Principles of Resolution

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Anacrusis. The polynomial $f(x,y,z) = (x^2 - y^3)^2 - (z^2 - y^2)^3$ defines as its real zeroset an algebraic surface X in Euclidean space $\mathbb{R}^3$. This surface cannot be a manifold, since the polynomial violates at certain points of X the assumptions of the Implicit Function Theorem. The points of failure represent the singularities of X. They are determined by equating the three partial derivatives to zero. The singular points occur along two curves Z_1 and Z_2 in X, defined in the two planes $y = \pm z$ by the equation $x^2 = y^3$. These cusps are parametrized by $t \to (t^3, t^2, \pm t^2)$; they meet at 0 with two limiting tangent lines in the direction of the vectors $(0, 1, \pm 1)$. Along the two cusps, at a point $a = (t^3, t^2, \pm t^2)$ different from 0, we may intersect the surface with a plane in $\mathbb{R}^3$ which is perpendicular to the respective cusp. Its equation is $t^3 x + t^2 y \pm t^2 z = 0$. The intersection curves turn out to be again cusps, say S_1 and S_2, with singularity at a. More precisely, the surface X is at its singular points locally analytically isomorphic to the Cartesian product of a smooth curve (the germ of Z_i at a) with the transversal cusp (the germ of S_i at a) – with the exception of the origin, where the geometry of X is much more involved.

Replacing x and z by $-x$ and $-z$ leaves the equation of X invariant, so that our surface is symmetric with respect to these reflections. Finally, $\mathbb{R}^*$ acts on X via $t \cdot (x, y, z) = (t^3 x, t^2 y, t^2 z)$, which shows that X is cone-like. All this makes it plausible to believe that the surface looks like the one depicted in Figure 1.

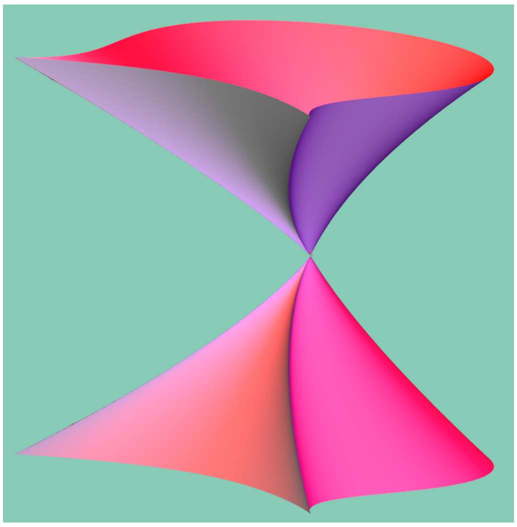

Figure 1: View of Daisy, the surface with equation $(x^2 - y^3)^2 = (z^2 - y^2)^3$.

Consider now the mapping π from $\mathbb{R}^3$ to $\mathbb{R}^3$ sending (x, y, z) to $(\psi^2 z^3,\ \psi y z^2,\ \psi z^2)$, where $\psi(x, y)$ is the polynomial $x(y^2 - 1) + y$. This map is an isomorphism from the complement of the four surfaces $y = \pm 1$, $z = 0$ and $\psi = 0$, onto the complement of Z_1 and Z_2. The inverse map is defined on $\mathbb{R}^3 \setminus (Z_1 \cup Z_2)$ by the triple of rational functions $((x^2 - yz^2)(y^2 - z^2)^{-1} z^{-1}, yz^{-1}, x^{-1} z^2)$. Therefore π is a birational

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morphism from $\mathbb{R}^3$ to $\mathbb{R}^3$. To compute the inverse image $X^* = \pi^{-1}(X)$ of X under π, substitute in the equation of X the variables x, y and z by the three components of π. This calculation gives

$$f(\psi^2z^3,\ \psi yz^2,\ \psi z^2) = (\psi^4z^6 - \psi^3y^3z^6)^2 - (\psi^2z^4 - \psi^2y^2z^4)^3 =$$
$$277x^2y^{10}z^{12} + 1240x^4y^8z^{12} - 482x^6y^{12}z^{12} + 322x^7y^7z^{12} - 56x^8y^{10}z^{12} + \cdots.$$

This is a polynomial of degree 36 in x, y and z, consisting of 67 monomials. Actually, it equals the product

$$f^*(x,y,z) = [(x-y)^2 + y^2 - 1] \cdot [y+1]^2 \cdot [y-1]^2 \cdot [x(y^2-1)+y]^6 \cdot z^{12}.$$

Therefore, X^* has five irreducible components, with multiplicities 1, 2, 2, 6 and 12, respectively. All components are smooth. The first component X' of equation $(x-y)^2 + y^2 = 1$ is a cylinder in the z-direction over an ellipse, to which the two planes $y = \pm 1$ are tangent. Aside from this tangency, all components meet pairwise transversally. The image of X' under π equals X, whereas the other four components contract to curves or points. The restriction of π to X' gives a parametrization $\pi : X' \to X$ of the singular surface X by the smooth surface X'. Such a morphism is called a resolution of the singularities of X.

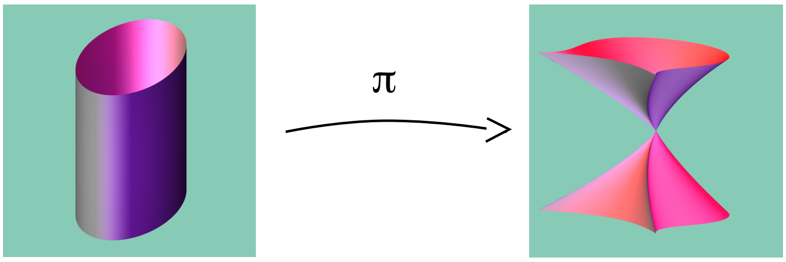

Figure 2: The parametrization $(x,y,z) \to (\psi^2z^3,\ \psi yz^2,\ \psi z^2)$ of Daisy ($\psi = x(y^2-1)+y$).

Adit. By a scheme we shall understand a separated scheme of finite type over an algebraically closed field k; you may think as well in terms of varieties; all ideal sheaves are assumed to be coherent. There are three main versions for the resolution of singularities, each stronger than the foregoing: Non-embedded resolution of reduced singular schemes, embedded resolution of reduced singular schemes, and principalization or monomialization of ideals. The present article (which is written for algebraic geometers and the interested non-expert) will explain the differences between these statements and develop the main ideas for proving them. Actually, all existing proofs resort to similar types of arguments and principles. It is our purpose here to make these common features explicit. This shall lead to an intuitive understanding of why the proofs do work in characteristic zero, and why and where they fail in positive characteristic. The exposition of these principles forms the main body of the article.

As a complement, it is then natural to look more closely at the two-dimensional case where we know that resolution also exists in prime characteristic. We will address the question of which extra arguments are necessary to make the reasoning go through in both cases. This is done by taking the characteristic zero induction invariant and to investigate its behaviour in positive characteristic. It turns out that the analysis is quite fruitful as it suggests a modification of the invariant which works in any characteristic (at least for surfaces of order equal to the characteristic). In the appendix, we collect for the convenience of the reader the definitions of some technical concepts used in the text.

I. Non-embedded Resolution of Singularities. *For any reduced scheme X there exists a smooth scheme $\widetilde{X}$ and a proper birational morphism $\pi_X : \widetilde{X} \to X$ which is an isomorphism over the smooth points of X.*

In addition, one may require that π_X is a sequence of blowups in smooth centers, and that π commutes with smooth base changes (thus preserving the symmetries of X).

In Figure 3, the successive centers of the blowups are (i) the most singular point in the center, (ii) the horizontal line at the intersection of the two "components", (iii) the horizontal "intersection" line forming the singular locus of the surface, and (iv) the two vertical singular lines of the surface. The two components of the singular locus of the initial surface are two plane cusps, which separate while transforming into two parabolas and a line meeting these transversally, then in three lines meeting at two points, then in a pair of two parallel lines.

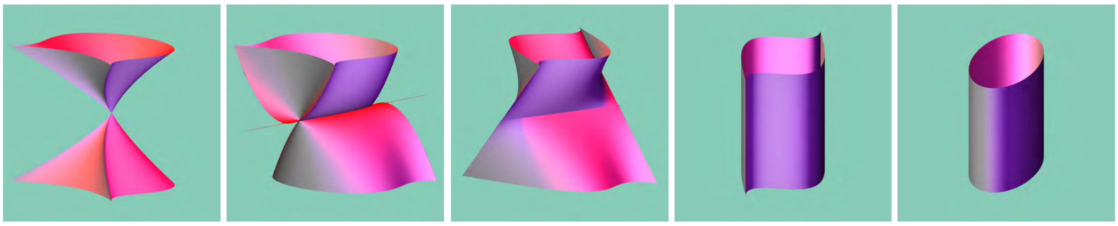

Figure 3: Resolution of Daisy by blowups in points and lines.

II. Embedded Resolution of Singularities. *For any reduced subscheme X of a smooth scheme W there exist a smooth scheme $\widetilde{W}$ and a proper birational morphism $\pi : \widetilde{W} \to W$ which is an isomorphism over the complement of the singular points of X so that the strict transform X^s of X is smooth and the total transform X^* has normal crossings.*

Again, we may want to achieve that π is a sequence of blowups in smooth centers and commutes with smooth base changes. Moreover, one may require that there exists (in an algorithmic way) for any pair $X \subset W$ an upper semicontinuous function $\mathrm{inv} : X \to \Gamma$ into a well ordered set Γ such that the centers of blowup are given as the locus of points of X where inv attains its maximal values. In general, such a function uses also part of the history of prior blowups and thus does not only depend on the scheme X. This dependence will be sketched in the section on transversality. As X^* is a normal crossings scheme and the strict transform X^s of X is smooth, it meets each component of the exceptional divisor transversally (in the sense that the scheme-theoretic intersection is smooth). Reading the sequence of blowups backwards, each step contracts one exceptional component to a point or a curve. This creates singularities on the surface along its intersection with the component which is contracted.

Figure 4 illustrates this contraction process. The exceptional components are depicted in different colours. The left side picture represents the surface one step before its embedded resolution is achieved (the two intersection lines of the surface with the red planes would still have to be blown up to get normal crossings for the total transform). In the second picture, the two red planes have been contracted to two vertical lines, the singular lines of the resulting surface. Next, the yellow plane is contracted to a line (the intersection of the green and blue plane). In the fourth picture, the green plane has been contracted to this line. It may be confusing that the inclined line of the singular locus of the surface transforms into a parabola. This is due to the chosen chart expression of the blowup. The last step contracts the blue plane to the most singular point.

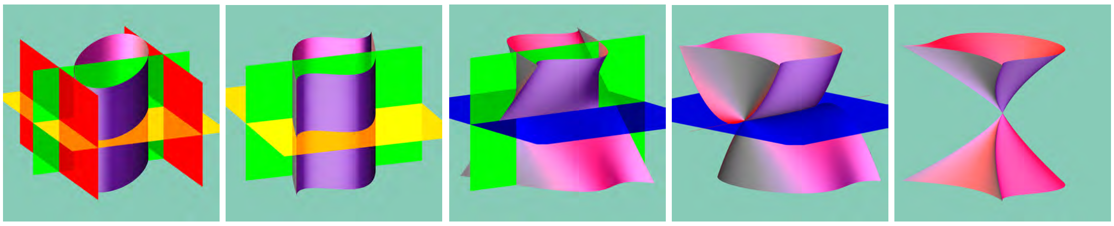

Figure 4: Successive contraction of a union of smooth surfaces.

III. Principalization of Ideals. *For any ideal sheaf J in a smooth ambient scheme W there exists a composition of blowups in smooth centers $\pi : \widetilde{W} \to W$ so that the total transform $J^* = \pi^{-1}(J)$ of J in $\widetilde{W}$ is a principal monomial ideal supported by the exceptional divisor.*

By a monomial ideal we understand an ideal which is, locally at each point, generated by monomials in local coordinates. The assertion of III is also called monomialization or log-resolution of J.

There are many further specifications which can be added to these results, see e.g. [EH]. Let us prove that III implies II and that II implies I. For the first, choose for J the ideal defining X in W. Then, by III, X^* is a normal crossings divisor in $\widetilde{W}$. Assume for simplicity that X is irreducible. As X^* is supported by the exceptional divisor, the strict transform of X in $\widetilde{W}$ is empty. This implies that at an earlier stage of the sequence of blowups, the strict transform of X was chosen as center, and was hence smooth. Stopping the blowups at that stage yields the required embedded resolution of X.

For II implies I it suffices to restrict the morphism $\pi : \widetilde{W} \to W$ to the strict transform X^s of X in $\widetilde{W}$. It is well known that X^s coincides with the blowup $\widetilde{X}$ of X in a (not necessarily smooth and reduced) center, hence the morphism $\pi_X = \pi_{|X^s}$ is a blowup map and thus proper and birational.

The principles of resolution treated in this text are: Choice of center; Upper semicontinuity; Resolution invariants; Chart expressions; Flags; Minima and maxima; Equiconstant points; Weak maximal contact; Cartesian induction; Transversality. Many important sites of the resolution landscape cannot be visited, due to space limitations. Among them are topics like alterations, normalization, Nash modification, polyhedral game, canonicity and functoriality, low dimensional techniques, arithmetic resolution, projections and elimination algebras, idealistic filtrations, algorithms and implementations. A selection of relevant articles is listed in the references.

Principle 1: Choice of center.

The first blowup of the surface Daisy (left two pictures in Figure 3) modifies the zeroset only in the origin. The projection map contracts the horizontal singular line of the second surface to the origin. Or, mutatis mutandis, the blowup replaces the most singular point of Daisy by a line. Outside the origin, the transformation is an isomorphism (even though it may not look like it). Alternatively, one could have tried to modify the surface directly at all points of its singular locus (the two cuspidal curves). This would yield a more complicated blowup map, but possibly a simpler surface. So there is some ambiguity on how to proceed for resolving a singular scheme. This section will discuss some of the possible options.

The blowup of a scheme is entirely determined by the selection of a closed subscheme, the center of the blowup. This is the locus of points where the scheme is actually modified and where the geometry will change when passing to the transform of the scheme. Outside the center, the scheme remains untouched.

It is a delicate matter how to choose the center of a blowup in order to simplify the singularities of the scheme. The canonical way is by means of a partition or stratification of the scheme, taking then the smallest stratum as center. We start with the simplest partition given by the singularities themselves. Any reduced scheme X is naturally partitioned by the regular or smooth locus $\mathrm{Reg}(X)$ consisting of its smooth points, and its singular locus $\mathrm{Sing}(X)$ of singular points.

$$X = \mathrm{Reg}(X) \mathbin{\dot\cup} \mathrm{Sing}(X).$$

The singular locus is a Zariski-closed subscheme and nowhere dense. Obviously, we do not want to touch the smooth points when we try to resolve, so we will select our centers of blowups inside $\mathrm{Sing}(X)$. On the other hand, each point of $\mathrm{Sing}(X)$ has to be modified in a resolution process in order to achieve a smooth final scheme. Therefore it has to belong at least once to a center. This suggests, in a first naive approach, to take the whole singular locus as the center of blowup. It is easy to see in simple examples that this does not yield in general a (non-embedded) resolution of X. Moreover, for the embedded situation, such a center may produce, when it is singular, singularities in the transform W' of the smooth ambient scheme W in which X is embedded. It is a challenge to understand the ambient singularities which arise from a singular center.[1]

In any case, our control of blowups with singular centers is too poor to give them any further consideration here. From now on, all centers will be reduced, closed and smooth in $\mathrm{Sing}(X)$, and moreover required to be transversal to possibly already existing exceptional components (blowups in centers transversal to normal crossings schemes preserve these). This restriction implies in the case where the singular locus of X is itself singular that the center is strictly contained in $\mathrm{Sing}(X)$, i.e., a proper closed subscheme of it. The goal is to find a good candidate for such a smooth subscheme. The natural idea is to stratify $\mathrm{Sing}(X)$ further, e.g. by taking the filtration obtained by the successive singular loci, starting with $\mathrm{Sing}(\mathrm{Sing}(X))$. By Noetherianity, one will arrive at a singular locus $Z = \mathrm{Sing}^k(X)$ which is smooth and non-empty (it may be reduced to a single point). Taking this subscheme as center will modify X only along Z, and the inverse image Y' of Z in the strict transform $X' = X^s$ of X will again be smooth (Y' is the exceptional divisor of the blowup of X with center Z). Two things can happen: Either $\mathrm{Sing}^k(X')$ has become empty, so that $\mathrm{Sing}^{k-1}(X')$ is already smooth. In this case we can apply induction on the length k of the filtration by singular loci. Or $\mathrm{Sing}^k(X')$ is still non-empty, and then the procedure can be repeated.

This construction is geometrically appealing and thus very plausible. It has just one major drawback: It is not clear that the process terminates, i.e., that X becomes eventually smooth. The termination would require some kind of measure which reflects the improvement of the singularities under each of the blowups and which would thus allow us to apply induction. It seems that no such measure has been discovered yet. One difficulty lies in the control of $\mathrm{Sing}^k(X)$ under blowup: Its pullback to X' need not equal $\mathrm{Sing}^k(X')$, and $\mathrm{Sing}^k(X')$ may even be singular, cf. the examples in [Ha5]. Due to these problems this approach has not led to concrete results. Instead, it has become standard to define the center in a more algebraically inspired way as the stratum of points where a certain local singularity invariant of the scheme attains its maximal value. These points are then, in this more algebraic perspective, the worst points of X, and they will be modified first. Before following this approach, we need some preparations.

Principle 2: Upper semicontinuity.

The Taylor expansion of the polynomial $f = (x^2 - y^3)^2 - (z^2 - y^2)^3$ at various points a will have varying order according to the location of the point (the order or multiplicity at a is understood to be the degree of

[1] By the existence of a resolution as in I, it is known that there must exist a generally non-reduced structure on $\mathrm{Sing}(X)$ so that taking this subscheme as center one gets in one stroke a smooth transform of X. Nobody seems to know what such a structure should look like.

the first non-vanishing term of the expansion). Outside the zeroset of the polynomial, the order will be 0, at smooth points it will be 1. Along the singular locus, the order is at least 2. Actually, in this example it is equal to 2 at all singular points with the exception of the origin (where it is 4). This suggests to declare the origin as the most singular point, and to consider schemes together with the stratification induced by local functions as is the order.

Let $\Gamma = (\Gamma, \leq)$ be a well-ordered set and let W be a reduced scheme. A function $i : W \to \Gamma$ is upper semicontinuous if for all $c \in \Gamma$ the set $\mathrm{top}(i, c) = \{a \in W,\ i(a) \geq c\}$ is closed. This implies that the value of i at a non closed point b of W is the minimum of its values at the closed points a in the closure of b. We set $\mathrm{top}^+(i, c) = \{a \in W,\ i(a) > c\}$. In general it is not immediate how to find explicit equations for $\mathrm{top}(i, c)$.

Any upper semicontinuous function i induces a filtration of W by the closed sets $\mathrm{top}(i, c)$. This filtration is finite because W is Noetherian. In particular, i assumes only finitely many values in Γ, and attains its maximum on the closed set $\mathrm{top}(i) = \mathrm{top}(i, c_0)$ with $c_0 = \max_{a \in W} i(a)$. The differences $S_c = \mathrm{top}(i, c) \setminus \mathrm{top}^+(i, c) = \{a \in W,\ i(a) = c\}$ define a finite partition of W into locally closed sets (which may be singular). Occasionally it is convenient to consider local top loci, taking into account only points of a suitable small neighborhood.

Instead of defining a stratification geometrically as it was done in the last section with the iterated singular loci it is more practical to define first an upper semicontinuous function and to consider then the associated stratification. This is the classical procedure in resolution matters to define the center of blowup. An upper semicontinuous function i is called a local invariant of an ideal J or a closed subscheme X of W if its value $i(a)$ only depends on the isomorphism class of the completions of the local rings $\mathcal{O}_{W,a}/J_a$, respectively $\mathcal{O}_{X,a} = \mathcal{O}_{W,a}/I_W(X)_a$. Here, $J_a \subset \mathcal{O}_{W,a}$ denotes the stalk of J at a, and $I_W(X)$ is the ideal sheaf defining X in W.

The easiest local invariant of an ideal sheaf J on W (or the closed subscheme X of W defined by J) is its order $o(a) = \mathrm{ord}_a(J) = \max\{k \in \mathbb{N},\ J_a \subset m_a^k\}$, with $m_a = m_{W,a}$ the maximal ideal of $\mathcal{O}_{W,a}$. It only depends on the isomorphism class of the completion of $\mathcal{O}_{W,a}/J_a$, and it is upper semicontinuous since $\mathrm{top}(o, c)$ equals the zeroset $V(\partial^\alpha f,\ f \in J_a,\ 0 \leq |\alpha| \leq c - 1)$ of all partial derivatives of elements of J_a up to order $c - 1$. The top locus $\mathrm{top}(o) = \mathrm{top}(J) = \mathrm{top}(X)$ is often called the equimultiple locus of J or X. Notice that the order depends of the embedding of X into the smooth ambient space W.

For plane curves X, the induced stratification is just the decomposition of X into the dense regular locus $\mathrm{Reg}(X)$ and the singular locus $\mathrm{Sing}(X) = \mathrm{top}(o, 2)$. More generally, for hypersurfaces X, we have $\mathrm{Sing}(X) = \mathrm{top}(o, 2)$. For non-hypersurfaces, the order seems to be of little significance, in particular if X is not minimally embedded in W and lives in a smooth hypersurface V inside W. In this case, the equation of V has order 1 along X and belongs to the ideal J of X. Therefore J has order 1 along X, and the order is not even able to detect the singular locus of X.

This flaw of the order of ideals became relevant when, at the end of the fifties, people, and, more specifically, Hironaka, started to consider resolution problems for non-hypersurfaces. The first attempt to overcome the deficiency of the order for non-principal ideals was to consider the sequence of orders of a selected generator system of the ideal (called standard basis by Hironaka, nowadays known as Macaulay basis). Take a minimal generator system of the (stalk of the) ideal whose initial forms (homogeneous forms of lowest degree) generate the initial ideal. The orders of these generators, taken in an increasing manner, constitute then an upper semicontinuous invariant when considered with respect to the lexicographic ordering on the set $\mathbb{N}^{(\mathbb{N})}$ of finite sequences. This vector, called ν by Hironaka, had then to be considered under blowups in permissible centers (i.e., centers, along which the invariant takes a constant value). It does not increase if the center is permissible.

It was soon realized that the Hilbert-Samuel function of the ideal represents a more conceptual local invariant. Bennett established its upper semicontinuity [Bn]. He also showed that it does not increase under permissible blowups. The stratification underlying this invariant became known as normal flatness, and was dominant in the field until the nineties. Only then was it observed that also the order of ideals alone suffices for the induction to ensure the termination of the resolution process (but considering the so called weak transform instead of the strict transform of X, and taking an entire string of orders of different ideals as the measure of improvement).

The proof of the non-increase of the Hilbert-Samuel function under permissible blowups is quite involved. A much shorter proof for an almost equivalent statement can be given when replacing the Hilbert-Samuel function by the initial ideal of the ideal J with respect to a degree compatible monomial order on the set of exponents [Ha1]. The monomial order specifies for each element of the stalk of J, considered either in the local ring or in its completion, a (smallest) initial monomial (in contrast to the initial form from above), and the initial ideal is the ideal generated by all these monomials. It can be made coordinate independent when taking its minimal or maximal value (with respect to the natural ordering on monomial ideals) over all coordinate choices. This ideal, the minimal or maximal initial ideal, is then a local invariant which is very easy to control under blowup. It contains essentially the same information as the Hilbert-Samuel function. However, it too is not sufficient to prove termination of the resolution algorithm. Further invariants have to be constructed.

All proofs known nowadays for the resolution of singularities in arbitrary dimension over a field of characteristic zero therefore use a whole string of local invariants, considered lexicographically. The first entry is the order of the ideal J (or, as in [BM1], the Hilbert-Samuel function), the next entries are (usually) the orders of suitable coefficient ideals. These are ideals in less variables, living in successive local hypersurfaces of maximal contact. The ideals depend on the choice of the hypersurfaces, but their orders do not, and are moreover upper semicontinuous, since they are orders of ideals. Therefore the string of orders is a local invariant. (In reality, the construction of this string is more involved, since the descent in dimension to coefficient ideals is technically rather subtle).

In the recent approaches to resolution in positive characteristic, the upper semicontinuity of the proposed invariants is more delicate. In particular, some of the invariants are only upper semicontinuous when restricting to closed points. This is unpleasant but can be handled by redefining the value at non-closed points as the minimum of the values at the closed points in the Zariski-closure [Hi1].

The upper semicontinuity of the invariants is substantial because the centers of blowup are chosen as their top locus and must be *closed*. Moreover, the invariant being defined locally, it has to be ensured that the local constructions patch to give a global center. This is automatic if the local invariant does not depend on any choices (i.e., is an invariant in the sense defined above). Large parts of recent works on resolution are devoted to the proofs of these two properties [Ka, KM, BeV, BrV1].

Principle 3: Chart expressions.

The formula $(x, y) \to (x, xy)$ is known as the prototype of a blowup map. And indeed, a very favorable feature of blowups is their explicit expression in affine charts. This allows to work with defining equations of affine schemes in local cooordinates and to compute the transform of the ideals by substitutions of the variables. Besides, these substitutions are of a very simple kind, namely given by quadratic monomials (for smooth centers).

To be precise, let W be a smooth ambient scheme, $Z \subset W$ a smooth closed subscheme taken as the center of the induced blowup $\pi : W' \to W$, with exceptional divisor $Y' = \pi^{-1}(Z)$. Let a be a closed point of Z, and let $a' \in Y'$ be a closed point above a, say $\pi(a') = a$. There then exist local coordinates $x = (x_1, \ldots, x_n)$ in W at a (i.e., a regular system of parameters of the local ring $\mathcal{O}_{W,a}$ as defined in the appendix) such that the following holds. (i) The point a has components $(0, \ldots, 0)$ with respect to x. (ii) The center Z is defined locally at a by $x_1 = \ldots = x_k = 0$, where k is the codimension of Z in W at a. (iii) The substitution formula $x_i \to x_i x_k$ for $i \leq k-1$ and $x_i \to x_i$ for $i \geq k$ induces local coordinates for W' at a' (say, in the so called x_k-chart of W'). (iv) In particular, a' has components $(0, \ldots, 0)$ with respect to the induced coordinates. (v) The exceptional divisor Y' is given in this chart by $x_k = 0$.

In this situation we say that the local blowup $\pi : (W', a') \to (W, a)$ (defined as a map germ) is expressed in the affine x_k-chart with respect to the coordinates x in W at a. Notice that the chart expression $x_i \to x_i x_k$, respectively $x_i \to x_i$, of the blowup map interprets the map as a (local) morphism of $\mathbb{A}^n$ onto itself with the same system of coordinates in source and target. We then say that the local blowup is *monomial* in the chosen coordinates, and that a' is the origin of the x_k-chart of the blowup.

With this explicit description of blowups it is easy to compute locally transforms of ideals and of subschemes. It suffices to substitute in the involved equations the coordinates by the formulas from above. But this computation is local, and often one wants to know the transforms simultaneously at all points a' of Y' above a. It is then appropriate to partition the fibre $Y'_a = \pi^{-1}(a)$ into locally closed subsets. As W and Z are assumed to be smooth, Y'_a is isomorphic to projective space $\mathbb{P}^{k-1}$, which decomposes naturally into affine spaces $\mathbb{A}^l$, for l varying between 0 and $k-1$ (there is some freedom how to distribute these affine pieces over Y'_a). Let be given coordinates $x_1, \ldots, x_n$ at a so that Z is defined by $x_1 = \ldots = x_k = 0$ as above. Let a' be any point in Y' above a (not necessarily the origin of an induced chart). We wish to determine a coordinate change at a which makes the local blowup $\pi : (W', a') \to (W, a)$ at a' monomial in the transformed coordinates. If a' is the origin of the x_k-chart of the blowup, the local blowup is already monomial and no change is necessary. Otherwise, it lies in the union of the remaining affine pieces of Y'_a. In the x_{k-1}-chart, we consider the affine line $x_1 = \ldots = x_{k-2} = x_k = \ldots = x_n = 0$. Its point at infinity is the origin of the x_k-chart. A linear triangular change $x_i \to x_{k-1} + t_{k-1} x_k$ in W at a with a suitable constant t_{k-1} then makes the local blowup monomial. In the l-th chart, for $l \leq k$, the coordinate change is of form $x_i \to x_i + t_{l,i} x_k$ for i between l and $k-1$ and suitable $t_{l,i}$ in the ground field.

These formulas are useful because the local invariants are mostly given by the Taylor expansions of elements of the local rings at a and a'. They allow us to check whether the invariant decreased or remained constant (it should not go up).

Often, the invariants are defined first in a coordinate dependent manner (e.g., as the order of a coefficient ideal). Either one is able to ensure by additional arguments their actual independence of the coordinates (one such argument is known as Hironaka's trick via test-blowups), or one has to choose the minimal, respectively maximal value over all coordinate choices. This makes the invariant automatically coordinate independent. This procedure is especially effective in positive characteristic, where hypersurfaces of maximal contact no longer exist. Instances of maximizing coordinate choices can be found in [Ab3, Hi3, Ha2, Ha3, Ko1].

Principle 4: Flags.

Triangular coordinate changes $(x, y) \to (x + ty, y)$ in the plane, with t a constant in the ground field, allow to render the local chart expression of a point blowup monomial. The geometric device behind this trick is the introduction of local flags.

The choice of hypersurfaces of maximal contact in characteristic zero and the subsequent descent in dimension to a coefficient ideal break the local symmetry of the ambient space W at a given point a (this is the symmetry of affine space $\mathbb{A}^n$ at a given point). Iterating this descent corresponds essentially to ordering the local coordinates lexicographically. The ordering can be made intrinsic by the consideration of local flags. They serve to restrict substantially the allowable choices of coordinates while still keeping sufficient flexibility in defining the local invariants. Moreover they are compatible with blowups.

Let a be a closed point of the smooth ambient space W. A local flag in W at a is a chain $\mathcal{F} : W_0 = \{a\} \subset W_1 \subset \ldots \subset W_{n-1} \subset W_n = W$ of local smooth subschemes W_i of dimension i (i.e., the W_i are considered as germs at a). When varying the reference point a, certain semicontinuity conditions (to be specified below) can be required. A smooth subscheme Z of W is called transversal to $\mathcal{F}$ if all scheme-theoretic intersections $Z \cap W_i$ are smooth. In this case, the blowup $\pi : W' \to W$ of W with center Z admits at each point $a' \in Y'$ above a a local flag $\mathcal{F}'$ associated naturally to $\mathcal{F}$. Assume first for simplicity that $Z = \{a\}$ is a point. Then W_i' is set equal to the strict transform W_i^s if a' belongs to W_i^s. Otherwise, it is defined as the unique linear i-dimensional subspace of $Y' \cong \mathbb{P}^{k-1}$ containing a' and the intersection with Y' of the pullback of the tangent space $\mathrm{T}_a W_i$ of W_i at a.

If the center is positive dimensional, say of dimension k, the blowup map is locally above a equal to the Cartesian product of the point blowup of a k-codimensional, Z-transversal section V of W through a, together with the identity map on the scheme Z, and the definition of the induced flag $\mathcal{F}'$ is built on this Cartesian product. The details are left to the reader, or can be found in [Ha1]. See also the work of Panazzolo, where local flags, called axes, are studied for the resolution of vector fields in three dimensional space [Pa]. They are considered in families.

In dimension 2, a flag is just a smooth plane curve C passing through a. Its strict transform C^s under the point blowup with center a hits the exceptional divisor Y' in precisely one point. There, the induced flag is C^s. At all other points of Y', the induced flag is equal to Y'. This already reflects what should be meant by the semicontinuity of the flag on W': the local flags defined at each point of Y' vary continuously along the strata of a suitable stratification of Y' by locally closed sets.

In dimension 3, a flag consists of a smooth curve C contained in a smooth surface S in three-space. The exceptional divisor Y' of the point blowup in $\mathbb{A}^3$ is isomorphic to $\mathbb{P}^2$, and the induced flag is defined as follows: At the intersection point p of the strict transform C^s with Y', the induced flag is $C^s \subset S^s$. Outside this point, but still inside the intersection curve $Y' \cap S^s$, the flag at a' is $C' \subset S^s$, where C' is the line in $\mathbb{P}^2$ connecting p with a'. Finally, if $a' \in Y'$ does not lie in S^s, the flag consists of the line connecting p with a' and the plane containing a' and the intersection of Y' with the pullback of the tangent space $\mathrm{T}_a S$.

Coordinates x at a in W are called subordinate to a given flag $\mathcal{F}$ if the i-th component W_i is defined by $x_{i+1} = \ldots = x_n = 0$. Two systems of subordinate coordinates differ hence by a triangular automorphism. Moreover, if the center Z is transversal to the flag, and a' is a given point in Y' above a, the coordinates can be chosen so that the local blowup is monomial and that the induced coordinates at a' are again subordinate to the induced flag [Ha1]. This suggests to define local invariants as maxima over subordinate coordinates; they depend only on the flag and are then also defined after blowup. Finally, the coordinate description of the blowup map makes it possible to determine their transformation behaviour.

Principle 5: Resolution invariants.

The surface defined by $f = x^2 + y^{11} + y^3 z^4 + z^5$ has at the origin a point of order 2. Blowing up this point and looking at the origin of the y-chart, one quickly observes that the order is again 2 (the strict transform being $f' = x^2 + y^9 + y^5 z^4 + y^3 z^5$.) So, either the center was too small and a curve should have been taken as the center, Or the origin was the correct choice but the improvement is not detected by the order of f. A more refined measure is necessary. The natural candidate is the order of $g = y^{11} + y^3 z^4 + z^5$ at the origin. It is 5. Helas, the order of g increases from 5 to 8, because the transform $g' = y^3(y^6 + y^2 z^4 + z^5)$ of g accrues additional exceptional factors. Only after having deleted the factor y^3 from g', the resulting order yields a reliable measure. It does not increase, which is good, but as it equals again 5, we are stuck with our induction.

Resolution invariants are designed to overcome this difficulty. As the complete resolution is often composed by a long sequence of blowups, the improvement in each step may be small. Accordingly, a relatively fine measure has to be invented to detect and exhibit this improvement. Usually this is a string of numbers, considered with respect to the lexicographical ordering. Each entry is indexed by a certain ambient dimension, which is prescribed by the respective hypersurface of maximal contact. In all known proofs (for resolution in arbitrary dimension), the invariant defining the center (call it the stratifying invariant) is the same as the invariant measuring the improvement (call it the resolution invariant). This may be a coincidence, but is certainly not the most economic choice, for the following reason.

First, experimenting with blowups and the transform of ideals and schemes, it quickly becomes clear that the larger the center is the stronger is the improvement. In fact, if the center is strictly included in the locus where the resolution invariant attains its maximum, the invariant cannot drop under blowup for semicontinuity reasons: At a point of the top locus outside the center, the value of the invariant remains the same, since the blowup map is an isomorphism over the complement of the center, and since the invariant only depends on the isomorphism type of the (completed) local ring. The closure of the difference of the top locus and the center is the entire top locus. On the transformed ambient space, at points in the closure of the inverse image of this difference, the invariant will have, by the upper semicontinuity, the same value as at points inside the difference. Therefore it will have remained constant there.

This argument shows that the top locus of the resolution invariant must be contained in the center of the blowup (which is the top locus of the stratifying invariant). But, at least in principle, it could be strictly contained. We conclude that the resolution invariant should refine (or be equal to) the stratifying invariant.

The non-increase of the order of an ideal under blowup in a center along which the ideal has constant order is one of the basic facts used permanently in resolution articles. This holds for both the strict and the weak transform, but in general not for the total or controlled transforms (see the appendix for these notions). The overall strategy of resolution is to associate to the given ideal locally at each point an ideal in one variable less, the coefficient ideal. This descent in ambient dimension commutes with blowup (in characteristic zero) as long as the order of the original ideal does not drop, taking as the transform of the coefficient ideal the controlled transform. Resolving in lower dimension produces after a sufficiently long sequence of blowups a coefficient ideal which will be a principal monomial ideal supported on the exceptional divisor (on the transform of the hypersurface), but that will have, in general, huge order. At that stage, the strategy is changed. The original ideal is then resolved by a purely combinatorial method, taking into account the monomiality of the coefficient ideal. Still, the resolution of the coefficient ideal will use the order as the main resolution invariant, but as the ideal passes under blowup to the controlled transform, it will be necessary to factor from this ideal as many exceptional components as possible to get as the remaining factor the weak transform (whose order will not have increased by the permissibility of the center).

We will see later on how the non-increase of the order of an ideal under blowup (and taking the weak transform) paves the way to the descent in dimension and the Cartesian induction. Indeed, it suffices to consider on the exceptional divisor the points where the order has remained constant (called equiconstant points), and to establish a local descent to coefficient ideals only at these points. Here, the commutativity of the descent with taking the controlled transform is crucial. It fails in positive characteristic, which represents the main obstacle there.

There is a recent new approach to induction by Bravo and Villamayor [BrV1], taking up the classical method of Jung: Instead of restricting to hypersurfaces, the idea is to project onto hypersurfaces. One thus obtains a resolution problem in one dimension less which is of different type than coefficient ideals. The approach works marvelously well, both the upper semicontinuity of the invariant and a certain commutativity of the descent with blowups can be ensured. Again, by induction on the dimension, one can resolve the lower dimensional problem. The still unsolved difficulty here is how to exploit this subordinate resolution for the construction of a resolution of the original scheme. This is again a kind of a combinatorial problem, but reversing the descent in dimension seems to be much more challenging [BrV1].

Principle 6: Minima and maxima.

For the polynomial $f = x^2 + 2xyz + y^5 + y^2z^2 + z^5$ the order of the coefficient ideal[2] in the hypersurface $x = 0$ is 4. This number is not significant, since it depends on the choice of coordinates. Applying the coordinate change $x \to x - yz$ yields the expansion $f = x^2 + y^5 + z^5$ with coefficient ideal of order 5. It turns out that the change has maximized the order over all coordinate choices. It is hence now intrinsic.

It is a classical feature of resolution of singularities that many types of invariants are constructed by means of special choices of coordinates. Given a polynomial f (or a polynomial ideal J) in n variables and a point a in $\mathbb{A}^n$ one wishes to measure the complexity of the singularity of the zeroset X defined by f or J at a. The main such measure is the order of f at a, say, the local multiplicity. It gives some but not enough information. For non-principal ideals, one may extend the order by the increasing sequence ν of orders of a standard (= Macaulay) basis of J at a as suggested by Hironaka, the Hilbert-Samuel function of J at a, or the generic initial ideal of J at a with respect to a degree compatible monomial order on $\mathbb{N}^n$ [Hi4, Bn, Ha1]. All these are by definition coordinate independent (and actually only depend on the completion of the local ring of $\mathbb{A}^n$ modulo f or J). But they are not fine enough to exhibit the improvement of the singularity under blowup. At some points of the exceptional divisor, they may assume the same value as below. This suggests to consider there a secondary invariant which complements and refines the local multiplicity.

Already for plane curves, it is convenient (though not absolutely necessary) to define the secondary invariant through the choice of local coordinates. Usually one takes the slope of the first segment of the Newton polygon of the local defining equation f (the Newton polygon is the positive convex hull $\mathrm{conv}(A + \mathbb{R}^2_\geq)$ of the set A of exponents of the Taylor expansion of f in given coordinates at the point of reference). This slope depends on the choice of coordinates. It is made coordinate independent by taking its maximal value over all choices of local coordinates (assuming that f is already in Weierstrass form). If, for some coordinates, there is no first segment in the Newton polygon, f is already a monomial up to units in the local ring. This case is considered as being resolved and can thus be discarded. If the maximal slope exists and is finite, one may choose local coordinates x at a realizing this value. Then look at the transform of f under blowup (with center a point where the order of f is maximal). On the exceptional divisor, choose any point a' where the order of f has not changed (if there is no such point, we are done by induction on the order). We wish to show that the slope has decreased. This would imply that the pair (order, slope) has decreased with respect to the lexicographic ordering at *any* point of the exceptional divisor. As it belongs to a well-ordered set, induction applies.

To compute the slope, first compute the Taylor expansion of the transform f' of f at a' using the chart expressions for the blowup. In the induced coordinates x' at a', the slope may not attain yet its maximal value. It may be necessary to apply a local (formal) coordinate change to maximize it. One then shows – and this is the clue – that the change at a' stems from a local coordinate change at a, i.e., that the naturally induced diagram of blowups and local automorphisms commutes. In principle, the change at a could destroy

[2] The coefficient ideal is defined here ad hoc as $(yz)^2 + (y^5 + y^2z^2 + z^5)$, cf. the appendix.

the maximality of the slope there (it can be shown that it does not), but this would not do any harm because of the chain of inequalities

$$\max_{x'} \text{slope}_{a',x'}(f') = \text{slope}_{a',x'}(f') < \text{slope}_{a,x}(f) \leq \max_x \text{slope}_{a,x}(f).$$

Here, the first equality holds by the choice of the coordinates x' at a', the strict inequality is checked by a direct computation in the chart expression of the blowup (which may be assumed to be given by a monomial substitution of the variables), and the last inequality holds by definition of the maximum. This is, in short, the inductive argument for the resolution of plane curves.

Of course, in this reasoning it is very helpful to allow at a' only a small class of coordinate changes (e.g., those subordinate to a flag), because this improves the chances to have them commute with the chart expression of the blowup (a fact which is crucial for making explicit computations).

For surfaces and higher dimensional schemes, the pair (order, slope) is not sufficient for applying induction. One usually constructs a whole string of numbers, considered with respect to the lexicographic ordering. The components are often defined through choices of hypersurfaces or coordinates.

When a component of the string is defined as a maximum over some coordinates, this maximum has to be realized *after* the blowup at all points a' where the earlier components of the invariant have remained constant. The opposite happens when taking the minimum over all coordinate changes. For reasons analogous to those suggested by the chain of inequalities above, it has to be realized *before* the blowup at the points a. By the way, the generic initial ideal mentioned earlier can be interpreted as a minimum [Ha1].

In both cases it is important that the chosen measure extracts substantial information on the local complexity of the polynomial at a and a'. For instance, the minimum of the slope of the first segment of the Newton polygon is useless since it is always -1. As a general rule, one tries to make by coordinate changes the Newton polyhedron (i.e., the three-dimensional analogon of the polygon) as small as possible, cf. [Hi3, Ha2, Yo].

Principle 7: Equiconstant points.

The polynomial $f = x^2 + y^9 + y^3z^3 + z^9$ has order 2 at 0 and transforms under the blowup of the origin into $f' = x^2 + y^7 + y^4z^3 + y^7z^9$. This is the chart expression of the strict transform of f at the origin of the y-chart. It has again order 2 there. The next blowup with center this point gives $f'' = x^2+y^5+y^5z^3+y^{14}z^9$, again of order 2 at the origin of the y-chart. In this way the origins of the successive y-charts represent points in a sequence of blowups where the order did not decrease. These points will be of particular interest.

Take now more generally a hypersurface X in $W = \mathbb{A}^n$ defined by an equation $f = 0$, for some polynomial f in local coordinates. For any point a in W, the basic measure of the complexity of the singularity of X at a is the order of the Taylor expansion of f at a. It equals the maximal power k for which f is contained in m_a^k with m_a the maximal ideal of the local ring $\mathcal{O}_{W,a}$. Equivalently, it is the minimal order $|\alpha|$ of a partial derivative $\partial^\alpha f$ of f which does not vanish at a (in characteristic p, one has to take the Hasse-Schmidt derivative).

We have already seen that the top locus $\text{top}(X)$ of X in W of points where the local order of f is maximal is closed. Let Z be a smooth closed subscheme of $\text{top}(X)$ (i.e., permissible), and let $\pi : W' \to W$ be the induced blowup with center Z and exceptional divisor Y'. The order of X is constant along Z, say equal to a value o. Using the chart description of blowups it is quite immediate to show that at any point a' of Y' the order of the strict transform $X' = X^s$ of X cannot increase,

$$\text{ord}_{a'}(X') \leq \text{ord}_a(X).$$

Moreover, there exists, locally at any point a of Z, a smooth hypersurface V in W whose transform $V' = V^s$ contains all points a' of Y' where the order has remained constant. This fact, first observed by Zariski [Za1], holds in any characteristic. These points a' will be called equiconstant for X, they are also known as infinitely near points. If V is chosen transversal to the center (in characteristic zero, V can even be chosen to contain $\mathrm{top}(X)$ and hence also Z), the transform V' is again smooth. Unfortunately, it need not be suitable for a second blowup $W'' \to W'$ in the sense that its transform V'' may no longer contain all points a'' in Y'' where the order has remained constant. One may have to choose a new smooth $\widetilde{V}'$ at any a' to ensure this containment. This new choice, however, is only necessary in positive characteristic (and there are examples proving that it cannot be avoided). In characteristic zero, by a miraculous coincidence, one may already choose locally at a a smooth hypersurface V such that all its transforms under an arbitrary sequence of permissible blowups contain all equiconstant points above a.

The choice of such a hypersurface is based on so called Tschirnhausen transformations, which boil down to choose for the defining equation of V a partial derivative of f which has order 1 at a (and hence defines a smooth V).

Therefore, in characteristic zero, we have at our disposal along any sequence of equiconstant points in a composition of permissible blowups of a sequence of local smooth hypersurfaces which accompany the points. This immediately suggests to descent to these hypersurfaces, formulate there a resolution problem (in one dimension less) which, by induction on the dimension, must improve in each step (provided the centers of blowup are chosen appropriately). This descent is feasable, but there are several technical complications involved we wish to skip.

In positive characteristic, the situation is much more interesting. Again we are led to consider a sequence of equiconstant points above a. The final goal is to show that any such sequence is finite, i.e., that eventually the order of f drops. (By the upper semicontinuity, the lengths of the various sequences of equiconstant points will be bounded.) The method to show this consists in trying to introduce a second local upper semicontinuous invariant which drops after each blowup at an equiconstant point. When it reaches its minimal possible value there cannot be any more equiconstant point above, hence the order of f must drop.

In characteristic zero, this secondary invariant is constructed by induction on the dimension. We pass locally to a hypersurface of maximal contact and consider the order of the coefficient ideal of f there, show that it does not increase under blowup (when passing to the weak transform), and that the descent commutes with the blowup of f at any equiconstant point (here, maximal contact is used in an essential way). Now, by induction on the dimension, we know that the order of the coefficient ideal (with the exceptional factors deleted) must eventually drop. But it cannot drop infinitely often, at some stage it becomes zero. There, the coefficient ideal consists only of exceptional factors, and is hence a principal monomial ideal. As mentioned above, this situation is so special that one can then define a direct (combinatorially constructed) resolution of f.

In positive characteristic, this approach fails, because even though a secondary invariant can be defined similarly, it may go up and down in quite an uncontrolled way. Let us describe how such an invariant could be constructed and where the complication arises.

Principle 8: Weak maximal contact.

For the polynomial $f = x^2 + y^5 + y^2z^2 + z^5$ the hypersurface $x = 0$ maximizes the order of the coefficient ideal, with the exception of the characteristic 2 case, where we have to take instead the hypersurface $x + yz = 0$. This shows that the choice of the hypersurface is sensitive to the characteristic. However, the maximum of the orders of the various coefficient ideals in different hypersurfaces will not depend on any choice. We thus obtain an intrinsic secondary measure (after the order of the defining polynomial) for the complexity of a singularity.

Fix a sequence of equiconstant points above a. Our goal is to prove that it is necessarily finite, independently of the characteristic. Again, the only conceivable way to show this is to exhibit a repeated improvement of some secondary invariant along the sequence. But this invariant has to be more involved than before, since coordinate changes as above make polynomials quite intractable in prime characteristic. The first thing which comes to our mind is to consider the coefficient ideal of f in $some$ local smooth hypersurface at a and to consider its order at a. It will depend on the choice of V, and is hence neither intrinsic nor necessarily significant. After all we wish to measure the complexity of the singularity of f, and this should not depend on any choices. The resolution of plane curves (which works in any characteristic) already encounters this problem, and handles it by taking, as mentioned above, the maximal slope, which can be reinterpreted as the maximum of the order of the coefficient ideals in local hypersurfaces [Ab3, Gi1, Gi2, Co1, Co2, HR].

Hypersurfaces V which realize the maximal order of the coeffient ideal of f at a are called hypersurfaces of weak maximal contact [EH]. They were first considered by Abhyankar [Ab3, Cu2] and have seen a renaissance in the recent approaches to resolution in positive characteristic [BrV1, Hi1, Hi2]. Their strict transform V^s under blowup will contain all equiconstant points a' above a, but need not have again weak maximal contact with the weak transform f' of f. One may have to choose a new local hypersurface V' at any equiconstant point a'. This would, a priori, not be a drawback, if the order of the coefficient ideal behaved nicely. But it does not: Moh exhibited examples where the maximal order of the coefficient ideal increases under blowup. (The order of f must be a power of the characteristic for this to happen.) This destroys the induction. However, Moh was able to bound the increase of the order (it is at most p^{e-1} if the order of f at a is equal to p^e where p is the characteristic). For surfaces of order exactly p, Hauser and Wagner were able to show that this occasional increase is made up by stronger decreases of the order of the coefficient ideal before and after the critical blowup [HW]. It thus decreases in the long run, which reestablishes the induction. For three-folds, the problem of resolution in positive characteristic is still open (at least concerning embedded resolution and principalization of ideals).

Principle 9: Cartesian induction.

The polynomial $f = x^3+y^7+y^2z^2+z^5$ has as coefficient ideal in the hypersurface $x = 0$ the ideal generated by $g = y^7 + y^2z^2 + z^5$. The transform of f under the blowup of the origin is $f' = x^3 + y^4 + yz^2 + y^2z^5$ (at the origin of the y-chart), with coefficient ideal generated by $g' = y^4 + yz^2 + y^2z^5$. This polynomial can be directly computed from g. Take the total transform $g^* = y^7 + y^4z^2 + y^5z^5$ and factor from it the exceptional monomial y^3. This transform is the controlled transform of g with respect to the control $c = 3$ (cf. the appendix.) We conclude in this example that passing to coefficient ideals commutes with taking certain transforms under blowup. We shall see next that this commutativity is a general feature of coefficient ideals.

The method of Cartesian induction has much to do with calling a subroutine in a computer program. Whenever the resolution process runs into a problem – here, the non-decrease of the invariant, or the non-transversality of the center with the exceptional locus (see the next section) – it formulates this obstruction/failure as a new resolution problem of $smaller\ size$. By induction on the size (which has to be defined properly and which is almost always the dimension of the smooth ambient scheme) we can assume to know how to resolve this smaller problem. For this to work in general, a comprehensive notion of resolution has to be defined; part of it is the embeddedness, i.e., the normal crossings requirement for the total transform.

Other properties, such as the resolution morphism being a composition of blowups in smooth centers, can be appropriate.

So assume that we are able to solve (or improve) the subproblem by a sequence of blowups, referring e.g. to induction on the dimension. These blowups usually have centers in a smooth subscheme V of the ambient scheme W we started with. So the sequence of blowups of V also induces a sequence of blowups of W, and hence transforms our initial resolution problem. Denote by V' the scheme above V where the subproblem is solved (respectively, has improved), and by W' the respective transform of our ambient scheme.

Now, at this stage, we look again at the obstruction which forced us at the beginning to formulate and solve a subproblem. It will have transformed with the blowups. Two things can happen: Either the obstruction has disappeared and we can proceed with the next step, or it has remained. In the latter case we have to check that, at least, it improved in some sense. The only practical way to do this is to compare it with the transform of the original obstruction, say with the transform of the subproblem. This transform, as we know, is resolved or has improved.

All known formulations of such subproblems aim at establishing a *commutativity* of the passage to the subroutine with blowups: The transform of the obstruction for the original problem should equal the obstruction of the transfom of the original problem.

If this commutativity holds, we can be sure, by induction on the size of the resolution subproblem expressing the obstruction, that our obstruction has improved during the auxiliary blowups. It then suffices to iterate until it has disappeared (obviously, for this to work, the measure of the intricacy of the subproblem has to lie in a well ordered set).

We leave it to the reader to visualize this principle of induction by a Cartesian diagram of blowups and descents in dimension (see [Ha4] for more details.)

Principle 10: Transversality.

Assume that in a sequence of blowups there appears at a certain stage a pinch point singularity of equation $x^2 - y^2z = 0$. Its singular locus is the z-axis. Blowing up the origin reproduces in the z-chart the identical singularity. So we are led to take as center the entire axis. But it may happen that this axis is tangent to an already existing exceptional hypersurface, e.g. the one given by $x + yz = 0$. Then the blowup will destroy the smoothness of this hypersurface, thus violating the requirements of an embedded resolution. It is therefore necessary to treat the non-transversality of the axis with the exceptional divisor first. This is usually done by separating through auxiliary blowups the center of preference from the exceptional component.

The transversality issue is often omitted in presentations about resolution by declaring it as part of the more technical machinery. Nevertheless it is a very substantial ingredient which caused many complications and controversies in the field. Its treatment is often related to taking into account the history of the resolution process.

Each blowup in a smooth center Z inside the ambient scheme W produces an exceptional hypersurface $Y' = \pi^{-1}(Z)$ in the new ambient scheme W'. The iteration of blowups adds several such components. For many applications, but also for the inductive proof, it is important to know that these components meet transversally, i.e., form a normal crossings divisor. The only reasonable way to ensure this is to choose each center transversal to the already existing collection of exceptional components (in the sense that the union is a normal crossings scheme). The centers are usually given as the top locus of some local upper semicontinuous invariant on the singular scheme we wish to resolve. It thus does not take into account a priori the position of the exceptional components. The possible non-transversality may result to be fatal.

There are two ways to confront this problem. The first consists in adding to the stratifying invariant the configuration of the exceptional components. This is possible but causes the aforementioned technical complication. Some exceptional components are automatically transversal (the so called new ones) and need not be considered for the modification of the invariant. The old components are dangerous and have to be taken into account. Hence, some book-keeping is necessary (the relevant information one has to keep track of is assembled in the notion of mobile, cf. [EH]). This is often referred to as remembering the history of the resolution process.

The other option is to formulate a subordinate resolution problem whenever the center prescribed by the stratifying invariant (let us call it the virtual center) is not transversal to the exceptional locus. By construction of the invariant, this locus should be smooth, but cannot be taken yet as the actual center of blowup. Using induction on the dimension it is then possible, by the assumed existence of embedded resolution in smaller dimension, to separate, by auxiliary blowups, the virtual center from all exceptional components to which it is not transversal. It is easier notationally to achieve separation rather than transversality.[3] While doing so it has to be ensured that the original resolution problem does not get worse. But as the auxiliary centers are smaller than the virtual center (along which the invariant is constant), the invariant remains constant by its upper semicontinuity (it does not change outside the center, and remains the same at points of the closure of a locally closed stratum along which it is constant, by the same argument as earlier).

Once the critical components are separated from (the transform of) the virtual center, this subscheme can be taken as the $actual$ center of the next blowup. The resolution invariant should now (hopefully) decrease at all points of the new exceptional component. On the transform of the singular scheme, the story repeats itself.

Sample case: Surface resolution.

Resolution of surfaces in positive characteristic was first established by Abhyankar [Ab2]. Today, the proofs for the embedded resolution of surfaces in arbitrary characteristic are often based on a resolution invariant proposed by Hironaka [Hi3, Co3, Ha2]. This invariant is extracted from the Newton polygon of a singularity. It seems to be kind of hand knitted, but it is substantial. It is used in Cutkosky's proof (following Abhyankar) of non-embedded resolution of three-folds in characteristic > 5, as well as in Cossart-Piltant's proof for the same case but over fields of arbitrary characteristic [Cu1, Cu2, CP1, CP2]. The invariant has been extended to the embedded resolution in the non-hypersurface case for arbitrary two-dimensional excellent schemes by Cossart-Jannsen-Saito [CJS].

For two-dimensional hypersurfaces of order equal to the characteristic of the ground field (where the purely inseparable case is known to be the tough one) the invariant has been replaced recently by Hauser and Wagner by a simpler and more conceptual invariant [HW], proposed originally by Zeillinger [Ze1]. We will briefly describe its definition and how it is used for the induction argument.

So let X be a (reduced) surface in a smooth, three-dimensional ambient space W. When working locally at closed points a of W, we will tacitly pass to the completion of the local ring $\mathcal{O}_{W,a}$ (say, work in an étale neighborhod) and assume that X is defined at a by a formal power series f in three variables.

The singular locus of X is given locally by the vanishing of the partial derivatives of f. It is a closed subscheme consisting of a finite number of points and irreducible curves. Blowing up the singular points of these curves provides, after finitely many iterations, a surface X where the one dimensional components of the singular locus have become smooth curves. Additional point blowups will then allow us to assume that these curves meet transversally [Za1]. We will suppose throughout that we are in this situation. Moreover

[3] There seems to be no genuine invariant measuring the distance of scheme from having normal crossings at a point. The option of factoring the maximal monomial from the defining equation and taking the order of the remaining factor falls short in positive characteristic when one descends in dimension.

we will assume for ease of exposition that locally at a given point a, the defining equation of X is in purely inseparable form

$$f(x,y,z) = x^{p^k} + g(y,z),$$

for some power series g in two variables of order $\geq p^k$ at 0. Here, $p > 0$ will be the characteristic of the ground field, k is ≥ 1, and x, y, z are chosen local coordinates at a, so that a becomes the origin of the chart. It is known that this type of equations poses the hardest obstacle for the resolution in positive characteristic. We will restrict to the case $k = 1$, i.e., $\mathrm{ord}\, f = p$, so that

$$f(x,y,z) = x^p + g(y,z),$$

with $\mathrm{ord}\, g \geq p$. Note that g, which generates, up to raising to a suitable power, just the coefficient ideal of f in $x = 0$, is only given up to the addition of p-th powers, since x can be replaced by $x + n(y,z)$ without changing the geometry of X. We say that g is monomial, or that we are in the monomial case, if g is, up to addition of p-th powers and up to a suitable multiplication of a unit in the power series ring, equal to a monomial $y^r z^s$ in the variables. In this special case one can render, by a sequence of point blowups, the curves defined by $y = 0$ and $z = 0$ in the plane $V : x = 0$ transversal to the possibly already existing exceptional curves. There is then a direct procedure to make the order of f drop below p: If $r \geq p$ or $s \geq p$ blow up the curves $y = 0$, respectively $z = 0$. It is easily seen by computations in local coordinates that this yields in finitely many steps $r < p$ and $s < p$. Either the order of f has meanwhile dropped, or $r + s$ is still at least p. In the latter case, apply a point blowup. Again, direct computations show that eventually the order of f drops below p.

These local blowups globalize since we had made at the beginning the singular locus of X normal crossings. And as the centers are transversal to the existing exceptional curves, the exceptional divisor stays normal crossings. So, to summarize, the monomial case permits a mostly combinatorial procedure to decrease the order of f at all points of X. We can therefore discard it in our further consideration. More precisely, in the general situation, we are led to transform g into a monomial (times a unit) by suitable blowups.

So let us assume henceforth that g is not monomial at a. The first thing to prove is that there are only finitely many such points. This is an argument on the upper semicontinuity of the coefficient ideal of f (in a suitable sense). First, it is shown that the locus of points where g is not monomial is algebraic, and then, that it cannot contain an entire curve. So the interesting locus consists of finitely many points. It is then heuristically clear (but has to be proven) that it is not necessary to modify X outside these points in order to obtain monomality of g everywhere (in which case we would be done). Hence point blowups should (and will) suffice to achieve this. Let us therefore place ourselves at one of these points where g is not monomial, call it a. Our objective now is to show that finitely many point blowups make g monomial (if the order of f does not drop meanwhile below p). This is the most interesting part of the whole argument, and we will be more explicit on this.

Let $g(y,z)$ be a power series in two variables, considered up to the addition of p-th powers. The blowups of X and f with centers isolated points have a specific impact on the transformation of g, and the chart expression of the transform of g can be directly computed (see below). We can therefore forget about f and concentrate on g. The outset is the search for a measure how far the series g is away from being a monomial (in the above sense.) This measure $\mathrm{inv}(g)$ should be an element of a well ordered set Γ, and it should decrease under blowup as long as g is not monomial. Then induction will show that finitely many point blowups yield monomiality.

The construction of a convenient candidate for $\mathrm{inv}(g)$ is built on the inspection of the transformation of

g under blowup. We shall work in local coordinates y, z at a, and denote by the same letters the induced coordinates after blowup at our selected reference point a'. It is well known that according to the location of a' on the exceptional divisor, there are three different chart expressions for the blowup map (we wish to use the same coordinates y, z at a for all points a' simultaneously):

(A) Translational move: $(y,z) \to (yz + tz, z)$ with $t \neq 0$,

(B) Horizontal move: $(y,z) \to (yz, z)$,

(C): Vertical move: $(y,z) \to (y, yz)$.

The naming is justified by the transformation of the Newton polygon of g under blowup. It is drawn with the y-axis vertically, the z-axis horizontally. As g is considered only up to the addition of p-th powers, all lattice points of $\mathbb{N}^2$ which are multiples of p, i.e., belong to $p \cdot \mathbb{N}^2$, are considered as "holes". A typical move of the Newton polygon is depicted in Figure 5.

It is not possible to use blindly the invariant $(\mathrm{ord}, \mathrm{slope})$ of the resolution of plane curves in the new context of polynomials modulo p-th powers. There are points a' above a where the order remains the same but the maximal slope increases. A more refined argument is necessary.

The first idea is to replace the order of g by a different local invariant, the *height*. This is not mandatory, but simplifies the construction of the second component of the invariant. The height is, roughly speaking, the vertical extension of the Newton polygon, minimized over all coodinates. For given coordinates y, z, consider the highest and the lowest vertex of the Newton polygon (i.e., those with largest and smallest y-component, not taking into account vertices which lie in holes). The difference of their y-components is the height of F with respect to the coordinates. Then take the minimal value $\mathrm{height}(g)$ over all coordinates at a.

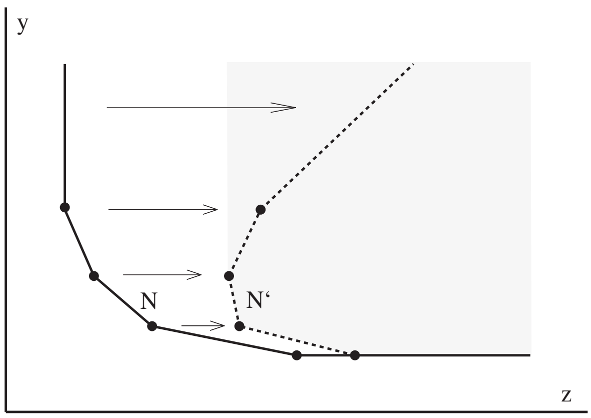

Figure 5 : Horizontal move of the Newton polygon under blowup.

This number is easy to control under the moves (B) and (C). Assume that g is not a monomial, i.e., that the height is not 0. Under horizontal moves the height remains constant, under vertical moves it decreases at least by two (or the transformed polynomial is already a monomial). The only delicate transformations are the translational moves. The height may increase, but, fortunately, it can increase at most by 1, adapting the argument of Moh (for equations of type $x^{p^k} + g(y,z)$ the increase is at most p^{k-1}). The points a' with an increase of the height are the so called kangaroo points (they are called metastatic points by Hironaka).

It is now immediate how to argue: Consider a sequence of point blowups. We wish to show that the height decreases in the long run until it reaches 0. There are two reasons for this. First, it can be shown, using a small portion of the characterization of kangaroo points [Ha3], that between two translational moves with an increase of the height in each of them there must always have been a vertical move. This results in a total drop of the height from one kangaroo point to the next.

Due to this we can conclude that there can only be a finite number of kangaroo points. Assume that we have already passed all of them. We are left to show that the height cannot remain constant infinitely many times. From the above we know that it can remain constant only under translational and horizontal moves. Using now the maximal slope of the first segment of the Newton polygon (the maximum being taken over all coordinates realizing the minimal height) it is not too hard to show that the height must eventually drop. This completes the induction argument.

Appendix

We collect the definitions of the technical terms used in this paper.

Schemes. All schemes appearing in the text are of finite type over an algebraically closed field. Locally in an affine chart, they are given as the spectrum $\mathrm{Spec}(A)$, where A is the quotient of a polynomial ring in n variables with coefficients in k by an ideal J. Readers who are not familiar with the language of schemes may equally work with algebraic varieties over k, not necessarily irreducible. When working locally at a point a, we may pass to the completion and thus place ourselves in the formal power series ring in n variables. This passage carries no harm as long as we ensure that our local constructions are sufficiently natural so as to produce (by patching) global objects.

If W is a scheme and a a point of W, we denote by $\mathcal{O}_W$ the sheaf of regular functions on W and by $\mathcal{O}_{W,a}$ the stalk of $\mathcal{O}_W$ at a. This is a local ring with maximal ideal m_a of functions vanishing at a. It is a regular ring if and only if a is a smooth point of W (the ground field is assumed to be algebraically closed, hence perfect).

Local coordinates. Local coordinates in a smooth scheme W at a point a are formed by a regular system of parameters of the local ring $\mathcal{O}_{W,a}$, i.e., by elements $x_1, \dots, x_n$ generating the maximal ideal m_a of $\mathcal{O}_{W,a}$, where n is the Krull-dimension of $\mathcal{O}_{W,a}$. Passing to the completion we get, by Cohen's Structure Theorem, a formal power series ring $\widehat{\mathcal{O}}_{W,a} \cong k[[x_1, \dots, x_n]]$ generated by $x_1, \dots, x_n$. Essentially all constructions in resolution of singularities are compatible with the passage to the completion, thus allowing us to work with formal power series and to use the Weierstrass Preparation Theorem. Grothendieck has shown that excellent schemes are the correct context for resolution problems.

Blowups. Let Z be a closed subscheme of W defined by the ideal sheaf $I(Z)$. The blowup of W with center Z is a morphism $\pi : W' \to W$ which is an isomorphism outside the exceptional divisor $Y' = \pi^{-1}(Z)$ onto $X \setminus Z$ and which contracts Y' to Z. It can be described in various ways, one is the following. Let $g_1, \dots, g_k$ be local generators of $I(Z)$, say on an open affine subset U of W. Consider then the map

$$\gamma : U \setminus Z \to \mathbb{P}^{k-1} : a \to (g_1(a) : \dots : g_k(a))$$

where $(u_1 : \dots : u_k)$ denote projective coordinates in $\mathbb{P}^{k-1}$. The graph $G(\gamma)$ of γ lives in $(U \setminus Z) \times \mathbb{P}^{k-1}$. We define U', the blowup of U in $Z \cap U$, as the Zariski closure of this graph

$$U' = \overline{G(\gamma)} \subset U \times \mathbb{P}^{k-1}.$$

It comes with a natural projection $\pi_U : U' \to U$, the blowup map, induced from the projection $U \times \mathbb{P}^{k-1} \to U$ on the first components. The equations of U' in $U \times \mathbb{P}^{k-1}$ are

$$u_i g_j(g) - u_j g_i(x) = 0 \qquad \text{for all } i \text{ and } j.$$

Different choices of the generators of $I(Z)$ yield isomorphic blowups of U. It is easy to see that these local constructions patch and give a scheme W' together with a (birational proper) morphism $\pi : W' \to W$.

The preimage $Y' = \pi^{-1}(Z)$ is a hypersurface (a Cartier divisor) in W' called the exceptional divisor. Blowups are characterized by a universal property, being a morphism which transforms the center into a Cartier divisor and is minimal with this property (any other such morphism factors through π). For more details, see [EiH].

Permissible centers. If X is a closed subscheme of a smooth ambient scheme W and if $\mathrm{inv} : X \to \Gamma$ is a local upper semicontinuous invariant on X, the center $Z \subset X$ of blowup is called permissible (with respect to inv) if inv is constant along Z, i.e., if Z is contained entirely in one stratum of the stratification of X defined by inv.

Transforms. Let $\pi : W' \to W$ be the blowup of a scheme W with center Z and exceptional divisor Y'. Let X be a closed subscheme of W defined by the ideal sheaf J of $\mathcal{O}_W$. The total transforms X^* and J^* of X and J are the pullbacks $\pi^{-1}(X)$ and $\pi^*(J)$ of X and J. The strict transform X^s of X is the closure of $\pi^{-1}(X \setminus Z)$ in W'. It is defined by the ideal J^s generated locally by the strict transforms f^s of elements of the stalks of J, where $f^s = f^* \cdot I_{W'}(Y')^{-\mathrm{ord}_Z(f)}$ (the negative exponent has to be understood in the sense that there exists, up to units in the local ring, a unique element f^sso that $f^* = f^s \cdot I_{W'}(Y')^{\mathrm{ord}_Z(f)}$). The weak transform $J^\curlyvee$ of J is the ideal $J^* \cdot I_{W'}(Y')^{-\mathrm{ord}_Z(J)}$; the controlled transform $J^!$ with respect to a control $c \leq \mathrm{ord}_Z(J)$ is defined as $J^* \cdot I_{W'}(Y')^{-c}$.

Transversality. Two subschemes U and V are said to meet transversally if they are both smooth along the intersection $U \cap V$ and if the intersection $U \cap V$ is scheme-theoretically smooth (in the sense that the sum of the ideals of U and V defines a smooth subscheme of W). Notice that the second condition implies the first. Transversality is equivalent to saying that the sum of the tangent spaces at an intersection point is of maximal possible dimension. Notice that in differential geometry, transversality has usually a different meaning (namely, that the sum of the tangent space equals the tangent space of the ambient space).

A reduced subscheme D of W has normal crossings at a, if it is, locally at a, analytically isomorphic to a union of coordinate subspaces of affine space $\mathbb{A}^n$. There is also an algebraic version of this, see [Ko2]. An arbitrary subscheme D of W has normal crossings at a, if it can be defined, locally at a, by a monomial ideal in a local coordinate system. A smooth subscheme V is said to meet D transversally, if the subscheme $D \cup V$ defined by the product of the ideals is a normal crossings subscheme. In particular, V will meet each component of D transversally.

Coefficient ideal. Let f be a formal power series in n variables, let o be the order of f at 0, and let $f = \sum_{i=0}^{\infty} a_{i,f}(y) x_n^i$ be the expansion of f with respect to x_n, where $a_{i,f}(y)$ are formal power series in $y = (x_1, \ldots, x_{n-1})$. The coefficient ideal $J_{x_n}(f)$ of f with respect to x_n is defined as

$$J_{x_n}(f) = \textstyle\sum_{i=o}^{o-1} \langle a_{i,f}(y)^{\frac{o!}{o-i}} \rangle.$$

In the literature the factorial is often omitted for notational reasons in the exponent, thus allowing rational exponents. This causes no problems as long as the exponent is interpreted correctly. For ideals K of order o at 0 we set

$$J_{x_n}(K) = \textstyle\sum_{i=o}^{o-1} \langle a_{i,f}(y)^{\frac{o!}{o-i}},\ f \in K \rangle.$$

If V is the local hypersurface defined in W by $x_n = 0$ we also say that $J_{x_n}(K)$ is the coefficient ideal of K in V, written by a slight abuse of notation as $J_V(K)$ (though it depends on the coordinates).

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