Sampling with derivatives in shift-invariant spaces generated by exponential B-splines

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We derive sufficient conditions for sampling with derivatives in shift-invariant spaces generated by an exponential B-spline. The sufficient conditions are expressed by a new notion of measuring the gap between consecutive points. As a consequence, we can construct sampling sets arbitrarily close to necessary conditions.

Sampling in shift-invariant spaces

We consider shift-invariant spaces: given a generator $\varphi \in L^p(\mathbb{R})$, we denote its integer translates by $T_\ell\varphi(x) = \varphi(x-\ell)$, $\ell \in \mathbb{Z}$ and with $V^p(\varphi) \subseteq L^p(\mathbb{R})$ the subspace

$$V^p(\varphi) := \Big\{ \sum_{\ell\in\mathbb{Z}} c_\ell T_\ell\varphi \in L^p(\mathbb{R}),\ c \in \ell^p(\mathbb{Z}) \Big\}. \qquad (1)$$

Let $X \subseteq \mathbb{R}$ be a δ-separated set, i.e., $0 < \delta \le |x-y|$ for all distinct $x, y \in X$, and $\mu_X : X \to \{0,\dots,S\}$ its *multiplicity function*. We call (X,μ_X) a *sampling set with multiplicities* for $V^p(\varphi)$ if there exist positive constants $0 < A_p \le B_p$ such that for all $f \in V^p(\varphi)$ holds

$$A_p\|f\|_p^p \le \sum_{x\in X}\sum_{s=0}^{\mu_X(x)} \left|f^{(s)}(x)\right|^p \le B_p\|f\|_p^p. \qquad (2)$$

The aim is to determine sufficient conditions for (X,μ_X) to be a sampling set with multiplicities.

Exponential B-splines

An exponential B-spline (EB-spline) $\mathcal{E}_{m,\alpha} : \mathbb{R} \longrightarrow \mathbb{R}$ of order m for parameters $\alpha \in \mathbb{R}^m$ is a function of the form

$$\mathcal{E}_{m,\alpha}(x) := \mathop{\circledast}_{s=1}^{m} e^{\alpha_s x}\chi_{[0,1)}(x), \qquad (3)$$

where $\circledast$ denotes the convolution product.

Theorem (Schoenberg-Whitney conditions)

Let φ be an EB-spline of order m. Further let $t_0 \le t_2 \le \dots \le t_D$ and set

$$d_i := \max\{\ell : t_i = \dots = t_{i-\ell}\}, \quad 0 \le i \le D. \qquad (4)$$

The collocation matrix

$$M\begin{pmatrix} t_0,\dots,t_D \\ \varphi,\dots,T_D\varphi \end{pmatrix} := \left(L_{d_i}T_\ell\varphi(t_i)\right)_{0\le i,\ell\le D} \qquad (5)$$

has a non-negative determinant. The collocation matrix is invertible if and only if for all $0 \le i \le D$ holds

$$t_i \in \begin{cases} (i, i+m), & d_i < m-1 \\ [i, i+m), & d_i = m-1. \end{cases} \qquad (6)$$

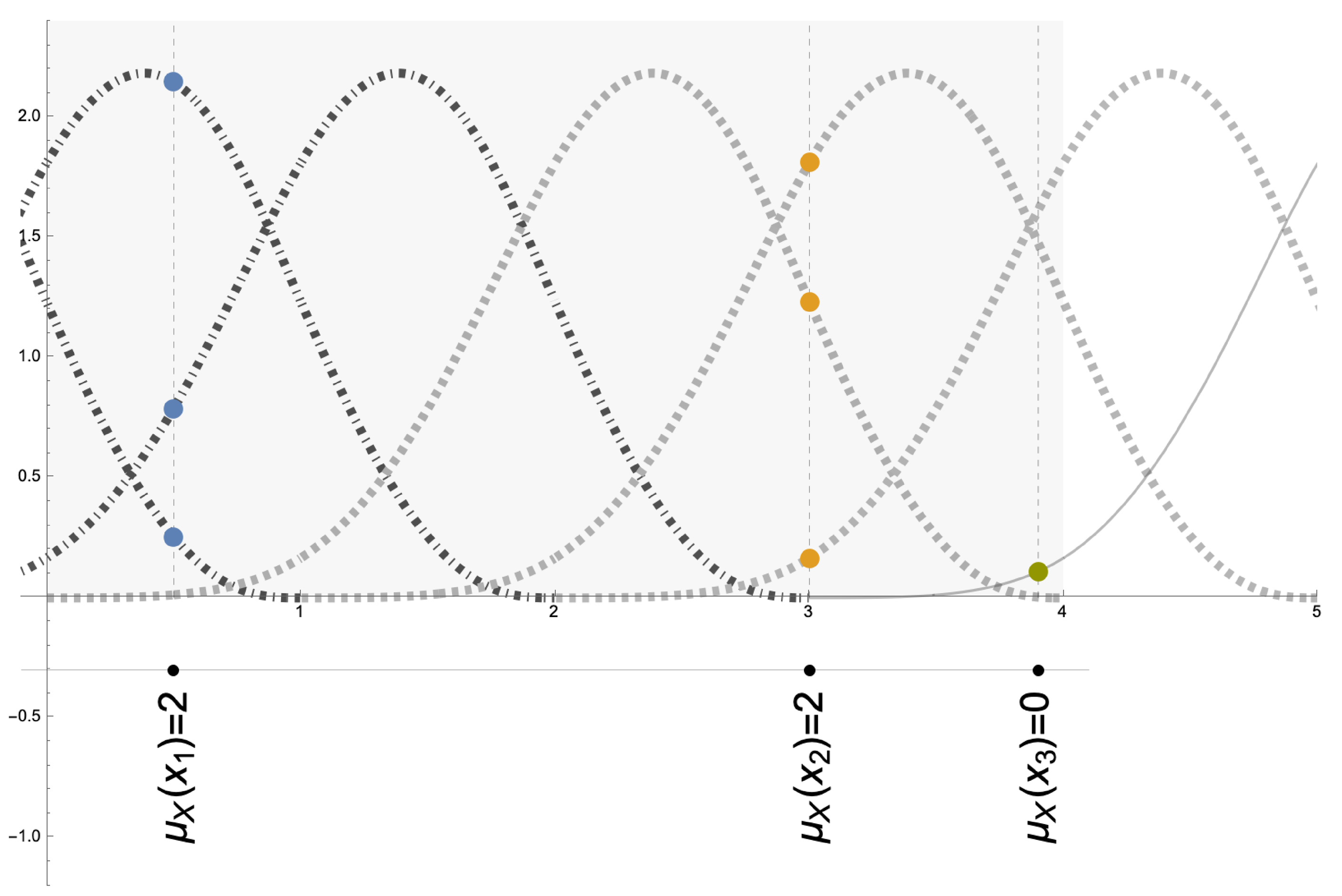

Figure: Non-vanishing shifts of $\varphi(x) = \circledast_{j=1}^4 e^x\chi_{[0,1)}(x)$ on $[0,4]$. The sampling points are $x_1 = 0.5$, $x_2 = 3$, $x_3 = 3.9$, with multiplicities $\mu_X(x_1) = \mu_X(x_2) = 2$, $\mu_X(x_3) = 0$. The first sampling point lies in the support of the first three shifts of φ (dot-dashed), the second point is in the support of the next three shifts of φ (dashed), and the last point - inthe the support of the last shift of φ (solid).

Main contribution (Maximum gap theorem)

Let φ be an EB-spline of order m and let $X \subseteq \mathbb{R}$ be a separated set with $\mu_X : X \to \{0,\dots,m-1\}$. If the multiplicity function satisfies

$$dist\left(\{x \in X : \mu_X(x) = m-1\},\ \mathbb{Z}\right) > 0 \qquad (7)$$

and the weighted maximum gap satisfies

$$\mathfrak{mg}(X,\mu_X) := \sup_{j\in\mathbb{Z}} \frac{x_{j+1}-x_j}{1+\min\{\mu_X(x_j),\mu_X(x_{j+1})\}} < 1, \qquad (8)$$

then (X,μ_X) is a sampling set for $V^p(\varphi)$.

Proof sketch

(i) *Weak limits reduce the sampling problem to a uniqueness problem:*

A set X with a multiplicity function μ_X is a sampling set with multiplicities for $V^p(\varphi)$ if and only if any of its weak limits of integer translates is a uniqueness set for $V^\infty(\varphi)$.

(ii) *Solve the problem locally: On an interval $[M, M+L]$, $M, L \in \mathbb{Z}$, the restriction of a prototypical function $f \in V^\infty(\varphi)$ is given by*

$$f_{|[M,M+L]} = \sum_{\ell=M-m+1}^{M+L-1} c_\ell T_\ell\varphi, \qquad (9)$$

i.e., it is a linear combination of $L+m-1$ shifts of the EB-spline.

(iii) *Prove that for a sufficiently large L, there are $L+m-1$ samples available on $[M, M+L]$.*

(iv) *Prove that these samples satisfy the Schoenberg-Whitney conditions.*

(v) *Recover f on $\mathbb{R}$ by induction.*

Gabor frames

A time-frequency shift $\pi(\lambda)$ acts as $\pi(\lambda)f(t) = e^{2\pi i\omega t}f(t-x)$, $\lambda = (x,\omega) \in \mathbb{R}^2$. Given a separated set $\Lambda \subseteq \mathbb{R}^2$, we call the collection

$$\mathcal{G}(\varphi,\Lambda) := \{\pi(\lambda)\varphi : \lambda \in \Lambda\} \qquad (10)$$

a *Gabor frame* if there exist positive constants $0 < A \le B$ such that for all $f \in L^2(\mathbb{R})$ holds

$$A\|f\|_2^2 \le \sum_{\lambda\in\Lambda} \left|\langle f, \pi(\lambda)\varphi\rangle\right|^2 \le B\|f\|_2^2. \qquad (11)$$

Corollary (Implications for Gabor frames)

Let φ be an EB-spline of order $m \ge 2$. Assume $X \subseteq \mathbb{R}$ is a discrete set with $\mathfrak{mg}(X,0) < 1$. Then $\mathcal{G}(\varphi, (-X)\times\mathbb{Z})$ is a Gabor frame. In particular, $\mathcal{G}(\varphi, a\mathbb{Z}\times\mathbb{Z})$ is a Gabor frame if and only if $0 < a < 1$.

References

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This work was supported by FWF: P33217.

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