Sampling theorems with derivatives in shift-invariant spaces generated by EB-splines

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Shift-invariant spaces

- Translation operator (time shift): $T_\ell f(x) = f(x-\ell)$.
- Modulation operator (frequency shift): $M_\omega f(x) = e^{2\pi i \omega x} f(x)$.

Definition (Shift-invariant spaces)

Let $\varphi \in L^p(\mathbb{R})$ be a ***nice*** function (e.g. in a Wiener-amalgam space). The L^p-associated shift-invariant space is given by

$$V^p(\varphi) := \Big\{ \sum_{\ell\in\mathbb{Z}} c_\ell T_\ell \varphi \in L^p(\mathbb{R}),\, c \in \ell^p(\mathbb{Z}) \Big\}.$$

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Example (The Paley-Wiener space)

$$PW^2(\mathbb{R}) = \Big\{ f \in L^2(\mathbb{R}) : \operatorname{supp}(\widehat{f}) \subseteq \big[\tfrac{1}{2}, \tfrac{1}{2}\big] \Big\} = V^2(\operatorname{sinc}).$$

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Definition (Stable integer translates)

$$\frac{1}{C_p} \|c\|_{\ell^p} \le \Big\| \sum_{\ell\in\mathbb{Z}} c_\ell T_\ell \varphi \Big\|_{L^p} \le C_p \|c\|_{\ell^p}, \qquad c \in \ell^p(\mathbb{Z}).$$

Sampling in SI-spaces

Classical sampling

A set X is ***sampling*** for $V^p(\varphi)$ if there exist constants $0 < A_p, B_p < \infty$ s.t. for all $f \in V^p(\varphi)$ holds

$$A_p \|f\|_p^p \leq \sum_{x \in X} |f(x)|^p \leq B_p \|f\|_p^p \,.$$

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Sampling with derivatives

A pair (X, μ_X), where X is a set and $\mu_X : X \to \{0, \dots, S\}$ is ***its multipilcity function***, is ***sampling*** for $V^p(\varphi)$ if there exist constants $0 < A_p, B_p < \infty$ s.t. for all $f \in V^p(\varphi)$ holds

$$A_p \|f\|_p^p \leq \sum_{x\in X} \sum_{s=0}^{\mu_X(x)} \left| f^{(s)}(x) \right|^p \leq B_p \|f\|_p^p .$$

EB-Splines

Definition

An exponential B-spline (EB-spline) $\mathcal{E}_{m,\alpha} : \mathbb{R} \longrightarrow \mathbb{R}$ of order m for $\alpha \in \mathbb{R}^m$ is a function of the form

$$\mathcal{E}_{m,\alpha}(x) := \prod_{s=1}^{m}{}^{*} e^{\alpha_s \bullet} \chi_{[0,1)}(x),$$

where $\prod^*$ denotes the convolution product.

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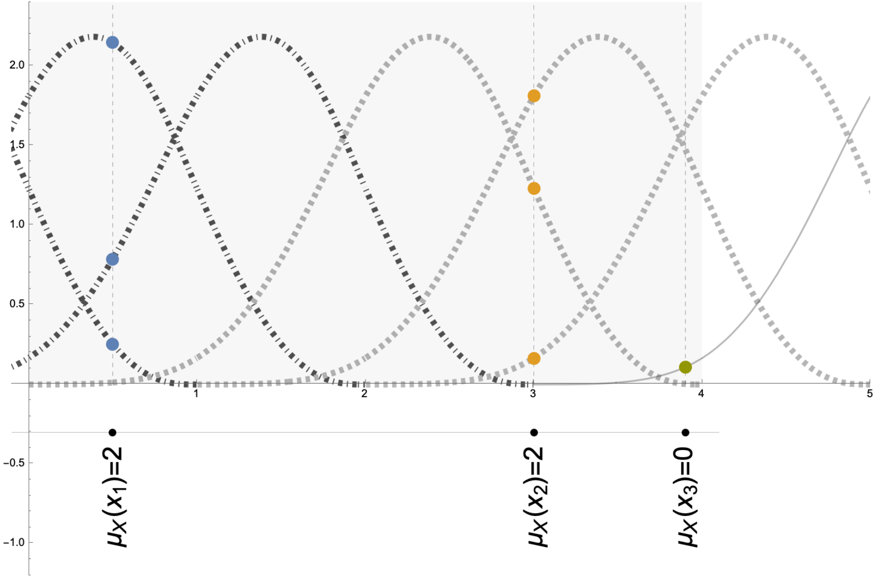
where $\prod^{*}$ denotes the convolution product.

Associated weights and operators:

- Exponential weights: $w_1(x) = e^{\alpha_1 x}$, $w_s = e^{(\alpha_s - \alpha_{s-1})x}$, $2 \leq s \leq m$,
- Differential operators: $D_0 f = f$ and for $0 \leq s \leq m$

$$D_s f := \frac{d}{dx} \frac{f}{w_s}, \quad L_s := D_s D_{s-1} \dots D_0.$$

EB-Splines



Schoenberg-Whitney conditions

Theorem

Let φ be an EB-spline of order m. Further let $t_0 \leq t_2 \leq \cdots \leq t_D$ and set

$$d_i := \max\{\ell : t_i = \cdots = t_{i-\ell}\}, \quad 0 \leq i \leq D.$$

The collocation matrix

$$M \begin{pmatrix} t_0, \ldots, t_D \\ \varphi, \ldots, T_D\varphi \end{pmatrix} := \Big(L_{d_i} T_\ell \varphi(t_i)\Big)_{0 \leq i, \ell \leq D}$$

has a non-negative determinant. The collocation matrix is invertible if and only if

$$t_i \in \begin{cases} (i, i+m), & d_i < m-1 \\ [i, i+m), & d_i = m-1, \end{cases}$$

for all $0 \leq i \leq D$.

Hermite interpolation problem

The collocation matrix is the matrix that describes the Hermite interpolation problem

$$f=\sum_{\ell=0}^{D} c_\ell T_\ell \varphi, \quad L_{d_i} f(t_i)=\xi_i, \quad 0\le i\le D.$$

The ***Schoenberg-Whitney conditions*** characterize when the Hermite interpolation problem has a unique solution.

Why Schoenberg-Whitney?

I. J. Schoenberg. *On Pólya frequency functions. I. The totally positive functions and their Laplace transforms.* J. Analyse Math., 1:331–374, 1951.

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Why Schoenberg-Whitney?

ON PÓLYA FREQUENCY FUNCTIONS. III. THE POSITIVITY OF TRANSLATION DETERMINANTS WITH AN APPLICATION TO THE INTERPOLATION PROBLEM BY SPLINE CURVES(1)

BY

I. J. SCHOENBERG AND ANNE WHITNEY

INTRODUCTION

1. A frequency function $\Lambda(x)$, i.e., a non-negative measurable function satisfying the inequalities

$$0 < \int_{-\infty}^{\infty} \Lambda(x)dx < \infty,$$

is called a Pólya frequency function provided(2) it satisfies the following condition: *For every two sets of increasing numbers*

(1) $$x_1 < x_2 < \cdots < x_n, \qquad y_1 < y_2 < \cdots < y_n, \qquad n = 1, 2, \cdots,$$

we have the inequality

(2) $$D \equiv \det \left\| \Lambda(x_i - y_j) \right\|_{1,n} \geqq 0.$$

Why Schoenberg-Whitney?

The present paper is divided into two sections. In §1 we answer (Theorem 1 below) the following question: *Given a Pólya frequency function* $\Lambda(x)$ *and a set of* $2n$ *numbers* (1), *how can we decide when the determinant* D, *defined by* (2), *is actually positive*? As an application of our answer to this question we solve in §2 the general problem of interpolation by so-called spline curves which were introduced in 1946 by one of us [5] for the purpose of approximation of infinitely many equidistant data.

Maximum gap theorem

Theorem (Gröchenig,S. "23 [4])

Let φ be an EB-spline of order m and let $X \subseteq \mathbb{R}$ be a separated set with $\mu_X : X \to \{0, \dots, m-1\}$. If the multiplicity function satisfies

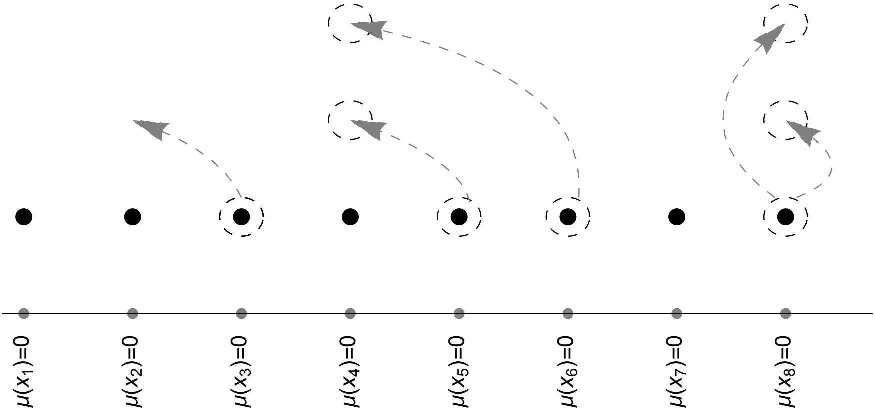
$$\operatorname{dist}\left(\{x \in X : \mu_X(x) = m-1\}\,,\ \mathbb{Z}\,\right) > 0$$

and the weighted maximum gap satisfies

$$\mathfrak{mg}(X, \mu_X) := \sup_{j \in \mathbb{Z}} \frac{x_{j+1} - x_j}{1 + \min\{\mu_X(x_j), \mu_X(x_{j+1})\}} < 1,$$

then (X, μ_X) is a sampling set for $V^p(\varphi)$. This holds in particular if $\mu_X \equiv S$ and $\mathfrak{mg}(X) < S$.

Constructing sampling sets



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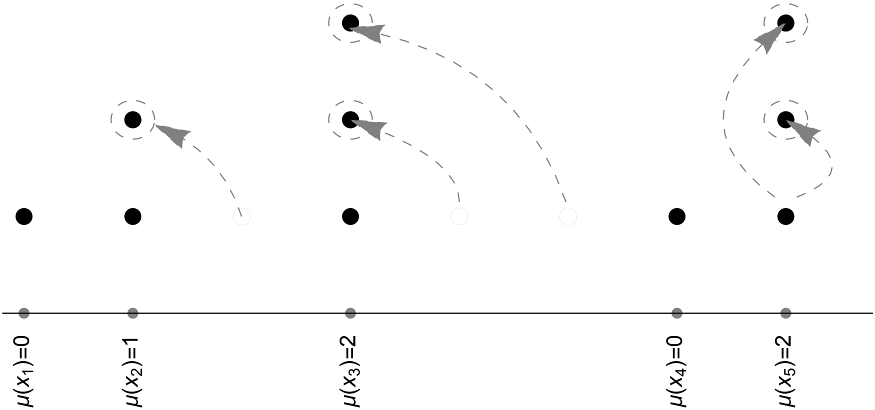

Proof idea

Solve the problem locally: On an interval $[M, M+L]$, $M, L \in \mathbb{Z}$, the restriction of a prototypical function $f \in V^\infty(\varphi)$ is given by

$$f_{|_{[M,M+L]}} = \sum_{\ell=M-m+1}^{M+L-1} c_\ell T_\ell \varphi,$$

i.e., it is a linear combination of $L+m-1$ shifts of the EB-spline.

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- Prove that for a sufficiently large L, there are $L+m-1$ samples available on $[M, M+L]$.

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- Prove that these samples satisfy the Schoenberg-Whitney conditions.

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- Prove that for a sufficiently large L, there are $L+m-1$ samples available on $[M, M+L]$.
- Prove that these samples satisfy the Schoenberg-Whitney conditions.
- Recover f on $\mathbb{R}$ by induction.

A major technical issue is the control of the frame constants!

Optimality of the results

Proposition (Gröchenig, Romero, Stöckler "19 [3])

Let $\varphi \in L^p(\mathbb{R})$ have stable integer translate. Then (X, μ_X) is a sampling set with derivatives if $D^-(X, \mu_X) \geq 1$, where $D^-(X, \mu_X)$ is the (weighted) lower Beurling density

$$D^-(X, \mu_X) := \liminf_{r\to\infty} \frac{1}{2r} \min_{y\in\mathbb{R}} \sum_{x\in X, |x-y|<r} (1 + \mu_X(x)).$$

[3] K. Gröchenig, J. L. Romero, and J. Stöckler. *Sharp Results on Sampling with Derivatives in Shift-Invariant Spaces and Multi-Window Gabor Frames.* Constr. Approx., 51(1):1–25, 2019.

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$$\mathfrak{mg}(X, \mu_X)^{-1} \geq D^-(X, \mu_X).$$

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Corollary

For all $\varepsilon > 1$ there exists a sampling set with multiplicity (X, μ_X) with weighted lower Beurling density $D^-(X, \mu_X) < 1 + \varepsilon$.

[3] K. Gröchenig, J. L. Romero, and J. Stöckler. *Sharp Results on Sampling with Derivatives in Shift-Invariant Spaces and Multi-Window Gabor Frames.* *Constr. Approx., 51(1):1–25, 2019.*

Application to Gabor frames

Definition

A set

$$\mathcal{G}(\varphi, X \times \Omega) = \{M_\omega T_x \varphi : x \in X, \omega \in \Omega\}$$

is called a ***Gabor frame*** if there exist constants $0 < A, B < \infty$ s.t.

$$A\|f\|_2^2 \leq \sum_{\omega \in \Omega} \sum_{x \in X} |\langle f, M_\omega T_x \varphi \rangle|^2 \leq B\|f\|_2^2, \qquad f \in L^2(\mathbb{R}).$$

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If the inequality is satisfied, there exist functions $\psi_{x,\omega} \in L^2(\mathbb{R})$ with

$$f = \sum_{\omega\in\Omega} \sum_{x\in X} \langle f, M_\omega T_x \varphi \rangle \, \psi_{x,\omega}, \qquad f \in L^2(\mathbb{R}).$$

Gabor frames and SI-spaces

Theorem (Gröchenig, Romero, Stöckler "19 [2])

Let $\varphi \in W_0(\mathbb{R})$ with stable integer translates. Let $X \subseteq \mathbb{R}$ be a separated set. Then the following statements are equivalent:

- *The family $\mathcal{G}(\varphi, (-X) \times \mathbb{Z})$ is a frame for $L^2(\mathbb{R})$.*
- *X is a sampling set for the space $V^p(\varphi)$ for some $p \in [1, \infty]$.*
- *X is a sampling set for the space $V^p(\varphi)$ for all $p \in [1, \infty]$.*

[2] K. Gröchenig, J. L. Romero, and J. Stöckler. *Sampling theorems for shift-invariant spaces, Gabor frames, and totally positive functions.* Invent. Math., 211(3):1119-1148, 2017.

Implications for Gabor frames

Corollary

Let φ *be an EB-spline of order* $m \geq 2$. *If* $X \subseteq \mathbb{R}$ *is separated with* $\mathfrak{mg}(X) < 1$, *then*

$$\mathcal{G}(\varphi, (-X) \times \mathbb{Z}) = \{M_\omega T_{-x}\varphi : x \in X, \omega \in \mathbb{Z}\}$$

is a Gabor frame. In particular,

$$\mathcal{G}(\varphi, a\mathbb{Z} \times \mathbb{Z}) = \{M_\omega T_{ak}\varphi : k, \omega \in \mathbb{Z}\}$$

is a Gabor frame if and only if $0 < a < 1$.

Thank you for your attention!

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Why EB-splines?

Proposition (Gröchenig "15)

Assume that $\varphi \in \mathrm{M}^1(\mathbb{R})$ generates a partition of unity

$$\sum_{\ell \in \mathbb{Z}} \varphi(x-\ell) = 1, \qquad x \in \mathbb{R}.$$

Let $m, n, r \in \mathbb{N}$, $j = 1, ..., r1$, such that

$$(r-1)m + 1 < rn + j < rm,$$

and $rn + j$ and $rm$are relatively prime. If $\alpha = \frac{1}{m}$ and $\beta = n + j$, then $\mathrm{G}(\varphi, \alpha, \beta)$ is not a frame.

K. Gröchenig. *Partitions of unity and new obstructions for Gabor frames.* *arXiv: https://doi.org/10.48550/arXiv.1507.08432, 2015.*