Phaseless sampling of the STFT

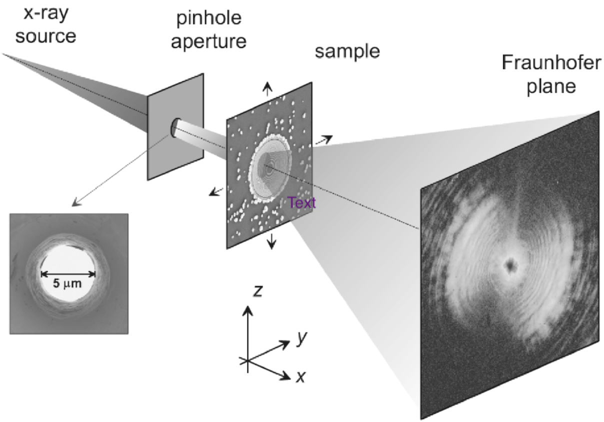
Irina Shafkulovska

Numerical Harmonic Analysis Group
University of Vienna

Joint work with Philipp Grohs and Lukas Liehr
University of Vienna

Der Wissenschaftsfonds.

Motivating example



Ptychography experiment. Image taken from [8].

[8]. Rodenburg et al. *Hard-X-Ray Lensless Imaging of Extended Objects. Phys. Rev. Lett., 98(3), 2007.*

The phase retrieval problem

Problem

Let $\mathcal{H}$ be a Banach space and T an injective linear operator T from $\mathcal{H}$ to a function space of measurable functions X.
The phase retrieval problem is determining whether the operator

$$\mathcal{A} : \mathcal{H} \to X, \qquad \mathcal{A}f := |Tf|$$

is injective on $\mathcal{H}/\!\sim$, where

$$f \sim h \qquad :\Leftrightarrow \quad \exists \tau \in \mathbb{T} : \; f = \tau h.$$

Notation

Fourier transform:

$$\mathcal{F}f(\omega) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i \omega \cdot t} dt, \qquad \omega \in \mathbb{R}^d$$

Time-frequency shifts:

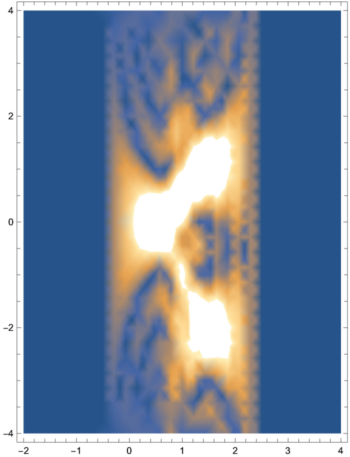
$$\pi(\lambda) f(t) = e^{2\pi i \omega t} f(t - x), \qquad \lambda = (x, \omega) \in \mathbb{R}^{2d}.$$

Definition (Short-time Fourier transform (STFT))

The short-time Fourier transform of a function $f \in L^2(\mathbb{R}^d)$ w.r.t. a *window* $\varphi \in L^2(\mathbb{R}^d) \setminus \{0\}$ is the function

$$\mathcal{V}_\varphi f(x, \omega) = \langle f, \pi(x, \omega)\varphi \rangle = \int_{\mathbb{R}^d} f(t) \overline{\varphi(t - x)} e^{-2\pi i \omega \cdot t} dt, \quad (x, \omega) \in \mathbb{R}^{2d}.$$

Example



$$f(t) = \begin{cases} e^{2\pi i\, 0\cdot t}, & t \in (0,1), \\ e^{2\pi i\, 1\cdot t} + e^{2\pi i\,(-2)\cdot t}, & t \in (1,2), \\ 0, & \text{else.} \end{cases}$$

Plot of the absolute value of the STFT $V_{b_0} f$ with respect to the box function $b_0 = \chi_{(-1/2,1/2)}$.

Sampling of the STFT

Theorem

The STFT $\mathcal{V}_\varphi$ w.r.t. to a window $\varphi \in L^2(\mathbb{R}^d) \setminus \{0\}$ is (up to scaling) an isometry. The associated Calderón-type reproducing formula is given by

$$f = \frac{1}{\|\varphi\|^2} \int_{\mathbb{R}^{2d}} \mathcal{V}_g f(\lambda) \pi(\lambda) \varphi \, d\lambda.$$

Sampling of the STFT

Given a window $\varphi \in L^2(\mathbb{R}^d) \setminus \{0\}$ and a separated set $\Lambda \subseteq \mathbb{R}^{2d}$, i.e., $\inf_{\lambda,\lambda'} |\lambda - \lambda'| > 0$, is the **sampling operator**

$$\mathcal{V}_\varphi\big|_\Lambda : L^2(\mathbb{R}^d) \to \mathbb{C}^\Lambda, \qquad f \mapsto (\mathcal{V}_\varphi f(\lambda))_{\lambda \in \Lambda}$$

a bounded injective operator?

Discretization of the STFT

Theorem

Let $\varphi \in L^2(\mathbb{R}^d) \setminus \{0\}$ *be a window with* $\mathcal{V}_\varphi \varphi \in L^1(\mathbb{R}^{2d})$*. Then for all sufficiently dense separated sets* $\Lambda \subseteq \mathbb{R}^{2d}$*, the restriction* $\mathcal{V}_\varphi\big|_\Lambda$ *is bounded injective operator.*

The deciding density is the *lower Beurling density*

$$D^-(\Lambda) := \liminf_{r\to\infty} \inf_{x\in\mathbb{R}^d} \frac{(\Lambda \cap B_r(x))}{\mathrm{vol}(B_r(x))}.$$

It can be seen as an asymptotic average number of points per unit area.
If Λ is a lattice, i.e., $\Lambda = A\mathbb{Z}^{2d}$, $A \in \mathrm{GL}(2d,\mathbb{R})$, then $D^-(\Lambda) = |\det A|^{-1}$.

Discretization barriers

Theorem (Discretization barriers, Grohs, Liehr 2022 [5])

Let $\varphi \in L^2(\mathbb{R})$ and $\Lambda \subseteq \mathbb{R}^2$ be a lattice. Then the phaseless (sampled) STFT

$$|\mathcal{V}_\varphi|\,|_\Lambda : L^2(\mathbb{R}) \to \mathbb{C}^\Lambda, \qquad f \mapsto \big(\,|V_\varphi f(\lambda)|\,\big)_{\lambda\in\Lambda}$$

is never injective.

$\implies$ One either has to consider sampling sets which aren't lattices or restrict to a proper subset of $L^2(\mathbb{R}^d)$.

[5]. P. Grohs and L. Liehr. *On Foundational Discretization Barriers in STFT Phase Retrieval.* *J. Fourier Anal. Appl., 28(39), 2022.*

[4]. P. Grohs and L. Liehr. *Non-uniqueness theory in sampled STFT phase retrieval.* *arXiv preprint: 2112.10136*

Phase retrieval in subspaces

Theorem (Wellershoff 2022 [9])

Let $B > 0$, $b \in \left(0, \frac{1}{4B}\right)$ *and* $p \in [1, \infty]$. *Further let* $\varphi(x) = 2^{1/4} e^{-\pi x^2}$ *be the standard Gaussian function. Then the following are equivalent for* $f, g \in L^2\left([-B, B]\right)$ *:*

1. $f \sim g$,
2. $|\mathcal{V}_\varphi f| = |\mathcal{V}_\varphi g|$ *on* $\mathbb{N} \times b\mathbb{Z}$.

[9]. M. Wellershoff. *Injectivity of sampled Gabor phase retrieval in spaces with general integrability conditions.* *arXiv preprint: 2112.10136, 2021.*

Phase retrieval with analytic windows

Theorem (P. Grohs, L. Liehr, I. S. 2022 [6])

Let $K \subseteq \mathbb{R}^d$ be a compact set, let $\varphi \in L^2(\mathbb{R}^d)$, $\varphi \neq 0$, be a product of a function of exponential type and a Gaussian. If $\Lambda, \Gamma \subseteq \mathbb{R}^d$ are sufficiently dense, then the following are equivalent for $f \in L^2(K)$:

1. $f \sim g$,
2. $|\mathcal{V}_\varphi f| = |\mathcal{V}_\varphi g|$ on $\Lambda \times \Gamma$.

[6]. P. Grohs, L. Liehr, I. Shafkulovska. *From completeness of discrete translates to phaseless sampling of the short-time Fourier transform.* *arXiv preprint: 2211.05687, 2022.*

Thank you for your attention!

References I

Gabor phase retrieval is severely ill-posed.
Appl. Comput. Harmon. Anal.

R. Alaifari and P. Grohs.
Phase Retrieval In The General Setting Of Continuous Frames For Banach Spaces.
SIAM J. Math. Anal., 49, 2016.
doi.org/10.1137/16M1071481.

J. Cahill, P. G. Casazza, and I. Daubechies.
Phase retrieval in infinite-dimensional Hilbert spaces.
Trans. Amer. Math. Soc. Ser. B, 3:63–76, 2016.
doi.org/10.1090/btran/12.

P. Grohs and L. Liehr.
Non-uniqueness theory in sampled STFT phase retrieval.
arXiv, 2022.
doi.org/10.48550/arXiv.2207.05628.

P. Grohs and L. Liehr.
On Foundational Discretization Barriers in STFT Phase Retrieval.
J. Fourier Anal. Appl., 28(39), 2022.