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Plain varieties

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Abstract

Algebraic varieties that are locally isomorphic to open subsets of affine space will be called *plain*. Plain varieties are smooth and rational. The converse is true for curves and surfaces, and is unknown in general. It is shown that plain varieties are stable under blowup in smooth centers.

1. *Introduction*

Differential manifolds are obtained by gluing open subsets of Euclidean space through differentiable isomorphisms. In the algebraic category, the situation is different: there are smooth algebraic varieties that cannot be covered by open subsets of affine space. Any non-rational variety provides an example.

It is nevertheless always possible to cover a smooth variety by open subsets that are isomorphic to smooth hypersurfaces of affine space. This is a generalization of the theorem of the primitive element (cf. Proposition 2.1). The isomorphisms can be obtained by generic projections onto linear subspaces of the appropriate dimension.

Those varieties that are locally isomorphic to *open* subsets of $\mathbb{A}^n$ will be called *plain*. They form a very natural class of varieties for which local computations are particularly efficient. The affine charts are not just any smooth subvarieties of affine space but open, and hence dense subsets. This allows us to work with affine coordinates. Moreover, the affine coordinate rings have unique factorization. The concept of plain varieties also appears in [**2**] as 'algebraic spaces' and in [**9**] as 'special varieties', where they are used for the definition of R-equivalence.

First examples of plain varieties are graphs of polynomial maps $\mathbb{A}^n \to \mathbb{A}^m$, projective spaces, Grassmannians, and their cartesian products – the class of plain varieties is obviously closed under finite products. More generally, it is easily seen that smooth toric varieties are plain (see Section 3).

We prove in Theorem 4.4 that the class of plain varieties is closed under blowup in smooth centers, provided that the base field is infinite. This is not immediate if the global structure of the center is complicated, for example, if the center has large genus. By Proposition 2.1, we may embed the center locally as a hypersurface in an affine subspace and then use the explicit ring-theoretic description of blowups by subrings of the function field of the variety.

An important application is the following: in characteristic zero, it is possible to prove the embedded resolution of singularities by blowing up the ambient space along smooth centers in the strict transform of the variety. In this way, we can compute embedded resolutions within plain ambient spaces. As these can be covered by open subsets of affine space, the number of ambient variables does not increase when passing to the transform of the variety. For instance, we can implement Villamayor's algorithm [**10**] or the algorithm of Bierstone and Milman [**3**] for the resolution of a singular hypersurface in such a way that the final (and all intermediate)

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results are covered by *hypersurfaces* in the literal sense, each of them given by one equation in independent variables. Indeed, this algorithm proceeds by blowing up the ambient space along smooth centers, and if the given ambient space is plain, then all subsequent ambient spaces are plain, too.

Giving necessary and sufficient criteria for a variety to be plain seems to be a much harder question. Obviously, plain varieties are rational and smooth. We do not know of an example of a smooth rational variety which is not plain.

2. *Smooth varieties are locally hypersurfaces*

Let k be a field, which is assumed to be infinite unless otherwise specified. We work in the category of varieties over k, as defined in [7]: a variety is an integral scheme over k of finite type (in particular, reduced and irreducible), and a morphism is a regular map. Affine space $\mathbb{A}^n_k$ is the spectrum of a polynomial ring R in n algebraically independent variables over k. Any generator system $x_1, \dots, x_n$ of R will be called a *coordinate system* in $\mathbb{A}^n_k$. Coordinate systems are transformed into each other by automorphisms of $\mathbb{A}^n_k$.

Let $F \subset \mathbb{A}^n$ be locally closed, $p \in F$. We say that $F \subset \mathbb{A}^n$ is a *local coordinate subvariety* at p if there is an open neighborhood $U \subset \mathbb{A}^n$ such that $U \cap F$ is the zeroset of a subset of a coordinate system. See also Example 4.2 below, where it is shown that the circle $x^2 + y^2 - 1 = z = 0$ in $\mathbb{A}^3$ is a local coordinate subvariety at any of its points.

PROPOSITION 2.1. *Any smooth variety Z admits a covering by open subsets isomorphic to smooth, locally closed hypersurfaces in affine space. If Z is locally closed in $\mathbb{A}^n$ and $q \in Z$, then there exists a local coordinate subvariety F at q containing Z as a hypersurface.*

Proof. Let $Z \subset U \subset \mathbb{A}^n$, $q \in Z$, and define $m := \dim(Z) + 1$. Let L be a generic element in the Grassmann variety $G(n-m, n)$, and let $\pi : \mathbb{A}^n \to \mathbb{A}^m$ be a linear projection map that maps L to a point (for example, the canonical map to the quotient space $\mathbb{A}^n/L \cong \mathbb{A}^m$). Then the Zariski closure H of $\pi(Z) \subset \mathbb{A}^m$ is a hypersurface. We claim that $\pi|_Z : Z \to H$ is a local isomorphism at q. This is a strengthening of the theorem of the primitive element, which says that Z is birationally equivalent to a hypersurface. (The strengthening is that we have a birational equivalence that is étale at a specific point.)

To prove the claim, we form the cone of Z at q, which is the Zariski closure of the union of Z-secants through q in $\mathbb{A}^n$. Its dimension is at most m; hence its intersection with L is just a single point (namely q). Let $p := \pi(q)$. As L is generic and k is infinite, the scheme-theoretic fiber $(\pi|_Z)^{-1}(p)$ is just q, and this also holds geometrically; that is, the residue field of q is equal to the residue field of p. Generic projections to subsets preserving the dimension are finite maps (compare with the proof in [8, p. 69] of the Nöther normalization lemma). Hence, the map $\pi|_Z$ corresponds to a local homomorphism $i : \mathcal{O}_{H,p} \hookrightarrow \mathcal{O}_{Z,q}$ such that $\mathcal{O}_{Z,q}$ is a finite $\mathcal{O}_{H,p}$-module. Because $\pi|_Z$ is étale at q, the completion $\widehat{\mathcal{O}}_{H,p} \hookrightarrow \widehat{\mathcal{O}}_{Z,q}$ is an isomorphism. However, completion is a faithfully exact functor on finitely generated $\mathcal{O}_{H,p}$-modules, and hence i is an isomorphism. It follows that $\pi|_Z$ is an isomorphism on suitably chosen open neighborhoods V of q and W of p.

Strictly speaking, the above argument shows only that $\pi|_Z$ gives an isomorphism after extending the base field to the residue field of q, because the construction of the cone at q already requires this extension. However, since the map $\pi|_Z$ is an isomorphism after base field extension, it must have also been an isomorphism before base field extension.

The map $\alpha := (\pi|_Z)^{-1}$ is defined in $W \subset H$ and can be extended to an open neighborhood W' of p in $\mathbb{A}^m$. The map

$$\phi : W' \times L \longrightarrow \pi^{-1}(W'), \quad (u, l) \longmapsto \alpha(u) + l,$$

is then an isomorphism mapping the coordinate subvariety $W' \times \{0\}$ to a subvariety F of dimension m containing Z. □

EXAMPLE 2.2. Consider the space curve Z given as the complete intersection of the two surfaces $S : y^2 = x^3 - x$ and $T : z^2 = y^3 - y$. Both S and T are cylinders over the elliptic curve and are therefore smooth. The intersection is transversal so that Z itself is smooth. Let U be the complement of $V_{\mathbb{A}^3}(x^3 - x - 1)$ in $\mathbb{A}^3$. This is an open neighborhood of the origin $p = 0$. We claim that inside U, the curve Z is isomorphic to a plane curve C. Indeed, substituting y^2 by $x^3 - x$ in the second equation yields the relation $y = z^2/(x^3 - x - 1)$ on $Z \cap U$. This equation defines a smooth surface (a graph) in $U \subseteq \mathbb{A}^3$ isomorphic to the open affine chart $\mathbb{A}^2 \setminus V_{\mathbb{A}^2}(x^3 - x - 1)$. And in this way, Z is isomorphic to its image curve there.

3. *Plain varieties*

Let V be a plain variety. The open subsets isomorphic to open subsets of affine space will be called *plain charts* of V.

Any smooth toric variety is plain: a toric variety is covered by affine toric varieties, and a smooth affine toric variety is the product of an affine space and a torus (see [**6**, p. 29]). Since the torus can be embedded into affine space as an open subset, the assertion follows.

The notion of plain varieties is clearly a local one; here is an algebraic criterion for being plain.

PROPOSITION 3.1. *A variety is plain if and only if the stalks of its structure sheaf are k-isomorphic to localizations of a polynomial algebra $k[x_1, \ldots, x_n]$ at prime ideals.*

Proof. The 'only if' condition is obvious. Conversely, assume that X is a variety, p a point in X, and that the local ring $\mathcal{O}_{X,p}$ is isomorphic to $\mathcal{O}_{\mathbb{A}^n,q}$ for some $q \in \mathbb{A}^n$. Then the quotient fields of these two local rings are isomorphic over k. Hence, there exist rational maps $\phi : X \to \mathbb{A}^n$ and $\psi : \mathbb{A}^n \to X$, inverse to each other, and defined at p and q, respectively, and open neighborhoods U of p and V of q between which ϕ and ψ induce k-isomorphisms. □

PROPOSITION 3.2. *Assume that k is algebraically closed. Then every rational smooth curve or surface is plain.*

Proof. Assume that the curve C is rational and smooth. Then C is an open subvariety of a complete rational smooth curve, which is obtained by projectivization and subsequent desingularization. As every birational map of complete smooth curves is an isomorphism (see [**7**, II.6.7, p. 136]), C is an open subvariety of $\mathbb{P}^1$. (see also [**7**, Example I.6.1, p. 46, Corollary I.6.10, p. 45, and IV.1.3.5, p. 297]).

Assume that the surface S is rational and smooth. As in the case of the curve, it suffices to prove the statement for complete surfaces (compare with [**7**, Remark II.4.10.2, p. 105]). It is well known that any complete rational surface can be obtained by repeated point blowups of a minimal rational surface. The minimal rational surfaces are $\mathbb{P}^2$ and the Hirzebruch surfaces F_n, $n = 0, 2, 3, \ldots$; these are all plain varieties. The blowup of a plain surface at a point is also plain, because the blowup of $\mathbb{A}^2$ at the origin can be covered by two open sets which are both isomorphic to $\mathbb{A}^2$. □

Obviously, plain varieties are smooth and rational. We conjecture that the converse also holds: any smooth and rational variety is plain.

In the following section, we will show that the class of plain varieties is closed under blowups along non-singular subvarieties. In order to prove the conjecture, it would suffice to show that plain varieties are also stable under the inverses of such blowups (blowdowns), at least in characteristic zero. The reason is that any birational map is a composition of such blowups and their inverses, by the Weak factorization theorem [1, 11]. The stability under blowdowns would follow if one could show that plain in codimension 1 (that is, the non-plain locus has codimension at least 2) implies plain.

Example 3.3. The surface S with defining equation $x-(x^2+z^2)y$ over $\mathbb{Q}$ is rational — the equation is linear in y — and non-singular at the origin. By the conjecture, there should be a plain chart containing the origin.

Such a chart does indeed exist, for instance, the intersection of S with the complement of $V_{\mathbb{A}^3}(1-xy)$. It is isomorphic to the complement of $V_{\mathbb{A}^2}(u^2v^2+1)$, by the isomorphism

$$(u,v)\longmapsto (u^2v/(u^2v^2+1), v, u/(u^2v^2+1))$$

with inverse

$$(x,y,z)\longmapsto (z/(1-xy), y).$$

4. *Blowups*

We recall the definition of blowups (following [5, 7]): let X be a variety, and let $\mathcal{I}\subseteq\mathcal{O}_X$ be a sheaf of ideals on X. Then the blowup $\widetilde{X}$ of X with center $\mathcal{I}$ is defined as the Proj of the sheaf of graded algebras $\mathcal{S}=\oplus_{i=0}^{\infty}\mathcal{I}^i$. If X is affine, say $X=\operatorname{Spec}(R)$, and $\mathcal{I}$ is the sheaf corresponding to the ideal $I\subseteq R$ that is generated by $a_1,\ldots,a_m\in R$, then $\widetilde{X}$ can be embedded into $\mathbb{P}_R^{m-1}$ as the closed subvariety defined by the homogeneous ideal $J=\ker(\varphi)$, where $\varphi: R[y_1,\ldots,y_m]\to R[t]$ is defined as the R-homomorphism sending y_i to a_it for $i=1,\ldots,m$. Here is a description of this situation in terms of affine charts.

Lemma 4.1. *Let X, I and $a_1,\ldots,a_m$ be as above. Then the blowup $\widetilde{X}$ is covered by affine open subsets $U_1,\ldots,U_m$ with U_i isomorphic to* $\operatorname{Spec}(R[a_1/a_i,\ldots,a_m/a_i])$.

Proof. For $i=1,\ldots,m$, we define U_i as the intersection of $\widetilde{X}$ with the affine patch $y_i\neq 0$ in $\mathbb{P}_R^m$. It has affine coordinates

$$x_1=\frac{y_1}{y_i},\ldots,x_{i-1}=\frac{y_{i-1}}{y_i},x_{i+1}=\frac{y_{i+1}}{y_i},\ldots,x_m=\frac{y_m}{y_i}$$

(in addition to the affine coordinates for X). Then the defining ideal of $U_i\subset\mathbb{A}_R^m$ is the image of J defined as above under the dehomogenization homomorphism $R[y_1,\ldots,x_m]\to R[x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_m]$ mapping y_i to 1 and y_j to x_j for $j\neq i$. However, this ideal is also the kernel of the ring homomorphism $R[x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_m]\to\operatorname{Quot}(R)$ mapping x_j to a_j/a_i; hence the quotient ring is isomorphic to $R[a_1/a_i,\ldots,a_m/a_i]$. □

Assume that $X=\mathbb{A}_k^n$ and that I is the ideal of a coordinate subvariety, for example, $I=(x_1,\ldots,x_r)$. Then, for $i=1,\ldots,r$, the ring $R[x_1/x_i,\ldots,x_r/x_i]$ is isomorphic to the polynomial ring $k[y_1,\ldots,y_n]$ by the isomorphism sending x_j/x_i to y_j for $j\neq i$ and x_i to y_i. Hence the spectra U_i of these rings are affine spaces, and $\widetilde{X}$ is plain. As being plain is a local property, this shows that the blowup of $\mathbb{A}_k^n$ along local coordinate subvarieties is plain.

EXAMPLE 4.2. The circle given by the ideal $I = (x^2 + y^2 - 1, z)$ in $\mathbb{A}^3$ is a local coordinate subvariety. For instance, let $q := (0, 1, 0)$. Then the rational map

$$(x, y, z) \longmapsto (u, v, w) = \left(\frac{x}{y+1}, \frac{x^2 + y^2 - 1}{(y+1)^2}, z \right)$$

is an isomorphism of an open neighborhood of q to an open neighborhood of $(0, 0, 0)$; its birational inverse is

$$(u, v, w) \longmapsto (x, y, z) = \left(\frac{2u}{u^2 - v + 1}, \frac{-u^2 + v + 1}{u^2 - v + 1}, w \right).$$

Any other point on the circle can be moved to q by a rotation. This shows directly (that is, without referring to the theorem below) that the blowup of the circle is plain (see also [**4**]).

In order to prove the stability of plain varieties under blowup in general, the observation above is not enough, for there are smooth subvarieties in $\mathbb{A}^n_k$ which are not local coordinate varieties, for example, non-rational curves.

THEOREM 4.3. *Let F be a coordinate subvariety of U open in $\mathbb{A}^n_k$, and let Z be a closed smooth hypersurface of F. Assume that k is infinite. Then the blowup of U along Z is plain.*

Proof. Assume that F is defined by $x_1 = \ldots = x_r = 0$, and that Z is defined within F by $f(x_{r+1}, \ldots, x_n) = 0$. Let $p \in Z$ be a point, and assume that $\partial_{x_n} f$ does not vanish at p. We will construct an open neighborhood V of p in U and a covering of the blowup $\widetilde{V}$ of V along $Z \cap V$ by plain charts.

Note first that $\mathrm{Spec}(k[U][x_1/f, \ldots, x_r/f])$ is a plain chart of the blowup. Indeed, the ring $k[x_1, \ldots, x_n, x_1/f, \ldots, x_r/f]$ is isomorphic to $k[y_1, \ldots, y_r, x_{r+1}, \ldots, x_n]$ by the isomorphism fixing $x_{r+1}, \ldots, x_n$ and sending x_i to $y_i f(x_{r+1}, \ldots, x_n)$ for $i = 1, \ldots, r$. Unfortunately, the other charts of the blowup as described in Lemma 4.1 need not be plain charts in general. The trick is now to choose other defining equations for Z, thus changing the charts on $\widetilde{V}$.

For $i = 1, \ldots, r$, we set $f_i = f(x_{r+1}, \ldots, x_{n-1}, x_n + x_i)$. We claim that there is an open neighborhood V of p where $f, f_1, \ldots, f_r$ are generators of the ideal of Z. This is a consequence of the Taylor expansion

$$f_i = f + x_i \cdot (\partial_{x_n} f + h_i)$$

for suitable polynomials h_i vanishing at p ($i \leqslant r$); we choose the affine open neighborhood V of p so that the polynomials $g_i := \partial_{x_n} f + h_i$ do not vanish in V. They, therefore, have multiplicative inverses in $k[V]$.

The first chart of $\widetilde{V}$ is isomorphic to the spectrum of $k[V][f_1/f, \ldots, f_r/f]$. From the fact that $f_i/f = 1 + g_i \cdot (x_i/f)$ for i from 1 to r and because all g_i are invertible on V, it follows that $k[V][f_1/f, \ldots, f_r/f] = k[V][x_1/f, \ldots, x_r/f]$. This algebra is a polynomial ring and therefore the chart is plain.

Any of the other charts of $\widetilde{V}$ is isomorphic to the spectrum of $k[V][f/f_i, f_1/f_i, \ldots, f_r/f_i]$ for some i between 1 and r. This algebra equals $k[V][(x_1/f_i), \ldots, x_r/f_i]$, because of the relations $f/f_i = 1 - g_i \cdot (x_i/f_i)$ and $f_j/f_i = f/f_i + g_j \cdot (x_j/f_i)$, respectively, $x_j/f_i = g_j^{-1} \cdot (f_j/f_i - f/f_i)$. Hence these other charts are also plain. The theorem is proved. □

As a consequence of Theorem 4.3 we obtain the stability of plain varieties under blowup.

THEOREM 4.4. *Over infinite fields, the blowup of a plain variety along a smooth subvariety is plain.*

Proof. As being plain is a local property, and using the local embedding of smooth varieties from Proposition 2.1, it suffices to consider blowups along smooth hypersurfaces in coordinate subvarieties; this is just the case settled in Theorem 4.3. □

EXAMPLE 4.5. We blow up $\mathbb{A}^3$ with center the plane elliptic curve $Z : z = x - x^3 + y^2 = 0$. We place ourselves in some neighborhood V of the origin of $\mathbb{A}^3$. Setting $f_2 = x - x^3 + y^2$, $f_1 = x + z - (x+z)^3 + y^2$, and $g = 1 - 3x^2 - 3xz - z^2$, we obtain $f_1 = f_2 + gz$. Hence f_1 and f_2 generate the ideal of Z in the principal open set V defined by $g \neq 0$.

In the first chart V_1, we adjoin the fraction $v = f_2/f_1$ to $k[V]$. We introduce the new variable w for $x + z$ and use it for eliminating x; in particular, we write g in the form $g(w, y, z) = 1 - 3w^2 + 3wz - z^2$. Let the variable s stand for the fraction $z/f_1 = g^{-1}(1 - v)$. Express now z as $tf_1(w, y)$ and v as $1 - tg(w, y, z) = 1 - tg(w, y, tf_1(w, y))$. This chart is isomorphic to the open set in $\mathbb{A}^3$ with coordinates w, y, t defined by the inequality $g(w, y, tf_1(w, y)) \neq 0$, say

$$1 - 3w^2 + 3wt(w - w^3 + y^2) - t^2(w - w^3 + y^2)^2 \neq 0,$$

and the exceptional divisor inside this chart has the following equation:

$$f_1(w, y) = w - w^3 + y^2 = 0.$$

In the second chart V_2 of the blowup, we adjoin the fraction $u := f_1/f_2$ to $k[V]$. We introduce a new variable s for the fraction $z/f_1 = g^{-1}(u - 1)$. Now, we can express z as $sf_2(x, y)$ and u as $1 + sg(x, y, z) = 1 + sg(x, y, sf_2(x, y))$. The chart is isomorphic to the open set of $\mathbb{A}^3$ with coordinates x, y, s defined by the inequality $g(x, y, sf_2(x, y)) \neq 0$, say

$$1 - 3x^2 - 3xs(x - x^3 + y^2) - s^2(x - x^3 + y^2)^2 \neq 0,$$

and the exceptional divisor inside this chart has the following equation:

$$f_2(x, y) = x - x^3 + y^2 = 0.$$

The intersection $V_1 \cap V_2$ is isomorphic to the open subset $V_2' \subset V_2$ defined by $u = 1 + sg(x, y, sf_2(x, y)) \neq 0$, say

$$1 + s - 3x^2s - 3xs^2(x - x^3 + y^2) - s^3(x - x^3 + y^2)^2 \neq 0,$$

and to the open subset $V_1' \subset V_1$ defined by $v = 1 - tg(w, y, tf_1(w, y)) \neq 0$, say

$$1 - t + 3w^2t - 3wt^2(w - w^3 + y^2) + t^3(w - w^3 + y^2)^2 \neq 0.$$

The chart change map $V_2' \to V_1'$ is given by

$$\begin{aligned}(x, y, s) &\longmapsto (w, y, t)\\ &= \left(x + sf_2(x, y), y, \frac{s}{1 + sg(x, y, sf_2(x, y))}\right)\\ &= \left(x + xs - x^3s + y^2s, y, \frac{s}{1 + s - 3x^2s - 3xs^2(x - x^3 + y^2) - s^3(x - x^3 + y^2)^2}\right).\end{aligned}$$

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