Metaplectic action on modulation spaces

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Notation

Translations: $T_x f(t) = f(t-x)$,

Modulations: $M_\omega f(t) = e^{2\pi i\omega\cdot t} f(t)$,

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Definition (Cross-ambiguity function)

$$\mathrm{A}(f,g)(x,\omega) = \int_{\mathbb{R}^d} f(t+\tfrac{x}{2})\overline{g(t-\tfrac{x}{2})} e^{-2\pi i\omega\cdot t}\, dt.$$

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$$\mathrm{A}(f,g)(x,\omega) = \langle f, \rho(\lambda)g\rangle\,.$$

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Alternative TF representations

Cross-ambiguity function

$$A(f,g)(x,\omega) = \int_{\mathbb{R}^d} f(t+\tfrac{x}{2})\overline{g(t-\tfrac{x}{2})}e^{-2\pi i\omega\cdot t}\,dt.$$

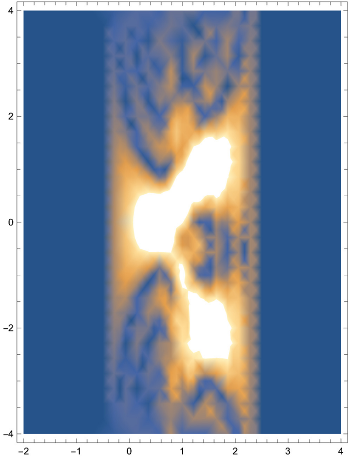
Short-time Fourier transform

$$V_g f(x,\omega) = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-2\pi i\omega\cdot t}\,dt.$$

Cross-Wigner transform

$$W(f,g)(x,\omega) = \int_{\mathbb{R}^d} f(x+\tfrac{t}{2})\overline{g(x-\tfrac{t}{2})}e^{-2\pi i\omega\cdot t}\,dt.$$

TF concentration



$$f(t) = \begin{cases} e^{2\pi i\, 0\cdot t}, & t \in (0,1), \\ e^{2\pi i\, 1\cdot t} + e^{2\pi i\, (-2)\cdot t}, & t \in (1,2), \\ 0, & \text{else.} \end{cases}$$

Plot of the spectrogram of f w.r.t. the box function $b_0 = \chi_{(-1/2,1/2)}$.

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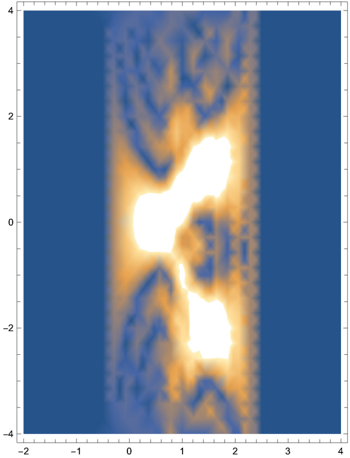

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Plot of the spectrogram of f w.r.t. the box function $b_0 = \chi_{(-1/2,1/2)}$.

We measure concentration in mixed $L^{p,q}$-norms.

Lebesgue spaces

Definition (Mixed-norm Lebesgue spaces $L^{p,q}(\mathbb{R}^{2d})$)

$F : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{C}$

$$\|F\|_{p,q} := \Big\| \omega \mapsto \|F(\bullet,\omega)\|_p \Big\|_q$$
$$= \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}_d} |F(x,\omega)|^p \, dx \right)^{q/p} d\omega \right)^{1/q}$$

Lebesgue spaces

Definition (Weighted mixed-norm Lebesgue spaces $L^{p,q}_{\mathbf{m}}(\mathbb{R}^{2d})$)

$F : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{C}$

$$\|F\|_{p,q,m} := \Big\| \omega \mapsto \|F(\bullet,\omega)\mathbf{m}(\bullet,\omega)\|_p \Big\|_q$$

$$= \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}_d} |F(x,\omega)\mathbf{m}(\mathbf{x},\omega)|^p \, dx \right)^{q/p} d\omega \right)^{1/q}$$

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$$\|F\|_{p,q,m} := \Big\| \omega \mapsto \|F(\bullet,\omega)\mathbf{m}(\bullet,\omega)\|_p \Big\|_q$$

$$= \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}_d} |F(x,\omega)\mathbf{m}(\mathbf{x},\omega)|^p \, dx \right)^{q/p} d\omega \right)^{1/q}$$

We assume that the weight is **moderate**, i.e., there is a submultiplicative weight $v : \mathbb{R}^{2d} \to \mathbb{R}_+$ and $C > 0$ such that for all $x, y \in \mathbb{R}^{2d}$

$$m(x+y) \leq C\, m(x)v(y), \qquad \mathbf{m}(\mathbf{x}+\mathbf{y}) \lesssim \mathbf{m}(\mathbf{x})\mathbf{v}(\mathbf{y}) \; .$$

Modulation spaces

Reminder:

$\mathrm{A}(\varphi, g)(x, \omega) = \langle \varphi, M_{\frac{\omega}{2}}\, T_x\, M_{\frac{\omega}{2}}\, g \rangle$ and $\|F\|_{p,q,m} = \Big\| \omega \mapsto \big\| F(\bullet, \omega) m(\bullet, \omega) \big\|_p \Big\|_q$.

Definition (Modulation spaces)

$$\mathrm{M}_m^{p,q}(\mathbb{R}^d) := \left\{ \varphi \in \mathcal{S}'(\mathbb{R}^d) \mid \mathrm{A}(\varphi, g_0) \in L_m^{p,q}(\mathbb{R}^{2d}) \right\}$$

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Modulation spaces

Reminder:

$$\mathrm{A}(\varphi, g)(x, \omega) = \langle \varphi, M_{\frac{\omega}{2}} T_x M_{\frac{\omega}{2}} g \rangle \text{ and } \|F\|_{p,q,m} = \left\| \omega \mapsto \left\| F(\bullet, \omega) m(\bullet, \omega) \right\|_p \right\|_q .$$

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$$\mathrm{M}_m^{p,q}(\mathbb{R}^d) := \left\{ \varphi \in \mathcal{S}'(\mathbb{R}^d) \mid \mathrm{A}(\varphi, g_0) \in L_m^{p,q}(\mathbb{R}^{2d}) \right\}$$

- $\mathrm{M}_m^{p,q}(\mathbb{R}^d)$ is independent of the choice of the window.
- Different windows induce equivalent norms

$$\|\varphi\|_{\mathrm{M}_m^{p,q}(\mathbb{R}^d)} \asymp \|\mathrm{A}(\varphi, g)\|_{p,q,m} .$$

Alternative definitions

Definition (Modulation spaces)

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Modulation spaces via the STFT

$$\mathrm{M}_m^{p,q}(\mathbb{R}^d) = \left\{\varphi \in \mathcal{S}'(\mathbb{R}^d) \mid \mathrm{V}_{g_0}\varphi \in L_m^{p,q}(\mathbb{R}^{2d})\right\}$$

Modulation spaces via the Wigner transform

$$\mathrm{M}_m^{p,q}(\mathbb{R}^d) = \left\{\varphi \in \mathcal{S}'(\mathbb{R}^d) \mid \mathrm{W}(\varphi, g_0) \in L_m^{p,q}(\mathbb{R}^{2d})\right\}$$

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Symmetric TF shifts

Notation:

$$\rho(x,\omega) = M_{\frac{\omega}{2}} T_x M_{\frac{\omega}{2}},$$

$$\lambda = (x,\omega)^t \in \mathbb{R}^{2d}$$
$$\nu = (\eta,\xi)^t \in \mathbb{R}^{2d},$$

$$M_\omega T_x = e^{2\pi i \omega \cdot x} T_x M_\omega$$

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$\nu = (\eta,\xi)^t \in \mathbb{R}^{2d}$,

$\mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in \mathbb{R}^{2d\times 2d}$

$$M_\omega T_x = e^{2\pi i\omega\cdot x} T_x M_\omega$$

$$\rho(\lambda)\rho(\nu) = e^{\pi i(\omega\cdot\eta - x\cdot\xi)}\rho(\lambda+\nu) = e^{-\pi i\lambda^t\mathcal{J}\nu}\rho(\lambda+\nu).$$

The symmetric time-frequency shifts are **not** a representation of $\mathbb{R}^{2d}$ on $L^2(\mathbb{R}^d)$!

Symplectic group: $\mathrm{Sp}(2d,\mathbb{R})$

Relations:

$$\rho(\lambda)\rho(\nu) = e^{-\pi i \lambda^t \mathcal{J} \nu} \rho(\lambda+\nu) \qquad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

Definition (The symplectic group $\mathrm{Sp}(2d,\mathbb{R})$)

$$\mathrm{Sp}(2d,\mathbb{R}) := \left\{ S \in \mathbb{R}^{2d\times 2d} : S^t \mathcal{J} S = \mathcal{J} \right\}.$$

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Examples (Building blocks)

Let $L \in \mathrm{GL}(d,\mathbb{R})$, $P, Q \in \mathbb{R}^{d\times d}$, such that $P = P^t,\ Q = Q^t$.

$$D_L := \begin{pmatrix} L & 0 \\ 0 & L^{-t} \end{pmatrix} \qquad U_P := \begin{pmatrix} I & P \\ 0 & I \end{pmatrix} \qquad V_Q := \begin{pmatrix} I & 0 \\ Q & I \end{pmatrix}$$

Metaplectic group: $\mathrm{Mp}(2d, \mathbb{R})$

Definition

The metaplectic group is the double cover of the symplectic group.

Theorem

For all $S \in \mathrm{Sp}(2d, \mathbb{R})$ there is a unitary operator $\widehat{S}$ with

$$\rho(S\lambda) = \widehat{S}\rho(\lambda)\widehat{S}^{-1}, \qquad \lambda \in \mathbb{R}^{2d}.$$

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For all $S \in \mathrm{Sp}(2d, \mathbb{R})$ *there is a unitary operator* $\widehat{S}$ *with*

$$\rho(S\lambda) = \tau \widehat{S}\, \rho(\lambda)\, \tau^{-1} \widehat{S}^{-1}, \qquad \lambda \in \mathbb{R}^{2d}, \quad \tau \in \mathbb{T}.$$

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$$\pi^{\mathrm{Mp}} : \mathrm{Mp}(2d, \mathbb{R}) \to \mathrm{Sp}(2d, \mathbb{R})$$

is a group homomorphism, with $\ker(\pi^{\mathrm{Mp}}) = \{\mathrm{id}, -\mathrm{id}\}$.

Examples

Examples

$$D_L := \begin{pmatrix} L & 0 \\ 0 & L^{-t} \end{pmatrix} \quad \rightsquigarrow \quad \mathcal{D}_L f(t) = |\det L|^{-1/2} f(L^{-1}t),$$

$$V_Q := \begin{pmatrix} I & 0 \\ Q & I \end{pmatrix} \quad \rightsquigarrow \quad \mathcal{V}_Q f(t) = e^{\pi i t \cdot Q t} f(t),$$

$$\mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad \rightsquigarrow \quad \mathcal{F}.$$

Symplectic covariance

Theorem

Let $f \in \mathcal{S}'(\mathbb{R}^d)$, $g \in \mathcal{S}(\mathbb{R}^d)$, *and* $\widehat{S} \in \mathrm{Mp}(2d, \mathbb{R})$ *be given. Let* $S \in \mathrm{Sp}(2d, \mathbb{R})$ *be the projection of* $\widehat{S}$ *onto* $\mathrm{Sp}(2d, \mathbb{R})$*. Then* $\widehat{S}f \in \mathcal{S}'(\mathbb{R}^d)$, $\widehat{S}g \in \mathcal{S}(\mathbb{R}^d)$, *and*

$$\mathrm{A}(\widehat{S}f, \widehat{S}g)(\lambda) = \mathrm{A}(f, g)(S^{-1}\lambda), \qquad \lambda \in \mathbb{R}^{2d}.$$

Problem

Symplectic covariance: $\mathrm{A}(\widehat{S}f,\widehat{S}g)(\lambda) = \mathrm{A}(f,g)(S^{-1}\lambda), \qquad \lambda \in \mathbb{R}^{2d}.$

Problem

Let $\widehat{S} \in \mathrm{Mp}(2d,\mathbb{R})$, $p,q \in [1,\infty]$ *be given. Is the operator*

$$\widehat{S} : M^{p,q}_m(\mathbb{R}^d) \to M^{p,q}_m(\mathbb{R}^d), \qquad f \mapsto \widehat{S}f$$

- *well-defined?*
- *bounded?*

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Let $g \in \mathcal{S}(\mathbb{R}^d)$. If $\mathrm{A}(f,g) \in L^{p,q}_m(\mathbb{R}^{2d})$, does this imply

$$L^{p,q}_m(\mathbb{R}^{2d}) \ni \mathrm{A}(\widehat{\mathbf{S}}f, h)$$

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Symplectic covariance: $A(\widehat{S}f, \widehat{S}g)(\lambda) = A(f,g)(S^{-1}\lambda), \qquad \lambda \in \mathbb{R}^{2d}.$

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Let $\widehat{S} \in \mathrm{Mp}(2d, \mathbb{R})$, $p, q \in [1, \infty]$ *be given. Is the operator*

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Let $g \in \mathcal{S}(\mathbb{R}^d)$. If $A(f,g) \in L^{p,q}_m(\mathbb{R}^{2d})$, does this imply

$$L^{p,q}_m(\mathbb{R}^{2d}) \ni A(\widehat{\mathbf{S}} f, h) = A(\widehat{S} f, \widehat{\mathbf{S}}\mathbf{g}) = A(f,g)(\mathbf{S}^{-1}\,\bullet)\ ?$$

$$\mathcal{D}_{\mathbf{S}} f(t) := f(\mathbf{S}^{-1} t).$$

Commutative diagram

$$\mathrm{A}(\widehat{S}f, \widehat{S}g)(\lambda) = \mathrm{A}(f,g)(S^{-1}\lambda), \qquad \lambda \in \mathbb{R}^{2d}.$$

$$\begin{array}{ccccc}
\mathrm{M}^{p,q}(\mathbb{R}^d) & \xrightarrow{\mathrm{A}(\cdot,g)} & \mathrm{A}(\mathrm{M}^{p,q}(\mathbb{R}^d),g) & \hookrightarrow & L^{p,q}(\mathbb{R}^{2d}) \\
\Big\downarrow \widehat{S} & & \Big\downarrow & & \Big\downarrow \mathcal{D}_S \\
\mathrm{M}^{p,q}(\mathbb{R}^d) & \xrightarrow[\mathrm{A}(\,\cdot\,,g)]{} & \mathrm{A}(\mathrm{M}^{p,q}(\mathbb{R}^d),g) & \hookrightarrow & L^{p,q}(\mathbb{R}^{2d})
\end{array}$$

$$\mathcal{D}_{\mathbf{S}} f(t) := f(\mathbf{S}^{-1}t).$$

The classification

Theorem (Führ, S., "22 [2])

Let $p, q \in [1, \infty]$, $\widehat{S} \in \mathrm{Mp}(2d, \mathbb{R})$ be given. Let $S \in \mathrm{Sp}(2d, \mathbb{R})$ be the projection of $\widehat{S}$ onto $\mathrm{Sp}(2d, \mathbb{R})$. Then the following statements are equivalent:

- (i) $\widehat{S} : M^{p,q}(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$ *is well-defined.*
- (ii) $\widehat{S} : M^{p,q}(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$ *is well-defined and bounded.*
- (iii) *One of the following conditions holds:*
  - (a) $p = q$, *or*
  - (b) $p \neq q$ *and* S *is a block upper triangular matrix.*

If one, hence all, of the statements hold, then $\mathcal{D}_S : L^{p,q}(\mathbb{R}^{2d}) \to L^{p,q}(\mathbb{R}^{2d})$ *is an isometric automorphism (up to a multiplicative constant), and* $\widehat{S}$ *has a bounded inverse.*

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Proof idea

$F = f \otimes g \in L^{p,q}(\mathbb{R}^{2d})$

$S^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

$$\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x)g(\omega)|^p \; dx \right)^{q/p} d\omega$$

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$$\int_{\mathbb{R}^d} |g(Cx + D\omega)|^q \left(\int_{\mathbb{R}^d} |f(Ax + B\omega)\,|^p \, dx \right)^{q/p} d\omega \quad < \infty?$$

Commutative diagram

$$
\begin{array}{ccccc}
\mathrm{M}^{p,q}_{\mathbf{m}}(\mathbb{R}^d) & \xrightarrow{\ \mathrm{A}(\cdot,g)\ } & \mathrm{A}(\mathrm{M}^{p,q}_{\mathbf{m}}(\mathbb{R}^d), g) & \hookrightarrow & L^{p,q}_{\mathbf{m}}(\mathbb{R}^{2d}) \\
\Big\downarrow \widehat{S} & & \Big\downarrow & & \Big\downarrow \mathcal{D}_S \\
\mathrm{M}^{p,q}_{\mathbf{m}}(\mathbb{R}^d) & \xrightarrow[\ \mathrm{A}(\cdot,\widehat{S}^{-1}g)\ ]{} & \mathrm{A}(\mathrm{M}^{p,q}_{\mathbf{m}}(\mathbb{R}^d), \widehat{S}^{-1}g) & \hookrightarrow & L^{p,q}_{\mathbf{m}}(\mathbb{R}^{2d})
\end{array}
$$

Lifting theorems

Theorem (Isomorphism relations)

The Lebesgue spaces $L^{p,q}_m(\mathbb{R}^{2d})$ and $L^{p,q}(\mathbb{R}^{2d})$ are isomorphic via

$$\Phi_m : L^{p,q}_m(\mathbb{R}^{2d}) \to L^{p,q}(\mathbb{R}^{2d}), \qquad f \to m \cdot f.$$

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$$\mathrm{M}^{p,q}_m(\mathbb{R}^d) \qquad \longleftrightarrow \qquad \mathrm{M}^{p,q}(\mathbb{R}^d) \quad \textbf{?}$$

Toeplitz operators

Definition

Let $a \in \mathcal{S}(\mathbb{R}^{2d})$ be a symbol and $g \in \mathcal{S}(\mathbb{R}^{2d})$ a fixed window. Then the Toeplitz operator $Tp_g(a)$ is defined by the formula

$$\langle Tp_g(a) f_1, f_2 \rangle_{L^2(\mathbb{R}^d)} = \langle a \, \mathrm{A}(f_1, g), \mathrm{A}(f_2, g) \rangle_{L^2(\mathbb{R}^{2d})}$$

for all $f_1, f_2 \in L^2(\mathbb{R}^d)$.

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for all $f_1, f_2 \in L^2(\mathbb{R}^d)$.

Theorem (Lifting theorem, Gröchenig, Toft, "11 [4])

Assume that m is an even, v-moderate weight function with $v(\lambda) = (1 + \|\lambda\|)^N$ for some $N \in \mathbb{N}$ and $g \in \mathcal{S}(\mathbb{R}^d)$. Then the Toeplitz operator $Tp_g(m)$ is an isomorphism from $\mathrm{M}^{p,q}_{m_0}(\mathbb{R}^d)$ onto $\mathrm{M}^{p,q}_{m_0/m}(\mathbb{R}^d)$ for every v-moderate even weight m_0 and every $p, q \in [1, \infty]$.

Theorem

Let $v(z) = (1 + \|z\|)^N$ *for some* $N \in \mathbb{N}$ *and* m *be a* v*-moderate, even weight,* $p, q \in [1, \infty]$. *If* $\widehat{S} \in \mathrm{Mp}(2d, \mathbb{R})$ *with projection* $\pi^{Mp}(\widehat{S}) = S$ *satisfies* $m \asymp m \circ S^{-1}$*, then the following statements are equivalent:*

1. $\widehat{S}$ *is a bounded operator from* $M^{p,q}_m(\mathbb{R}^d)$ *to* $M^{p,q}_m(\mathbb{R}^d)$.
2. $\widehat{S}$ *is a bounded operator from* $M^{p,q}(\mathbb{R}^d)$ *to* $M^{p,q}(\mathbb{R}^d)$.

Thank you for your attention!

References I

M. A. de Gosson.
Symplectic Methods in Harmonic Analysis and in Mathematical Physics.
Springer Basel, 2011.

H. Führ and I. Shafkulovska.
The metaplectic action on modulation spaces.
arXiv preprint: 2211.08389, 2022.

K. Gröchenig.
Foundations of Time-Frequency Analysis.
Birkhäuser Boston, 2001.

K. Gröchenig and J. Toft.
Isomorphism properties of Toeplitz operators and pseudo-differential operators between modulation spaces.
J. d'Analyse Math., 114(1), 2011.

References I

M. A. de Gosson.
Symplectic Methods in Harmonic Analysis and in Mathematical Physics.
Springer Basel, 2011.

H. Führ and I. Shafkulovska.
The metaplectic action on modulation spaces.
arXiv preprint: 2211.08389, 2022.

K. Gröchenig.
Foundations of Time-Frequency Analysis.
Birkhäuser Boston, 2001.

K. Gröchenig and J. Toft.
Isomorphism properties of Toeplitz operators and pseudo-differential operators between modulation spaces.
J. d'Analyse Math., 114(1), 2011.