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The metaplectic action on modulation spaces

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FWF

Der Wissenschaftsfonds.

Translations: $T_x f(t) = f(t - x)$,

Modulations: $M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)$,

Fourier transform: $\mathcal{F}f(\omega) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \omega \cdot x} dx$, $\omega \in \mathbb{R}^d$,

Symmetric time-frequency shifts: $\lambda = (x, \omega) \in \mathbb{R}^{2d}$,

$$\rho(\lambda) = M_{\omega/2} T_x M_{\omega/2} = T_{x/2} M_\omega T_{x/2}.$$

Definition (Cross-ambiguity function)

$$A(f, g)(x, \omega) = \langle f, \rho(x, \omega)g \rangle = e^{\pi i \omega \cdot x} V_g f(x, \omega)$$

Definition (Mixed-norm Lebesgue spaces $L^{p,q}(\mathbb{R}^{2d})$)

$$F : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{C}$$

$$\begin{aligned} \|F\|_{p,q} &:= \left\| \omega \mapsto \|F(\cdot, \omega)\|_p \right\|_q \\ &= \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |F(x, \omega)|^p dx \right)^{q/p} d\omega \right)^{1/q} \end{aligned}$$

Definition (Weighted mixed-norm Lebesgue spaces $L_{\mathbf{m}}^{p,q}(\mathbb{R}^{2d})$)

$$F : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{C}$$

$$\begin{aligned} \|F\|_{p,q,m} &:= \left\| \omega \mapsto \|F(\cdot, \omega) \mathbf{m}(\cdot, \omega)\|_p \right\|_q \\ &= \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |F(x, \omega) \mathbf{m}(\cdot, \omega)|^p dx \right)^{q/p} d\omega \right)^{1/q} \end{aligned}$$

We assume that the weight is **moderate**, i.e., there is a submultiplicative weight $v : \mathbb{R}^{2d} \rightarrow \mathbb{R}_+$ and $C > 0$ such that for all $x, y \in \mathbb{R}^{2d}$

$$m(x+y) \leq C m(x)v(y), \quad m(x+y) \lesssim m(x)v(y).$$

Reminder:

$$A(\varphi, g)(x, \omega) = \langle \varphi, M_{\frac{\omega}{2}} T_x M_{\frac{\omega}{2}} g \rangle \quad \text{and} \quad \|F\|_{p,q,m} = \left\| \omega \mapsto \|F(\cdot, \omega) m(\cdot, \omega)\|_p \right\|_q.$$

Definition (Modulation spaces)

$$M_m^{p,q}(\mathbb{R}^d) := \{ \varphi \in \mathcal{S}'(\mathbb{R}^d) \mid A(\varphi, g_0) \in L_m^{p,q}(\mathbb{R}^{2d}) \}$$

- $M_m^{p,q}(\mathbb{R}^d)$ is independent of the choice of the window.
- Different windows induce equivalent norms

$$\|\varphi\|_{M_m^{p,q}(\mathbb{R}^d)} \asymp \|A(\varphi, g)\|_{p,q,m}.$$

For all $\lambda, \nu \in \mathbb{R}^{2d}$ holds

$$\rho(\lambda)\rho(\nu) = e^{-\pi i \lambda^t \mathcal{J} \nu} \rho(\lambda + \nu),$$

where

$$\mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

is the standard skew-symmetric matrix.

Relations:

$$\rho(\lambda)\rho(\nu) = e^{-\pi i \lambda^t \mathcal{J} \nu} \rho(\lambda + \nu)$$

$$\mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

Definition (The symplectic group $\mathrm{Sp}(2d, \mathbb{R})$)

$$\mathrm{Sp}(2d, \mathbb{R}) := \{S \in \mathbb{R}^{2d \times 2d} : S^t \mathcal{J} S = \mathcal{J}\}.$$

Examples (Building blocks)

Let $L \in \mathrm{GL}(d, \mathbb{R})$, $P, Q \in \mathbb{R}^{d \times d}$, such that $P = P^t$, $Q = Q^t$.

$$D_L := \begin{pmatrix} L & 0 \\ 0 & L^{-t} \end{pmatrix} \quad U_P := \begin{pmatrix} I & P \\ 0 & I \end{pmatrix} \quad V_Q := \begin{pmatrix} I & 0 \\ Q & I \end{pmatrix}$$

Definition

The metaplectic group is the double cover of the symplectic group.

Theorem

For all $S \in \mathrm{Sp}(2d, \mathbb{R})$ there is a unitary operator \widehat{S} with

$$\rho(S\lambda) = \widehat{S}\rho(\lambda)\widehat{S}^{-1}, \quad \lambda \in \mathbb{R}^{2d}.$$

$$\pi^{\mathrm{Mp}} : \mathrm{Mp}(2d, \mathbb{R}) \rightarrow \mathrm{Sp}(2d, \mathbb{R})$$

is a group homomorphism, with $\ker(\pi^{\mathrm{Mp}}) = \{\mathrm{id}, -\mathrm{id}\}$.

Definition

The metaplectic group is the double cover of the symplectic group.

Theorem

For all $S \in \mathrm{Sp}(2d, \mathbb{R})$ there is a unitary operator \widehat{S} with

$$\rho(S\lambda) = \tau \widehat{S} \rho(\lambda) \tau^{-1} \widehat{S}^{-1}, \quad \lambda \in \mathbb{R}^{2d}, \quad \tau \in \mathbb{T}.$$

$$\pi^{\mathrm{Mp}} : \mathrm{Mp}(2d, \mathbb{R}) \rightarrow \mathrm{Sp}(2d, \mathbb{R})$$

is a group homomorphism, with $\ker(\pi^{\mathrm{Mp}}) = \{\mathrm{id}, -\mathrm{id}\}$.

Examples

$$D_L := \begin{pmatrix} L & 0 \\ 0 & L^{-t} \end{pmatrix} \rightsquigarrow \mathcal{D}_L f(t) = |\det L|^{-1/2} f(L^{-1}t),$$

$$V_Q := \begin{pmatrix} I & 0 \\ Q & I \end{pmatrix} \rightsquigarrow \mathcal{V}_Q f(t) = e^{\pi i t \cdot Q t} f(t),$$

$$\mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \rightsquigarrow \mathcal{F}.$$

Theorem

For all $f \in \mathcal{S}'(\mathbb{R}^d)$, $g \in \mathcal{S}(\mathbb{R}^d)$ and all $\widehat{S} \in \text{Mp}(2d, \mathbb{R})$, it holds that

$$A(\widehat{S}f, \widehat{S}g)(\lambda) = A(f, g)(S^{-1}\lambda), \quad \lambda \in \mathbb{R}^{2d}.$$

Symplectic covariance: $A(\widehat{S}f, \widehat{S}g)(\lambda) = A(f, g)(S^{-1}\lambda), \quad \lambda \in \mathbb{R}^{2d}.$

Problem

Let $\widehat{S} \in \text{Mp}(2d, \mathbb{R})$, $p, q \in [1, \infty]$ be given. Is the operator

$$\widehat{S} : M_m^{p,q}(\mathbb{R}^d) \rightarrow M_m^{p,q}(\mathbb{R}^d), \quad f \mapsto \widehat{S}f$$

- well-defined?
- bounded?

Let $g \in \mathcal{S}(\mathbb{R}^{2d})$. If $A(f, g) \in L_m^{p,q}(\mathbb{R}^{2d})$, does this imply

$$L_m^{p,q}(\mathbb{R}^{2d}) \ni A(\widehat{S}f, h) = A(\widehat{S}f, \widehat{S}g) = A(f, g)(S^{-1} \cdot) ?$$

$$\widetilde{S}f(t) := f(S^{-1}t).$$

$$A(\widehat{S}f, \widehat{S}g)(\lambda) = A(f, g)(S^{-1}\lambda), \quad \lambda \in \mathbb{R}^{2d}.$$

$$\begin{array}{ccccc}
 M^{p,q}(\mathbb{R}^d) & \xrightarrow{A(\cdot, g)} & A(M^{p,q}(\mathbb{R}^d), g) & \hookrightarrow & L^{p,q}(\mathbb{R}^{2d}) \\
 \downarrow \widehat{S} & & \downarrow \text{---} & & \downarrow \widetilde{S} \\
 M^{p,q}(\mathbb{R}^d) & \xrightarrow{A(\cdot, g)} & A(M^{p,q}(\mathbb{R}^d), g) & \hookrightarrow & L^{p,q}(\mathbb{R}^{2d})
 \end{array}$$

$$\widetilde{S}f(t) := f(S^{-1}t).$$

Theorem (Führ, S., "22 [2])

Let $p, q \in [1, \infty]$, $\widehat{S} \in \text{Mp}(2d, \mathbb{R})$ be given. Let $S \in \text{Sp}(2d, \mathbb{R})$ be the projection of \widehat{S} onto $\text{Sp}(2d, \mathbb{R})$. Then the following statements are equivalent:

- (i) $\widehat{S} : \mathbb{M}^{p,q}(\mathbb{R}^d) \rightarrow \mathbb{M}^{p,q}(\mathbb{R}^d)$ is well-defined.
- (ii) $\widehat{S} : \mathbb{M}^{p,q}(\mathbb{R}^d) \rightarrow \mathbb{M}^{p,q}(\mathbb{R}^d)$ is well-defined and bounded.
- (iii) One of the following conditions holds:
 - (a) $p = q$, or
 - (b) $p \neq q$ and S is a block upper triangular matrix.

If one, hence all, of the statements hold, then $\widetilde{S} : L^{p,q}(\mathbb{R}^{2d}) \rightarrow L^{p,q}(\mathbb{R}^{2d})$ is an isometric automorphism (up to a multiplicative constant), and \widehat{S} has a bounded inverse.

$$F = f \otimes g \in L^{p,q}(\mathbb{R}^{2d})$$

$$S^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x)g(\omega)|^p dx \right)^{q/p} d\omega = \int_{\mathbb{R}^d} |g(\omega)|^q \left(\int_{\mathbb{R}^d} |f(x)|^p dx \right)^{q/p} d\omega$$

$$\int_{\mathbb{R}^d} |g(Cx + D\omega)|^q \left(\int_{\mathbb{R}^d} |f(Ax + B\omega)|^p dx \right)^{q/p} d\omega < \infty?$$

$$\begin{array}{ccccc}
 M_{\mathbf{m}}^{p,q}(\mathbb{R}^d) & \xrightarrow{A(\cdot, g)} & A(M_{\mathbf{m}}^{p,q}(\mathbb{R}^d), g) & \hookrightarrow & L_{\mathbf{m}}^{p,q}(\mathbb{R}^{2d}) \\
 \downarrow \widehat{S} & & \downarrow & & \downarrow \widetilde{S} \\
 M_{\mathbf{m}}^{p,q}(\mathbb{R}^d) & \xrightarrow{A(\cdot, \widehat{S}^{-1}g)} & A(M_{\mathbf{m}}^{p,q}(\mathbb{R}^d), \widehat{S}^{-1}g) & \hookrightarrow & L_{\mathbf{m}}^{p,q}(\mathbb{R}^{2d})
 \end{array}$$

Theorem (Isomorphism relations)

The Lebesgue spaces $L_m^{p,q}(\mathbb{R}^{2d})$ and $L^{p,q}(\mathbb{R}^{2d})$ are isomorphic via

$$\Phi_m : L_m^{p,q}(\mathbb{R}^{2d}) \rightarrow L^{p,q}(\mathbb{R}^{2d}), \quad f \rightarrow m \cdot f.$$

$$M_m^{p,q}(\mathbb{R}^d) \quad \longleftrightarrow \quad M^{p,q}(\mathbb{R}^d) \quad ?$$

Definition

Let $a \in \mathcal{S}(\mathbb{R}^{2d})$ be a symbol and $g \in \mathcal{S}(\mathbb{R}^{2d})$ a fixed window. Then the Toeplitz operator $Tp_g(a)$ is defined by the formula

$$\langle Tp_g(a)f_1, f_2 \rangle_{L^2(\mathbb{R}^d)} = \langle a A(f_1, g), A(f_2, g) \rangle_{L^2(\mathbb{R}^{2d})}$$

for all $f_1, f_2 \in L^2(\mathbb{R}^d)$.

Theorem (Lifting theorem, Gröchenig, Toft, "11 [4])





Assume that m is an even, v -moderate weight function with $v(\lambda) = (1 + \|\lambda\|)^N$ for some $N \in \mathbb{N}$ and $g \in \mathcal{S}(\mathbb{R}^d)$. Then the Toeplitz operator $Tp_g(m)$ is an isomorphism from $M_{m_0}^{p,q}(\mathbb{R}^d)$ onto $M_{m_0/m}^{p,q}(\mathbb{R}^d)$ for every v -moderate even weight m_0 and every $p, q \in [1, \infty]$.

Theorem

Let $v(z) = (1 + \|z\|)^N$ for some $N \in \mathbb{N}$ and m be a v -moderate, even weight, $p, q \in [1, \infty]$. If $\widehat{S} \in \text{Mp}(2d, \mathbb{R})$ with projection $\pi^{Mp}(\widehat{S}) = S$ satisfies $m \asymp m \circ S^{-1}$, then the following statements are equivalent:

- 1 \widehat{S} is a bounded operator from $M_m^{p,q}(\mathbb{R}^d)$ to $M_m^{p,q}(\mathbb{R}^d)$.
- 2 \widehat{S} is a bounded operator from $M^{p,q}(\mathbb{R}^d)$ to $M^{p,q}(\mathbb{R}^d)$.

Thank you for your attention!

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