Gabor frame bound optimizations

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Determining whether a Gabor system $\mathcal{G}(g,\Lambda)$ *is a frame* is a difficult task even when the sampling set Λ is as simple as a separable lattice $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$. Once though confirmed, the second challenge is to *find the optimal one* under a given constraint, with a suitable definition of optimality. A reasonable condition, both for applications and the theory alike, is the density $\delta = (ab)^{-1}$ of the lattice. Here, we explore separable lattices of integer density $n \in \mathbb{N}$, with respect to three different *optimality criteria*: maximum lower frame bound A, minimal upper frame bound B and minimal condition number $\kappa = B/A$. The last ratio is the condition number of reconstructing f from its STFT's samples and is likely the most important of the three criteria. The examples illustrate that various cases might appear, from non-existence to uniqueness and the three optimality definitions are not necessarily equivalent, i.e.,

there might not be a lattice optimal with respect to all three criteria.

Initial observations and general problems

The constraint on the density makes the problem univariate. Choosing a good parameterization would give a hint at what the optimal lattice is. For instance, to rely fully on the time or on the frequency properties of the window would mean choosing $(a, n^{-1}a^{-1})$, or $(n^{-1}b^{-1}, b)$, respectively. To address symmetries about the density condition, $(a,b) = (\eta\, n^{-1}, \eta^{-1})$ is more suitable, as the square lattice $n^{-1/2}\mathbb{Z} \times n^{-1/2}\mathbb{Z}$ corresponds to $\eta = \sqrt{n}$.
It suffices to only consider the *optimal lattice bounds* instead of all possible frame bounds.

In the statements below, A, B and κ will denote the optimal frame bounds and their ratio.

There are several immediate questions which arise with the optimization problem.
- What is the set of admissible parameters (a,b)?
- Are there closed forms for A and B with respect to the lattice parameters?
- Is there an optimizer? If so, is it unique?
- Do any of the optimizers of A, B and κ coincide?

The windows we consider are the hyperbolic secant, cut-off exponentials, the one-sided and the two-sided exponential. In terms of the first two problems, they were solved in [3]. For each density $n \in \mathbb{N}$, the free parameter can be freely chosen in $(0,\infty)$. The only exception is made in the case of the hyperbolic secant and the two-sided exponential, which are well localized, so they cannot admit frames at the critical density $n = 1$.

The hyperbolic secant

Let g the denote the (dilated) hyperbolic secant

$$g(t) = \operatorname{sech}(\pi t) = \frac{1}{\cosh(\pi t)}.$$

This window belongs to Feichtinger's algebra, it is an eigenfunction of the Fourier transform and a totally positive function of infinite type. The frame bounds of $\mathcal{G}(g, \eta\, n^{-1}\mathbb{Z} \times \eta^{-1}\mathbb{Z})$ are given by

$$\begin{aligned} A(\tfrac{\eta}{n},\tfrac{1}{\eta}) &= \tfrac{n\pi}{2} \sum_{k=-\infty}^{\infty} \tfrac{\eta}{n} \operatorname{sech}(\tfrac{2k+1}{2}\tfrac{\pi\eta}{n})^2 + \tfrac{n\pi}{2} \sum_{k=-\infty}^{\infty} \tfrac{1}{\eta} \operatorname{sech}(\tfrac{2k+1}{2}\tfrac{\pi}{\eta})^2 - n \\ &= n\pi\, f_A(\tfrac{\eta}{n}) + n\pi\, f_A(\tfrac{1}{\eta}) - n, \end{aligned} \tag{1}$$

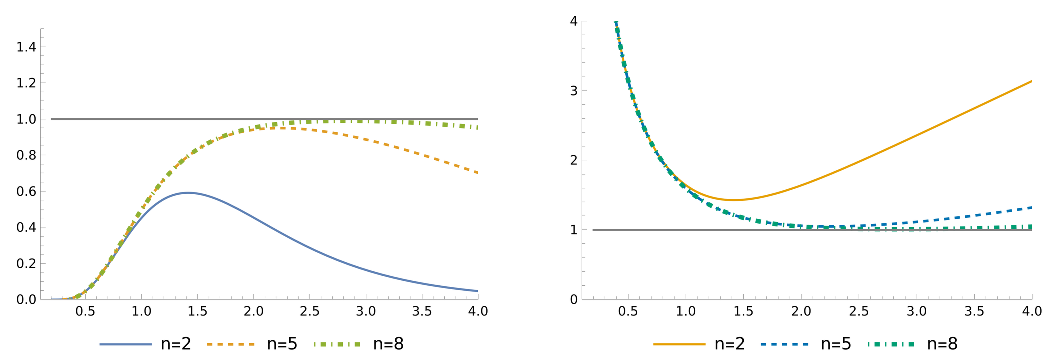
$$\begin{aligned} B(\tfrac{\eta}{n},\tfrac{1}{\eta}) &= \tfrac{n\pi}{2} \sum_{k=-\infty}^{\infty} \tfrac{\eta}{n} \operatorname{sech}(\tfrac{\pi k \eta}{n})^2 + \tfrac{n\pi}{2} \sum_{k=-\infty}^{\infty} \tfrac{1}{\eta} \operatorname{sech}(\tfrac{\pi k}{\eta})^2 - n \\ &= n\pi\, f_B(\tfrac{n}{\eta}) + n\pi\, f_B(\eta) + n. \end{aligned} \tag{2}$$

The problem reduces to solving

$$0 = \tfrac{\partial}{\partial\eta}\left(f_*(\tfrac{\eta}{n}) + f_*(\tfrac{1}{\eta})\right) \quad\Longleftrightarrow\quad \tfrac{\eta}{n} f_*'(\tfrac{\eta}{n}) = \tfrac{1}{\eta} f_*'(\tfrac{1}{\eta}). \tag{3}$$

In both cases, the solution is only the trivial $\frac{\eta}{n} = \frac{1}{\eta}$, hence

the square lattice is the unique optimal lattice for A, B and κ.



(a) Lower bounds. (b) Upper bounds.

Figure: The re-normalized optimal bounds $n^{-1}A(\eta\, n^{-1}, n^{-1})$ and $n^{-1}B(\eta\, n^{-1}, n^{-1})$ for densities $n \in \{2,5,8\}$. As n grows, the bounds become extremely flat around the optimum.

Cut-off exponentials (1/b)

We consider families of Gabor frames $\mathcal{G}(g_{b,\gamma}, a\mathbb{Z} \times b\mathbb{Z})$, where $(ab)^{-1} \in \mathbb{N}$ and

$$g_{b,\gamma}(t) = C_{b,\gamma} e^{-\gamma t} \chi_{[0,1/b]}(t),$$

where $C_{b,\gamma}$ is a normalizing constant. *The support condition is due to Janssen's representation of the frame operator.* The frame bounds of $\mathcal{G}\left(g, a\mathbb{Z} \times n^{-1}a^{-1}\mathbb{Z}\right)$ for $\gamma > 0$ are given by

$$A = n\, \tfrac{2\gamma a}{1-e^{-2\gamma a}}\, e^{-2\gamma a},$$

$$B = n\, \tfrac{2\gamma a}{1-e^{-2\gamma a}}.$$

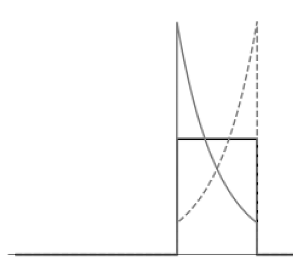
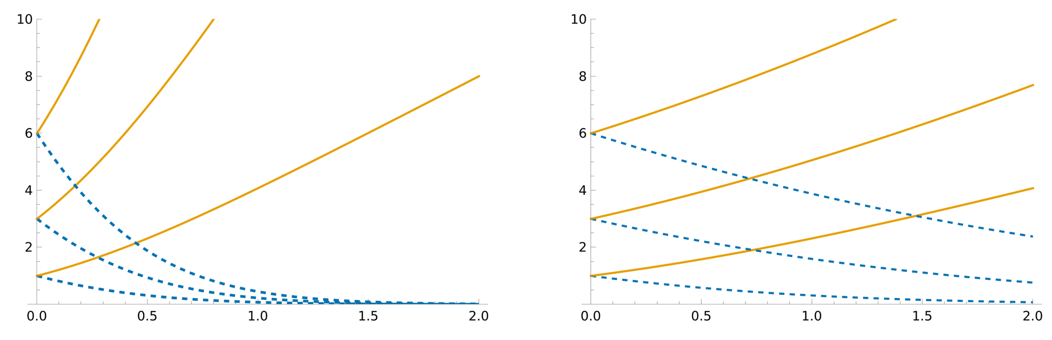


Figure: $g_{b,\gamma}$ for $\gamma = 0$ (black), $\gamma = 2$ (gray) and $\gamma = -2$ (gray, dashed).

- The degenerate case $\gamma = 0$, i.e., the box function, can be included by a limiting procedure (see figure above).
- The negative case $\gamma < 0$ can be observed through the positive one with a simple flipping argument (see figure above).
- The condition number and the frame bounds are strictly monotonic in $\gamma \in (0,\infty)$, making the degenerate case optimal.



(a) $\mathbb{Z} \times n^{-1}\mathbb{Z}$. (b) $n^{-1/2}\mathbb{Z} \times n^{-1/2}\mathbb{Z}$.

Figure: Frame bounds of the Gabor systems $\mathcal{G}(g_{b,\gamma}, a\mathbb{Z} \times b\mathbb{Z})$ for $(ab)^{-1} = n \in \{1,3,6\}$ and $a = 1$, $b = n^{-1}$ and $a = b = n^{-1/2}$. For $\gamma = 0$, we always have a tight frame with bounds $A = B = n$. The upper frame bounds (orange) are increasing with γ whereas the lower frame bounds (blue, dashed) are decreasing.

There is no optimal lattice for the frame bounds or the condition number.

Cut-off exponentials (2/b)

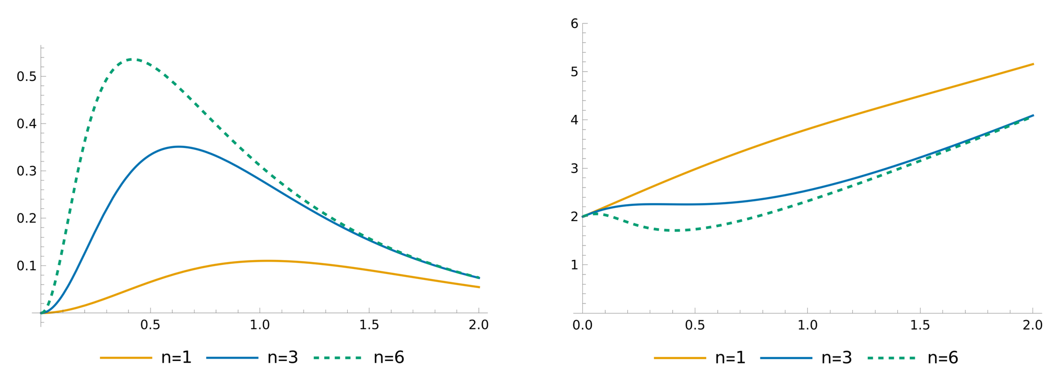
We consider families of Gabor frames $\mathcal{G}(g_{b/2,\gamma}, a\mathbb{Z} \times b\mathbb{Z})$, where $(ab)^{-1} \in \mathbb{N}$ and $g_{b/2,\gamma}$ is the cut-off exponential as in the previous section with $\gamma > 0$. The frame bounds are given by

$$A = \frac{2\gamma a(1-e^{-2\gamma n a})(1-e^{-\gamma n a})^2}{(1-e^{-2\gamma a})(1-e^{-4\gamma n a})}\, e^{-2a\gamma}, \tag{4}$$

$$B = \frac{2\gamma a(1-e^{-2\gamma n a})(1+e^{-\gamma n a})^2}{(1-e^{-2\gamma a})(1-e^{-4\gamma n a})}. \tag{5}$$

The bounds depend on $\gamma \cdot a$, amalgamating the two parameters.

In stark contrast to $\mathcal{G}(g_{b,\gamma}, a\mathbb{Z} \times b\mathbb{Z})$, we do not have a frame at the limiting cases $\gamma \to 0$ and $\gamma \to \infty$.



(a) Lower frame bounds. (b) Upper frame bounds.

Figure: Lower frame bounds and upper frame bounds in dependence of the lattice parameter a for $\gamma = 1$ and $n \in \{1,3,6\}$. We see that A has a unique maximum attained at the positive solution whereas B may not have a minimizing lattice (the lattice is degenerated for $a = 0$).

For fixed n and $\gamma > 0$, we can conclude the following:
- The lower frame bound has a unique global maximum attained at the unique positive solution of the equation

$$-1 + \frac{1}{\gamma a} - \coth(\gamma a) = n \coth\left(\frac{\gamma n a}{2}\right) \operatorname{sech}(\gamma n a). \tag{6}$$

- The upper frame bound is strictly monotonically increasing when $n = 1$ or $n = 2$.
- For $n \geq 3$, B has a local maximum and a local minimum, which *may be global*. If its value at the local minimum is strictly less than 2, then it is unique.
- The condition number is minimal if and only if

$$\gamma a = \frac{\log\left(n+\sqrt{n^2+4}\right) - \log(2)}{n}.$$

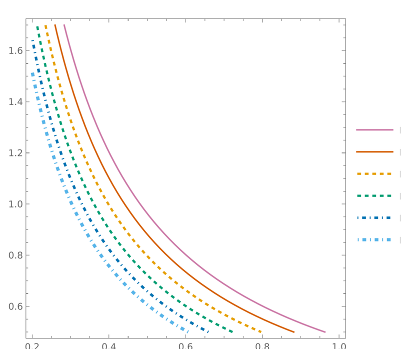


Figure: Curves of optimal parameters for (a,γ) and different lattice densities $n \in \{1,2,3,4,5,6\}$.

One-sided exponential

Let g denote the one-sided exponential

$$g(t) = \sqrt{2}\, e^{-t} \chi_{(0,\infty)}(t).$$

This window is a totally positive function of order 1, *does not* belong to Feichtinger's algebra and we do obtain *a frame at critical density* $n = 1$. The frame bounds of $\mathcal{G}(g, \eta\, n^{-1}\mathbb{Z} \times \eta^{-1}\mathbb{Z})$ are given by

$$A(\eta) = 2\eta \tanh(\tfrac{\eta}{2}) \operatorname{csch}(\tfrac{\eta}{n})\, e^{-\frac{\eta}{n}}, \tag{7}$$

$$B(\eta) = 2\eta \coth(\tfrac{\eta}{2}) \operatorname{csch}(\tfrac{\eta}{n})\, e^{\frac{\eta}{n}}. \tag{8}$$

- The unique minimum of κ is attained at $\eta_\kappa = \operatorname{arcsinh}(n)$.
- Both frame bounds have a unique global extremum in $(0,\infty)$. Evaluating at η_κ indicates that the maximum of A is attained at $\eta_A < \eta_\kappa$ and the minimum of B is attained at $\eta_B > \eta_\kappa$.

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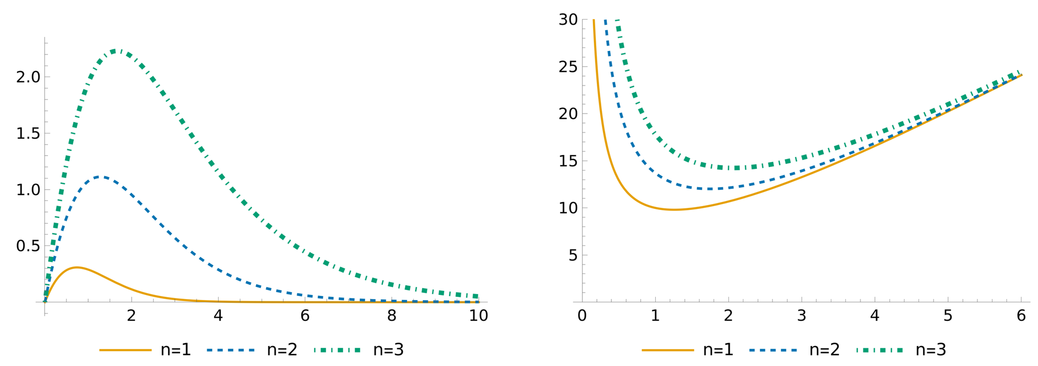


(a) Lower frame bounds. (b) Upper frame bounds.

Figure: Lower and upper bounds for density $n \in \{1,2,3\}$.

Two-sided exponential

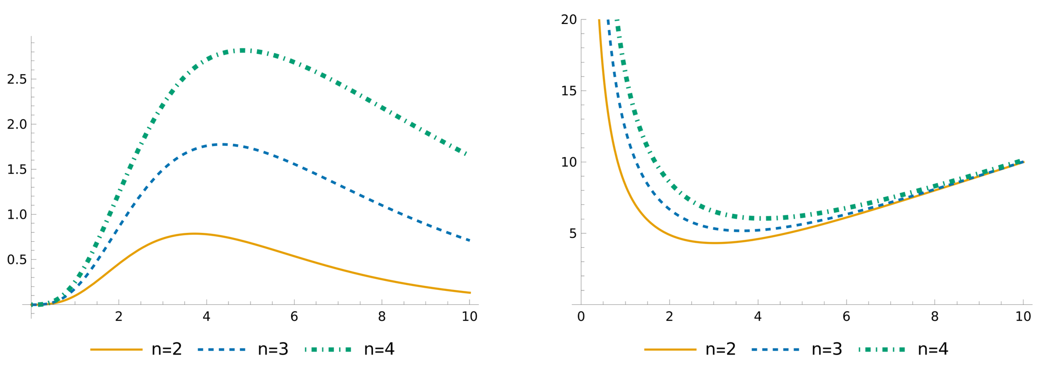
Let g be the two-sided exponential

$$g(t) = \sqrt{2}e^{-|t|}.$$

It is a totally positive function of order 2, hence belongs to Feichtinger's algebra and there are no frames at critical density $n = 1$. The frame bounds of $\mathcal{G}(g, \eta\, n^{-1}\mathbb{Z} \times \eta^{-1}\mathbb{Z})$ are given by

$$A(\tfrac{\eta}{n},\tfrac{1}{\eta}) = \tanh(\tfrac{\eta}{2})\Big(\tfrac{\eta}{n}\operatorname{csch}(\tfrac{\eta}{n}) - \eta \operatorname{csch}(\eta)\Big), \tag{9}$$

$$B(\tfrac{\eta}{n},\tfrac{1}{\eta}) = \coth(\tfrac{\eta}{2})\Big(\tfrac{\eta}{n}\coth(\tfrac{\eta}{n}) + \eta \operatorname{csch}(\eta)\Big). \tag{10}$$



(a) Lower frame bounds. (b) Upper frame bounds.

Figure: Lower and upper bounds for densities $n \in \{2,3,4\}$.

The optimal points are unique, with $\eta_B < \eta_\kappa < \eta_A$. Asymptotically, they are close to the reference point $\eta_n := 2\operatorname{arccosh}(n)$.

There is an optimal lattice for all optimization problems, though no simultaneously optimal lattice.

Despite the visual clarity, the bounds' derivatives are too fragile for global estimates. We propose the following approach:
- Compare the exact value of the bound at η_n with a rough estimate of the bound on $(0, 0.5\,\eta_n]$ (left cut-off).
- Compare the exact value of the bound at η_n with a rough estimate of the bound on $[2\,\eta_n, \infty)$ (right cut-off).
- Show that the objective function $f \in \{A, B, \kappa\}$ has a unique maximum / minimum on the remaining interval.

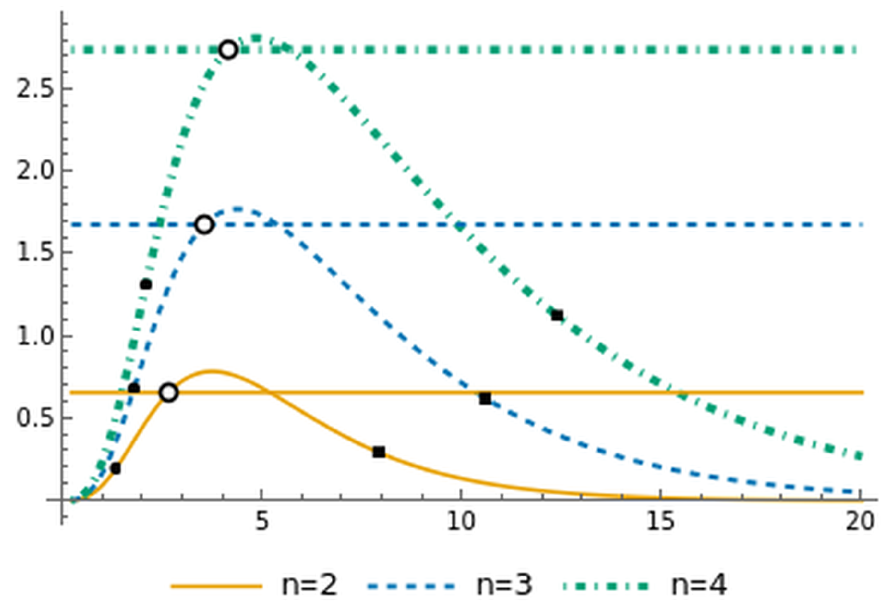


Figure: The idea of the cut-offs for the lower bound and density $n \in \{2,3,4\}$. The • and ■ mark left and right cut-offs, respectively. The ◦ marks η_n.

References

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For further details and interactive plots, scan the QR-code!