

# Web Appendix for “A Canonical Representation of Block Matrices with Applications to Covariance and Correlation Matrices”\*

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## **Abstract**

This is a web appendix with supplementary material for the paper “A Canonical Representation of Block Matrices with Applications to Covariance and Correlation Matrices” by Archakov and Hansen (2021). We present additional results for the empirical analysis discussed in Section 5 of the paper, namely, tables with estimation summary statistics and plots with estimated correlation matrices for U.S. equity returns for calendar years 1995-2018. We also outline how conditional covariances and correlations are simple to obtain from a block covariance matrix using the canonical representation. We report the partial block correlations for U.S. equity returns computed from the sector-block structure for each of the calendar years, 1995-2014. The results for 2015-2020 are presented in the main paper.

*Keywords:* Block Matrices, Block Covariance Matrix, Block Correlation Matrix, Equicorrelation, Covariance Regularization, Covariance Modeling, High Dimensional Covariance Matrices, Matrix Logarithm

*JEL Classification:* C10; C22; C58

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# 1 Additional Tables and Figures

| Block structure                                       | Summary statistics of estimated block correlations |       |        |            |            |            |       | $-\frac{2\ell}{nT}$ |         | $\frac{1}{nT}$ BIC |       | # blocks |                    |
|---|--|-------|--------|------------|------------|------------|-------|---------------------|---------|--------------------|-------|----------|--------------------|
|   | Mean   | Std.  | Min    | $Q_{10\%}$ | $Q_{50\%}$ | $Q_{90\%}$ | Max   |                     |         |                    |       | $K$      | $\frac{K(K+1)}{2}$ |
| <i>U.S. market in 1995 (6076 stocks and 252 days)</i> |  |       |        |            |            |            |       |                     |         |                    |       |          |                    |
| Equicorrelation                                       | 0.020  | 0     | 0.020  | 0.020      | 0.020      | 0.020      | 0.020 | 2.81830             | 2.81831 | 1                  | 1     |          |                    |
| Sectors   | 0.022  | 0.011 | 0.012  | 0.015      | 0.019      | 0.029      | 0.091 | 2.81271             | 2.81332 | 11                 | 66    |          |                    |
| Groups  | 0.022  | 0.012 | 0.007  | 0.013      | 0.019      | 0.032      | 0.165 | 2.81122             | 2.81449 | 26                 | 351   |          |                    |
| Industries  | 0.023  | 0.013 | -0.009 | 0.012      | 0.021      | 0.035      | 0.190 | 2.80727             | 2.83309 | 74                 | 2775  |          |                    |
| Sub-industries  | 0.024  | 0.020 | -0.177 | 0.007      | 0.021      | 0.044      | 0.343 | 2.79213             | 2.94350 | 180                | 16290 |          |                    |
| <i>U.S. market in 1996 (6340 stocks and 254 days)</i> |  |       |        |            |            |            |       |                     |         |                    |       |          |                    |
| Equicorrelation                                       | 0.034  | 0     | 0.034  | 0.034      | 0.034      | 0.034      | 0.034 | 2.80442             | 2.80443 | 1                  | 1     |          |                    |
| Sectors   | 0.038  | 0.013 | 0.025  | 0.027      | 0.035      | 0.048      | 0.122 | 2.79811             | 2.79869 | 11                 | 66    |          |                    |
| Groups  | 0.037  | 0.013 | 0.017  | 0.024      | 0.035      | 0.049      | 0.150 | 2.79637             | 2.79949 | 26                 | 351   |          |                    |
| Industries  | 0.038  | 0.016 | -0.016 | 0.021      | 0.037      | 0.056      | 0.227 | 2.79227             | 2.81756 | 75                 | 2850  |          |                    |
| Sub-industries  | 0.039  | 0.024 | -0.147 | 0.016      | 0.037      | 0.066      | 0.424 | 2.77776             | 2.92381 | 181                | 16471 |          |                    |
| <i>U.S. market in 1997 (6637 stocks and 253 days)</i> |  |       |        |            |            |            |       |                     |         |                    |       |          |                    |
| Equicorrelation                                       | 0.044  | 0     | 0.044  | 0.044      | 0.044      | 0.044      | 0.044 | 2.79358             | 2.79359 | 1                  | 1     |          |                    |
| Sectors   | 0.049  | 0.014 | 0.035  | 0.039      | 0.047      | 0.057      | 0.136 | 2.78695             | 2.78752 | 11                 | 66    |          |                    |
| Groups  | 0.047  | 0.014 | 0.031  | 0.035      | 0.044      | 0.060      | 0.170 | 2.78529             | 2.78829 | 26                 | 351   |          |                    |
| Industries  | 0.050  | 0.018 | -0.023 | 0.031      | 0.048      | 0.071      | 0.252 | 2.78093             | 2.80525 | 75                 | 2850  |          |                    |
| Sub-industries  | 0.052  | 0.030 | -0.157 | 0.022      | 0.048      | 0.087      | 0.601 | 2.76616             | 2.90663 | 181                | 16471 |          |                    |
| <i>U.S. market in 1998 (6408 stocks and 252 days)</i> |  |       |        |            |            |            |       |                     |         |                    |       |          |                    |
| Equicorrelation                                       | 0.072  | 0     | 0.072  | 0.072      | 0.072      | 0.072      | 0.072 | 2.76358             | 2.76359 | 1                  | 1     |          |                    |
| Sectors   | 0.073  | 0.019 | 0.035  | 0.048      | 0.072      | 0.090      | 0.143 | 2.75531             | 2.75589 | 11                 | 66    |          |                    |
| Groups  | 0.076  | 0.019 | 0.014  | 0.055      | 0.073      | 0.100      | 0.177 | 2.75328             | 2.75638 | 26                 | 351   |          |                    |
| Industries  | 0.075  | 0.028 | -0.013 | 0.043      | 0.073      | 0.108      | 0.312 | 2.74688             | 2.77277 | 76                 | 2926  |          |                    |
| Sub-industries  | 0.079  | 0.038 | -0.143 | 0.036      | 0.077      | 0.125      | 0.522 | 2.72956             | 2.87686 | 182                | 16653 |          |                    |
| <i>U.S. market in 1999 (5951 stocks and 252 days)</i> |  |       |        |            |            |            |       |                     |         |                    |       |          |                    |
| Equicorrelation                                       | 0.023  | 0     | 0.023  | 0.023      | 0.023      | 0.023      | 0.023 | 2.81583             | 2.81584 | 1                  | 1     |          |                    |
| Sectors   | 0.028  | 0.017 | 0.010  | 0.019      | 0.023      | 0.039      | 0.125 | 2.80591             | 2.80654 | 11                 | 66    |          |                    |
| Groups  | 0.026  | 0.013 | 0.006  | 0.016      | 0.023      | 0.037      | 0.130 | 2.80323             | 2.80655 | 26                 | 351   |          |                    |
| Industries  | 0.026  | 0.017 | -0.075 | 0.013      | 0.023      | 0.043      | 0.279 | 2.79734             | 2.82507 | 76                 | 2926  |          |                    |
| Sub-industries  | 0.027  | 0.023 | -0.119 | 0.008      | 0.024      | 0.050      | 0.566 | 2.78021             | 2.93456 | 180                | 16290 |          |                    |
| <i>U.S. market in 2000 (5738 stocks and 252 days)</i> |  |       |        |            |            |            |       |                     |         |                    |       |          |                    |
| Equicorrelation                                       | 0.048  | 0     | 0.048  | 0.048      | 0.048      | 0.048      | 0.048 | 2.78947             | 2.78948 | 1                  | 1     |          |                    |
| Sectors   | 0.047  | 0.029 | 0.009  | 0.025      | 0.039      | 0.077      | 0.168 | 2.76560             | 2.76625 | 11                 | 66    |          |                    |
| Groups  | 0.048  | 0.031 | 0.006  | 0.023      | 0.039      | 0.085      | 0.304 | 2.75915             | 2.76259 | 26                 | 351   |          |                    |
| Industries  | 0.047  | 0.031 | -0.045 | 0.020      | 0.041      | 0.081      | 0.304 | 2.75183             | 2.78052 | 76                 | 2926  |          |                    |
| Sub-industries  | 0.047  | 0.035 | -0.124 | 0.013      | 0.041      | 0.088      | 0.589 | 2.73318             | 2.89110 | 179                | 16110 |          |                    |
| <i>U.S. market in 2001 (5418 stocks and 248 days)</i> |  |       |        |            |            |            |       |                     |         |                    |       |          |                    |
| Equicorrelation                                       | 0.068  | 0     | 0.068  | 0.068      | 0.068      | 0.068      | 0.068 | 2.76873             | 2.76874 | 1                  | 1     |          |                    |
| Sectors   | 0.069  | 0.029 | 0.035  | 0.045      | 0.063      | 0.093      | 0.206 | 2.74714             | 2.74783 | 11                 | 66    |          |                    |
| Groups  | 0.071  | 0.032 | 0.025  | 0.041      | 0.063      | 0.107      | 0.347 | 2.73939             | 2.74307 | 26                 | 351   |          |                    |
| Industries  | 0.071  | 0.035 | -0.014 | 0.036      | 0.065      | 0.114      | 0.424 | 2.73145             | 2.76138 | 75                 | 2850  |          |                    |
| Sub-industries  | 0.072  | 0.046 | -0.170 | 0.026      | 0.066      | 0.128      | 0.590 | 2.71096             | 2.87442 | 176                | 15576 |          |                    |
| <i>U.S. market in 2002 (5017 stocks and 252 days)</i> |  |       |        |            |            |            |       |                     |         |                    |       |          |                    |
| Equicorrelation                                       | 0.093  | 0     | 0.093  | 0.093      | 0.093      | 0.093      | 0.093 | 2.74128             | 2.74129 | 1                  | 1     |          |                    |
| Sectors   | 0.103  | 0.032 | 0.061  | 0.074      | 0.097      | 0.130      | 0.289 | 2.72513             | 2.72587 | 11                 | 66    |          |                    |
| Groups  | 0.102  | 0.035 | 0.041  | 0.065      | 0.095      | 0.146      | 0.336 | 2.71696             | 2.72086 | 26                 | 351   |          |                    |
| Industries  | 0.103  | 0.042 | -0.029 | 0.056      | 0.096      | 0.157      | 0.426 | 2.70799             | 2.73883 | 74                 | 2775  |          |                    |
| Sub-industries  | 0.108  | 0.062 | -0.160 | 0.042      | 0.100      | 0.186      | 0.659 | 2.68686             | 2.85018 | 171                | 14706 |          |                    |

Table 1: (Continued on next page).

| Block structure                                       | Summary statistics of estimated block correlations |       |        |            |            |            |       |                     | # blocks           |     |                    |
|---|--|-------|--------|------------|------------|------------|-------|---------------------|--------------------|-----|--------------------|
|   | Mean   | Std.  | Min    | $Q_{10\%}$ | $Q_{50\%}$ | $Q_{90\%}$ | Max   | $-\frac{2\ell}{nT}$ | $\frac{1}{nT}$ BIC | $K$ | $\frac{K(K+1)}{2}$ |
| <i>U.S. market in 2003 (4534 stocks and 252 days)</i> |  |       |        |            |            |            |       |                     |                    |     |                    |
| Equicorrelation                                       | 0.100  | 0     | 0.100  | 0.100      | 0.100      | 0.100      | 0.100 | 2.73401             | 2.73402            | 1   | 1                  |
| Sectors   | 0.106  | 0.030 | 0.063  | 0.076      | 0.103      | 0.132      | 0.273 | 2.71857             | 2.71938            | 11  | 66                 |
| Groups  | 0.108  | 0.032 | 0.053  | 0.072      | 0.102      | 0.149      | 0.287 | 2.71195             | 2.71624            | 26  | 351                |
| Industries  | 0.105  | 0.040 | -0.013 | 0.062      | 0.100      | 0.155      | 0.413 | 2.70335             | 2.73721            | 74  | 2775               |
| Sub-industries  | 0.111  | 0.056 | -0.135 | 0.051      | 0.104      | 0.181      | 0.586 | 2.68196             | 2.85720            | 169 | 14365              |
| <i>U.S. market in 2004 (4488 stocks and 252 days)</i> |  |       |        |            |            |            |       |                     |                    |     |                    |
| Equicorrelation                                       | 0.104  | 0     | 0.104  | 0.104      | 0.104      | 0.104      | 0.104 | 2.72950             | 2.72951            | 1   | 1                  |
| Sectors   | 0.110  | 0.036 | 0.067  | 0.079      | 0.104      | 0.144      | 0.274 | 2.70722             | 2.70803            | 11  | 66                 |
| Groups  | 0.104  | 0.032 | 0.040  | 0.069      | 0.103      | 0.144      | 0.317 | 2.69987             | 2.70419            | 26  | 351                |
| Industries  | 0.109  | 0.043 | 0.010  | 0.059      | 0.104      | 0.161      | 0.385 | 2.69051             | 2.72468            | 74  | 2775               |
| Sub-industries  | 0.110  | 0.056 | -0.128 | 0.046      | 0.106      | 0.175      | 0.672 | 2.66845             | 2.84327            | 168 | 14196              |
| <i>U.S. market in 2005 (4424 stocks and 252 days)</i> |  |       |        |            |            |            |       |                     |                    |     |                    |
| Equicorrelation                                       | 0.093  | 0     | 0.093  | 0.093      | 0.093      | 0.093      | 0.093 | 2.74164             | 2.74165            | 1   | 1                  |
| Sectors   | 0.109  | 0.050 | 0.063  | 0.072      | 0.095      | 0.155      | 0.340 | 2.71768             | 2.71850            | 11  | 66                 |
| Groups  | 0.098  | 0.033 | 0.041  | 0.064      | 0.093      | 0.136      | 0.340 | 2.71270             | 2.71709            | 26  | 351                |
| Industries  | 0.102  | 0.046 | -0.049 | 0.057      | 0.095      | 0.157      | 0.459 | 2.70344             | 2.73715            | 73  | 2701               |
| Sub-industries  | 0.103  | 0.059 | -0.107 | 0.039      | 0.097      | 0.172      | 0.764 | 2.68083             | 2.85800            | 168 | 14196              |
| <i>U.S. market in 2006 (4380 stocks and 251 days)</i> |  |       |        |            |            |            |       |                     |                    |     |                    |
| Equicorrelation                                       | 0.104  | 0     | 0.104  | 0.104      | 0.104      | 0.104      | 0.104 | 2.72945             | 2.72946            | 1   | 1                  |
| Sectors   | 0.115  | 0.043 | 0.068  | 0.082      | 0.105      | 0.148      | 0.350 | 2.70411             | 2.70494            | 11  | 66                 |
| Groups  | 0.105  | 0.038 | 0.017  | 0.063      | 0.103      | 0.146      | 0.350 | 2.69795             | 2.70239            | 26  | 351                |
| Industries  | 0.108  | 0.051 | -0.090 | 0.058      | 0.105      | 0.167      | 0.460 | 2.68790             | 2.72205            | 73  | 2701               |
| Sub-industries  | 0.113  | 0.064 | -0.155 | 0.044      | 0.106      | 0.187      | 0.749 | 2.66530             | 2.84475            | 168 | 14196              |
| <i>U.S. market in 2007 (4193 stocks and 251 days)</i> |  |       |        |            |            |            |       |                     |                    |     |                    |
| Equicorrelation                                       | 0.131  | 0     | 0.131  | 0.131      | 0.131      | 0.131      | 0.131 | 2.69855             | 2.69856            | 1   | 1                  |
| Sectors   | 0.158  | 0.047 | 0.078  | 0.109      | 0.150      | 0.213      | 0.357 | 2.67637             | 2.67723            | 11  | 66                 |
| Groups  | 0.143  | 0.043 | 0.029  | 0.087      | 0.141      | 0.195      | 0.357 | 2.67064             | 2.67527            | 26  | 351                |
| Industries  | 0.146  | 0.057 | -0.006 | 0.082      | 0.139      | 0.221      | 0.523 | 2.65919             | 2.69379            | 72  | 2628               |
| Sub-industries  | 0.159  | 0.073 | -0.062 | 0.075      | 0.149      | 0.253      | 0.756 | 2.63463             | 2.81279            | 164 | 13530              |
| <i>U.S. market in 2008 (4083 stocks and 253 days)</i> |  |       |        |            |            |            |       |                     |                    |     |                    |
| Equicorrelation                                       | 0.230  | 0     | 0.230  | 0.230      | 0.230      | 0.230      | 0.230 | 2.57825             | 2.57826            | 1   | 1                  |
| Sectors   | 0.264  | 0.064 | 0.165  | 0.201      | 0.252      | 0.331      | 0.499 | 2.53980             | 2.54068            | 11  | 66                 |
| Groups  | 0.243  | 0.058 | 0.098  | 0.170      | 0.243      | 0.312      | 0.499 | 2.52951             | 2.53422            | 26  | 351                |
| Industries  | 0.248  | 0.074 | 0.043  | 0.160      | 0.243      | 0.340      | 0.641 | 2.51482             | 2.55005            | 72  | 2628               |
| Sub-industries  | 0.266  | 0.101 | -0.064 | 0.140      | 0.262      | 0.392      | 0.867 | 2.48249             | 2.66596            | 165 | 13695              |
| <i>U.S. market in 2009 (3931 stocks and 252 days)</i> |  |       |        |            |            |            |       |                     |                    |     |                    |
| Equicorrelation                                       | 0.209  | 0     | 0.209  | 0.209      | 0.209      | 0.209      | 0.209 | 2.60494             | 2.60495            | 1   | 1                  |
| Sectors   | 0.239  | 0.059 | 0.141  | 0.170      | 0.233      | 0.312      | 0.440 | 2.57415             | 2.57507            | 11  | 66                 |
| Groups  | 0.219  | 0.059 | 0.055  | 0.138      | 0.221      | 0.290      | 0.440 | 2.56347             | 2.56836            | 26  | 351                |
| Industries  | 0.226  | 0.071 | 0.050  | 0.135      | 0.224      | 0.319      | 0.598 | 2.54999             | 2.58661            | 72  | 2628               |
| Sub-industries  | 0.248  | 0.095 | -0.032 | 0.137      | 0.240      | 0.371      | 0.870 | 2.51893             | 2.70738            | 164 | 13530              |
| <i>U.S. market in 2010 (3741 stocks and 252 days)</i> |  |       |        |            |            |            |       |                     |                    |     |                    |
| Equicorrelation                                       | 0.226  | 0     | 0.226  | 0.226      | 0.226      | 0.226      | 0.226 | 2.58303             | 2.58304            | 1   | 1                  |
| Sectors   | 0.250  | 0.060 | 0.164  | 0.185      | 0.242      | 0.335      | 0.500 | 2.55752             | 2.55849            | 11  | 66                 |
| Groups  | 0.238  | 0.056 | 0.065  | 0.167      | 0.237      | 0.309      | 0.500 | 2.54715             | 2.55228            | 26  | 351                |
| Industries  | 0.248  | 0.075 | 0.058  | 0.158      | 0.242      | 0.347      | 0.689 | 2.53267             | 2.57101            | 72  | 2628               |
| Sub-industries  | 0.265  | 0.095 | -0.115 | 0.151      | 0.260      | 0.389      | 0.880 | 2.50358             | 2.69849            | 163 | 13366              |

Table 1: (Continued on next page).

| Block structure                                       | Summary statistics of estimated block correlations |       |        |            |            |            |       |                     | # blocks           |     |                    |
|---|--|-------|--------|------------|------------|------------|-------|---------------------|--------------------|-----|--------------------|
|   | Mean   | Std.  | Min    | $Q_{10\%}$ | $Q_{50\%}$ | $Q_{90\%}$ | Max   | $-\frac{2\ell}{nT}$ | $\frac{1}{nT}$ BIC | $K$ | $\frac{K(K+1)}{2}$ |
| <i>U.S. market in 2011 (3626 stocks and 252 days)</i> |  |       |        |            |            |            |       |                     |                    |     |                    |
| Equicorrelation                                       | 0.304  | 0     | 0.304  | 0.304      | 0.304      | 0.304      | 0.304 | 2.47774             | 2.47776            | 1   | 1                  |
| Sectors   | 0.327  | 0.064 | 0.238  | 0.256      | 0.314      | 0.419      | 0.598 | 2.44774             | 2.44873            | 11  | 66                 |
| Groups  | 0.313  | 0.070 | 0.098  | 0.221      | 0.319      | 0.393      | 0.598 | 2.43616             | 2.44143            | 26  | 351                |
| Industries  | 0.327  | 0.085 | 0.085  | 0.227      | 0.321      | 0.434      | 0.761 | 2.41848             | 2.45795            | 72  | 2628               |
| Sub-industries  | 0.342  | 0.109 | 0.004  | 0.210      | 0.340      | 0.478      | 0.857 | 2.38519             | 2.58339            | 162 | 13203              |
| <i>U.S. market in 2012 (3488 stocks and 250 days)</i> |  |       |        |            |            |            |       |                     |                    |     |                    |
| Equicorrelation                                       | 0.160  | 0     | 0.160  | 0.160      | 0.160      | 0.160      | 0.160 | 2.66588             | 2.66590            | 1   | 1                  |
| Sectors   | 0.170  | 0.045 | 0.112  | 0.128      | 0.161      | 0.206      | 0.404 | 2.64005             | 2.64109            | 11  | 66                 |
| Groups  | 0.157  | 0.046 | 0.042  | 0.108      | 0.156      | 0.210      | 0.404 | 2.63263             | 2.63813            | 26  | 351                |
| Industries  | 0.162  | 0.054 | 0.011  | 0.103      | 0.155      | 0.228      | 0.530 | 2.61855             | 2.65977            | 72  | 2628               |
| Sub-industries  | 0.172  | 0.075 | -0.078 | 0.092      | 0.165      | 0.265      | 0.705 | 2.58876             | 2.79320            | 161 | 13041              |
| <i>U.S. market in 2013 (3406 stocks and 252 days)</i> |  |       |        |            |            |            |       |                     |                    |     |                    |
| Equicorrelation                                       | 0.153  | 0     | 0.153  | 0.153      | 0.153      | 0.153      | 0.153 | 2.67328             | 2.67330            | 1   | 1                  |
| Sectors   | 0.169  | 0.053 | 0.099  | 0.117      | 0.156      | 0.224      | 0.493 | 2.64706             | 2.64811            | 11  | 66                 |
| Groups  | 0.155  | 0.052 | -0.100 | 0.099      | 0.155      | 0.214      | 0.493 | 2.63946             | 2.64505            | 26  | 351                |
| Industries  | 0.170  | 0.063 | -0.100 | 0.106      | 0.161      | 0.247      | 0.702 | 2.62287             | 2.66470            | 72  | 2628               |
| Sub-industries  | 0.171  | 0.070 | -0.100 | 0.092      | 0.164      | 0.259      | 0.703 | 2.59407             | 2.79391            | 158 | 12561              |
| <i>U.S. market in 2014 (3437 stocks and 252 days)</i> |  |       |        |            |            |            |       |                     |                    |     |                    |
| Equicorrelation                                       | 0.162  | 0     | 0.162  | 0.162      | 0.162      | 0.162      | 0.162 | 2.66285             | 2.66286            | 1   | 1                  |
| Sectors   | 0.164  | 0.050 | 0.107  | 0.119      | 0.153      | 0.195      | 0.447 | 2.61981             | 2.62086            | 11  | 66                 |
| Groups  | 0.153  | 0.058 | -0.036 | 0.111      | 0.156      | 0.209      | 0.447 | 2.60928             | 2.61481            | 26  | 351                |
| Industries  | 0.164  | 0.061 | -0.043 | 0.102      | 0.162      | 0.230      | 0.663 | 2.59054             | 2.63201            | 72  | 2628               |
| Sub-industries  | 0.168  | 0.067 | -0.153 | 0.093      | 0.165      | 0.249      | 0.706 | 2.56043             | 2.75858            | 158 | 12561              |
| <i>U.S. market in 2015 (3516 stocks and 252 days)</i> |  |       |        |            |            |            |       |                     |                    |     |                    |
| Equicorrelation                                       | 0.152  | 0     | 0.152  | 0.152      | 0.152      | 0.152      | 0.152 | 2.67437             | 2.67439            | 1   | 1                  |
| Sectors   | 0.162  | 0.048 | 0.094  | 0.120      | 0.157      | 0.204      | 0.439 | 2.62392             | 2.62494            | 11  | 66                 |
| Groups  | 0.157  | 0.046 | 0.002  | 0.110      | 0.158      | 0.203      | 0.439 | 2.61323             | 2.61863            | 26  | 351                |
| Industries  | 0.164  | 0.056 | -0.021 | 0.109      | 0.158      | 0.230      | 0.710 | 2.59377             | 2.63437            | 72  | 2628               |
| Sub-industries  | 0.176  | 0.072 | -0.079 | 0.098      | 0.168      | 0.264      | 0.879 | 2.56135             | 2.75782            | 159 | 12720              |
| <i>U.S. market in 2016 (3412 stocks and 252 days)</i> |  |       |        |            |            |            |       |                     |                    |     |                    |
| Equicorrelation                                       | 0.165  | 0     | 0.165  | 0.165      | 0.165      | 0.165      | 0.165 | 2.65958             | 2.65960            | 1   | 1                  |
| Sectors   | 0.164  | 0.056 | 0.054  | 0.104      | 0.166      | 0.220      | 0.432 | 2.60372             | 2.60477            | 11  | 66                 |
| Groups  | 0.171  | 0.048 | 0.031  | 0.119      | 0.167      | 0.230      | 0.432 | 2.59150             | 2.59667            | 25  | 325                |
| Industries  | 0.171  | 0.063 | -0.002 | 0.100      | 0.165      | 0.246      | 0.720 | 2.57110             | 2.61172            | 71  | 2556               |
| Sub-industries  | 0.179  | 0.080 | -0.153 | 0.087      | 0.173      | 0.280      | 0.859 | 2.53813             | 2.73262            | 156 | 12246              |
| <i>U.S. market in 2017 (3344 stocks and 251 days)</i> |  |       |        |            |            |            |       |                     |                    |     |                    |
| Equicorrelation                                       | 0.092  | 0     | 0.092  | 0.092      | 0.092      | 0.092      | 0.092 | 2.74348             | 2.74350            | 1   | 1                  |
| Sectors   | 0.089  | 0.052 | 0.021  | 0.040      | 0.078      | 0.147      | 0.331 | 2.68577             | 2.68684            | 11  | 66                 |
| Groups  | 0.090  | 0.041 | 0.012  | 0.050      | 0.084      | 0.141      | 0.331 | 2.67312             | 2.67840            | 25  | 325                |
| Industries  | 0.091  | 0.048 | -0.034 | 0.043      | 0.086      | 0.143      | 0.522 | 2.65410             | 2.69564            | 71  | 2556               |
| Sub-industries  | 0.094  | 0.060 | -0.154 | 0.033      | 0.087      | 0.164      | 0.783 | 2.62228             | 2.82113            | 156 | 12246              |
| <i>U.S. market in 2018 (3340 stocks and 251 days)</i> |  |       |        |            |            |            |       |                     |                    |     |                    |
| Equicorrelation                                       | 0.153  | 0     | 0.153  | 0.153      | 0.153      | 0.153      | 0.153 | 2.67314             | 2.67316            | 1   | 1                  |
| Sectors   | 0.150  | 0.057 | 0.053  | 0.088      | 0.142      | 0.204      | 0.426 | 2.62111             | 2.62218            | 11  | 66                 |
| Groups  | 0.150  | 0.047 | 0.039  | 0.097      | 0.147      | 0.203      | 0.426 | 2.60772             | 2.61261            | 24  | 300                |
| Industries  | 0.151  | 0.058 | -0.006 | 0.087      | 0.145      | 0.220      | 0.544 | 2.58841             | 2.62770            | 69  | 2415               |
| Sub-industries  | 0.160  | 0.074 | -0.123 | 0.076      | 0.151      | 0.250      | 0.795 | 2.55653             | 2.74559            | 152 | 11628              |

Table 1: Summary statistics for the estimated block correlation matrices by calendar year (1995 - 2018).



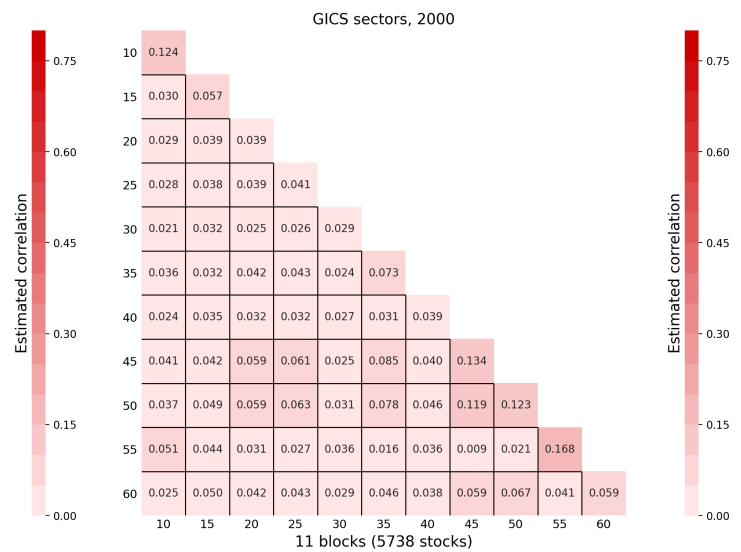
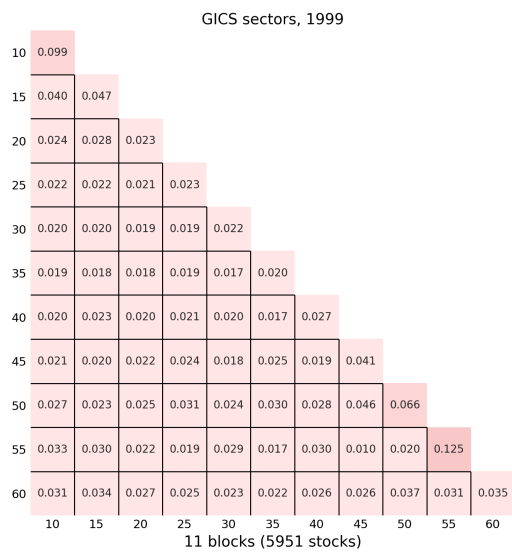
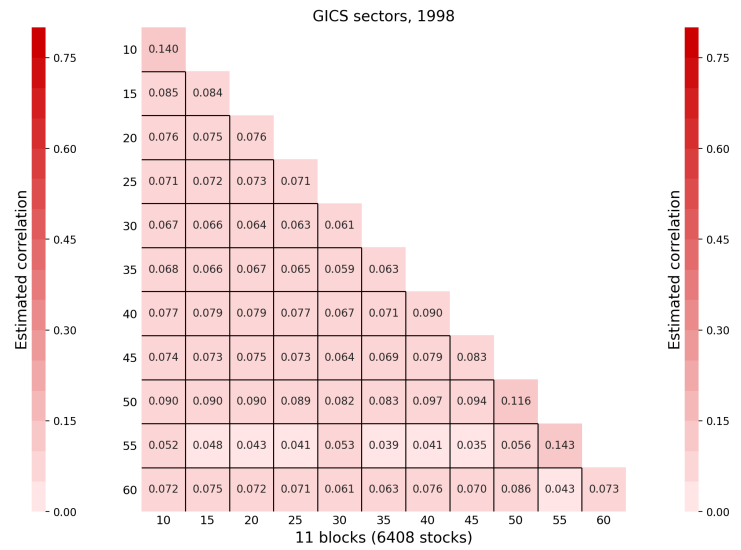
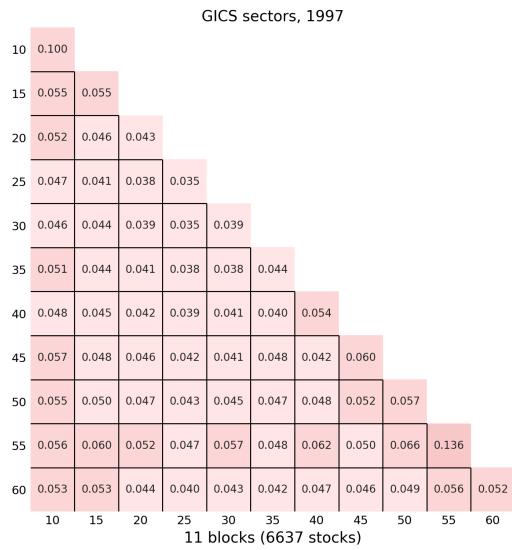
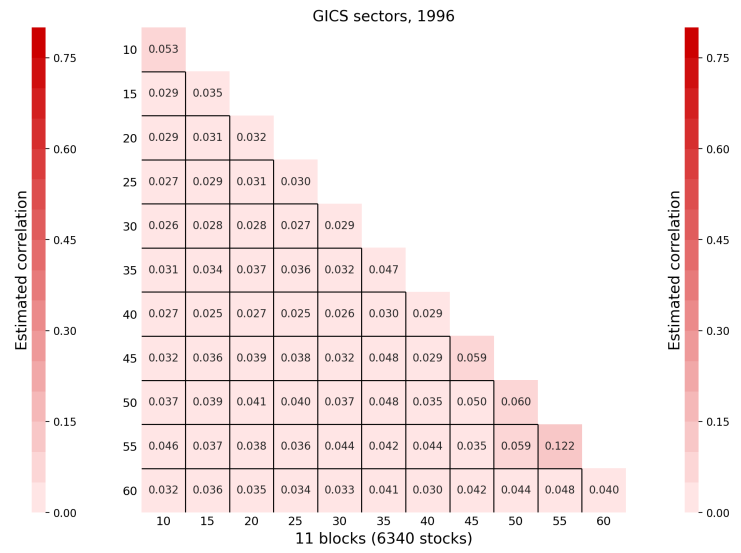
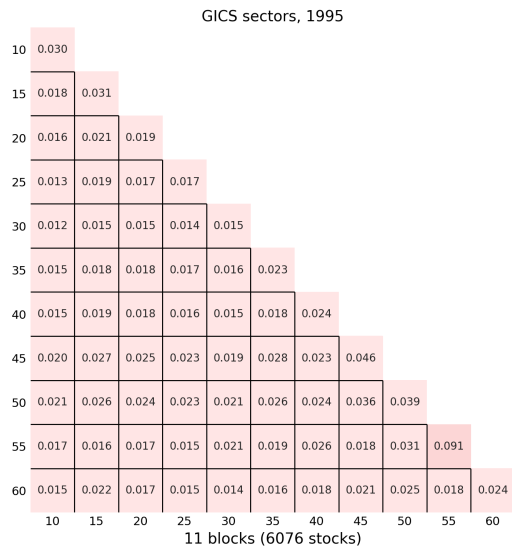


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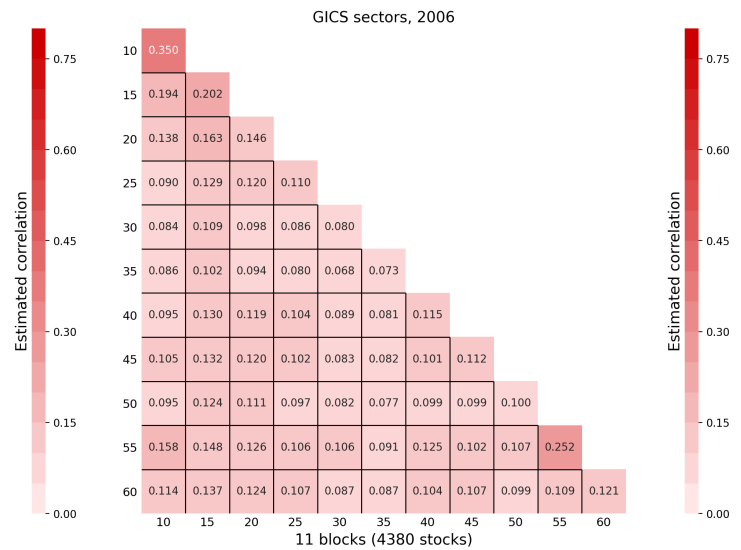
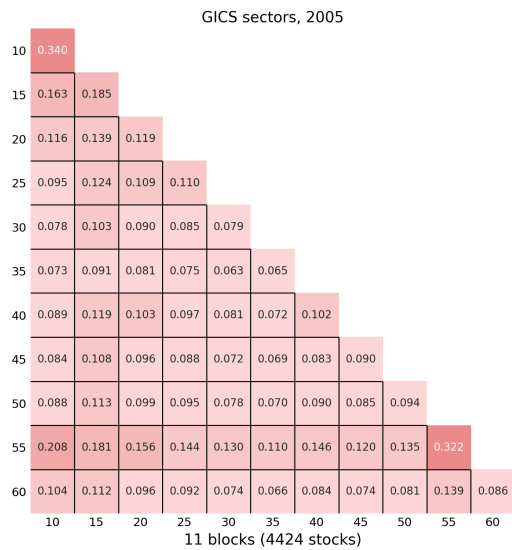
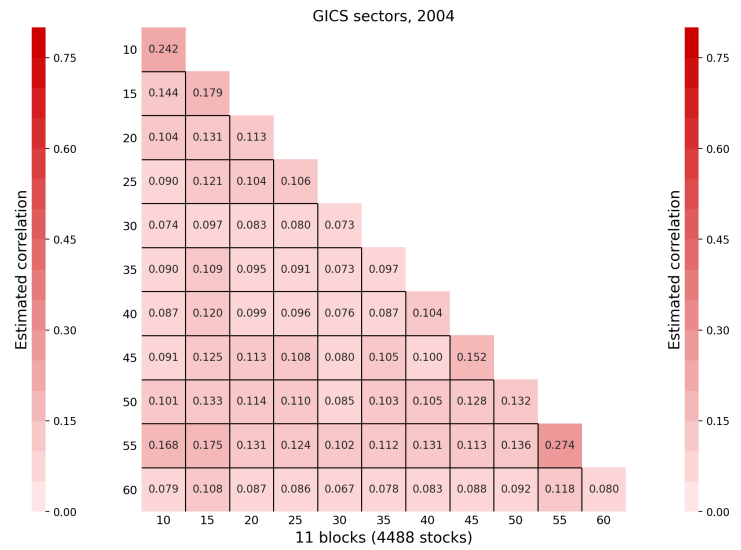
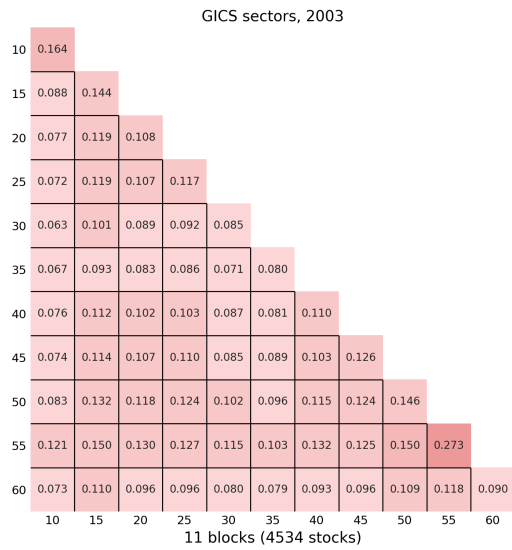
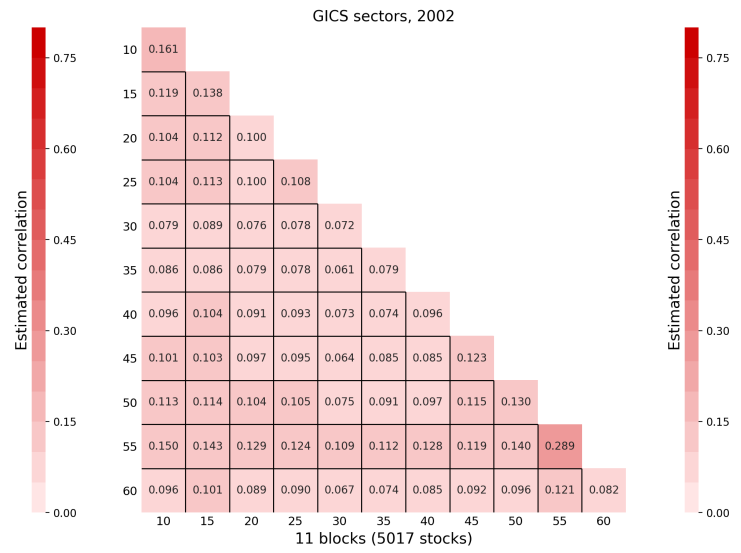
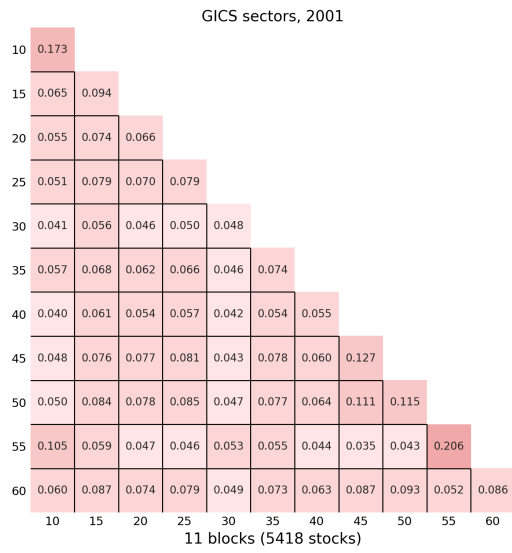


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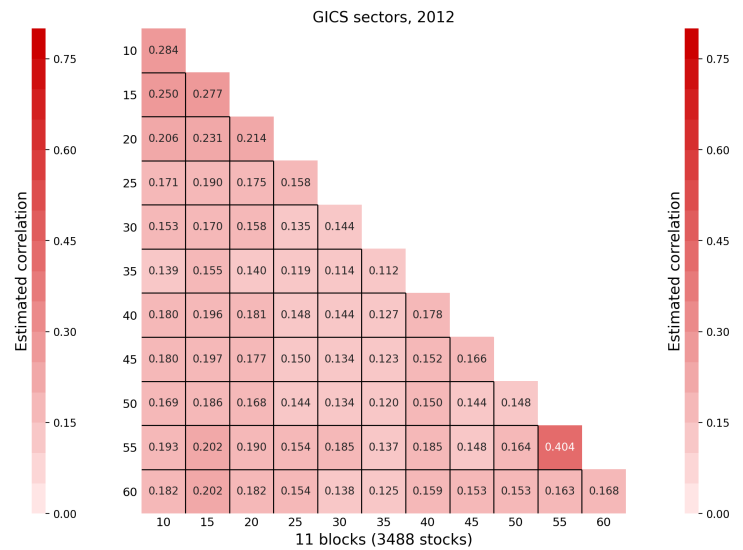
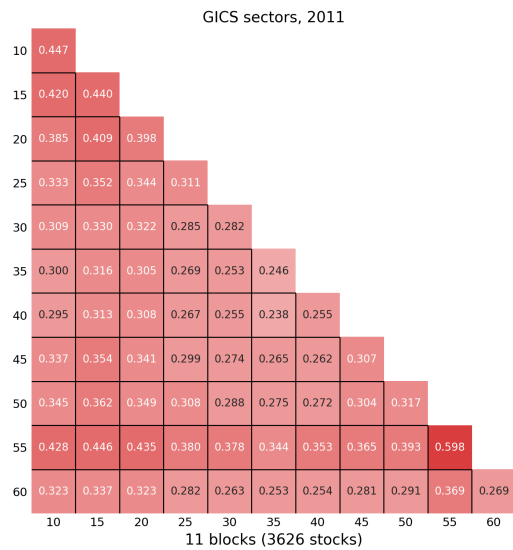
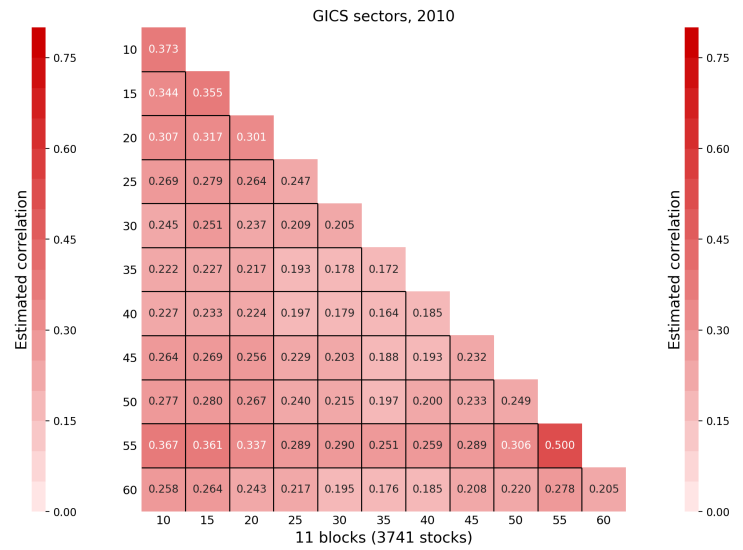
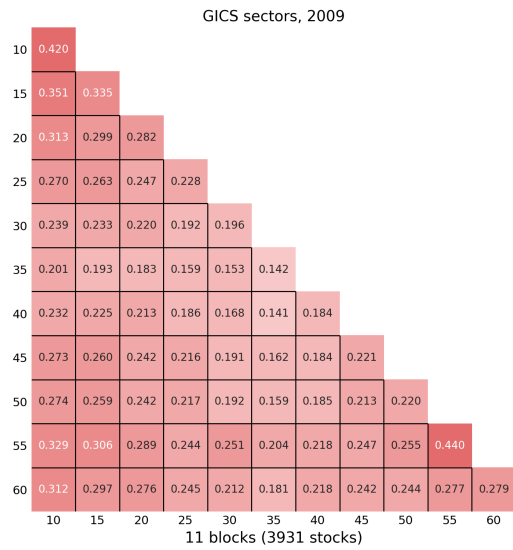
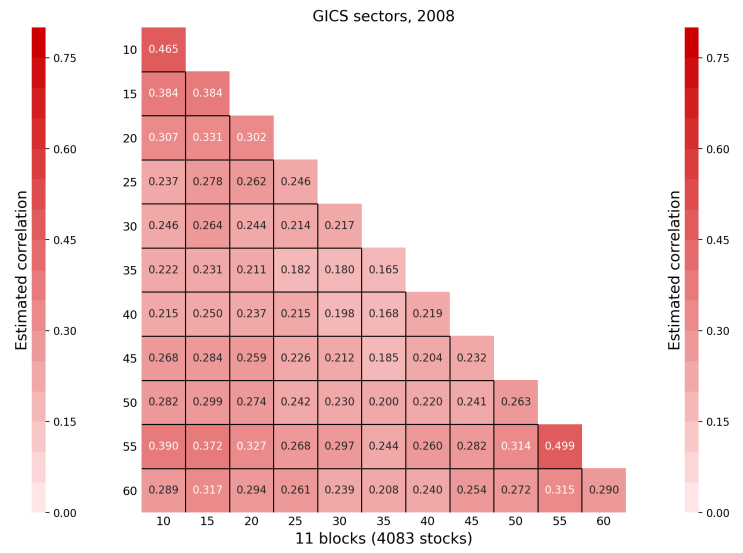
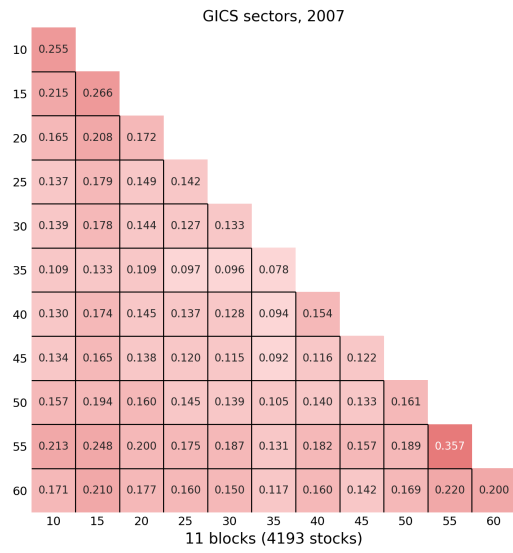


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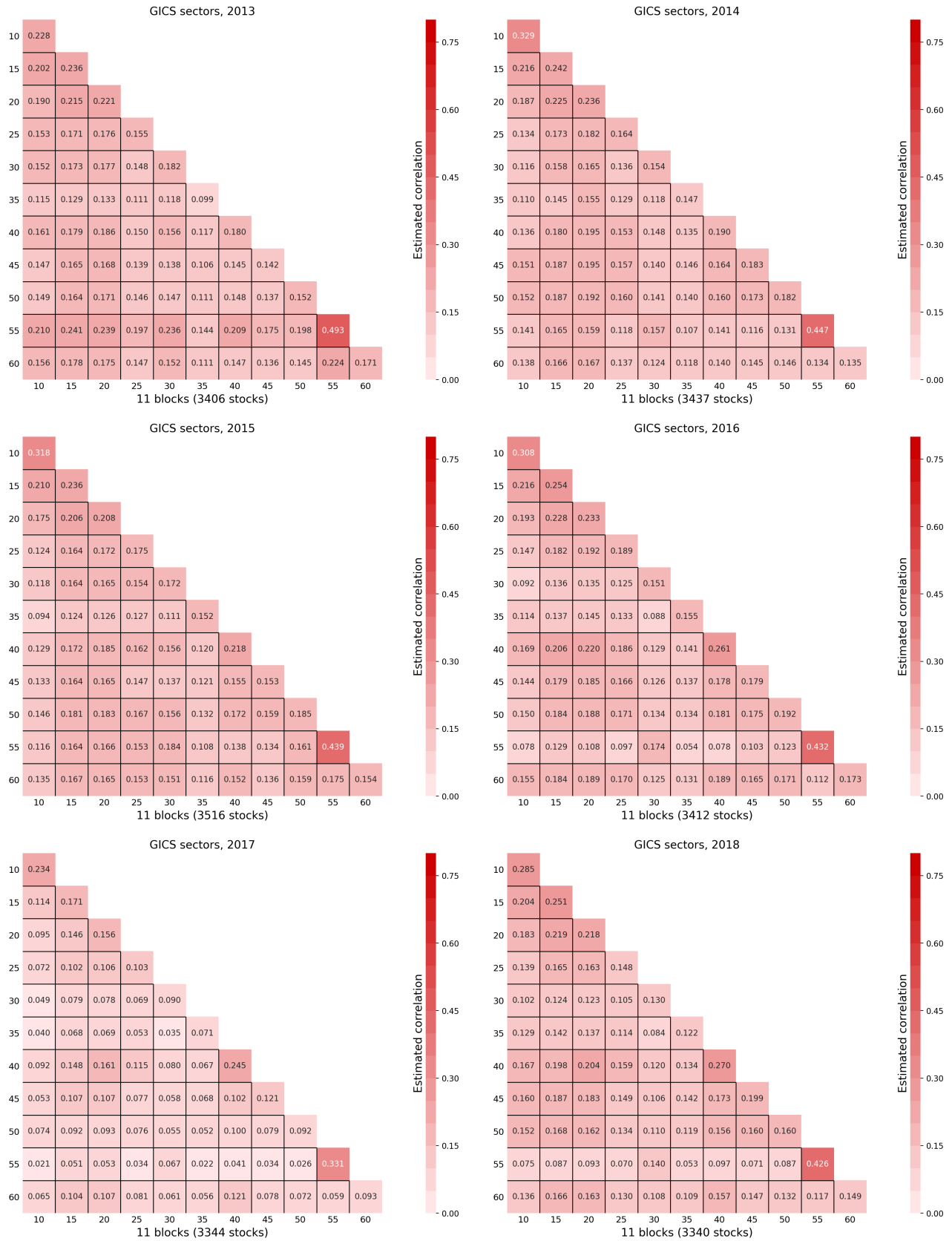


Figure 1: Estimated correlations for a block structure based on GICS sectors by calendar year (1995 - 2018).

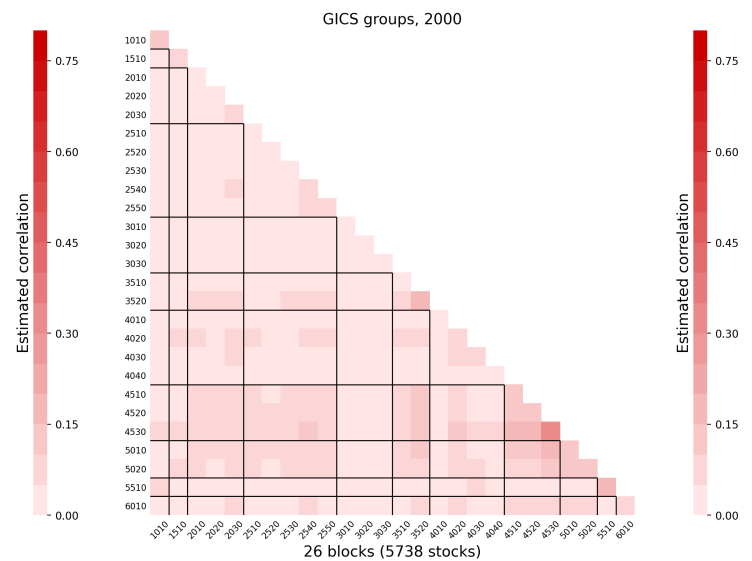
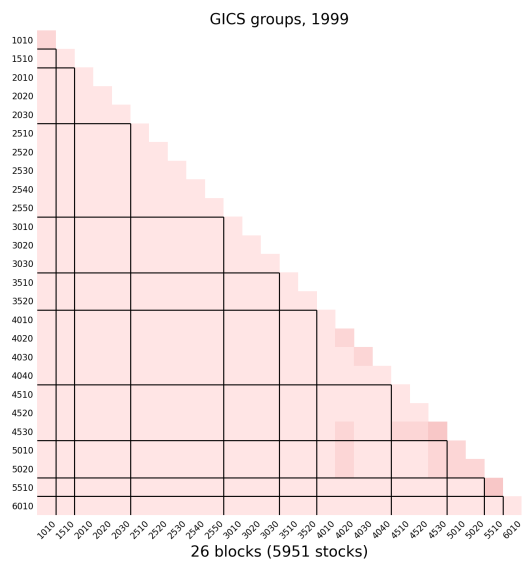
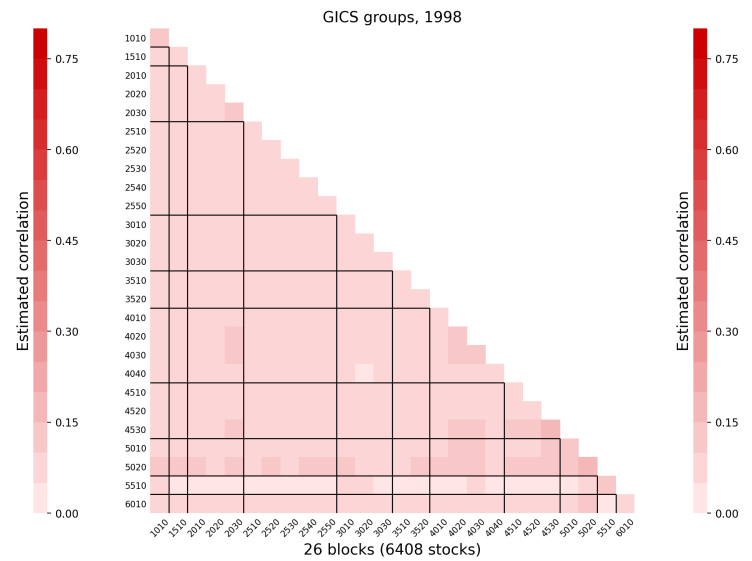
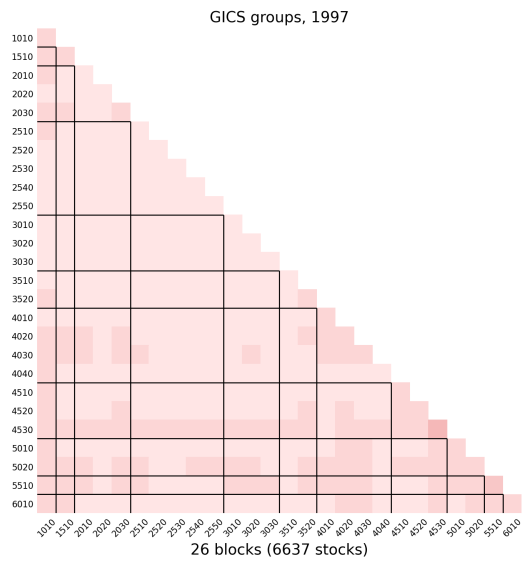
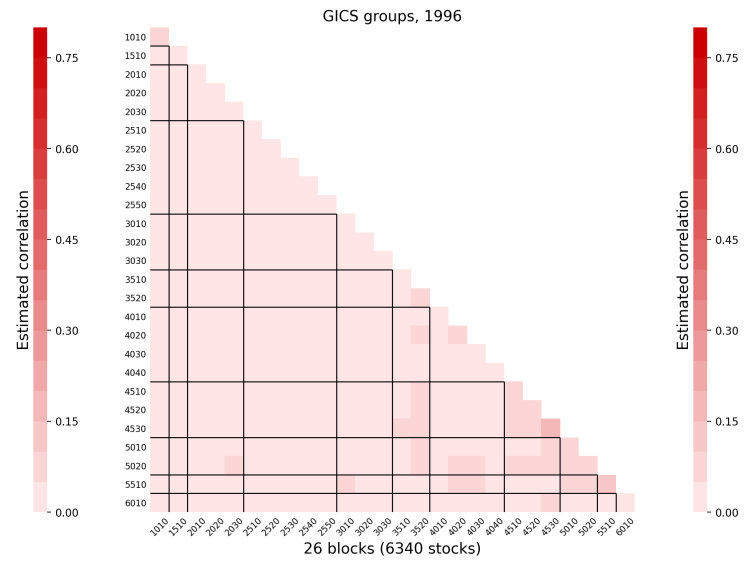
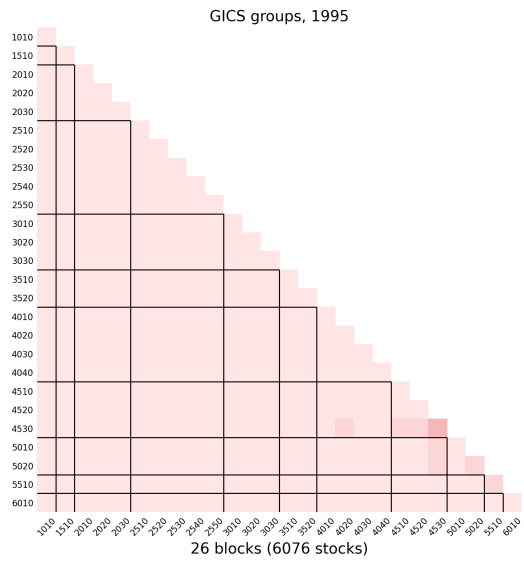


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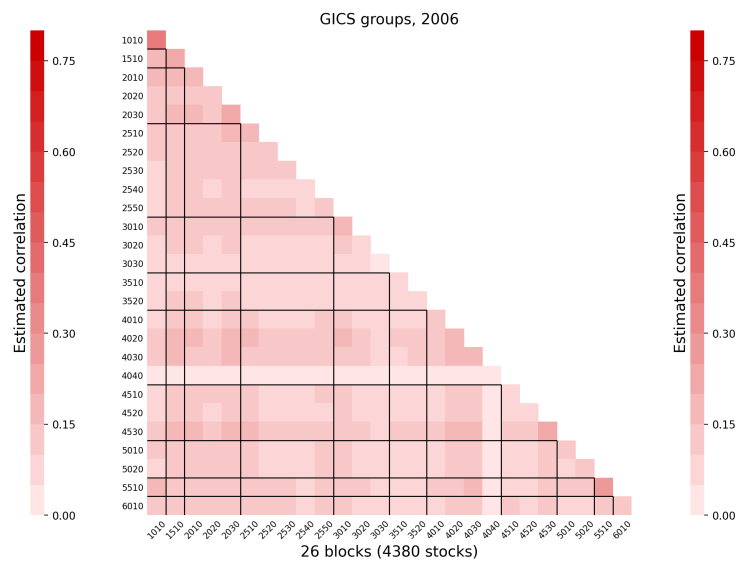
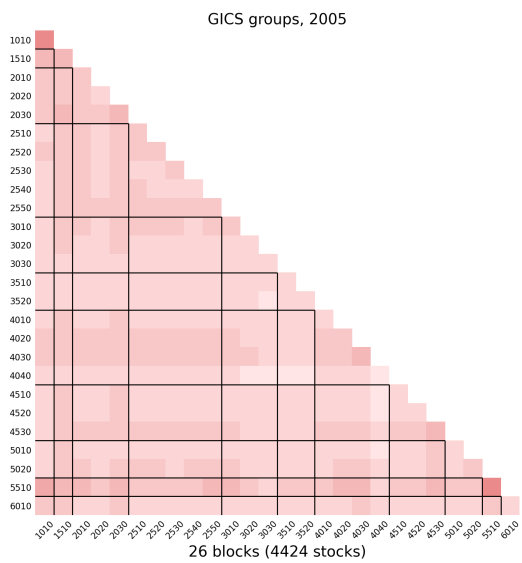
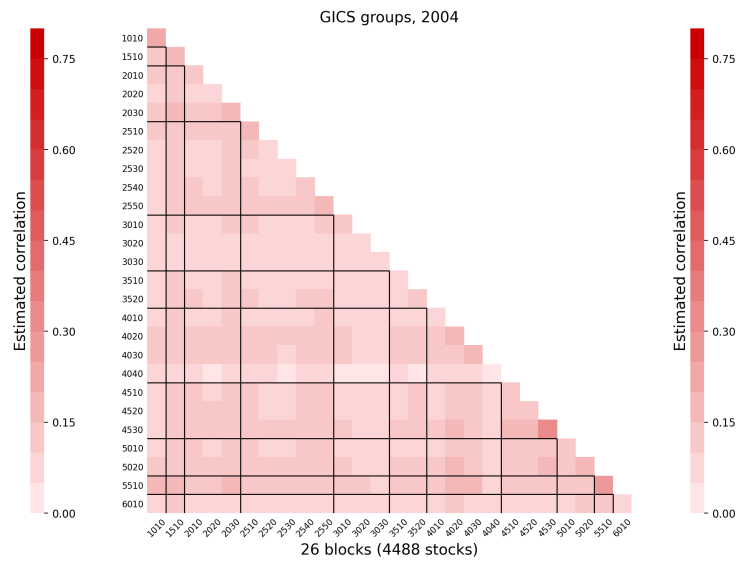
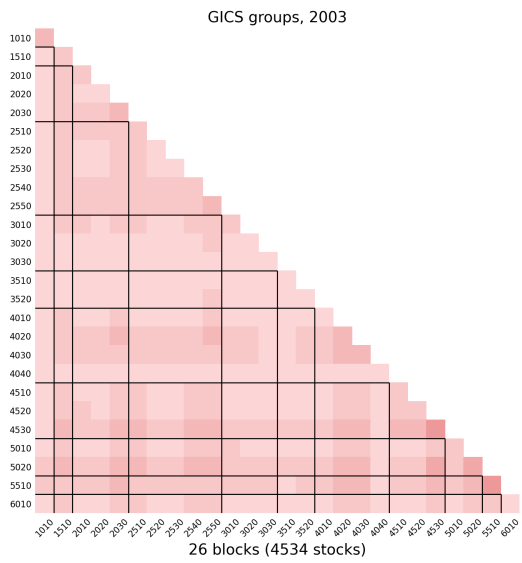
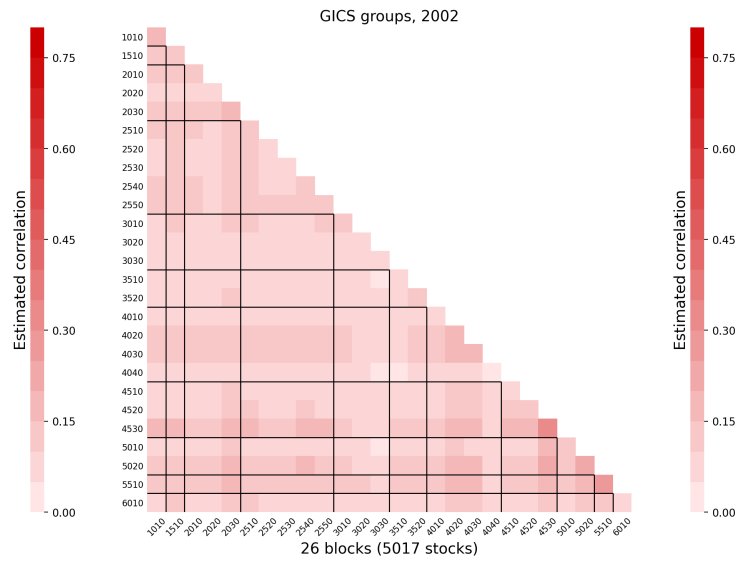
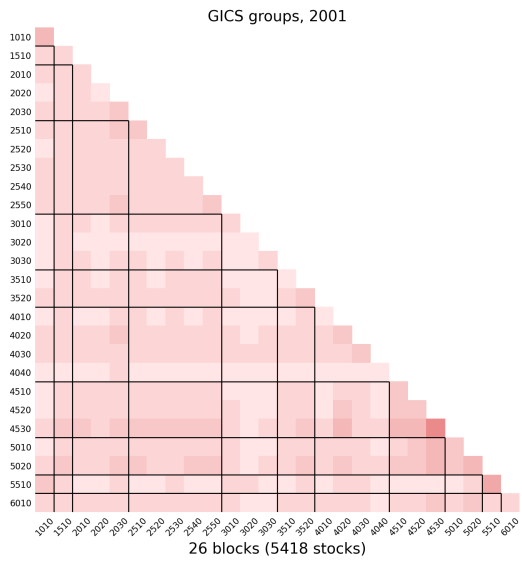


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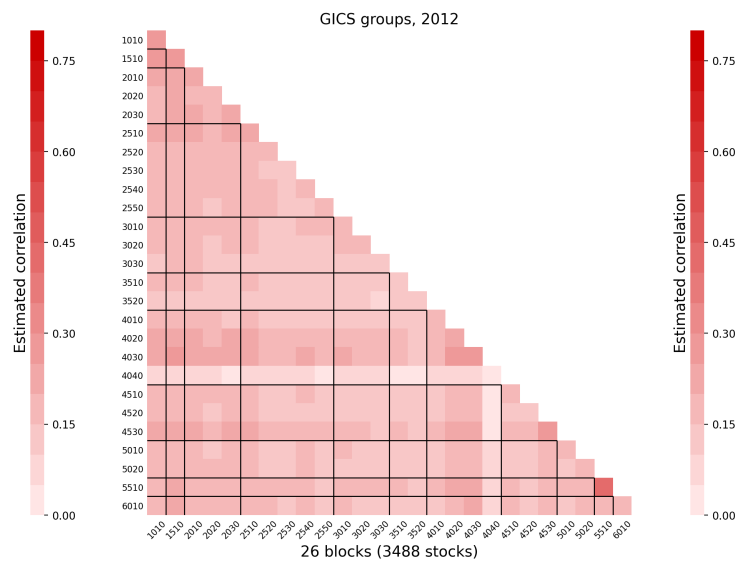
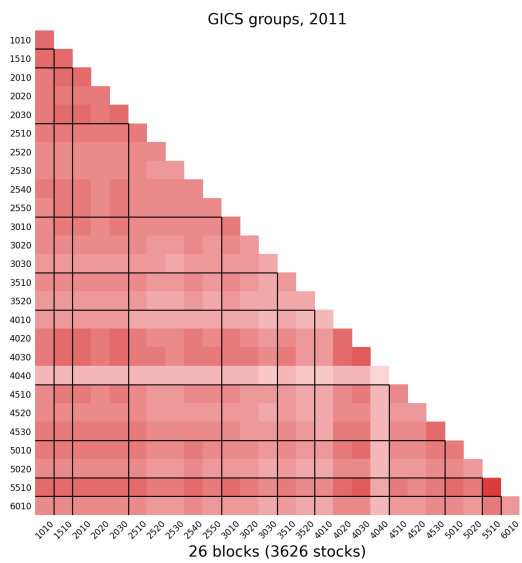
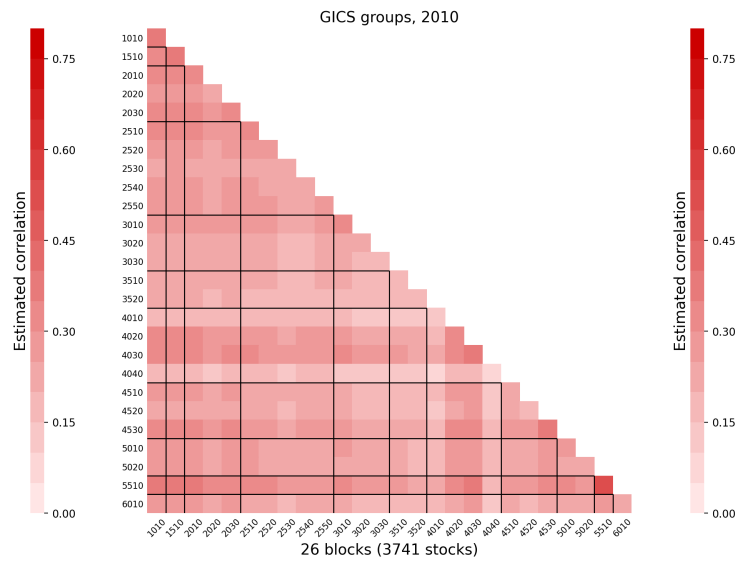
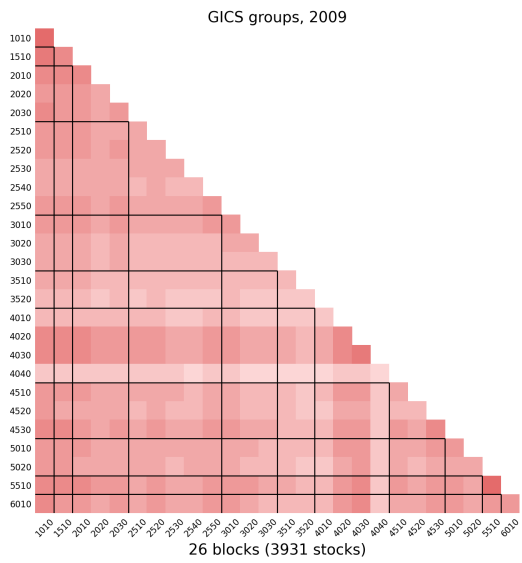
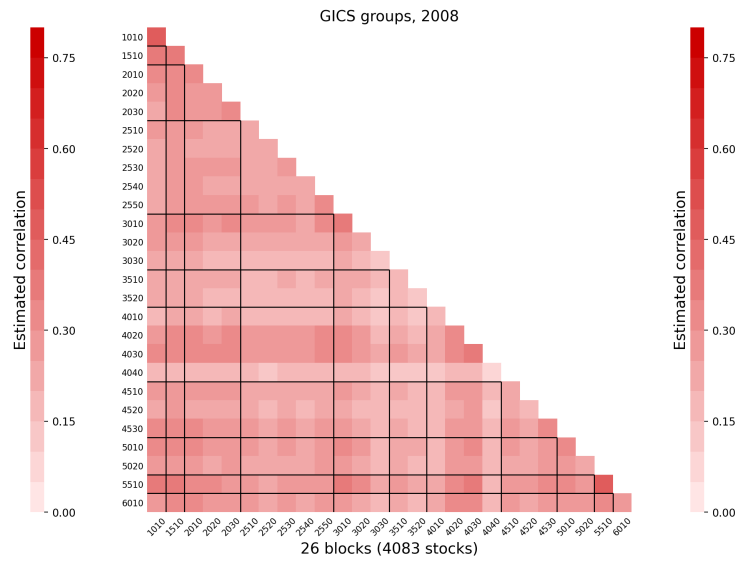
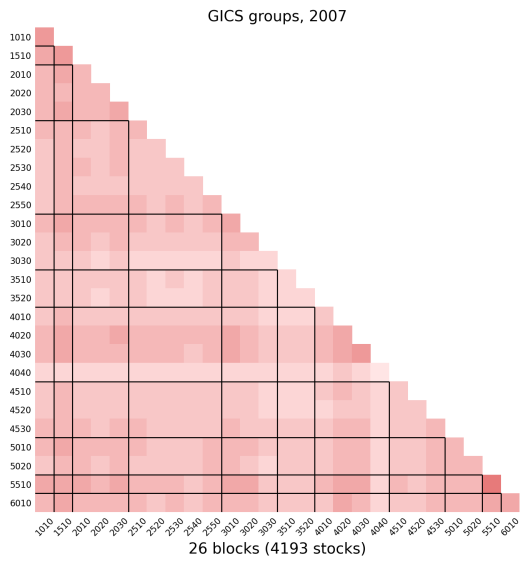


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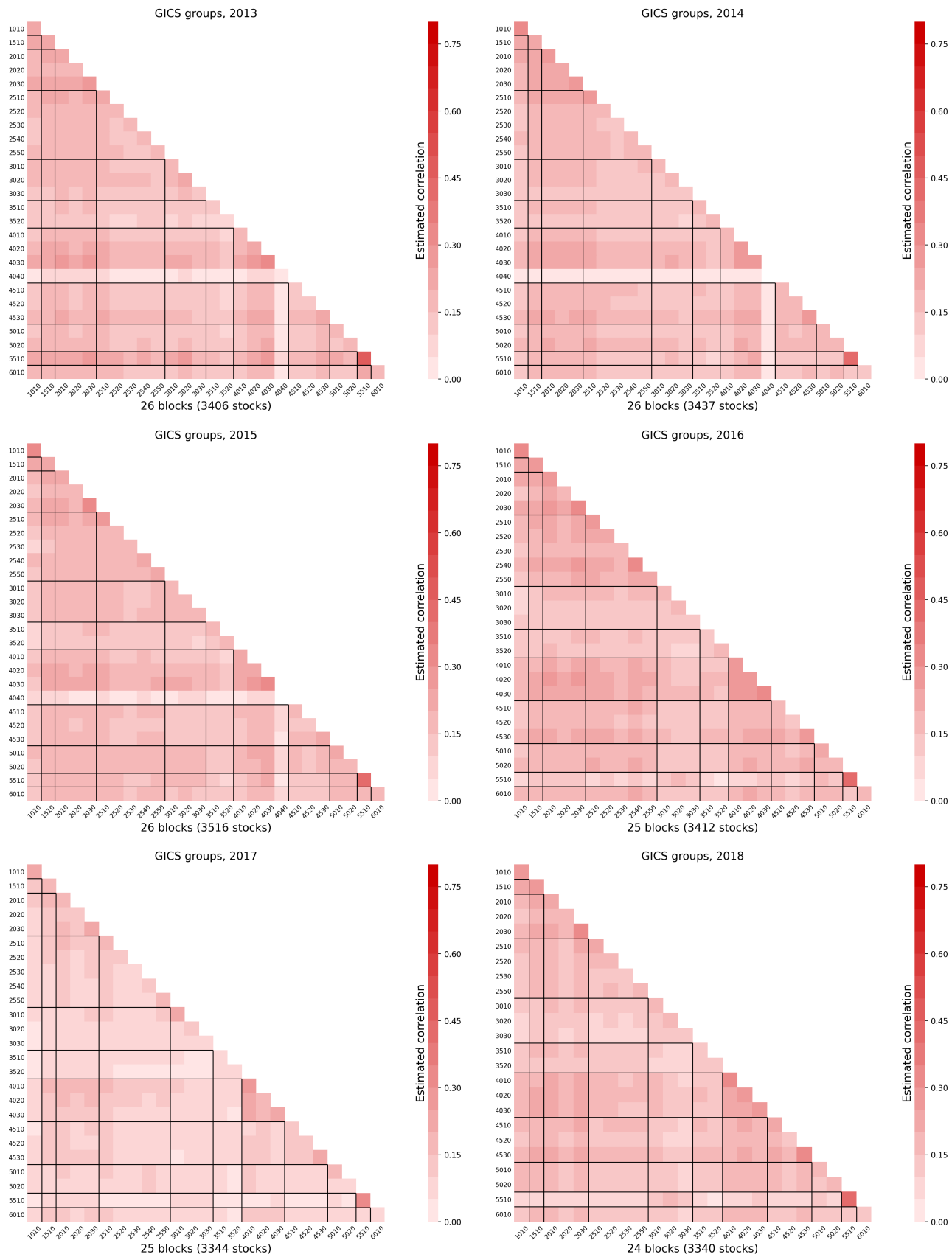


Figure 2: Estimated correlations for a block structure based on GICS groups for by calendar year (1995 - 2018).



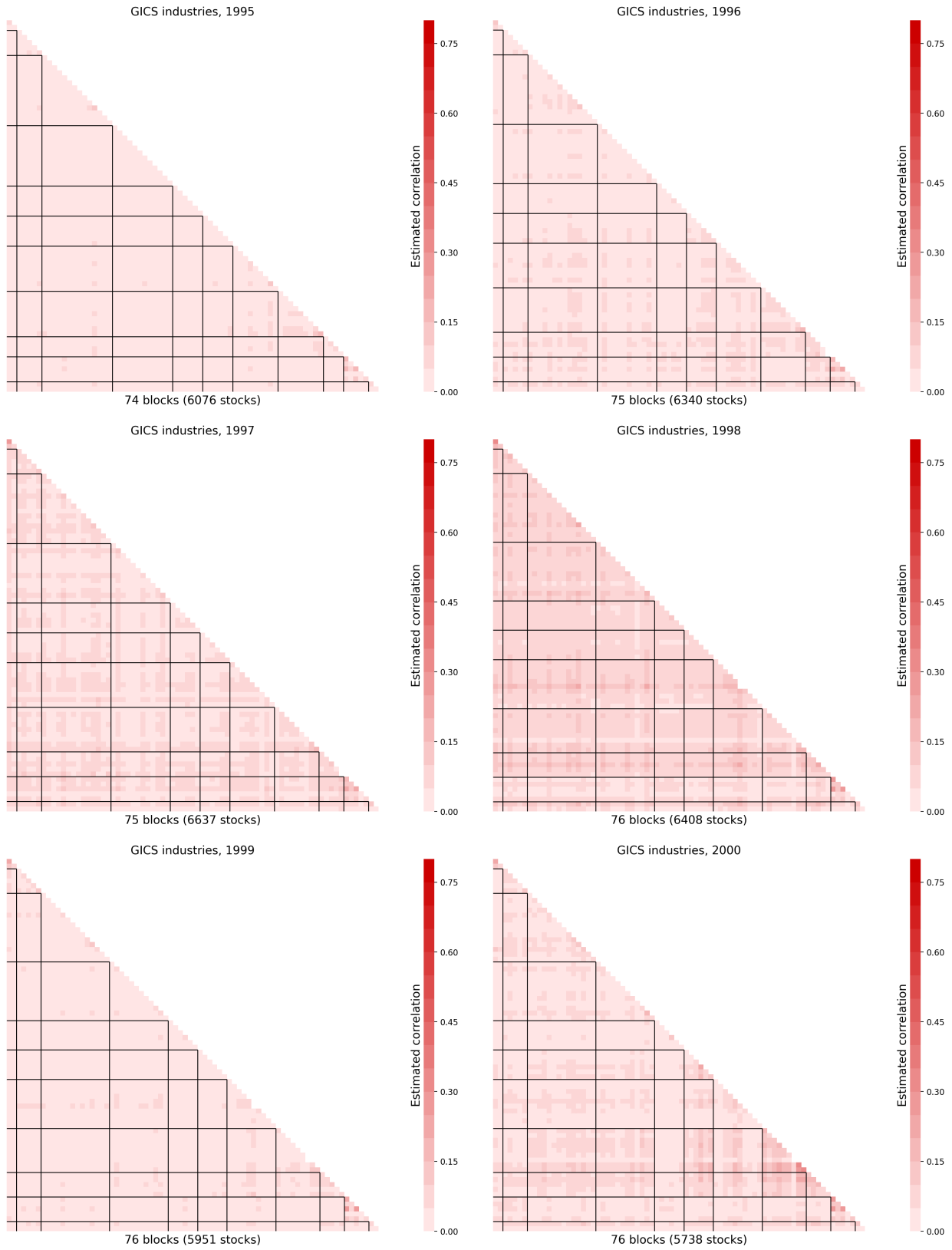


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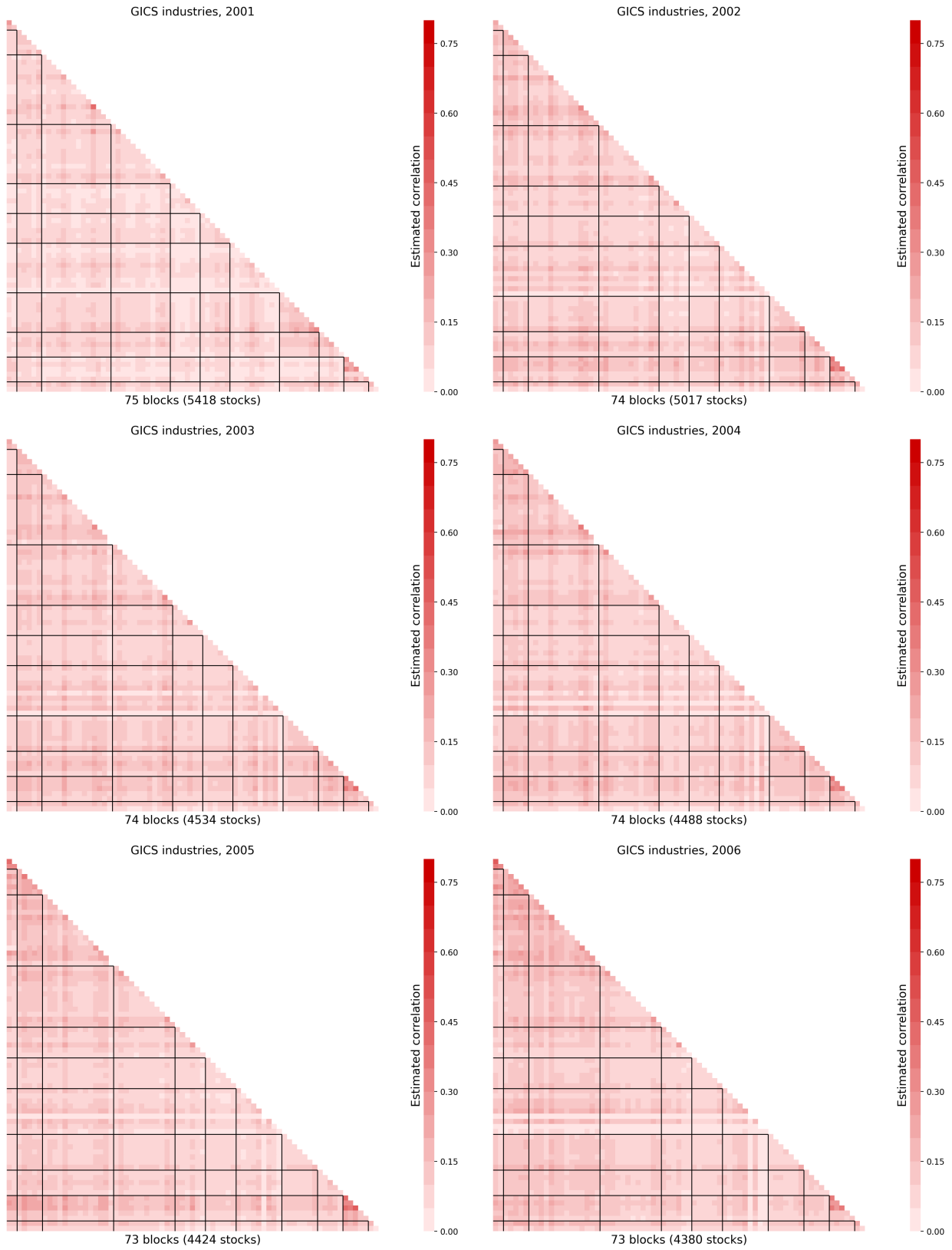


Figure 3: (Continued on next page).

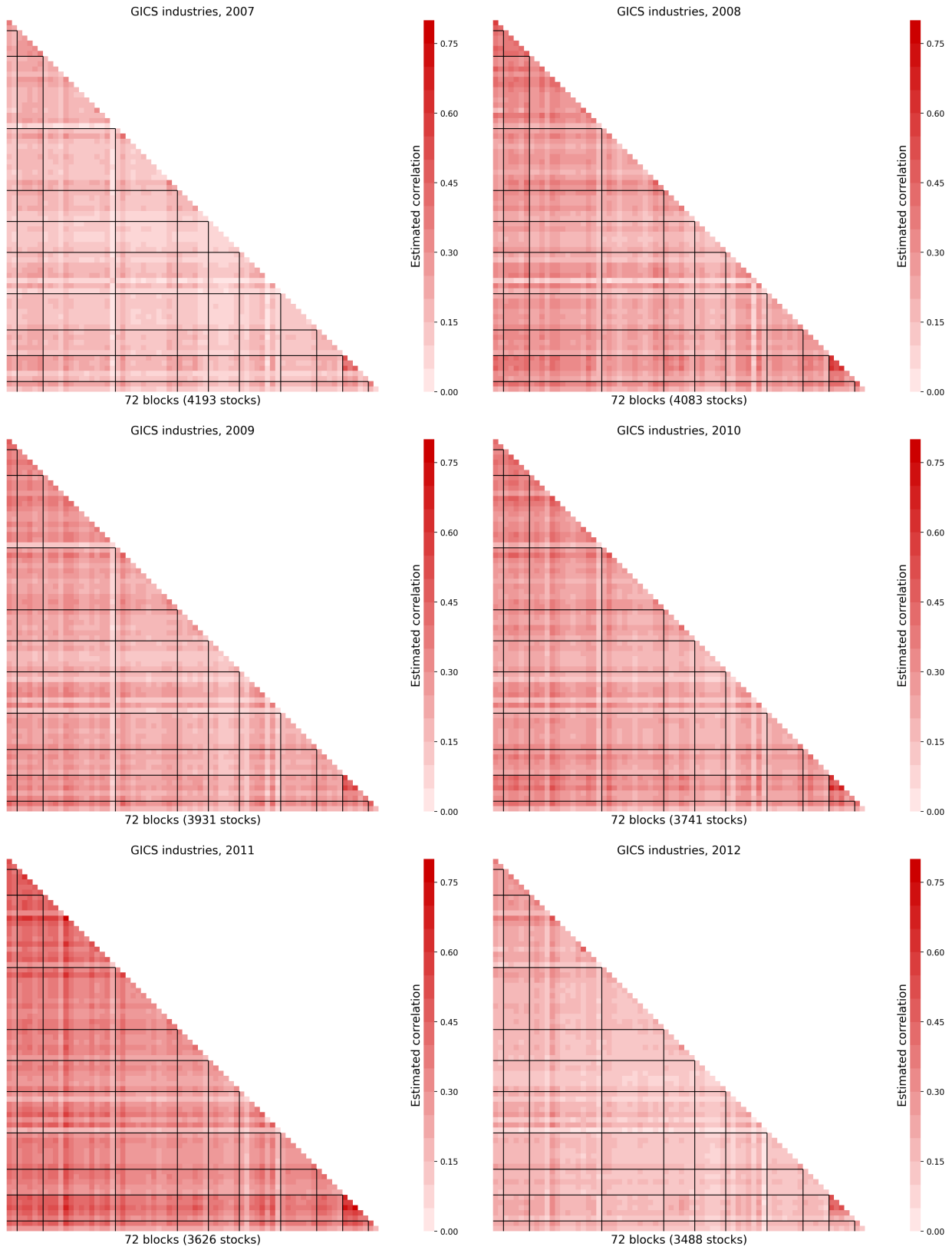


Figure 3: (Continued on next page).

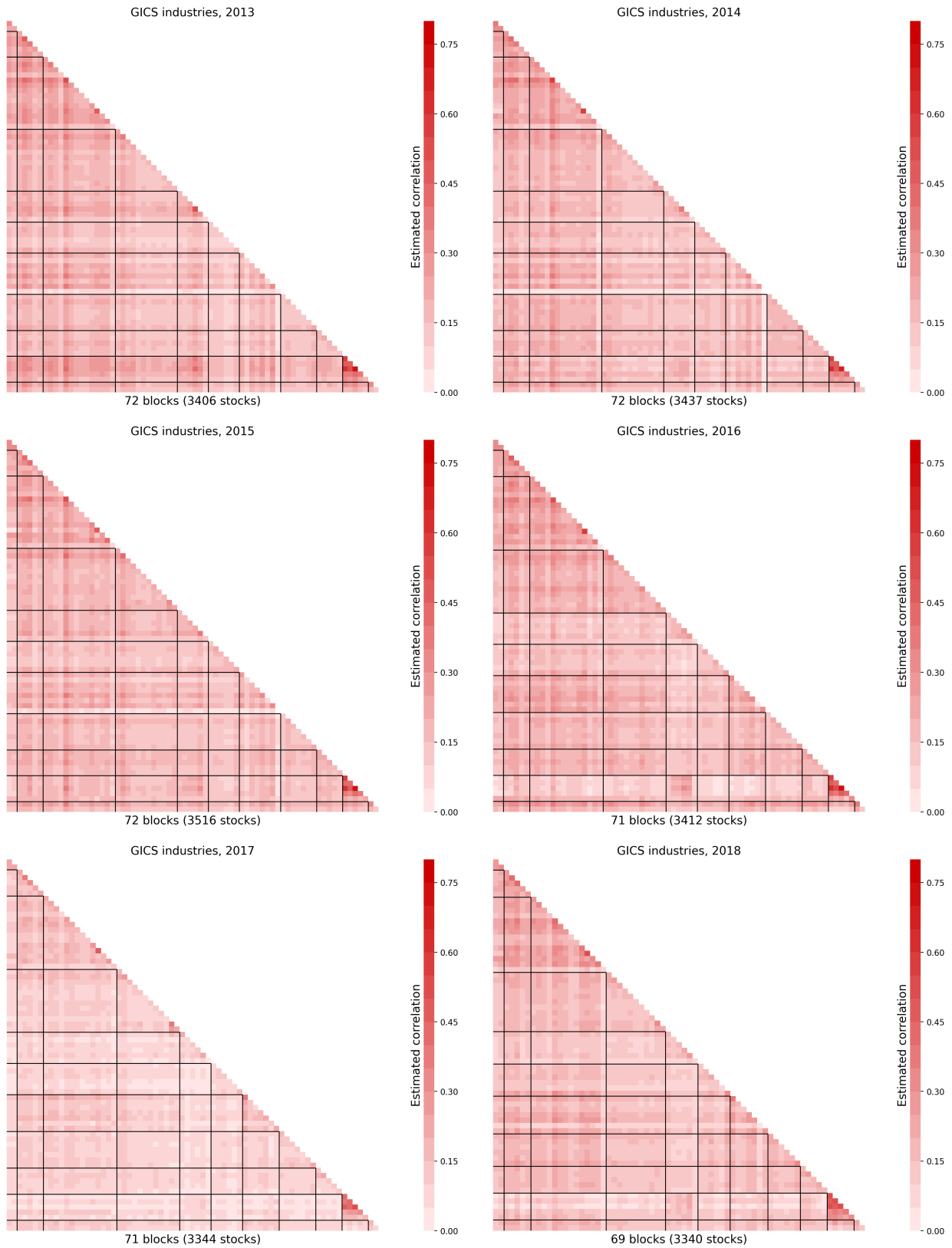


Figure 3: Estimated correlations for a block structure based on GICS industries for by calendar year (1995 - 2018).

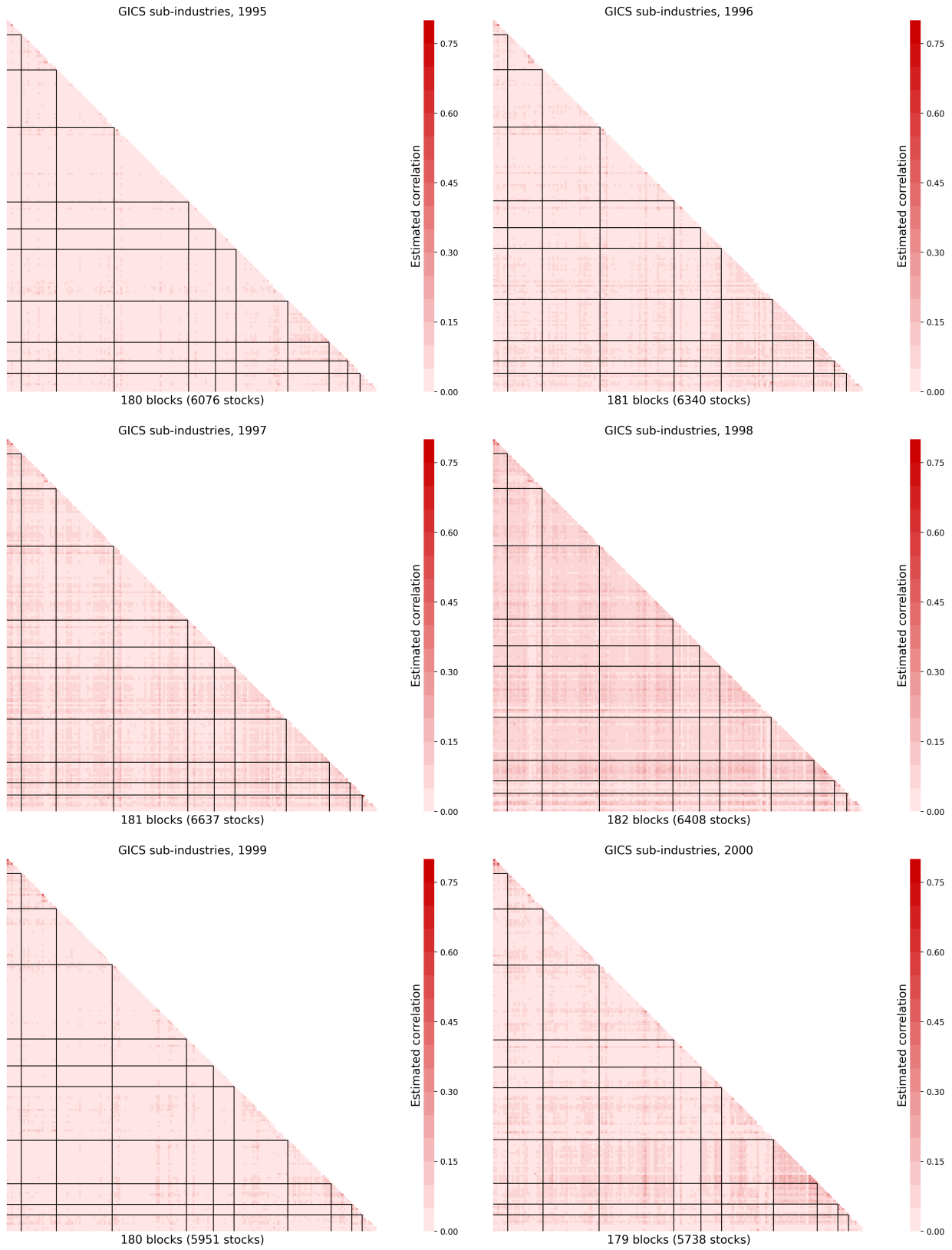


Figure 4: (Continued on next page).

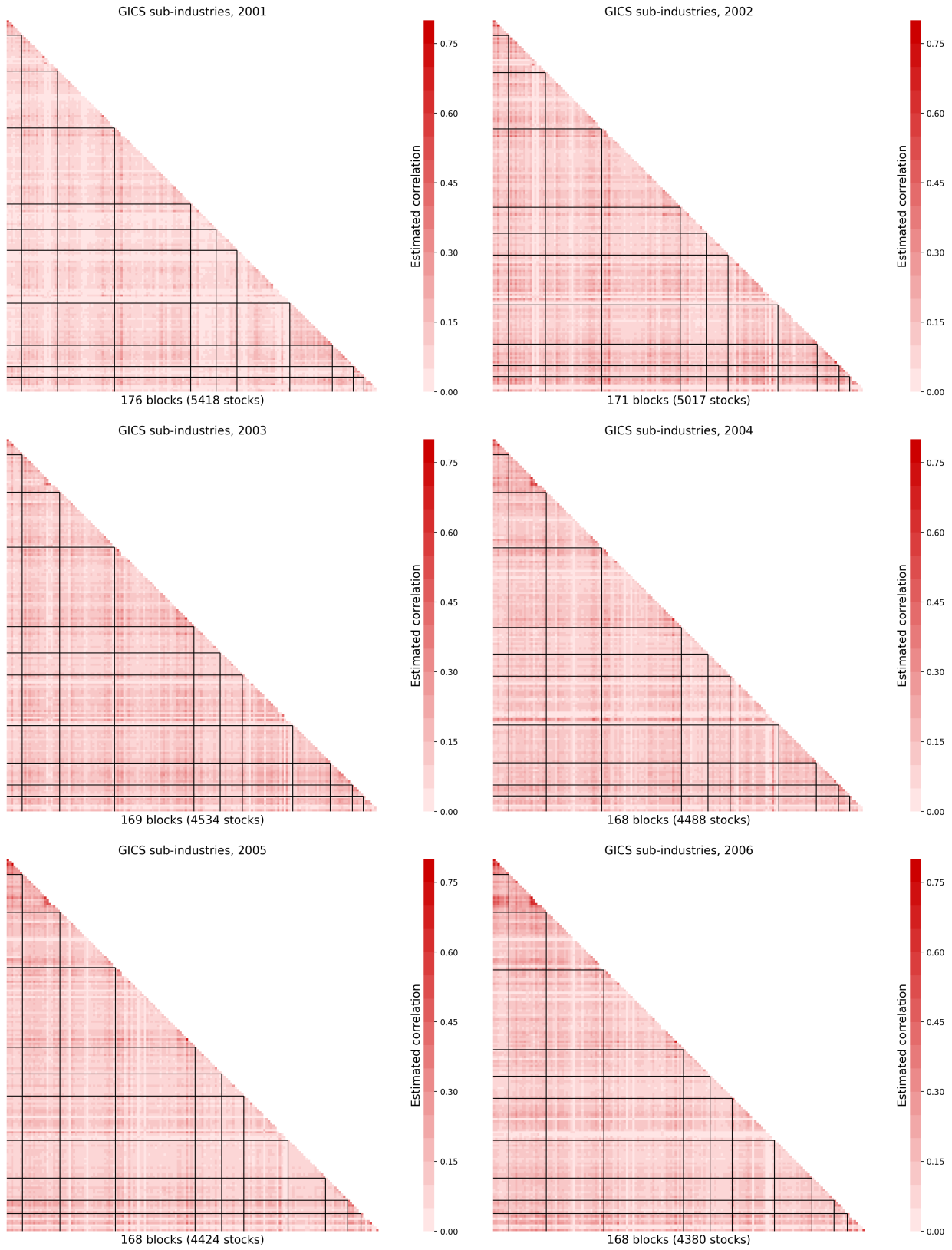


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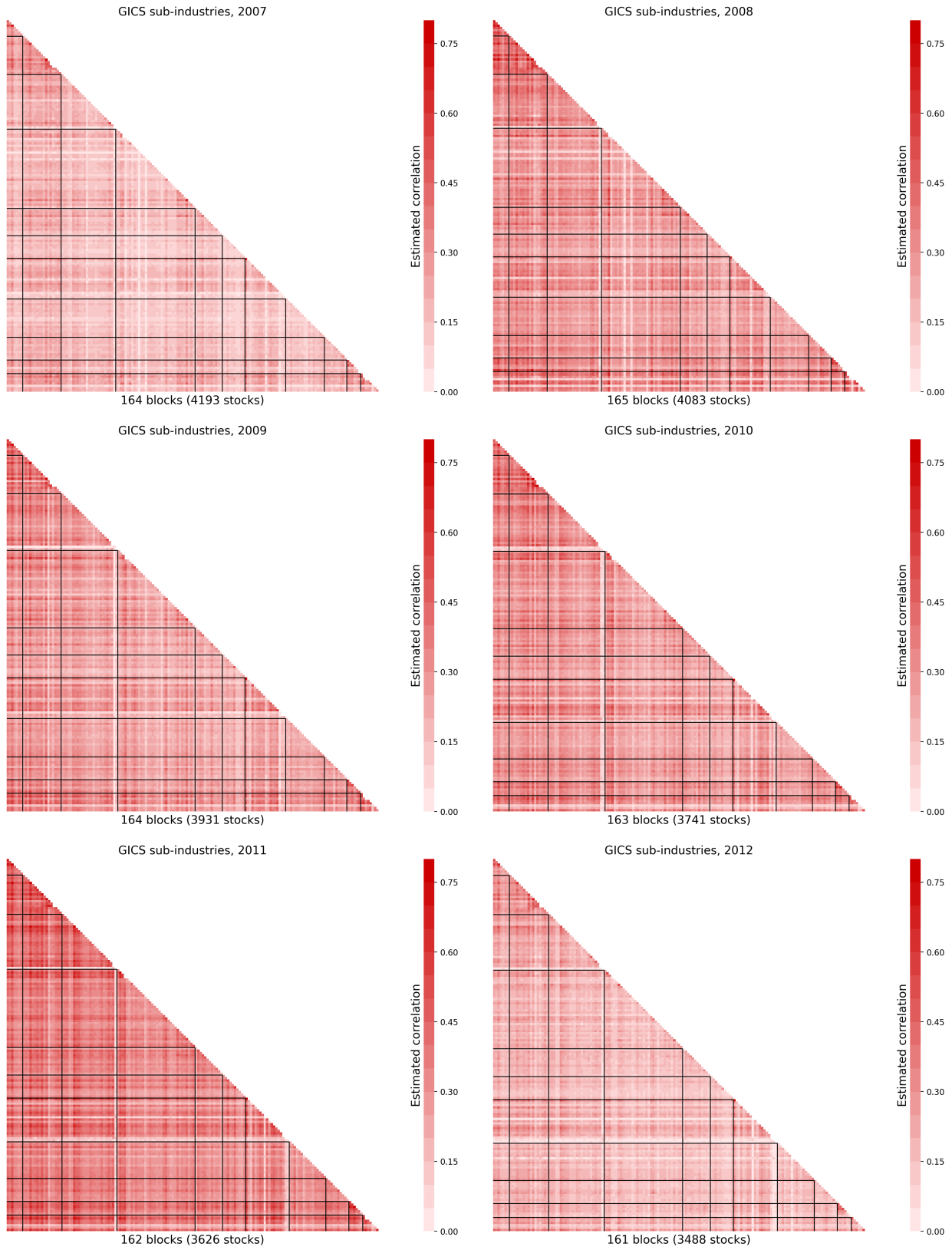


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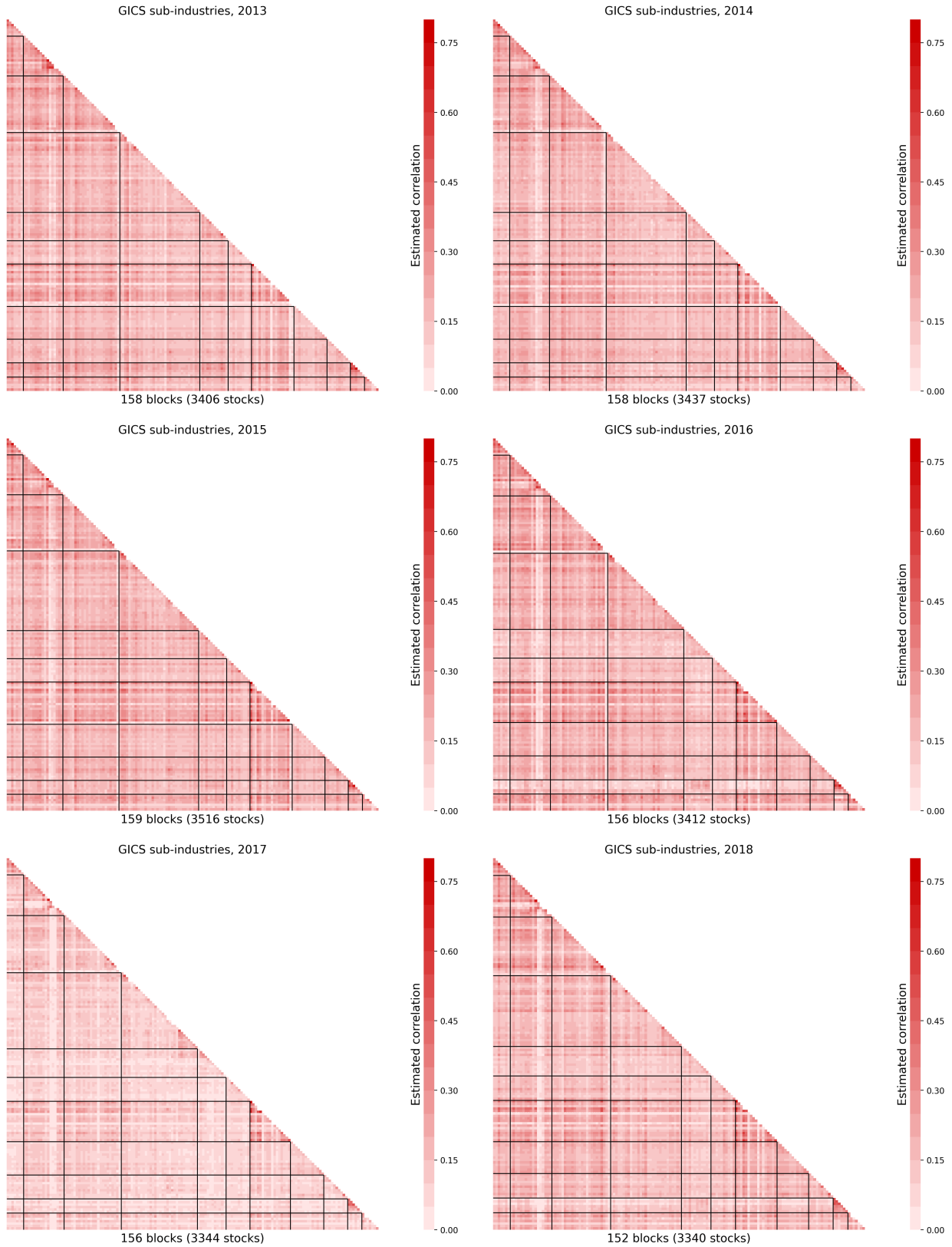


Figure 4: Estimated correlations for a block structure based on GICS sub-industries by calendar year (1995 - 2018).



## 2 Conditional Covariances and Correlations

The canonical representation simplifies computations of conditional variances and covariances from block covariance matrices. In this section we provide details behind these calculations.

Suppose that  $X \sim N_n(0, \Sigma)$ , where  $\Sigma$  is a block covariance matrix with  $K \times K$  blocks. Thus  $X = (x'_1, \dots, x'_K)'$ , where  $x_k \in \mathbb{R}^{n_k}$ ,  $k = 1, \dots, K$ , is a sub-vector corresponding to  $k$ -th block. Denote by  $\mathcal{L}$  a collection of  $L \geq 1$  blocks, which is a subset of  $\{1, 2, \dots, K\}$ , and introduce vector  $X_{\mathcal{L}} = (x'_{\mathcal{L}(1)}, \dots, x'_{\mathcal{L}(L)})'$ , which consists of the sub-vectors of  $X$  for the blocks in  $\mathcal{L}$ . Similarly, we define by  $\mathcal{M}$  a collection of  $M \geq 1$  other blocks, which is a subset of  $\{1, 2, \dots, K\}/\mathcal{L}$ , so  $L+M \leq K$ , and the vector  $X_{\mathcal{M}} = (x'_{\mathcal{M}(1)}, \dots, x'_{\mathcal{M}(M)})'$  consists of the sub-vectors of  $X$  from the blocks in  $\mathcal{M}$ . Then the covariance matrix of  $X_{\mathcal{L}}$  conditionally on  $X_{\mathcal{M}}$  has an  $L \times L$  block structure with partition  $(n_{\mathcal{L}(1)}, \dots, n_{\mathcal{L}(L)})$  and is given by

$$\Sigma_{\mathcal{L}|\mathcal{M}} = \Sigma_{\mathcal{L},\mathcal{L}} - \Sigma_{\mathcal{L},\mathcal{M}}\Sigma_{\mathcal{M},\mathcal{M}}^{-1}\Sigma_{\mathcal{M},\mathcal{L}},$$

where  $\Sigma_{\mathcal{L},\mathcal{M}}$  is  $L \times M$  block sub-matrix of  $\Sigma$  defined by

$$\Sigma_{\mathcal{L},\mathcal{M}} = \begin{pmatrix} \Sigma_{[\mathcal{L}(1),\mathcal{M}(1)]} & \cdots & \Sigma_{[\mathcal{L}(1),\mathcal{M}(M)]} \\ \vdots & & \vdots \\ \Sigma_{[\mathcal{L}(L),\mathcal{M}(1)]} & \cdots & \Sigma_{[\mathcal{L}(L),\mathcal{M}(M)]} \end{pmatrix},$$

and  $\Sigma_{\mathcal{L},\mathcal{L}}$ ,  $\Sigma_{\mathcal{M},\mathcal{M}}$  and  $\Sigma_{\mathcal{M},\mathcal{L}}$  are defined similarly.

Let the canonical representation of  $\Sigma$  is  $\Sigma = QDQ'$ , where  $D = \text{diag}(A, \lambda_1 I_{n_1-1}, \dots, \lambda_K I_{n_K-1})$ . The canonical representation of  $\Sigma_{\mathcal{L}|\mathcal{M}}$  can be then obtained directly from the canonical representation of  $\Sigma$ ,

$$\Sigma_{\mathcal{L}|\mathcal{M}} = Q_{\mathcal{L}} D_{\mathcal{L}|\mathcal{M}} Q'_{\mathcal{L}}, \tag{1}$$

where  $Q_{\mathcal{L}}$  is an orthogonal matrix consistent with the block partition of  $\Sigma_{\mathcal{L}|\mathcal{M}}$ , namely  $(n_{\mathcal{L}(1)}, \dots, n_{\mathcal{L}(L)})$ . The block-diagonal matrix is given by  $D_{\mathcal{L}|\mathcal{M}} = \text{diag}(A_{\mathcal{L}|\mathcal{M}}, \lambda_{\mathcal{L}(1)} I_{n_{\mathcal{L}(1)}}, \dots, \lambda_{\mathcal{L}(L)} I_{n_{\mathcal{L}(L)}})$ , where  $\lambda_{\mathcal{L}(l)}$  is the  $\mathcal{L}(l)$ -th eigenvalue entry from the canonical representation of  $\Sigma$ , for  $l = 1, \dots, L$ , and

$$A_{\mathcal{L}|\mathcal{M}} = A_{\mathcal{L},\mathcal{L}} - A_{\mathcal{L},\mathcal{M}}A_{\mathcal{M},\mathcal{M}}^{-1}A_{\mathcal{M},\mathcal{L}},$$

where  $A_{\mathcal{L},\mathcal{M}}$  is an  $L \times M$  sub-matrix of  $A$  such that  $[A_{\mathcal{L},\mathcal{M}}]_{l,m} = A_{\mathcal{L}(l),\mathcal{M}(m)}$ , for  $l = 1, \dots, L$  and

$m = 1, \dots, M$ . The matrices,  $A_{\mathcal{L},\mathcal{L}}$ ,  $A_{\mathcal{M},\mathcal{M}}$ , and  $A_{\mathcal{M},\mathcal{L}}$ , are defined similarly. Thus, the canonical representation simplifies computation of conditional covariance matrix  $\Sigma_{\mathcal{L}|M}$  by bypassing a brute-force inversion of the (possibly large) submatrix  $\Sigma_{\mathcal{M},\mathcal{M}}$ .

Let  $a_{kl}^\diamond$  denote the  $kl$ -th entry of  $A_{\mathcal{L}|M}$  and by  $b_{kl}^\diamond$  the off-diagonal entry from  $kl$ -th block of  $\Sigma_{\mathcal{L}|M}$ , for  $k, l \in \mathcal{L}$ . Then, as it follows from Theorem 1,  $b_{kl}^\diamond$  is given by

$$b_{kl}^\diamond = \begin{cases} (n_{\mathcal{L}(k)})^{-1}(a_{kk}^\diamond - \lambda_{\mathcal{L}(k)}) & \text{for } k = l, \\ (n_{\mathcal{L}(k)}n_{\mathcal{L}(l)})^{-\frac{1}{2}}a_{kl}^\diamond & \text{for } k \neq l. \end{cases}$$

The diagonal entry from  $k$ -th diagonal block of  $\Sigma_{\mathcal{L}|M}$  equals  $d_k^\diamond = b_{kk}^\diamond + \lambda_{\mathcal{L}(k)}$ . To obtain the conditional correlation matrix, we need to rescale  $\Sigma_{\mathcal{L}|M}$ , and the off diagonal entry from  $kl$ -th block of the corresponding conditional correlation matrix will be equal to  $\frac{b_{kl}^\diamond}{\sqrt{d_k^\diamond d_l^\diamond}}$ .

As an illustration, we calculate conditional (partial) correlations between equity returns. For this we use the block correlation matrices estimated on a yearly basis for the block structure implied by 11 GICS sectors. We focus on a partial correlation between a pair of stocks conditionally on all stocks from the other sectors. So, for a pair of stocks from the same sector  $k$  we set  $\mathcal{L} = \{k\}$  and  $\mathcal{M} = \{1, 2, \dots, K\}/\mathcal{L}$ . Then we proceed to compute  $\Sigma_{\mathcal{L}|M}$  as described in ((1)) by using the estimated block correlation matrix,  $\hat{\Sigma}$ . The resulting matrix consist of a single block and it represents a conditional covariance matrix of the standardized returns. To obtain the desired partial correlation, we rescale the matrix appropriately. For a pair of stocks from different sectors,  $k$  and  $l$ , we set  $\mathcal{L} = \{k, l\}$  and  $\mathcal{M} = \{1, 2, \dots, K\}/\mathcal{L}$ . In this case we compute  $\Sigma_{\mathcal{L}|M}$ , which is a  $2 \times 2$  block matrix, and the partial correlation is obtained by rescaling the entry from the off-diagonal block of  $\Sigma_{\mathcal{L}|M}$ .

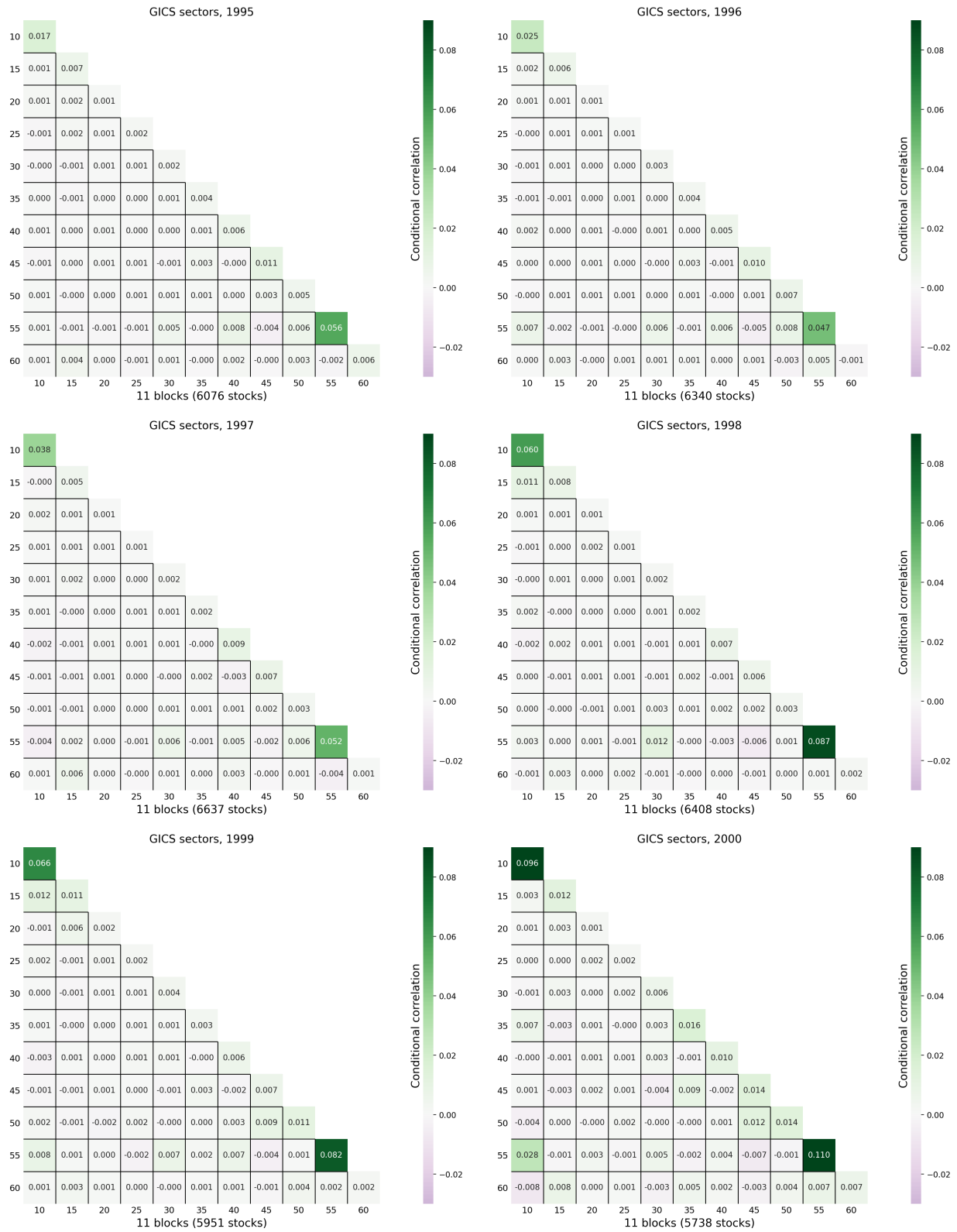


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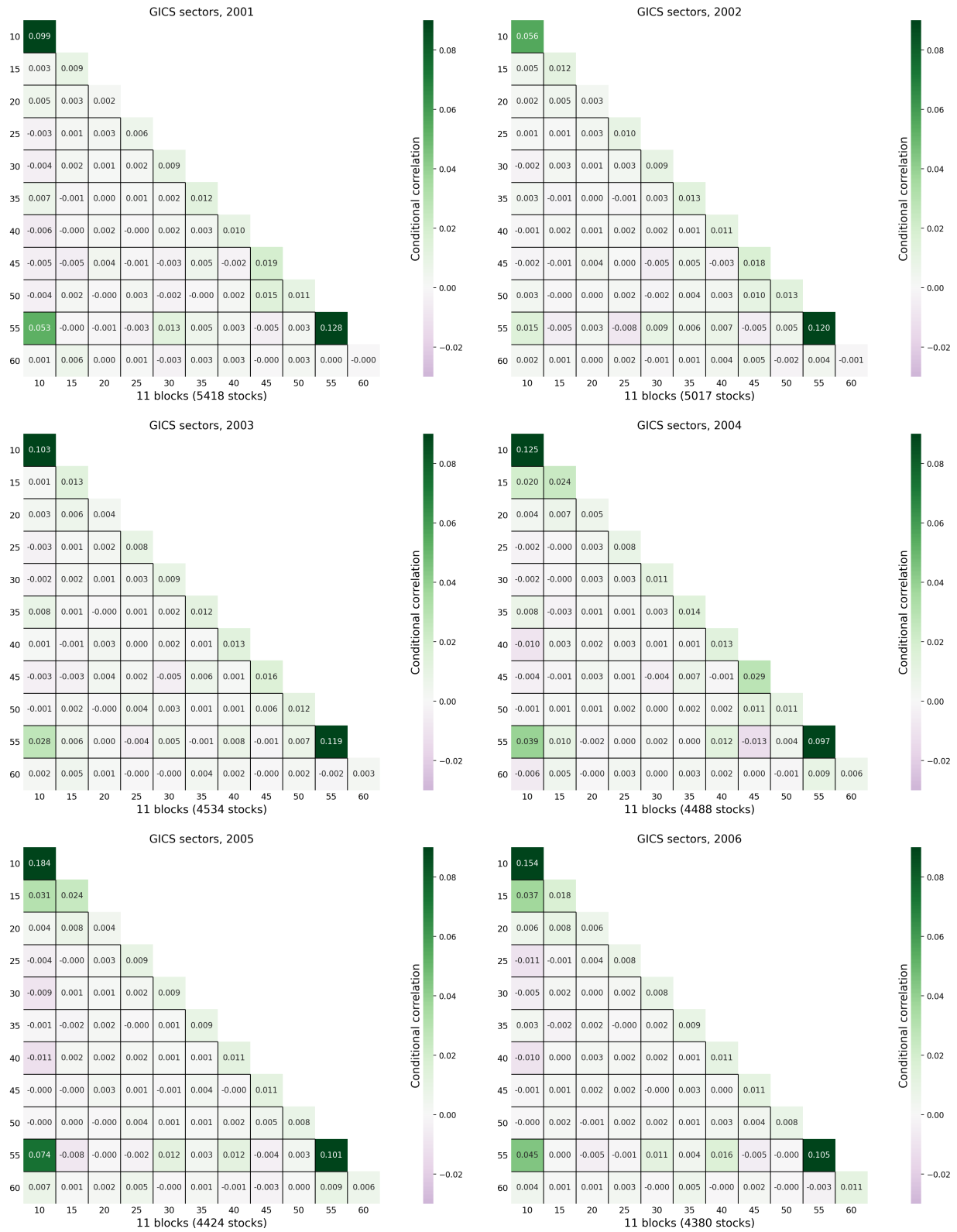


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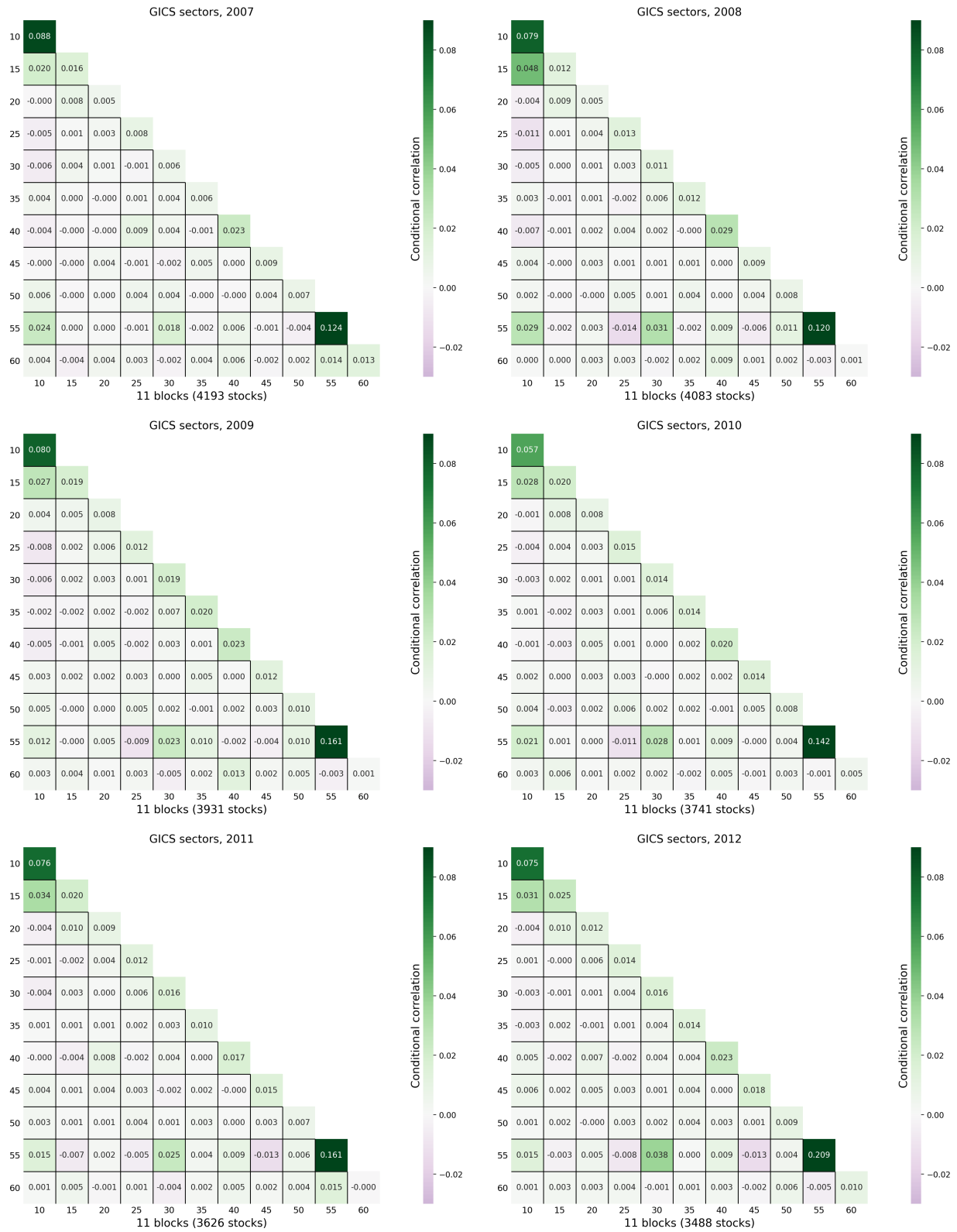


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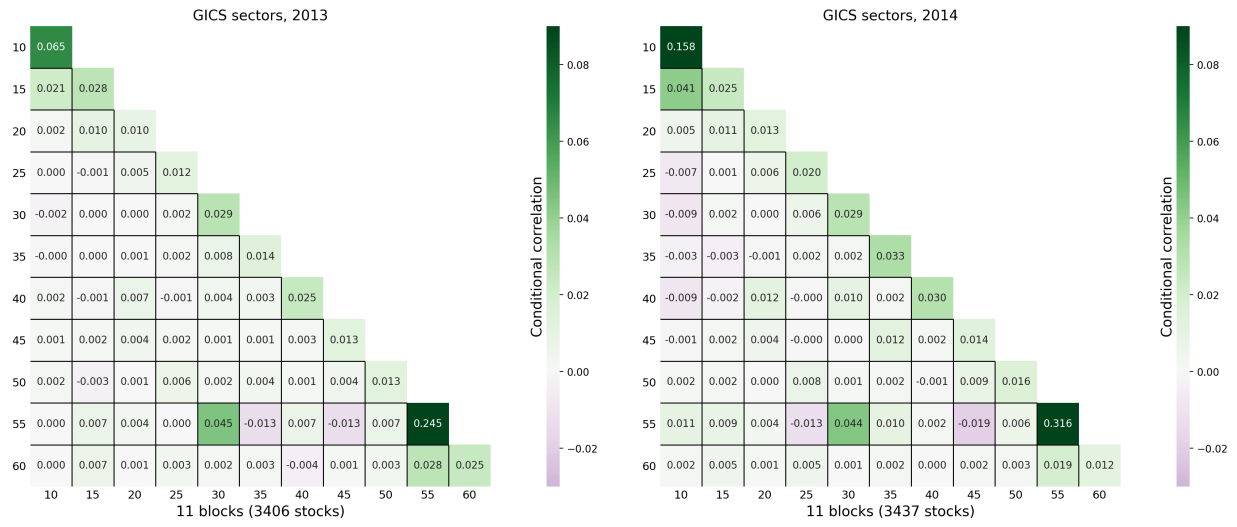


Figure 5: Partial correlations based on the sector-block correlation matrix, by calendar year (1995-2014).

### 3 MLE for Block Covariance Matrix in Proposition 2 (MATLAB)

```

function [A_hat,Lambda_hat,Q] = MLE_block_cov(X,vB)

% Inputs:
% X - T x n data array of n-dimensional vectors, where T is a sample size.
% It is assumed that  $X_t \sim i.i.d.\mathcal{N}(0,\Sigma)$ ,  $t=1,\dots,T$ , and  $\Sigma$  has  $K \times K$  block structure.
% vB - K-dimensional vector of block sizes, where K is a number of blocks (clusters).
% Entries of vB must be positive integers and sum up to n.
% Note: data in matrix X must be pre-ordered in such a way that variables
% in the first vB(1) columns correspond to the first cluster,
% variables in the next vB(2) columns are from the second cluster, etc.

% Output:
% A_hat - ML estimate of  $K \times K$  matrix A from the canonical representation
% Lambda_hat - K-dimensional vector of ML estimates of  $\lambda_1, \dots, \lambda_K$ 
% Q -  $n \times n$  orthogonal matrix from the canonical representation

[T,n] = size(X); K = length(vB);

% Check if input is of proper format: vB consists of positive integers and sum(vB) equals n
if all(vB > 0) && all(mod(vB,1) == 0) && (sum(vB) == n)

    vB_ind = [0;cumsum(vB)]; % Get indicies of the block partition implied by vB
    Q = zeros(n); % Allocate n x n matrix for Q

    % Estimate A
    YO = zeros(T,K);
    for k = 1:K
        % Construct k-th column of Q
        Qk = zeros(n,1); vk = ones(vB(k),1)/sqrt(vB(k));
        Qk(vB_ind(k)+1:vB_ind(k+1)) = vk;

        Q(:,k) = Qk; % Update Q
        YO(:,k) = X*Qk; % Transform data vectors
    end
    A_hat = YO'*YO/T; % MLE of A

    % Estimate  $\lambda_1, \dots, \lambda_K$ 
    Lambda_hat = zeros(K,1);
    for k = 1:K
        % Explicitly get an orthogonal complement to vk and restore remaining columns of Q
        Qk_ort = zeros(n,vB(k)-1); vBk_lin = (0:vB(k))';
        vk_ort = triu(ones(vB(k))*diag([0;1./sqrt(vBk_lin(2:end-1).*vBk_lin(3:end))]))';
        vk_ort(eye(vB(k), 'logical')) = -diag(vk_ort).*vBk_lin(1:end-1);
        Qk_ort(vB_ind(k)+1:vB_ind(k+1),:) = vk_ort(:,2:end);

        Q(:,K-k+vB_ind(k)+2:K-k+vB_ind(k+1)) = Qk_ort; % Update Q
        Yk = X*Qk_ort; % Transform data vectors
        Lambda_hat(k) = trace(Yk*Yk')/((vB(k)-1)*T); % MLE of  $\lambda_k$ 
    end
end

else
    fprintf('Error: input is of wrong format');
end

```