Dynamic Stochastic Accumulation Model with Application to Pension Savings Management

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joint work with I. Melicherčík, S. Kilianová, Z. Macová
Slovak pension system overview
Dynamic stochastic accumulation model
  - Time discrete version - stochastic dynamic programming problem
  - Time continuous version - Hamilton–Jacobi–Bellman equation
  - Computational results
  - Qualitative behavior of optimal portfolio selection function

Conclusions
The system is based on the 3-pillar World Bank Model

- **1st PILLAR - PUBLIC AND MANDATORY**
  - Pay-as-you-go type (contributions immediately redistributed to current pensioners)
  - Defined benefit
  - Social Insurance Agency

- **2nd PILLAR - FUNDED, MANDATORY**
  - Private pension funds
  - Defined contribution
  - The savings will be converted into perpetual annuity

- **3rd PILLAR - FUNDED, VOLUNTARY**
  - Supplementary Pension Savings
28.75% from the gross wage
Include:
old-age (18%), disability+survival (6%) insurance, reserve fund (4.75%)
old-age contributions (18% = 9% + 9%) are redirected to the First and Second pillar

D. Ševčovič Dynamic Stochastic Accumulation Model
Slovak pension system - Second pillar limitations

<table>
<thead>
<tr>
<th>Fund type</th>
<th>Stocks</th>
<th>Bonds and money market instruments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Growth Fund</td>
<td>up to 80%</td>
<td>at least 20%</td>
</tr>
<tr>
<td>Balanced Fund</td>
<td>up to 50%</td>
<td>at least 50%</td>
</tr>
<tr>
<td>Conservative Fund</td>
<td>no stocks</td>
<td>100%</td>
</tr>
</tbody>
</table>

Slovak governmental regulations

- in the last 15 years before retirement the savings may not be in the Growth Fund
- in the last 7 years before retirement the savings must be in the Conservative Fund
Future pensioner with expected retirement time $T$ deposits once a year $\varepsilon = 0.09 = 9\%$ (defined contribution) of his yearly salary $w_t$ (with expected wage growth rate $\beta_t$) to a pension fund having $\theta_t \in [0, 1]$ proportion of stocks to bonds in its portfolio with return $r_t(\theta_t)$ and variance $\sigma^2_t(\theta_t)$.

- $\gamma_t$ — Accumulated sum at time $t = 1, ..., T$

$$\gamma_{t+1} = \gamma_t (1 + r_t(\theta_t)) + w_{t+1}\varepsilon, \quad \gamma_1 = w_1\tau.$$  

New state variable: $y_t = \frac{\gamma_t}{w_t}$  $w_{t+1} = (1 + \beta_t)w_t$

- $y_t$ is the ratio of the accumulated sum to yearly salary

$$y_{t+1} = y_t \frac{1 + r_t(\theta_t)}{1 + \beta_t} + \varepsilon, \quad y_1 = \varepsilon.$$  

**Problem:** choose optimal stocks to bonds proportion $\theta_t$ for each time $t$
• Returns on the pension fund portfolio as a stochastic variable

\[ r_t(\theta) \sim N(\mu_t(\theta), \sigma_t^2(\theta)), \quad \text{i.e.} \quad r_t(\theta) = \mu_t(\theta) + \sigma_t(\theta)Z \]

where \( Z \sim N(0, 1) \)

• Expected values of stocks and bonds returns \( \mu_t^{(s)}, \mu_t^{(b)} \) and volatilities \( \sigma_t^{(s)}, \sigma_t^{(b)} \) \( \Rightarrow \)

\[ \mu_t(\theta) = \theta \mu_t^{(s)} + (1 - \theta)\mu_t^{(b)} \]

\[ \sigma_t^2(\theta) = \theta^2[\sigma_t^{(s)}]^2 + (1 - \theta)^2[\sigma_t^{(b)}]^2 + 2\theta(1 - \theta)\sigma_t^{(s)}\sigma_t^{(b)}\rho_t, \]

where \( \rho_t \in [-1, 1] \) is a correlation coefficient between the returns on stocks and bonds at time \( t \)
Aim is to determine the optimal strategy $\theta_t$ at each time $t$ maximizing the saver’s utility (expressed by the saver’s utility function $U$) from the terminal wealth allocated on their pension account

$$\max_{S} \mathbb{E}(U(y_T))$$

subject to the constraint where the maximum in the stochastic dynamic problem is taken over all non-anticipative strategies, time sequences of $\theta_t$ stocks proportions, $S = \{(t, \theta_t) \mid t = 1, \ldots, T\}$. 
Using the tower law of iterated expectations the optimal strategy is the solution to the Bellman equation

\[
W(t, y) = \begin{cases} 
U(y), & t = T, \\
\max_{\theta \in \Delta_t} \mathbb{E}_Z \left( W(t + 1, F_t^1(\theta, y, Z)) \right), & t < T,
\end{cases}
\]

where \( F_t^1(\theta, y, z) = y \frac{1 + \mu_t(\theta) + \sigma_t(\theta)z}{1 + \beta_t} + \varepsilon. \)

- The admissible set \( \Delta_t \subset [0, 1] \)
- \( \Delta_t \subset \{0\%, 50\%, 80\%\} \) for the second pillar of the Slovak pension system

**Conservative, Balanced, Growth fund**
Dynamic stochastic accumulation model
Model parameters

- time horizon $T = 40$ – from the age 25 to 65
- based on historical data of S&P500 and 10-years US government bonds (Jan 1996 - June 2002)
  - $r^{(s)} = 0.1028$, $\sigma^{(s)} = 0.1690$, $r^{(b)} = 0.0516$, $\sigma^{(b)} = 0.0082$, $\rho = -0.1151$
- expected wage growths profile (Páleník et al. 2006)
according to Arrow and Pratt the attitude to risk can be expressed in terms of the so-called coefficient of relative risk aversion defined as $C(y) = -yU''(y)/U'(y)$

constant relative risk aversion $C(y) \equiv d > 0 \Rightarrow$ increasing CRRA utility function $U(y) = -y^{1-d}$

typical value of the coefficient $d$ of relative risk aversion, $d \approx 9$
risk aversion $d = 9$

$\Delta_t \subset \{0\%, \ 50\%, \ 80\%\}$ for the Slovak pension system

Conservative – $\text{F3}$, Balanced – $\text{F2}$, Growth – $\text{F1}$ fund

Results of 10000 Monte-Carlo simulations of the path $y_t$.

Mean $E(y_t)$ and $\pm \sigma(y_t)$
Dynamic stochastic accumulation model
Computational results – Various risk aversions

- Higher risk aversion $\Rightarrow$ earlier transition to less risky funds
Dynamic stochastic accumulation model
Computational results – Various risk aversions

- Sensitivity of the optimal stock to bonds proportion with respect to the saver’s risk aversion $d > 0$ at $t = T$

- Higher risk aversion $\Rightarrow$ lower expected level of savings
Dynamic stochastic accumulation model
Computational results – Various stock mean returns

- lower stock returns (-2 pp)
- 0pp
- higher stock returns (+2 pp)

- Higher stock returns
  ⇒ higher savings and later transition to less risky funds
Dynamic stochastic accumulation model
Computational results – Various stock mean returns

lower wage growth $\beta_t$ (-1 pp) 0pp higher wage growth $\beta_t$ (+1 pp)

Higher wage growth
$\Rightarrow$ lower savings and later transition to less risky funds
Dynamic stochastic accumulation model
Computational results $\Delta_t = [0, 1]$

- the admissible set $\Delta_t = [0, 1]$
- the risk aversion coefficient $d = 9$

- the expected terminal accumulated sum $E(y_T)$ is larger than the one corresponding to the restricted set $\Delta_t = \{0, 0.5, 0.8\}$
- the optimal stocks to bonds proportion $\theta = \theta_t(y)$ is decreasing function w.r. to time $t$ to retirement as well as w.r. to the amount of saved yearly salaries $y$
the goal is to derive a continuous version of the discrete accumulation model

instead of yearly based time intervals \([t, t + 1], t = 1, \ldots, T - 1\)

\[\Rightarrow\]

the proportion of the size \(\varepsilon\tau\) of saving deposits is transferred to the saver account on short time intervals \([0, \tau], [\tau, 2\tau], \ldots, [T - \tau, T]\), where \(0 < \tau \ll 1\) is a small time increment

the increase of the saver’s account at time \(t + \tau\) can be expressed (using Itô’s lemma) as:

\[
y_{t+\tau} = F_{t}^{\tau}(\theta, y_{t}, Z), \quad \text{where} \quad Z \sim N(0, 1), \quad \text{and}
\]

\[
F_{t}^{\tau}(\theta, y_{t}, z) = y_{t} \exp \left( \left( \mu_{t}(\theta) - \beta_{t} - \frac{1}{2} \sigma_{t}^{2}(\theta) \right) \tau + \sigma_{t}(\theta) z \sqrt{\tau} \right) + \varepsilon\tau
\]

for \(0 < t \leq T\)
in the time continuous limit $\tau \equiv dt \to 0^+$ we obtain

$$\max_{\theta \in \Delta_t} \mathbb{E} \left( \frac{V(t + dt, y_{t+dt}) - V(t, y_t)}{dt} \Bigg| y_t = y \right) = 0$$

again by Itô's lemma we have:

$$dy_t = A_\varepsilon(\theta_t, t, y_t)dt + B(\theta_t, t, y_t)dW_t$$

where $A_\varepsilon(\theta, t, y) = \varepsilon + [\mu_t(\theta) - \beta_t]y$ and $B(\theta, t, y) = \sigma_t(\theta)y$, and

$$V(t + dt, y_{t+dt}) - V(t, y_t) =$$

$$\left[ \frac{\partial V}{\partial t}(t, y_t) + A_\varepsilon(\theta_t, t, y_t)\frac{\partial V}{\partial y}(t, y_t) + \frac{1}{2}B^2(\theta_t, t, y_t)\frac{\partial^2 V}{\partial y^2}(t, y_t) \right]dt$$

$$+ B(\theta, t, y_t)\frac{\partial V}{\partial y}(t, y_t)dW_t$$
the intermediate utility function $V = V(t, y)$ satisfies the following fully nonlinear partial differential Hamilton–Jacobi–Bellman equation, for $y > 0$, $t \in [0, T)$:

$$
\frac{\partial V}{\partial t}(t, y) + \max_{\theta \in \Delta_t} \left\{ A_\varepsilon(\theta, t, y) \frac{\partial V}{\partial y}(t, y) + \frac{1}{2} B^2(\theta, t, y) \frac{\partial^2 V}{\partial y^2}(t, y) \right\} = 0
$$

and the terminal condition

$$
V(T, y) = U(y) \quad \text{for } y > 0
$$
we allow $\Delta_t = [0, \infty)$ – short position on stocks are allowed

the intermediate utility function – a solution $V = V(t, y)$ to HJB equation with $\Delta_t = [0, \infty)$ is larger (superoptimal) than the one with $\Delta_t = [0, 1]$

the HJB equation with $\Delta_t = [0, \infty)$ is analytically tractable

the unique argument $\tilde{\theta}(t, y) \in \Delta_t = [0, \infty)$ of the maximum in the HJB equation is given by

$$\tilde{\theta}(t, y) = \frac{b_t}{a_t} - \frac{\Delta \mu_t}{a_t} \frac{\partial V}{\partial y}(t, y)$$

where the negative stocks to bonds correlation $\varrho_t < 0$ implies

- $b_t := \sigma_t^{(b)}[\sigma_t^{(b)} - \rho_t \sigma_t^{(s)}] > 0$
- $a_t := [\sigma_t^{(s)}]^2 + [\sigma_t^{(b)}]^2 - 2\rho_t \sigma_t^{(s)} \sigma_t^{(b)} > b_t$
- $\Delta \mu_t := \mu_t^{(s)} - \mu_t^{(b)} > 0$
the Hamilton–Jacobi–Bellman equation can be rewritten in terms of a fully nonlinear parabolic PDE as follows:

\[ 0 = \frac{\partial V}{\partial t}(t, y) + [\varepsilon + y(\mu_t^{(b)} - \beta_t + \frac{b_t}{a_t} \Delta \mu_t)] \frac{\partial V}{\partial y}(t, y) \]

\[ + \frac{1}{2a_t} \left[ \sigma_t^{(b)} \right]^2 \left[ \sigma_t^{(s)} \right]^2 (1 - \rho_t^2) y^2 \frac{\partial^2 V}{\partial y^2}(t, y) - \frac{1}{2} \frac{(\Delta \mu_t)^2}{a_t} \left[ \frac{\partial V}{\partial y}(t, y) \right]^2 \]

\[ \frac{\partial^2 V}{\partial y^2}(t, y) \]

for \( y > 0, t \in [0, T) \)

and the terminal condition

\[ V(T, y) = U(y) \quad \text{for} \quad y > 0 \]
introducing the Ricatti-like transformation

$$\psi(s, x) = -\gamma y \frac{\partial^2 V(t, y)}{\partial y^2}(t, y),$$

where

- $x = \ln y \in \mathbb{R}$, $s = T - t \in (0, T]$
- $\gamma = \frac{c \sqrt{a}}{\Delta \mu}$
- $\alpha = \mu^{(b)} - \beta + \frac{b}{a} \Delta \mu$ and $c = \sigma^{(b)} \sigma^{(s)} \sqrt{\frac{1 - \rho^2}{a}}$

fully nonlinear parabolic PDE for the transformed function $\psi$

$$\frac{\partial \psi}{\partial s} = \frac{c^2}{2} \frac{\partial}{\partial x} \left( \left[ 1 + \frac{\partial}{\partial x} \right] \left( \psi - \frac{1}{\psi} \right) + \psi \left( \frac{2}{c^2} (\varepsilon e^{-x} + \alpha) - \frac{\psi}{\gamma} \right) \right)$$

for $s \in (0, T)$, $x \in \mathbb{R}$.

initial condition $\psi(0, x) = -\gamma \frac{U''(e^x)}{U'(e^x)} e^x = \gamma d$, for $x \in \mathbb{R}$
Dynamic stochastic accumulation model
Time continuous model – Asymptotic expansion

- the first order asymptotic expansion of the solution $\psi(s, x)$ for small enough values of the parameter $\varepsilon$

$$
\psi(s, x) = d\gamma + \varepsilon\psi_1(s, x) + O(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0^+
$$

\[\downarrow\]

$$
\psi(s, x) = \psi_0 + \varepsilon\psi_1(s, x) + O(\varepsilon^2) = d\gamma\left\{1 + \varepsilon\frac{e^{-\delta s} - 1}{\delta}e^{-x}\right\} + O(\varepsilon^2)
$$

where $\delta = \alpha - dc^2$

- using backward transformation to $t, y$ variables we obtain the first order approximation of the optimal stocks to bonds proportion $\hat{\theta}(t, y)$ in the portfolio

$$
\hat{\theta}(t, y) = \frac{b}{a} + \frac{\Delta \mu}{ad} \left[1 + \frac{\varepsilon}{y} \frac{1 - e^{-\delta(T-t)}}{\delta}\right] + O(\varepsilon^2).
$$
Dynamic stochastic accumulation model
Time continuous model – Asymptotic expansion

- 3D graph (left) and contour plot (right) of the optimal value $\hat{\theta}(t, y)$ (cut by 1 from above) computed from the first order approximation of $\psi$. 
zero limit of the rate of contributions $\varepsilon \to 0^+$

$$\psi_0(s, x) = \gamma d \quad \text{for any } s \in [0, T], \ x \in \mathbb{R}.$$ 

$$\Downarrow$$

$$\hat{\theta}(t, y) = \frac{\sigma(b)(\sigma(b) - \rho \sigma(s)) + \frac{\mu(s) - \mu(b)}{d}}{[\sigma(s)]^2 + [\sigma(b)]^2 - 2\rho \sigma(s) \sigma(b)} \equiv \text{const}$$

for any $t \in [0, T], y > 0$.

this observation is in agreement with Merton and Samuelson’s result stating that the stock to bond proportion is constant and it depends on saver’s risk aversion only
Dynamic stochastic accumulation model

Time continuous model – Sensitivity analysis

- The first order approximation of the optimal stocks to bonds proportion

\[ \hat{\theta}(t, y) = \frac{b}{a} + \frac{\Delta \mu}{a d} \left[ 1 + \frac{\epsilon}{y} \frac{1 - e^{-\delta(T-t)}}{\delta} \right] + O(\epsilon^2). \]

\[ \downarrow \]

\[ \frac{\partial \hat{\theta}}{\partial y}(t, y) < 0 \quad \text{and} \quad \frac{\partial \hat{\theta}}{\partial t}(t, y) < 0 \]

for any \( t \in [0, T), y > 0. \)

- It means that the optimal stock to bonds proportion is a decreasing function with respect to time \( t \) as well as to the amount \( y > 0 \) of yearly saved salaries.

- Higher amount of saved yearly salaries \( y \Rightarrow \) less risk assets in the portfolio.

- Closer time \( t \) to retirement \( T \Rightarrow \) less risk assets in the portfolio.
sensitivity of the optimal stock to bonds proportion with respect to the small parameter $\varepsilon$ – the percentage of contributions

$$\frac{\partial \hat{\theta}}{\partial \varepsilon}(t, y) = \frac{\Delta \mu}{ady} \frac{1 - e^{-\delta(T-t)}}{\delta} > 0$$

for any $0 \leq t < T$ and $y > 0$

higher percentage $\varepsilon$ of salary transferred each year to a pension fund $\Rightarrow$ higher optimal stock to bond proportion $\hat{\theta}$. 
- sensitivity of the optimal stock to bonds proportion with respect to the saver's risk aversion $d > 0$

$$\frac{\partial \hat{\theta}}{\partial d}(t, y) = -\frac{\Delta \mu}{ad^2} \left[ 1 + \frac{\varepsilon}{y} \omega(T - t) \right]$$

where $\omega(s) = \frac{1 - e^{-\delta s}}{\delta} \left( 1 - \frac{d c^2}{\delta} \right) + s d c^2 \frac{e^{-\delta s}}{\delta} \Rightarrow$

$$\frac{\partial \hat{\theta}}{\partial d}(t, y) < 0$$

- higher risk aversion $\Rightarrow$ less amount of stocks in saver's portfolio, as expected.
sensitivity of the optimal stock to bonds proportion with respect to the average stock returns

\[
\frac{\partial \hat{\theta}}{\partial \mu(s)}(t, y) = \frac{1}{a d} \left[ 1 + \frac{\varepsilon}{y} \omega(T - t) \right]
\]

where \( \omega(s) = \frac{1 - e^{-\delta s}}{\delta} \left( 1 - \frac{\Delta \mu b}{a} \right) + \Delta \mu \frac{b s e^{-\delta s}}{\delta} \).

as \( \omega(T - t) > 0 \) we have

\[
\frac{\partial \hat{\theta}}{\partial \mu(s)}(t, y) > 0
\]

optimal stocks to bonds proportion \( \hat{\theta} \) is an increasing function with respect to the average stock return \( \mu(s) \).
sensitivity of the optimal stock to bonds proportion with respect to the expected wage growth

\[
\frac{\partial \hat{\theta}}{\partial \beta}(t, y) = \frac{\Delta \mu \varepsilon}{ad} \omega(T - t)
\]

where \(\omega(s) = \frac{1 - e^{-\delta s}}{\delta^2} - \frac{s e^{-\delta s}}{\delta} \).

as \(\omega(T - t) > 0\) we have

\[
\frac{\partial \hat{\theta}}{\partial \beta}(t, y) > 0
\]

optimal stock to bond proportion \(\hat{\theta}\) is an increasing function with respect to the wage growth \(\beta\).

