computational study we compare a compact hop-indexed tree model against the two cut based models on the simplest layered graph. We provide results for instances with up to 1300 nodes and 115000 arcs.

Keywords: Hop constrained Steiner trees, Connected Facility Location, Mixed Integer Programming Models

1 Introduction

In the field of designing the last mile of telecommunication networks the Fiber-to-the-Curb strategy can be modeled as the Connected Facility Location problem (ConFL) [1]: Fiber optic cables run to a cabinet serving a neighborhood. End users connect to this cabinet using the existing copper connections. Expensive switching devices are installed in these cabinets. The problem is to minimize the costs by determining positions of cabinets, deciding which customers to connect to them, and how to reconnect cabinets among each other

[1] Email: `ivana.ljubic@univie.ac.at`
[2] Email: `stefan.gollowitzer@univie.ac.at`

and to the backbone.
In such simply connected graphs reliability against single arc failures is not provided. Economic arguments do not allow the installation of 2-connected last mile networks. Therefore, the reliability of end-user connections is maintained by limiting the number of nodes between them and the 2-connected backbone network. We model these reliability constraints within the Fiber-to-the-Curb strategy by generalizing the ConFL to the Hop Constrained ConFL (HC ConFL).

Problem Definition

We assume that a root node (i.e., a central office) is given in advance and needs to be included in any feasible solution. In our previous work about the ConFL [1] we describe a transformation of undirected instances into directed ones. Since Mixed Integer Programming models provide stronger lower bounds when defined on bidirected graphs, we consider the *rooted Hop Constrained Connected Facility Location problem on directed graphs*:

Definition 1.1 [Rooted HC ConFL on directed graphs] We are given a directed graph (V, A) with edge costs $c_{ij} \geq 0, ij \in A$, facility opening costs $f_i \geq 0, i \in F$ and a disjoint partition $\{S, R\}$ of V with $R \subset V$ being the set of *customers*, $S \subset V$ the set of possible *Steiner nodes*, $F \subset S$ the set of *facilities*, and the *root* node $r \in F$. Find a subset of open facilities such that

- each customer is assigned to exactly one open facility,
- a Steiner arborescence rooted in r connects all open facilities,
- the cost defined as the sum of assignment, facility opening and Steiner arborescence cost, is minimized and
- there are at most H hops between the root and any open facility.

Note that facilities incur costs only if customers are assigned to them. Customers will be leaves in any optimal solution, thus we do not consider arcs emanating from customer nodes. In [1] we have shown that w.l.o.g. we can assume that the sets of *core nodes* (S) and customers (R) are disjoint.
In this article we will use the following notation: $A_R = \{ij \in A \mid i \in F, j \in R\}$, $A_S = \{ij \in A \mid i, j \in S\}$. We will refer to A_R as *assignment arcs* and to A_S as *core arcs*. Furthermore, for any $W \subset V$ we denote by $\delta^-(W) = \{ij \in A \mid i \notin W, j \in W\}$.

The remainder of this paper is organized as follows: In the following section we will describe a compact model for HC ConFL. In Section 3 we study three different transformations of HC ConFL into ConFL on layered graphs.

In Section 4 we present a computational comparison of some of the models described.

2 A Compact Model for HC ConFL

Gouveia [2] proposes a hop-indexed tree model for the Hop Constrained STP. The LP relaxation of this disaggregated model provides better lower bounds than the classical Miller-Tucker-Zemlin formulation and comprises only slightly more variables and constraints for small values of H.
In our study on ConFL models in [1], we have shown that the practical usage of compact flow based models is very limited. Not even the LP relaxations of flow based models were solvable in reasonable time for larger instance sizes. Therefore, we do not consider flow models in this short abstract.

Let X_{ij}^p indicate whether arc $ij \in A_S$ is used at the p-th position from the root node. Let x_{jk} denote, whether arc $jk \in A_R$ belongs to the solution (1) or not (0). Variable z_i indicates whether the possible facility $i \in F$ is used as such in the solution. Using this notation we can model HC ConFL as follows:

$$(HOP) \ \min \sum_{p=1}^{H} \sum_{ij \in A_S} c_{ij} X_{ij}^p + \sum_{jk \in A_R} c_{jk} x_{jk} + \sum_{i \in F} f_i z_i$$

$$\sum_{\substack{i \in S \setminus \{k\}: \\ ij \in A_S}} X_{ij}^{p-1} \geq X_{jk}^p \qquad \forall jk \in A_S,\ j \neq r,\ p = 2, \dots, H \tag{1}$$

$$\sum_{ij \in A_S} \sum_{p=1}^{H} X_{ij}^p \geq z_j \qquad \forall j \in F \setminus \{r\} \tag{2}$$

$$X_{ij}^p = 0 \qquad ij \in A_S, \begin{cases} i = r,\ p = 2, \dots, H \\ i \neq r,\ p = 1 \end{cases} \tag{3}$$

$$\sum_{jk \in A_R} x_{jk} = 1 \qquad \forall k \in R \tag{4}$$

$$x_{jk} \leq z_j \qquad \forall jk \in A_R \tag{5}$$

$$z_r = 1 \tag{6}$$

$$X_{ij}^p \in \{0, 1\} \qquad \forall ij \in A_S,\ p = 1, \dots, H \tag{7}$$

$$x_{jk} \in [0, 1] \qquad \forall jk \in A_R \tag{8}$$

$$z_i \in \{0, 1\} \qquad \forall i \in F \tag{9}$$

Constraints (1) are connectivity constraints. Since X_{ij}^p are integer variables, they eliminate cycles as well. Inequalities (2) link opening facilities to their in-degree. Equations (3) fix some of the X_{ij}^p to zero: Arcs emanating from the root can only be 1 hop away from it. Conversely, all other arcs are at least two hops away from the root. To ensure that each customer is assigned to exactly one facility, we use constraints (4). If facility serves a customer, it needs to be open (constraints (5)), and the root node is an open facility (equality (6)).

3 Layered Graph Models

Gouveia et al. [3] model the Minimum Spanning Tree problem with hop constraints (HCMST) as Steiner tree problem (STP) on a so-called layered graph. This allows to apply all algorithms developed for the STP to the HCMST. Additionally, the directed cut model on this layered graph turns out to be stronger than the models considered before.
We extend this idea and develop three variants of a layered graph to model the HC ConFL as ConFL on a directed graph. In the first one we transform only the core graph into the layered graph, define nodes at the level H as potential facilities and leave the assignment graph unchanged. We denote the models on this graph by LG_x. For the second variant we build a layered graph in a similar fashion, but now we disaggregate the assignment graph by allowing assignments between a customer and each potential facility at level h, $1 \leq h \leq H$. The models on this graph are denoted by $LG_{x,z}$. Finally, we build a layered graph by introducing facilities and customer nodes at each level $1 \leq h \leq H$ and $1 \leq h \leq H+1$, respectively. The latter models we denote by $LG_{x,z,x}$.

Layered Core Graph LG_x

Consider a graph $LG_x = (V_x, A_x)$ defined as an instance of directed ConFL with the set of potential facilities F_x and the set of core nodes S_x as follows:

$$
\begin{aligned}
&V_x := \{r\} \cup S_x \cup R \text{ where} \\
&\quad F_x = \{(i,H) : i \in F \setminus \{r\}\}, \\
&\quad S_x = F_x \cup \{(i,p) : 1 \leq p \leq H-1, i \in S\} \text{ and} \\
&A_x := \bigcup_{i=1}^{6} A_i \text{ where} \\
&\quad A_1 = \{(r,(j,1)) : rj \in A_S\}, \\
&\quad A_2 = \{((i,p),(j,p+1)) : 1 \leq p \leq H-2, (i,j) \in A_S\}, \\
&\quad A_3 = \{((i,H-1),(j,H)) : ij \in A_S, i \in S \setminus \{r\}, j \in F \setminus \{r\}\}, \\
&\quad A_4 = \{((i,p),(i,H)) : 1 \leq p \leq H-1, i \in F \setminus \{r\}\},
\end{aligned}
$$

$$A_5 = \{((j,H),k) : jk \in A_R, j \neq r\} \text{ and}$$
$$A_6 = \{rk : rk \in A_R\}$$

The facility opening and assignment costs are left unchanged. The arc costs between (i,p) and $(j,p+1)$ are given as c_{ij}. Finally, arcs between (i,p) and (i,H) are assigned costs of 0 for all $p = 1,\ldots,H-1$ and i in F.

Lemma 3.1 (Ljubić and Gollowitzer [4]) *Any HC ConFL instance can be transformed into an equivalent directed ConFL instance on the layered graph LG_x as described above.*

We link binary variables to the arcs in A_x as follows: X^1_{rj} corresponds to $(r,(j,1)) \in A_1$, X^p_{ij} to $((i,p-1),(j,p)) \in A_2$, X^H_{ij} to $((i,H-1),(j,H)) \in A_3$, X^p_{ii} to $((i,p-1),(i,H)) \in A_4$, X_{jk} to $((j,H),k) \in A_5$ and X^1_{rk} to $rk \in A_6$.

Let $X[V_x \setminus W, W]$ denote the sum of all variables X^p_{ij} and X_{jk} in the cut $\delta^-(W)$ defined in LG_x by $W \subset V_x$ and $r \notin W$. In our previous work [1] we have described two cut set based formulations for ConFL, CUT_F and CUT_R. In the former, connectivity is ensured by cuts between the root and the facilities, while assignment constraints guarantee, that each customer is served. Model CUT_R comprises connectivity constraints between the root and each customer and facility variables are linked to assignment edges. Based on these models we derive two formulations, LG_xCUT_F and LG_xCUT_R, as follows:

$$(LG_xCUT_F)\ \min \sum_{rj \in A} c_{rj} X^1_{rj} + \sum_{ij \in A, i \neq r} c_{ij} \sum_{p=2}^{H} X^p_{ij} + \sum_{jk \in A_R, j \neq r} c_{jk} X_{jk} + \sum_{i \in F} f_i z_i$$

$$X[V_x \setminus W, W] \geq z_i \qquad \forall W \in S_x,\ r \notin W,\ (i,H) \in W,\ i \in F \setminus \{r\} \tag{10}$$

$$X^1_{rk} + \sum_{jk:((j,H),k) \in A_5} X_{jk} = 1 \qquad \forall k \in R \tag{11}$$

$$X_{jk} \leq z_j \qquad \forall ((j,H),k) \in A_5 \tag{12}$$

$$\boldsymbol{X} \in \{0,1\}^{|A_x|} \tag{13}$$

$$(6), (9)$$

Constraints (10) are cuts on LG_x between sets containing the root and a facility i respectively. These cuts ensure connectivity between the root and each open facility $i \in F_x$. Equalities (11) ensure each customer is assigned to a facility. Inequalities (12) necessitate a facility to be open if customers are assigned to it.

If we replace constraints (10) and (11) by the following ones, we obtain a stronger formulation that we denote by LG_xCUT_R:

$$X[V_x \setminus W, W] \geq 1 \quad \forall W \subset V_x \setminus \{r\}, W \cap R \neq \emptyset \tag{14}$$

Inequalities (14) are cuts on LG_x between sets that contain the root and at least one customer respectively.

Layered Core and Assignment Graph $LG_{x,z}$

In graph $LG_{x,z} = (V_x, A_{x,z})$, the set of potential facilities is defined as $F_{x,z} = \{(i,p) : i \in F \setminus \{r\}, 1 \leq p \leq H\}$. The arc set $A_{x,z} = \bigcup_{i=1}^{3} A_i \cup A_6 \cup A_7$ with $A_7 = \{((i,p),k) \mid (i,p) \in F_{x,z}, k \in R\}\}$. The arc costs for the latter set are defined as c_{ik} for all $i \in F \setminus \{r\}$ and $k \in R$.

Layered Graph with Split Customers $LG_{x,z,x}$

Graph $LG_{x,z,x}$ equals $(\{r\} \cup S_x \cup R_x, A_{x,z,x})$, where the set of customers R_x is disaggregated as follows: $R_x = \{(k,p) : k \in R, 1 \leq p \leq H+1\}$. The set of potential facilities is $F_{x,z}$ as defined above. The set of arcs in this graph is $A_{x,z,x} = \bigcup_{i=1}^{3} A_i \cup \bigcup_{i=8}^{10} A_i$ where

$$\begin{aligned}
A_8 &= \{(r,(k,1)) : rk \in A_R\}, \\
A_9 &= \{((j,p),(k,p+1)) : (j,p) \in F_L, jk \in A_R\} \text{ and} \\
A_{10} &= \{((k,p),(k,H+1)) : k \in R, 1 \leq p \leq H\}.
\end{aligned}$$

The arc costs for A_8 and A_9 are given by the respective facility-customer pairs, the costs for arcs in A_{10} are 0.

Let $\upsilon_{LP}(.)$ denote the optimal solution value of the LP relaxation of a given model. The LP relaxations of the latter two formulations do not lead to improved lower bounds, compared to the respective formulations on LG_x:

Lemma 3.2 (Ljubić and Gollowitzer [4])

a) $\upsilon_{LP}(LG_xCUT_R) = \upsilon_{LP}(LG_{x,z}CUT_R) = \upsilon_{LP}(LG_{x,z,x}CUT_R)$ *and* $\upsilon_{LP}(LG_xCUT_F) = \upsilon_{LP}(LG_{x,z}CUT_F) = \upsilon_{LP}(LG_{x,z,x}CUT_F)$

b) $\upsilon_{LP}(HOP) \leq \upsilon_{LP}(LG_xCUT_F)$. *There are instances for which the strict inequality holds.*

Therefore, in our computational study we only considered LG_x. It comprises the least number of arcs and nodes.

4 Computational Results

In our computational study we used a set of instances derived from OR-library [3] and UflLib [4]. These instances consist of up to 1000 facilities, 300 customers and 115000 edges. A detailed description can be found in [1].
Our experiments were performed on a Intel Core2 Quad 2.33 GHz machine with 3.25 GB RAM, where each run was performed on a single processor. For solving the linear programming relaxations and for a generic implementation of the branch-and-cut approach, we used the commercial packages IBM CPLEX (version 11.2) [5] and ILOG Concert Technology (version 2.7).
In our settings we deactivated CPLEX cuts and set a time limit of 1 hour. We set the highest branching priority to variables z.
The first three columns in Table 1 show the hop limit and the respective instance group. Then, for each of the three models and the instance group we report the average LP gap ($gap_{LP} = (OPT - v_{LP})/OPT$), the number of instances that were solved to optimality and the average running time until the optimum was reached. Hyphens denote that not all instances of the group could be solved.

Preprocessing: In all instances considered, we removed arcs jk with $j \in F \setminus \{r\}$ and $k \in R$ if $c_{rk} < c_{jk}$. From the layered graph we recursively removed nodes different from the root node with in-degree 0, starting from level 1. Also, Steiner nodes with out-degree 0 were removed recursively, starting from level $H - 1$.

LP Gaps: Between the LP gaps obtained for CUT_F and HOP there were almost negligible differences. For CUT_R the obtained results were slightly better. The largest gap obtained was 3.42% for instance `d15-mp1` and model HOP with $H = 3$. Note, that not for all instances the LP relaxation of HOP could be solved due to the large memory consumption.
For models $CUT_{F/R}$ and HOP 9 and 8 LP solutions were found to be integer, respectively.

Running times: CUT_F solves all instances to optimality in less than 110 seconds on average. The maximum running time was 666 seconds for instance `d15-mq2` with a hop limit of 5.
Using model HOP we solved all but 7 instances to optimality. The average (maximum) running time for these instances was 81.5 (350.7) seconds. For the remaining 7 instances the model was too large and could not be solved

[3] http://people.brunel.ac.uk/~mastjjb/jeb/orlib/steininfo.html
[4] http://www.mpi-inf.mpg.de/departments/d1/projects/benchmarks/UflLib/
[5] http://www.ilog.com/products/cplex/

due to memory limitations.
CUT_R solved 82 models to optimality within the given time limit of one hour. For the remaining 14 instances we obtained a gap between 0.14 and 1.66%.

			CUT_F			HOP			CUT_R		
H			gap_{LP}	$\#_{OPT}$	t	gap_{LP}	$\#_{OPT}$	t	gap_{LP}	$\#_{OPT}$	t
3	c	mp	1.43	**8**	26.3	1.45	**8**	**25.8**	**1.28**	**8**	107.0
3	c	mq	2.11	**8**	116.5	2.12	**8**	**87.0**	**2.01**	**8**	955.3
3	d	mp	1.04	**8**	**21.0**	1.04	**8**	31.3	**0.97**	**8**	139.9
3	d	mq	1.36	**8**	**68.8**	1.36	**8**	75.5	**1.30**	**8**	620.8
4	c	mp	1.35	**8**	42.9	1.37	**8**	**39.2**	**1.12**	**8**	297.9
4	c	mq	2.06	**8**	211.7	2.07	**8**	**167.0**	**1.91**	5	2058.6
4	d	mp	1.71	**8**	**41.1**	1.72	**8**	52.2	**1.53**	**8**	222.6
4	d	mq	2.21	**8**	**167.2**	-	6	-	**2.10**	6	1561.8
5	c	mp	1.67	**8**	66.6	1.71	**8**	**64.5**	**1.41**	7	1207.9
5	c	mq	2.38	**8**	**235.9**	-	6	-	**2.24**	3	2702.8
5	d	mp	1.55	**8**	**87.6**	-	7	-	**1.33**	**8**	305.4
5	d	mq	2.26	**8**	**229.9**	-	6	-	**2.10**	5	2279.8
				96			89			82	

Table 1
Average results per instance group and hop limit.

The obtained results indicate the computational advantage of layered graph models in comparison to compact models for HC ConFL. The facility-based cut set model with weaker lower bounds (CUT_F) computationally outperforms its customer-based counterpart (CUT_R) on layered graphs.

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