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"Resolution of plane algebraic curves via geometric invariants"

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1 Introduction

The problem of the existence and construction of a resolution of singularities is one of the major questions in algebraic geometry. The question of existence was solved in general, for algebraic varieties defined over fields in characteristic zero, by Heisuke Hironaka in his famous paper in the year 1964. But Hironaka's proof is not constructive. And it took more than 25 years to develop a constructive way from this. However, the resolution of singularities of algebraic curves is not so difficult and it was already well known in the 19th century. These days different constructive methods of proving resolution of singularities of algebraic curves are known.

The purpose of this thesis is to present a new constructive method of resolving singularities of plane algebraic curves over $\mathbb{C}$. To get some new ideas how to resolve the singularities of a particular polynomial equation defining a plane algebraic curve, we work with parametrizations of the branches of the curve in each of its singularities. From each such parametrization we get additional information about the singularity itself. With help of this information we construct from the implicit equation the so-called modified higher curvatures that will play the key role in resolving the singularities of the curve and in some cases they will even completely determine the resolution.

To be more precise, let $X \subseteq \mathbb{A}^2_{\mathbb{C}}$ be a plane algebraic curve with a singularity at the origin. Consider an analytic parametrization $\gamma(t) = (\mathsf{x}(t), \mathsf{y}(t))$ of a singular branch of X at the origin. The aim is to construct an analytic function $\mathsf{z}(t)$ from γ such that the following three conditions are satisfied:

1) The triple $(\mathsf{x}(t), \mathsf{y}(t), \mathsf{z}(t))$ parametrizes a branch of an algebraic space curve X_{z},
2) the branch of X_{z} parametrized by $(\mathsf{x}(t), \mathsf{y}(t), \mathsf{z}(t))$ is regular at the point lying over $(0, 0)$,
3) X_{z} is birationally equivalent to X.

As for the first and third condition, we introduce the concept of a geometric invariant of X. Let $\mathsf{z}(t)$ be a meromorphic function that can be written as a rational function in $\mathsf{x}(t), \mathsf{y}(t)$ and their higher derivatives. Let us write $\mathsf{z}(t) = \mathsf{z}(\mathsf{x}(t), \mathsf{y}(t))$ to indicate that z depends on x and y. We call $\mathsf{z}(t)$ a geometric invariant if it is invariant under reparametrization, i.e., if for every reparametrization $\varphi \in \mathrm{Aut}(\mathbb{C}\{t\})$ we have $\mathsf{z}((\mathsf{x} \circ \varphi)(t), (\mathsf{y} \circ \varphi)(t)) = [\mathsf{z}(\mathsf{x}(t), \mathsf{y}(t))]|_{t=\varphi(t)}$. For example the expression

$$\frac{\mathsf{x}''(t) \cdot \mathsf{y}'(t) - \mathsf{x}'(t) \cdot \mathsf{y}''(t)}{(\mathsf{x}'(t) + \mathsf{y}'(t))^3}$$

is a geometric invariant. It turns out that every geometric invariant z is indeed a rational function in x and y, i.e. $\mathsf{z}(t) = \frac{g(\mathsf{x}(t),\mathsf{y}(t))}{h(\mathsf{x}(t),\mathsf{y}(t))}$ with polynomials $g, h \in \mathbb{C}[x, y]$. Thus, $(\mathsf{x}(t), \mathsf{y}(t), \mathsf{z}(t))$ parametrizes a branch of an algebraic space curve $X_{\mathsf{z}} \subseteq V(f, g - z \cdot h)$, where $f \in \mathbb{C}[x, y]$ is the defining polynomial of X. In our example of a geometric invariant the equality

$$\frac{\mathsf{x}''(t)\cdot\mathsf{y}'(t)-\mathsf{x}'(t)\cdot\mathsf{y}''(t)}{(\mathsf{x}'(t)+\mathsf{y}'(t))^3}=\left(\frac{f_{xx}f_y^2-2f_{xy}f_xf_y+f_{yy}f_x^2}{(f_x-f_y)^3}\right)(\mathsf{x}(t),\mathsf{y}(t))$$

holds. In particular, the map $X\to X_{\mathsf{z}},(x,y)\mapsto\left(x,y,\frac{g(x,y)}{h(x,y)}\right)$ is birational. Furthermore, we will see that the polynomials g and h are always polynomials in the higher partial derivatives of f, so each geometric invariant of a plane algebraic curve is completely determined by modified differential operators.

To fulfill the second condition we use a criterion for the regularity of parametrized curves that tells us that a branch at the origin of an algebraic space curve $Y\subseteq\mathbb{A}^n$ is regular if and only if the branch admits a parametrization $\eta(t)=(\mathsf{y}_1(t),\ldots,\mathsf{y}_n(t))$ for which the minimum of the t-adic orders of the $\mathsf{y}_i(t)$, denoted by $\mathrm{ord}(\eta)$, is equal to one. For the construction of a geometric invariant $\mathsf{z}(t)$ of t-adic order one we proceed as follows. We construct from the parametrization γ a finite sequence of geometric invariants $\mathsf{z}_1,\ldots,\mathsf{z}_k$ of X with

$$0<\mathrm{ord}(\mathsf{z}_{i+1}(t))<\mathrm{ord}(\mathsf{z}_i(t))<\max\{\mathrm{ord}(\mathsf{x}(t)),\mathrm{ord}(\mathsf{y}(t))\}$$

for all $i=1,\ldots,k-1$ and

$$\mathrm{ord}(\mathsf{z}_k(t))<\min\{\mathrm{ord}(\mathsf{x}(t)),\mathrm{ord}(\mathsf{y}(t))\}.$$

We call them the modified higher curvatures of X. Let us denote the corresponding modified differential operators by κ_i. Here

$$\kappa_i:=\frac{\kappa_i^{(1)}}{\kappa_i^{(2)}}$$

with the modified differential operators $\kappa_i^{(1)},\kappa_i^{(2)}$ corresponding to $\mathsf{z}_i(t)$, i.e.

$$(\kappa_i(f))(\mathsf{x}(t),\mathsf{y}(t))=\left(\frac{\kappa_i^{(1)}(f)}{\kappa_i^{(2)}(f)}\right)(\mathsf{x}(t),\mathsf{y}(t))=\mathsf{z}_i(t),$$

where $(\kappa_i(f))(\mathsf{x}(t),\mathsf{y}(t))$ denotes the modified differential operator κ_i applied to f and consecutive substitution of variables $(x,y)\mapsto(\mathsf{x}(t),\mathsf{y}(t))$. Taking the z_k, which is of smallest t-adic order, and adding this as the third component to the parametrization of X produces a parametrization $(\mathsf{x}(t),\mathsf{y}(t),\mathsf{z}_k(t))$ of an algebraic space curve X_{z_k}. After the projection $X_{\mathsf{z}_k}\to\mathbb{A}^2,(x,y,z)\mapsto(y,z)$ we get, as image of this projection, a plane algebraic curve that has one branch parametrized by $\gamma_k(t)=(\mathsf{y}(t),\mathsf{z}_k(t))$. Then the inequality

$$\mathrm{ord}(\gamma_k)<\mathrm{ord}(\gamma)$$

holds and we proceed by induction on the order of the parametrization γ_k.

The modified differential operators κ_i are universal and can be applied to an arbitrary polynomial in two variables. Even more, for an arbitrary plane algebraic curve $V(g)$ parametrized by $(\mathsf{u}(t), \mathsf{v}(t))$ at the origin we have

$$\operatorname{ord}((\kappa_{i+1}(g))(\mathsf{u}(t), \mathsf{v}(t))) < \operatorname{ord}((\kappa_i(g))(\mathsf{u}(t), \mathsf{v}(t)))$$

for all $i \in \mathbb{N}$ and

$$0 < \operatorname{ord}((\kappa_j(g))(\mathsf{u}(t), \mathsf{v}(t))) < \min\{\operatorname{ord}(\mathsf{u}(t)), \operatorname{ord}(\mathsf{v}(t))\}$$

for some $j \in \mathbb{N}$. Therefore, the process of constructing the modified higher curvatures of a plane algebraic curve with increasing t-adic order does not depend on the curve itself and can be used for an arbitrary plane algebraic curve. But the length of the finite sequence of modified higher curvatures depends on the parametrization of the curve and so on the curve itself and therefore varies from case to case.

Applying the above procedure to each singular branch of X at the origin produces an algebraic space curve $\widetilde{X}$ with regular branches at the origin. Furthermore, by construction of the modified higher curvatures, the curve $\widetilde{X}$ is birationally equivalent to X.

The whole procedure of making singular branches regular can be described as a blowup of X with a suitable center. Here, the center is completely determined by the modified differential operators corresponding to the modified higher curvatures that define the curve $\widetilde{X}$. Finally, for the separation of the regular branches of $\widetilde{X}$ we draw inspiration from the modified higher curvatures of X and generate a new system of geometric invariants from which we then select those which are relevant for us. Again, the modified differential operators corresponding to these new geometric invariants determine the center of the blowup of $\widetilde{X}$ which describes the process of separating the regular branches of $\widetilde{X}$.

Repeatedly applying the procedure described above to all singularities of X produces a resolution of singularities of X.

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3 Preparation for resolution of singularities

3.1 Basic concepts of algebraic geometry

In this thesis, some familiarity with the basic concepts of algebraic geometry is assumed. However, the most important definitions and statements will be recalled in this section. Some of them will be for example the definition of the singular and regular locus of an algebraic variety, the definition of a blowup of an algebraic variety and the definition of a parametrization of an algebraic curve and its branches.

Let $Z = V(I) \subseteq \mathbb{A}^n$ be an algebraic variety and $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ its defining ideal. In this thesis $\mathbb{A}^n$ denotes $\mathbb{A}^n_{\mathbb{C}}$, the affine space over $\mathbb{C}$. Assume that $0 \in Z$ and consider the primary decomposition of I in the convergent power series ring $\mathbb{C}\{x_1, \ldots, x_n\}$ locally at the origin, $I = I_1 \cap \cdots \cap I_m$. For each $j = 1, \ldots, m$, let $Z_j = V(I_j)$ be the analytic variety defined by the ideal I_j. The germs $(Z_j, 0)$ of the zero sets Z_j at 0 are called the *branches* of Z at the origin. An ideal $J \subseteq \mathbb{C}\{x_1, \ldots, x_n\}$ is called *defining ideal* of $(Z_j, 0)$ if the analytic variety $V(J)$ is a representative of the germ $(Z_j, 0)$. If Z has more than one branch at the origin, it is called analytically reducible at the origin. Otherwise, we call Z analytically irreducible at the origin.

Let us now consider an algebraic space curve $X \subseteq \mathbb{A}^n$ and a point $a \in X$. Let (Y, a) be a branch of X at a and $J \subseteq \mathbb{C}\{x_1, \ldots, x_n\}$ a defining ideal of this branch. Then a map

$$\begin{aligned} \gamma : D(\mathsf{x}_1(t)) \cap ... \cap D(\mathsf{x}_n(t)) &\to X \\ b &\mapsto (\mathsf{x}_1(b), ..., \mathsf{x}_n(b)) \end{aligned}$$

that is defined by convergent power series $\mathsf{x}_i(t) \in \mathbb{C}\{t\}$ is called a *parametrization of the branch* (Y, a) if $g(\mathsf{x}_1(t), ..., \mathsf{x}_n(t)) = 0$ in $\mathbb{C}\{t\}$ for all elements $g \in J$ and if there exists a point b in the interior of $(D(\mathsf{x}_1(t)) \cap ... \cap D(\mathsf{x}_n(t)))$ so that $(\mathsf{x}_1(b), ..., \mathsf{x}_n(b)) = a$. Here $D(\mathsf{x}_i(t))$ is the area of convergence of $\mathsf{x}_i(t)$. We say that γ parametrizes X at a if γ is a parametrization of one of the branches of X at a and the Zariski-closure of $\mathrm{Im}(\gamma)$ equals X. Sometimes we also say that the n-tuple fo convergent power series $\gamma(t) = (\mathsf{x}_1(t), ..., \mathsf{x}_n(t))$ parametrizes the branch (Y, a) or the curve X at a, respectively.

Remark 3.1.1. For each branch (Y, a) of X, a parametrization of (Y, a) can be constructed according to the Newton-Puiseux algorithm. Each parametrization of (Y, a) even parametrizes the curve itself. For more details see the appendix.

Definition 3.1.2. *Let $Y \subseteq \mathbb{C}^n$ be an analytic variety and $a \in Y$ a point. Assume that $\dim_a(Y) = k$. The point a is called a regular point of Y if there exists $U \subseteq \mathbb{C}^n$ an open neighbourhood of a and an ideal $J = (g_1, \ldots, g_m) \subseteq \mathbb{C}\{x_1, \ldots, x_n\}$ so that $Y \cap U = V_U(J)$ and the Jacobian matrix of J evaluated in a,*

$$D_J(a) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(a) & \dots & \frac{\partial g_1}{\partial x_n}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1}(a) & \dots & \frac{\partial g_m}{\partial x_n}(a) \end{pmatrix},$$

has rank $n-k$. Otherweise, a is said to be a singular point of Y. Here $V_U(J) = \{b \in U | g_i(b) = 0$ for all $i = 1, \dots, m\}$ is the zero set of J in U.

There are more equivalent definitions of regular and singular points of analytic varieties. One of them is for example the following:

Definition 3.1.3. *Let $Y \subseteq \mathbb{C}^n$ be an analytic variety and $a \in Y$ a point on Y. The point a is called a regular point of Y if there exists $U \subseteq \mathbb{C}^n$ an open neighbourhood of a and $V \subseteq \mathbb{C}^n$ an open neighbourhood of 0 and biholomorphic map $\varphi : U \to V$ sending $Y \cap U$ to $L \cap V$ for some linear subspace $L \subseteq \mathbb{C}^n$.*

Proofs for the equivalence of both definitions can be found in most differential geometry books.

A branch (Y, a) at the point a of an algebraic curve $X \subseteq \mathbb{A}^n$ is called *regular* if each representative of (Y, a) is as an analytic variety regular at a. Otherwise we call (Y, a) a *singular* branch of X.

Proposition 3.1.4. *Let $X \subseteq \mathbb{A}^n$ be an algebraic curve with $0 \in X$. Let $(Y, 0)$ be a branch of X at the origin. Then the branch $(Y, 0)$ is regular if and only if $(Y, 0)$ admits a parametrization $\gamma(t) = (\mathsf{x}_1(t), \dots, \mathsf{x}_n(t))$ with $\gamma(0) = 0$ and $\min\{\mathrm{ord}_0(\mathsf{x}_i)\} = 1$. Here $\mathrm{ord}_0(\mathsf{x}_i)$ denotes the t-adic order of $\mathsf{x}_i(t)$.*

Proof. $\Rightarrow$: Assume that $(Y, 0)$ is regular. Let $\widetilde{Y}$ be a representative of $(Y, 0)$. Let $U, V \subseteq \mathbb{C}^n$ be open neighbourhoods of 0, $L \subseteq \mathbb{C}^n$ a linear subspace and $\varphi : U \to V$ a biholomorphic map sending $\widetilde{Y} \cap U$ to $L \cap V$ as in Definition 3.1.3. We may w.l.o.g. assume that L is the line parametrized by $\gamma(t) = (t, \dots, t)$ going through the origin. Then $\varphi^{-1}(\gamma)$ has components of t-adic order equal to one and parametrizes $(Y, 0)$.
$\Leftarrow$: Let $\gamma(t) = (\mathsf{x}_1(t), \dots, \mathsf{x}_n(t))$ be a parametrization of $(Y, 0)$ so that $\gamma(0) = 0$ and $\mathrm{ord}(\mathsf{x}_1) = 1$. Then for each representative $\widetilde{Y}$ of $(Y, 0)$ the map

$$\varphi : \widetilde{Y} \to L$$

$$(y_1, \dots, y_n) \mapsto (y_1, 0, \dots, 0),$$

with $L = V(x_i, i = 2, \dots, n) \subseteq \mathbb{C}^n$ one-dimensional linear subspace, is biholomorph with the inverse map defined by $\varphi^{-1}(t, 0, \dots, 0) = (\mathsf{x}_1(t), \dots, \mathsf{x}_n(t))$. Therefore, the conditions from Definition 3.1.3 are satisfied and $\widetilde{Y}$ is regular at the origin. $\square$

A Noetherian local ring R with maximal ideal m is called *regular* if m can be generated by d elements, where d is the Krull-dimension of R. The field $K = R/m$ is called the *residue field* of R and a minimal system of generators of m is called a *regular parameter system for R.*

Definition 3.1.5. *A point a of an algebraic variety $Z \subseteq \mathbb{A}^n$ is said to be regular if the local ring $\mathcal{O}_{Z,a} := \mathbb{C}[Z]_{m_{Z,a}}$ of Z at a is a regular local ring. Here $m_{Z,a}$ is a maximal ideal of the coordinate ring $\mathbb{C}[Z]$ defined as $m_{Z,a} = \{g \in \mathbb{C}[Z] | g(a) = 0\}$.*

There is also another equivalent definition of the singular and regular locus.

Definition 3.1.6. *Let $Z \subseteq \mathbb{A}^n$ be an algebraic variety defined by the ideal $I := (f_1, \ldots, f_m) \subseteq \mathbb{C}[x_1, ..., x_n]$ with $I = \sqrt{I}$. A point $a \in Z$ is called singular if for the rank of the Jacobian matrix D_I evaluated at the point a the following inequality holds:*

$$\operatorname{rk}(D_I(a)) < \operatorname{codim}_a(Z).$$

Here rk *denotes the rank of the matrix and* $\operatorname{codim}_a(Z)$ *the local codimension of Z in a. Otherwise the point a is said to be a regular point. The set of all singular points of Z is denoted by* $\operatorname{Sing}(Z)$ *and called the singular locus of Z. Its complement in Z,* $\operatorname{Reg}(Z) := Z \setminus \operatorname{Sing}(Z)$*, is called the regular locus of Z. The variety Z is called regular if each point on Z is a regular point.*

For the proof of the equivalence of both definitions see [Har77] Thm.5.1., p.32.

Theorem 3.1.7. *The ring $\mathcal{O}_{Z,a}$ is regular if and only if its completion $\widehat{\mathcal{O}_{Z,a}}$ is a regular local ring.*

Proof. [AM69] Prop.11.24., p.124. □

However, it is not clear how to decide based only on a parametrization at a point a of an algebraic curve $X \subseteq \mathbb{A}^n$ whether a is a regular or singular point of X. We will discuss this problem in the next section.

Let R be a Noetherian ring and I an ideal in R. The *height* of I is defined as the maximal length k of a chain of prime ideals $I_0 \subsetneq I_1 \subsetneq ... \subsetneq I_k = I$.

Krull's principal ideal theorem 3.1.8. *If R is a Noetherian ring and I is a principal, proper ideal of R, then I has height at most one. Moreover, if $I = (f)$ and f is a non-zero divisor in R, then I has height 1.*

The geometrical meaning of the height of an ideal I in the polynomial ring $\mathbb{C}[x_1, ..., x_n]$ is the following: Let $Z = V(I) \subseteq \mathbb{A}^n$ be the algebraic variety defined by I. Then the height of I coincides with the codimension of Z. We have $Z = Z_k \subsetneq Z_{k-1} \subsetneq ... \subsetneq Z_0 = V(0) = \mathbb{A}^n$, where $Z_i = V(I_i), I_i$ as in the definition above. Furthermore, the Krull-dimension of Z equals the Krull-dimension of the coordinate ring of Z, $\mathbb{C}[Z]$. Namely the chain of prime ideals $(\overline{0}) = p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_k \subseteq \mathbb{C}[Z]$ corresponds to the chain of irreducible varieties $Z_k \subsetneq Z_{k-1} \subsetneq \cdots \subsetneq Z_0 \subseteq Z = V(I)$ with $Z_i = V(p_i)$.

Let $\mathbb{C}(Z)$ be the function field of Z. Then we say that $\mathbb{C}(Z)$ has *transcendence degree* d over $\mathbb{C}$ if d is the maximal number of elements of $\mathbb{C}(Z)$ that are algebraically independent over $\mathbb{C}$.

Lemma 3.1.9. *For an algebraic variety $Z \subseteq \mathbb{A}^n$ the equality*

$$\dim(Z) = \text{transdeg}_{\mathbb{C}}(\mathbb{C}(Z))$$

holds.

Proof. [CLO15] Thm.7., p.511. $\square$

Corollary 3.1.10. *Let $Z_1 \subseteq \mathbb{A}^n$ and $Z_2 \subseteq \mathbb{A}^m$ be irreducible algebraic varieties which are birationally equivalent. Then $\dim(Z_1) = \dim(Z_2)$.*

Proof. [CLO15] Cor.8., p.512. $\square$

Lemma 3.1.11. *Let $\varphi : \mathbb{A}^n \to \mathbb{A}^m$ be a morphism. Let $Z \subseteq \mathbb{A}^n$ be an algebraic variety of Krull-dimension d. Then*

$$d \geq \dim(\overline{\varphi(Z)}),$$

where $\dim(\overline{\varphi(Z)})$ denotes the Krull-dimension of $\overline{\varphi(Z)}$.

Proof. If $Y := \overline{\varphi(Z)}$ is reducible, we consider for each irreducible component $Y_i \subseteq Y$ the restricted morphism $\varphi^{-1}(Y_i) \to Y$ and reduce the problem in this way to the case that $\varphi : Z \to Y$ is a dominant map with Y an irreducible variety.
Let Z_j be the irreducible components of Z. Then we have $Y = \bigcup \overline{\varphi(Z_j)}$, as Z is the union of finitely many Z_j's, and so we have already $Y = \overline{\varphi(Z_j)}$ for one j. Therefore, considering the restriction $\varphi|_{Z_j} : Z_j \to Y$, the problem is reduced to the case where $\varphi : Z \to Y$ is a dominant morphism between two irreducible varieties.
We proceed now by induction on the Krull-dimension of Y. The case $k = 0$ is clear. Let us assume that the statement is true for some $d \in \mathbb{N}$. Let Y be an algebraic variety of Krull-dimension $d + 1$. We construct a chain of maximal length of irreducible algebraic subvarieties of Y,

$$Y_{d+1} \subsetneq \cdots \subsetneq Y_0 = Y.$$

The preimages $\varphi^{-1}(Y_i)$ are algebraic varieties and we have

$$\varphi^{-1}(Y_{d+1}) \subsetneq \cdots \subsetneq \varphi^{-1}(Y_0) = \varphi^{-1}(Y) = Z.$$

We apply the induction hypothesis to the irreducible subvariety Y_1 of Y of Krull-dimension d. For the dominant morphism $\varphi^{-1}(Y_1) \to Y_1$ and the varieties $\varphi^{-1}(Y_1)$ and Y_1 we then have

$$\dim(\varphi^{-1}(Y_1)) \geq \dim(Y_1)$$

from which then, using the irreducibility of Z, follows

$$\dim(Z) \geq \dim(\varphi^{-1}(Y_1)) + 1 \geq \dim(Y_1) + 1 = \dim(Y).$$

$\square$

Lemma 3.1.12. *Let $Z \subseteq \mathbb{A}^n$ be an algebraic variety and $\mathbb{C}[Z]$ its coordinate ring. If $\dim(Z) = 0$, then $\mathbb{C}[Z]$ is a finite-dimensional $\mathbb{C}$-vector space.*

Proof. [AM69] Prop.6.10., p.78 and Thm.8.5., p.90. $\square$

Proposition 3.1.13. *Let $X_1, X_2 \subseteq \mathbb{A}^2$ be plane algebraic curves defined by polynomials $f_1, f_2 \in \mathbb{C}[x,y]$ of total degrees n_1 and n_2, respectively. Then X_1 and X_2 intersect in finitely many points if and only if f_1 and f_2 have no common irreducible factor. Even more, if X_1 and X_2 intersect in finitely many points, then we have*

$$|X_1 \cap X_2| \leq \dim_{\mathbb{C}} \mathbb{C}[x,y]/(f_1, f_2) \leq n_1 \cdot n_2.$$

Proof. $\Rightarrow$: Let us assume that $g \not\equiv 0$ is a common irreducible factor of f_1 and f_2. Then $\emptyset \neq V(g) \subseteq X_1 \cap X_2$ and so $|X_1 \cap X_2| = \infty$.
$\Leftarrow$: If X_1 and X_2 have no common irreducible component, then $\mathbb{C}[x,y]/(f_1, f_2)$ has Krull-dimension zero and according to Lemma 3.1.12 it is a finite-dimensional $\mathbb{C}$-vector space. For each finite set $\{p_1, \ldots, p_k\}$ of common points of X_1 and X_2 we can define the polynomials

$$h_i = \prod_{j \neq i} (x - p_{j_1}) \cdot \prod_{j \neq i} (y - p_{j_2}), i = 1, \ldots, k$$

that satisfy $h_i(p_i) \neq 0$ and $h_i(p_j) = 0$ for all $j \neq i$. Here $p_j = (p_{j_1}, p_{j_2})$. We then have that if

$$\sum_{i=1}^{k} c_i \cdot h_i = uf_1 + vf_2,$$

with some polynomials $u, v \in \mathbb{C}[x,y]$ and constants $c_i \in \mathbb{C}$, then after substituing the points p_i we get $c_i \cdot h_i(p_i) = 0$ wich implies $c_i = 0$ for all $i = 1, \ldots, k$. Hence, the images of $h_i, i = 1, \ldots, k$ in $\mathbb{C}[x,y]/(f_1, f_2)$ are linearly independent. And so the inequality

$$|X_1 \cap X_2| \leq \dim_{\mathbb{C}} \mathbb{C}[x,y]/(f_1, f_2)$$

was shown.
Let $\mathbb{C}[x,y]_d$ be the $\mathbb{C}$-vector space of polynomials of total degree at most d. Then $\dim_{\mathbb{C}}(\mathbb{C}[x,y]_d) = 1 + \cdots + (d+1) = \frac{1}{2}(d+1)(d+2)$. For $d \geq n_1 + n_2$ we consider the following sequence of linear maps

$$\mathbb{C}[x,y]_{d-n_1} \times \mathbb{C}[x,y]_{d-n_2} \to^{\alpha} \mathbb{C}[x,y]_d \to^{\pi} \mathbb{C}[x,y]_d/(f_1, f_2) \to 0,$$

where $\alpha(u,v) = uf_1 + vf_2$ and π is the quotient map. Since f_1 and f_2 have no common factor, the kernel of α consists of the pairs $(wf_2, -wf_1)$ with $w \in \mathbb{C}[x,y]_{d-n_1-n_2}$. Hence, $\dim_{\mathbb{C}}(\ker(\alpha)) = \frac{1}{2}(d-n_1-n_2+1)(d-n_1-n_2+2)$. Using the Rank-nullity theorem we get $\dim_{\mathbb{C}}(\mathrm{Im}(\alpha)) = \frac{1}{2}(d-n_1+1)(d-n_1+2) + \frac{1}{2}(d-n_2+1)(d-n_2+2) - \frac{1}{2}(d-n_1-n_2+1)(d-n_1-n_2+2)$. The surjectivity of π together with $\mathrm{Im}(\alpha) \subseteq \ker(\pi)$ implies that

$$\dim_{\mathbb{C}}(\mathbb{C}[x,y]_d/(f_1, f_2)) \leq \dim_{\mathbb{C}}(\mathbb{C}[x,y]_d) - \dim(\mathrm{Im}(\alpha)) =$$

$$= \frac{1}{2}(d+1)(d+2) - \frac{1}{2}(d-n_1+1)(d-n_1+2) -$$

$$-\frac{1}{2}(d-n_2+1)(d-n_2+2) - \frac{1}{2}(d-n_1-n_2+1)(d-n_1-n_2+2) = n_1 \cdot n_2.$$

And so the inequality $\dim_{\mathbb{C}}(\mathbb{C}[x,y]_c/(f_1,f_2)) \leq n_1 \cdot n_2$ is fulfilled for all $c \in \mathbb{N}$ and consequently $\dim_{\mathbb{C}}(\mathbb{C}[x,y]/(f_1,f_2)) \leq n_1 \cdot n_2$. $\square$

A sequence $r_0, \dots, r_d$ in a commutative ring R is called *regular* if r_i is a non-zero-divisor in $R/(r_1, \dots, r_{i-1})$ for all $i = 1, \dots, d$.

Definition 3.1.14. *Let $X \subseteq \mathbb{A}^n$ be an algebraic variety with the coordinate ring $\mathbb{C}[X]$ and let Z be a subvariety of X. Let $I = (g_1, \dots, g_k) \subseteq \mathbb{C}[X]$ be the defining ideal of Z. Assume that the set $X \backslash Z$ is Zariski-dense in X. The morphism*

$$\begin{aligned} \delta : X \backslash Z &\to \mathbb{P}^{k-1} \\ a &\mapsto (g_1(a) : \dots : g_k(a)) \end{aligned}$$

is welldefined. The Zariski-closure $\widetilde{X}$ of the graph Δ of δ inside $X \times \mathbb{P}^{k-1}$ together with the restriction $\pi|_{\widetilde{X}} : \widetilde{X} \to X$ of the projection map $\pi : X \times \mathbb{P}^{k-1} \to X$ is called a blowup of X along Z. Sometimes also called a blowup of X with center Z.

Remark 3.1.15. The definition of a blowup does not depend, up to an isomorphism over X, on the choice of the generators g_i of I. It can be shown that a blowup of X along Z is unique up to a unique isomorphism. Therefore, it is called the blowup of X along Z.

For more details see [Hau12].

3.2 Rational and geometric invariants of parametrized plane curves

The aim of this section is to introduce the concept of geometric invariants which will then play the key role by the process of resolving singularities of plane algebraic curves. Furthermore, we will establish in this section a criterion that works only with parametrizations of algebraic curves and that distinguishes between the regular and singular locus of algebraic curves.

A *reparametrization*

$$\begin{aligned} \varphi : \mathbb{C}\{t\} &\to \mathbb{C}\{t\} \\ t &\mapsto \varphi(t) \end{aligned}$$

is a $\mathbb{C}$-algebra automorphism of the convergent power series ring $\mathbb{C}\{t\}$. The image $\varphi(t)$ of t is then a power series of t-adic order equal to one and so φ is a local map, i.e., we have $\varphi(m) \subseteq m$ for $m = (t)$ the maximal ideal of the local ring $\mathbb{C}\{t\}$. Furthermore, $\varphi(m^k) = \varphi(m)^k \subseteq m^k$ for all $k \in \mathbb{N}$ and so φ is continuous with respect to the t-adic topology. Every convergent power series g can be written as limit of polynomials $g_i \in \mathbb{C}[t], i \in \mathbb{N}$, $g = \lim g_i$, where $g_i \equiv g \mod m^i$. As φ is continous, we have the equalities

$$\varphi(g(t)) = \varphi(\lim g_i(t)) = \lim \varphi(g_i(t))$$

and the map φ is completely determined by the image of t. The automorphism group of $\mathbb{C}\{t\}$, $(\mathrm{Aut}(\mathbb{C}\{t\}), \circ)$, acts on $\mathbb{C}\{t\}^2$ via

$$\begin{aligned} \mathrm{Aut}(\mathbb{C}\{t\}) \times \mathbb{C}\{t\}^2 &\to \mathbb{C}\{t\}^2 \\ (\varphi, (x, y)) &\mapsto \varphi * (x, y) := (\varphi(x), \varphi(y)) = (x(\varphi(t)), y(\varphi(t))). \end{aligned}$$

This induces the following left group action of $\mathrm{Aut}(\mathbb{C}\{t\})$ on $\mathbb{C}\{\{t\}\}$:

$$\begin{aligned} \mathrm{Aut}(\mathbb{C}\{t\}) \times \mathbb{C}\{\{t\}\} &\to \mathbb{C}\{\{t\}\} \\ \left(\varphi, \frac{x}{y}\right) &\mapsto \varphi * \left(\frac{x}{y}\right) := \frac{\varphi(x)}{\varphi(y)} = \frac{x(\varphi(t))}{y(\varphi(t))}. \end{aligned}$$

Let us associate to every rational function in an even number of variables

$$R = \frac{P}{Q} \in \mathbb{C}(u_0, v_0, \ldots, u_k, v_k)$$

the map

$$\begin{aligned} \psi_R : \mathbb{C}\{t\}^2 &\to \mathbb{C}\{\{t\}\} \\ (x, y) &\mapsto R \star (x, y) := \frac{P \star (x, y)}{Q \star (x, y)}. \end{aligned}$$

For a polynomial in an even number of variables

$$P(u_0,\dots,v_{k-1}) = \sum c_\alpha \cdot u_0^{\alpha_1} \cdot v_0^{\alpha_2} \cdots u_{k-1}^{\alpha_{2k-1}} \cdot v_{k-1}^{\alpha_{2k}},$$

the image of the map ψ_P is defined as

$$P \star (x(t), y(t)) = \sum c_\alpha \cdot x(t)^{\alpha_1} \cdot y(t)^{\alpha_2} \cdots x^{(k-1)}(t)^{\alpha_{2k-1}} \cdot y^{(k-1)}(t)^{\alpha_{2k}}.$$

Here for a convergent power series $z(t) \in \mathbb{C}\{t\}$, $z(t)^{(i)}$ denotes the i-th derivative of $z(t)$ with respect to t.

Remark 3.2.1. In general, an n-tuple of convergent power series does not necessarily parametrize an algebraic curve. For example the pair $(t, e^t) \in \mathbb{C}\{t\}^2$ cannot parametrize any plane algebraic curve. As e^t is not an algebraic power series, for every polynomial in two variables $f \in \mathbb{C}[x,y]$ we have $f(t,e^t) \neq 0$ in $\mathbb{C}\{t\}$. However, in the differential geometry, analytic curves in $\mathbb{C}^n$ are defined via their parametrizations by n-tuples of convergent power series. So the pair (t, e^t) parametrizes a plane analytic curve.

A rational function $R \in \mathbb{C}(u_0,\dots,v_k)$ in $2(k+1)$ variables, for any $k \in \mathbb{N}$, is called a *rational invariant of order* k if the associated map ψ_R is $\mathrm{Aut}(\mathbb{C}\{t\})$-equivariant. In other words, $R \in \mathbb{C}(u_0,\dots,v_k)$ is a rational invariant if and only if for every reparametrization $\varphi \in \mathrm{Aut}(\mathbb{C}\{t\})$ and every pair of convergent power series $(x,y) \in \mathbb{C}\{t\}^2$ the equality

$$\psi_R(\varphi * (x(t), y(t))) = \varphi * \psi_R(x(t), y(t))$$

holds in $\mathbb{C}\{\{t\}\}$. Note that we have the equalities

$$\psi_R(\varphi * (x(t), y(t))) = R \star (x(\varphi(t)), y(\varphi(t)))$$

and

$$\varphi * \psi_R(x(t), y(t)) = [R \star (x(t), y(t))]\,|_{t=\varphi(t)}.$$

Notice that to be a rational invariant does not depend on the pair of convergent power series $(x(t), y(t))$. For each non-negative integer $k \in \mathbb{N}$ we set

$$\Lambda_k := \{R \in \mathbb{C}(u_0, v_0 \dots, u_k, v_k) | R \text{ is a rational invariant}\}$$

as the set of all rational invariants of order k. And furthermore, we define the set of all rational invariants (of an arbitrary order)

$$\Lambda := \bigcup_{k\in\mathbb{N}} \Lambda_k.$$

Remark 3.2.2. The sets Λ and $\Lambda_k, k \in \mathbb{N}$ are fields.

Example 3.2.3. 1) For any polynomial in two variables $p \in \mathbb{C}[u,v]$, $R_1(u,v) = p(u,v)$ is a rational invariant of order 0.

2) $R_2(u_0, v_0, u_1, v_1) = \dfrac{u_1}{v_1}$ is a rational invariant of order 1 because of

$$R_2 \star (x(\varphi(t)), y(\varphi(t))) = \frac{\frac{\partial}{\partial t}x(\varphi(t))}{\frac{\partial}{\partial t}y(\varphi(t))} = \frac{x'(\varphi(t)) \cdot \varphi'(t)}{y'(\varphi(t)) \cdot \varphi'(t)} = [R_2 \star (x(t), y(t))]\,|_{t=\varphi(t)}.$$

3) $R_3(u_0, v_0, \ldots, u_2, v_2) = \dfrac{u_2 v_1 - u_1 v_2}{v_1^2}$ is not a rational invariant. We have

$$R_3 \star (x(\varphi(t)), y(\varphi(t))) =$$

$$= \frac{\frac{\partial^2}{\partial t}x(\varphi(t)) \cdot \frac{\partial}{\partial t}y(\varphi(t)) - \frac{\partial}{\partial t}x(\varphi(t)) \cdot \frac{\partial^2}{\partial t}y(\varphi(t))}{\frac{\partial}{\partial t}y(\varphi(t))^2} =$$

$$= \frac{x''(\varphi(t)) \cdot y'(\varphi(t)) - x'(\varphi(t)) \cdot y''(\varphi(t))}{y'(\varphi(t))^2} \cdot \varphi'(t) = \frac{\partial}{\partial t}\left(\frac{x'(\varphi(t))}{y'(\varphi(t))}\right)$$

but

$$[R_3 \star (x(t), (t))]\,|_{t=\varphi(t)} = \frac{x''(\varphi(t)) \cdot y'(\varphi(t)) - x'(\varphi(t)) \cdot y''(\varphi(t))}{y'(\varphi(t))^2}.$$

4) $R_4(u_0, v_0, \ldots, u_2 v_2) = \dfrac{u_2 v_1 - u_1 v_2}{v_1^3}$ is a rational invariant of order 2 since

$$R_4 \star (x(\varphi(t)), y(\varphi(t))) = \frac{x''(\varphi(t)) \cdot y'(\varphi(t)) - x'(\varphi(t)) \cdot y''(\varphi(t))}{y'(\varphi(t))^3 \varphi'(t)} \cdot \varphi'(t) =$$

$$= \frac{x''(\varphi(t)) \cdot y'(\varphi(t)) - x'(\varphi(t)) \cdot y''(\varphi(t))}{y'(\varphi(t))^3} = [R_4 \star (x(t), y(t))]\,|_{t=\varphi(t)}.$$

Notice that there is a relationship between R_2 and R_3, namely for each pair of convergent power series $(x,y) \in \mathbb{C}\{t\}^2$ we have:

$$R_3 \star (x,y) = \frac{\partial}{\partial t} R_2 \star (x,y).$$

And we introduce the concept of the modified derivative. A rational function in an even number of variables $R \in \mathbb{C}(u_0, v_0 \ldots, u_k, v_k)$ is called the *modified derivative of* $S \in \mathbb{C}(u_0, v_0 \ldots, u_{k-1}, v_{k-1})$ if for each pair of convergent power series $(x,y) \in \mathbb{C}\{t\}^2$ the equality

$$R \star (x,y) = \frac{\partial}{\partial t} S \star (x,y)$$

holds in $\mathbb{C}\{\{t\}\}$. We denote the modified derivative of S by ∂S. For a polynomial in an even number of variables

$$P = \sum_{(\alpha,\beta)\in N} c_{\alpha,\beta} \cdot u_0^{\alpha_0} \cdot \dots \cdot u_{k-1}^{\alpha_{k-1}} \cdot v_0^{\beta_0} \cdots v_{k-1}^{\beta_{k-1}},$$

where N is a finite subset of $\mathbb{N}^{2k}$, the modified derivative has the form

$$\begin{aligned}\partial P = & \sum_{(\alpha,\beta)\in N} \sum_{i=0}^{k-1} \alpha_i \cdot u_0^{\alpha_0} \cdots u_i^{\alpha_i - 1} \cdot u_{i+1}^{\alpha_{i+1}+1} \cdots u_{k-1}^{\alpha_{k-1}} \cdot v_0^{\beta_0} \cdots v_{k-1}^{\beta_{k-1}} + \\ & + \sum_{(\alpha,\beta)\in N} \sum_{i=0}^{k-1} \beta_i \cdot u_0^{\alpha_0} \cdots u_{k-1}^{\alpha_{k-1}} \cdot v_0^{\beta_0} \cdots v_i^{\beta_i - 1} \cdot v_{i+1}^{\beta_{i+1}+1} v_{k-1}^{\beta_{k-1}}.\end{aligned}$$

And using

$$\partial R = \frac{\partial P \cdot Q - P \cdot \partial Q}{Q^2}$$

for a rational function $R = \frac{P}{Q}$, with P, Q polynomials, this extends to a formula of the modified derivative of an arbitrary rational function in an even number of variables.

For a plane algebraic curve $X \in \mathbb{A}^2$ we define the set of the *geometric invariants of X of order k* as

$$\Lambda_{X,k} := \{R \star (\mathsf{x}(t), \mathsf{y}(t)) | R \in \Lambda_k, (\mathsf{x}(t), \mathsf{y}(t)) \text{ is a parametrization of } X\}$$

and the set of all *geometric invariants of X* to be the set

$$\Lambda_X := \{R \star (\mathsf{x}(t), \mathsf{y}(t)) | R \in \Lambda, (\mathsf{x}(t), \mathsf{y}(t)) \text{ is a parametrization of } X\}.$$

Remark 3.2.4. As the geometric invariants of a plane algebraic curve are invariant under reparametrization they depend only on the local geometry at a certain point of the curve itself.

Let us consider a parametrization $\gamma(t) = (\mathsf{x}(t), \mathsf{y}(t))$ of a plane algebraic curve $X \subseteq \mathbb{A}^2$. Let $\mathsf{z} \in \Lambda_Y$ be a geometric invariant of a plane algebraic curve $Y \subseteq \mathbb{A}^2$. Then adding z as the third component to γ gives the parametrization $(\mathsf{x}(t), \mathsf{y}(t), \mathsf{z}(t))$ of new analytic curve $X_{\mathsf{z}} \subseteq \mathbb{C}^3$. However, it is not clear yet whether this triple parametrizes an algebraic space curve or not. To clarify this problem, the following can be helpful.

Lemma 3.2.5. *Let $X = V(f) \subseteq \mathbb{A}^2$ be a plane algebraic curve and $(\mathsf{x}(t), \mathsf{y}(t))$ a parametrization of X. Then there exists some $p(t) \in \mathbb{C}\{\{t\}\}$ so that the equality*

$$(\mathsf{x}'(t), \mathsf{y}'(t)) = p(t) \cdot (-f_y(\mathsf{x}(t), \mathsf{y}(t)), f_x(\mathsf{x}(t), \mathsf{y}(t)))$$

holds in $\mathbb{C}\{\{t\}\}$.

Proof. We apply the chain rule to the equality $f(\mathsf{x}(t), \mathsf{y}(t)) = 0$ and get $\frac{\mathsf{y}'(t)}{\mathsf{x}'(t)} = -\frac{f_x(\mathsf{x}(t),\mathsf{y}(t))}{f_y(\mathsf{x}(t),\mathsf{y}(t))}$ in $\mathbb{C}\{\{t\}\}$. This implies the existence of a factor $p(t) \in \mathbb{C}\{\{t\}\}$ as in the lemma. $\square$

Theorem 3.2.6. *Let R be a rational function in $2(k+1)$ variables. Then the following are equivalent:*

(i) $R \in \Lambda_k$.
(ii) There exist two modified differential operators P, Q of order k,

$$P(h) = \sum_{\alpha \in N} a_\alpha \prod_{0 \le i,j \le k} \left(\frac{\partial^i}{\partial x} \frac{\partial^j}{\partial y} h \right)^{\alpha_{i,j}},$$

$$Q(h) = \sum_{\beta \in M} b_\beta \prod_{0 \le i,j \le k} \left(\frac{\partial^i}{\partial x} \frac{\partial^j}{\partial y} h \right)^{\beta_{i,j}},$$

with constants $a_\alpha, b_\beta \in \mathbb{C}$ and N, M finite subsets of $\mathbb{N}^{(k+1)^2}$, so that for each plane algebraic curve $X = V(f) \subseteq \mathbb{A}^2$ and every parametrization $(\mathsf{x}(t), \mathsf{y}(t))$ of X the equality

$$R \star (\mathsf{x}(t), \mathsf{y}(t)) = \left(\frac{P(f)}{Q(f)} \right) (\mathsf{x}(t), \mathsf{y}(t))$$

holds in $\mathbb{C}\{\{t\}\}$. Here $\left(\frac{P(f)}{Q(f)}\right)(\mathsf{x}(t), \mathsf{y}(t))$ denotes the modified differential operators P and Q applied to f and consecutive substitution of variables $(x, y) \mapsto (\mathsf{x}(t), \mathsf{y}(t))$.

Proof. We proceed by induction on the order k. We set

$$f_x := f_x(\mathsf{x}(t), \mathsf{y}(t)), f_y := f_y(\mathsf{x}(t), \mathsf{y}(t)) \text{ etc. and } \mathsf{x}' = \mathsf{x}(t), \mathsf{y}' = \mathsf{y}(t) \text{ etc.}$$

Recall the existence of a factor $p \in \mathbb{C}\{\{t\}\}$ so that the equality

$$(\mathsf{x}', \mathsf{y}') = p \cdot (-f_y, f_x)$$

holds in $\mathbb{C}\{\{t\}\}$. Furthermore, for each pair of convergent power series $(r(t), s(t)) \in \mathbb{C}\{t\}^2$ we have

$$\left(\frac{\partial}{\partial t} r(\varphi(t)), \frac{\partial}{\partial t} s(\varphi(t)) \right) = \varphi' \cdot (r'(\varphi(t)), s'(\varphi(t))).$$

And we see from the above two equalities that for each rational function in 4 variables $S_1(u_0, v_0, u_1, v_1) \in \mathbb{C}(u_0, v_0, u_1, v_1)$ and every parametrization $(\mathsf{x}(t), \mathsf{y}(t))$ of X the following are equivalent:

(*i*) $S_1 \star (r(\varphi(t)), s(\varphi(t))) = [S_1 \star (r,s)]\,|_{t=\varphi(t)}$, i.e. $S_1 \in \Lambda_1$.
(*ii*) $S_1(\mathsf{x},\mathsf{y},\mathsf{x}',\mathsf{y}') = S_1(\mathsf{x},\mathsf{y},-f_y,f_x)$.

And the claim is shown for each rational invariant of order 1. To see how the mechanism of the proof works, let us now discuss the case of rational invariants of order 2. Let us consider a rational function in 6 variables $S_2(u_0,...,v_2) \in \mathbb{C}(u_0,...,v_2)$. For every parametrization γ of X, S_2 induces a rational function in γ, γ' and γ'', $S_2\star(\mathsf{x},\mathsf{y})$. Similarly as before, the derivative of the above equalities with respect to t and the substitution of $(\mathsf{x}',\mathsf{y}') = p\cdot(-f_y,f_x)$ into the first one yields

$$(\mathsf{x}'',\mathsf{y}'') = p'\cdot(-f_y,f_x) + p^2\cdot(f_{xy}f_y - f_{yy}f_x, -f_{xx}f_y + f_{xy}f_x)$$

and

$$\left(\frac{\partial^2}{\partial t}r(\varphi(t)), \frac{\partial^2}{\partial t}s(\varphi(t))\right) = \varphi''\cdot(r'(\varphi(t)), s'(\varphi(t))) + \varphi'^2\cdot(r''(\varphi(t)), s''(\varphi(t))).$$

Successively, using the chain rule and substituing $(\mathsf{x}',\mathsf{y}') = p\cdot(-f_y,f_x)$ after each derivative gives for $\gamma^{(k)}$ and $\frac{\partial^k}{\partial t}(r(\varphi(t)), s(\varphi(t)))$ again two equalities which are completely symmetric in the derivatives of p and φ. Thus, for an arbitrary rational function $S_k, k \in \mathbb{N}$ in $2(k+1)$ variables the following are equivalent:

(*i*) $S_k \star (r(\varphi(t)), s(\varphi(t))) = [S_k \star (r,s)]\,|_{t=\varphi(t)}$, i.e. $S_k \in \Lambda_k$.
(*ii*) $S_k(\mathsf{x},\mathsf{y},\mathsf{x}',\mathsf{y}',\ldots,\mathsf{x}^{(k)},\mathsf{y}^{(k)}) = S_k(\mathsf{x},\mathsf{y},-f_y,f_x,f_{xy}f_y - f_{yy}f_x, -f_{xx}f_y + f_{xy}f_x,\ldots)$,

and the claim follows. □

Theorem 3.2.7. *Let $X \subseteq \mathbb{A}^2$ be an irreducible plane algebraic curve parametrized by $\gamma(t) = (\mathsf{x}(t),\mathsf{y}(t))$. Let $R_i \in \Lambda, i = 1,\ldots,k$ be rational invariants and $\mathsf{z}_i(t) = R_i \star (\mathsf{x}(t),\mathsf{y}(t))$ the corresponding geometric invariants of X. If $\mathrm{ord}_0(\mathsf{z}_i(t)) \geq 0$ for all i, then $(\mathsf{x}(t),\mathsf{y}(t),\mathsf{z}_1(t),\ldots,\mathsf{z}_k(t))$ parametrizes an algebraic space curve X_{z} that is birationally equivalent to X. Here $\mathrm{ord}_0(\mathsf{z}_i(t))$ denotes the t-adic order of the power series $\mathsf{z}_i(t)$.*

Proof. Let $f \in \mathbb{C}[x,y]$ be the defining polynomial of X. Let $P_{\mathsf{z}_i}, Q_{\mathsf{z}_i}$ be the to z_i corresponding modified differential operators, i.e.

$$\mathsf{z}_i(t) = \left(\frac{P_{\mathsf{z}_i}(f)}{Q_{\mathsf{z}_i}(f)}\right)(\mathsf{x}(t),\mathsf{y}(t)).$$

Then the map

$$\tau : X \to \mathbb{A}^{k+2}$$
$$(x,y) \mapsto (x, y, \left(\frac{P_{\mathsf{z}_1}(f)}{Q_{\mathsf{z}_1}(f)}\right)(x,y), \ldots, \left(\frac{P_{\mathsf{z}_k}(f)}{Q_{\mathsf{z}_k}(f)}\right)(x,y))$$

is defined on $X\backslash(\bigcup V(Q_{\mathsf{z}_i}(f)))$. But since $Q_{\mathsf{z}_i}(f)$ is not divisible by the polynomial f for all $i = 1, \ldots, k$, otherwise z_i would not be well-defined, according to Proposition 3.1.13 the set $X \cap (\bigcup V(Q_{\mathsf{z}_i}(f)))$ is finite and so τ is defined on a dense subset of X. Hence, X and X_{z} are birationally equivalent. It follows then from te Corollary 3.1.10 that X_{z} is an algebraic space curve. □

Corollary 3.2.8. *Let $X \subseteq \mathbb{A}^2$ be an irreducible plane algebraic curve parametrized by $\gamma(t) = (\mathsf{x}(t), \mathsf{y}(t))$. Let $\mathsf{z}_1(t), \ldots, \mathsf{z}_m(t) \in \Lambda_X$ be geometric invariants of X that stem all from the parametrization $\gamma(t)$. Then the m-tuple $(\mathsf{z}_1(t), \ldots, \mathsf{z}_m(t))$ parametrizes an algebraic space curve $Z \subseteq \mathbb{A}^m$.*

Proof. According to Theorem 3.2.7, $(\mathsf{x}(t), \mathsf{y}(t), \mathsf{z}_1(t), \ldots, \mathsf{z}_m(t))$ parametrizes an algebraic space curve $\widetilde{X}$. Now apply Lemma 3.1.11 to $\overline{\pi(\widetilde{X})}$, where $\pi : \mathbb{A}^{m+2} \to \mathbb{A}^m$, $(x, y, z_1 \ldots, z_m) \mapsto (z_1, \ldots, z_m)$. □

Let $X \subseteq \mathbb{A}^2$ be a plane algebraic curve. Assume that X has a singular branch $(Y, 0)$ at the origin. Let (x, y) be a parametrization of $(Y, 0)$. W.l.o.g. we may assume that $(\mathsf{x}(0), \mathsf{y}(0)) = 0$. The goal is to find a rational invariant R such that the triple $(\mathsf{x}, \mathsf{y}, \mathsf{z}) := (\mathsf{x}, \mathsf{y}, R \star (\mathsf{x}, \mathsf{y}))$ parametrizes a regular branch $(Y_{\mathsf{z}}, (0, 0, \mathsf{z}(0)))$ of a space curve X_{z}. In general, even though we know the modified differential operators defining z, it is not easy to find the defining equations of the curve X_{z}. Thus, the definition of the singular locus that uses Jacobian matrix is not helpful at the moment. And so, for further work, it is necessary to develop a criterion which can read off from parametrizations of algebraic curves how their singular locus looks like.

Theorem 3.2.9. *Let $X = V(I) \subseteq \mathbb{A}^n$ be an algebraic curve. Let $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be a radical ideal. Assume that $0 \in X$ and that X is analytically irreducible at the origin. If X can be parametrized at the origin by an n-tuple of convergent power series $(\mathsf{x}_1, \ldots, \mathsf{x}_n) \in \mathbb{C}\{t\}^n$ with $\mathsf{x}_i(0) = 0$ for all $i = 1, \ldots, n$ and $\mathrm{ord}_0(\mathsf{x}_i(t)) = 1$ for at least one $i \in \{1, \ldots, n\}$, then X is regular at the origin.*

Proof. Let $(\mathsf{x}_1, \ldots, \mathsf{x}_n)$ be a parametrization of X with $\mathsf{x}_i(0) = 0$ for all $i = 1, \ldots, n$ and $\min\{\mathrm{ord}_0(\mathsf{x}_i(t))\} = 1$. We can assume $\mathrm{ord}_0(\mathsf{x}_1(t)) = 1$. Using Proposition 5.1.16, the density of the image of the parametrization

$$\gamma : \bigcap_{i=1}^{n} D(\mathsf{x}_i(t)) \to X$$

$$a \mapsto (\mathsf{x}_1(a), \ldots, \mathsf{x}_n(a))$$

implies the injectivity of the map

$$\gamma^* : \mathbb{C}[x_1, \ldots, x_n]/I \to \mathbb{C}\{t\}$$

$$\overline{x_i} \mapsto \mathsf{x}_i(t).$$

The map γ^* induces

$$\widehat{\gamma^*} : \widehat{\mathcal{O}_{X,0}} = \mathbb{C}[[x_1, \ldots, x_n]]/I \to \mathbb{C}[[t]]$$

$$\overline{x_i} \mapsto \mathsf{x}_i(t).$$

Since X is analytically irreducible at the origin, the ring $\widehat{\mathcal{O}_{X,0}}$ is an integral domain. As $\widehat{\gamma^*}$ maps between two integral domains, $\ker(\widehat{\gamma^*})$ must be a prime ideal. Then either $\ker(\widehat{\gamma^*}) = (\overline{x}_i, i = 1, \ldots, n)$, because of $0 \in X$, or $\ker(\widehat{\gamma^*}) = (\overline{0})$. But the case $\ker(\widehat{\gamma^*}) = (\overline{x}_i, i = 1, \ldots, n)$ is not possible because the image of γ is Zariski-dense in X. Hence, $\ker(\widehat{\gamma^*}) = (\overline{0})$ and $\widehat{\gamma^*}$ is injective. We now show the regularity of $\widehat{\mathcal{O}_{X,0}}$ which implies, according to the Theorem 3.1.7, the regularity of $\mathcal{O}_{X,0}$. Our claim is that for each $i = 2, \ldots, n$ there exist power series $F_i(\overline{x}_1)$ so that the equality $\overline{x}_i = F_i(\overline{x}_1) \cdot \overline{x}_1$ holds. This implies then that the maximal ideal $\widehat{m_{X,0}}$ can be generated by a single element and as the the Krull-dimension of $\mathbb{C}[[x_1, \ldots, x_n]]/I$ equals one, the statement follows.
As $\mathrm{ord}_0(\mathsf{x}_1(t)) = 1$ it follows that there exists a power series $g(t)$ with $\mathrm{ord}_0(g(t)) = 1$ so that the equality $g(\mathsf{x}_1) = t$ is fulfilled. And so we have $G_i(\mathsf{x}_1) := (\mathsf{x}_i(g(\mathsf{x}_1)) = \mathsf{x}_i$ for all $i = 2, \ldots, n$. Because of $\mathsf{x}_i(0) = 0$, we get $G_i(0) = 0$ for all $i = 1, \ldots, n$. Thus, we can write $G_i(\mathsf{x}_1) = \tilde{G}_i(\mathsf{x}_1) \cdot \mathsf{x}_1^m$ for a suitable power $m > 0$ and $\tilde{G}_i(\mathsf{x}_1) \in \mathbb{C}[[\mathsf{x}_1]]^*$. Let us consider the term $G_i(\overline{x}_1) - \overline{x}_i \in \mathbb{C}[[x_1, \ldots, x_n]]/I$. Evidently, we have the equality $\widehat{\gamma^*}(G_i(\overline{x}_1) - \overline{x}_i) = G_i(\mathsf{x}_1) - \mathsf{x}_i = 0$. From the injectivity of $\widehat{\gamma^*}$ follows $G_i(\overline{x}_1) - \overline{x}_i = 0$ and from the equality $\tilde{G}_i(\overline{x}_1) \cdot \overline{x}_1^m = \overline{x}_i$ we get $\overline{x}_i \in (\overline{x}_1)$ for all $i = 2, \ldots, n$ which finishes the proof. $\square$

For a parametrization $\gamma = (\mathsf{x}_1, ..., \mathsf{x}_n)$ of an algebraic curve $X \subseteq \mathbb{A}^n$ at a point $a \in X$, with $(\mathsf{x}_1(b), ..., \mathsf{x}_n(b)) = a$ for some $b \in \mathbb{C}$, we call the value

$$\mathrm{ord}_b(\gamma) := \min_{i \in [n]} \{\mathrm{ord}_b(\mathsf{x}_i(t))\}$$

the *order of* γ *at* b. Here $\mathrm{ord}_b(\mathsf{x}_i(t))$ is the $(t-b)$-adic order of $\mathsf{x}_i(t)$.

Remark 3.2.10. Let $X \subseteq \mathbb{A}^n$ be an algebraic curve and $a \in X$ a point of X.
(i) The regularity of X at a does not imply that each parametrization γ of X at a with $\gamma(b) = a$ has order 1 at b.
(ii) Only the existence of a parametrization γ at a of X with $\gamma(b) = a, \mathrm{ord}_b(\gamma) = 1$ does not automatically imply the regularity of X at a.

To illustrate the problem of the remark let us consider the following three examples:

Example 3.2.11. 1) Let X be the plane algebraic curve that is parametrized by $\gamma(t) = (t, t^2)$. The defining equation of X is $f(x, y) = x^2 - y$. As X is analytically irreducible at the origin, $\gamma(0) = (0, 0)$ and $\mathrm{ord}_0(\gamma) = 1$, the proposition applies and tells us that X is regular.

2) Let X be the plane algebraic curve parametrized by $\gamma(t) = (t^2, t^4)$. Then X is defined by the equation $f(x, y) = x^2 - y$ as well and γ parametrizes the same curve as in example 1). Obviously the curve X is regular. But $\mathrm{ord}_0(\gamma) > 1$.

3) Let X be the plane algebraic curve parametrized by $\gamma(t) = (t^2 - 1, t^3 - t)$. The defining

polynomial of X is $f(x,y) = x^2+x^3-y^2$. We have $\gamma(-1) = \gamma(1) = (0,0)$ and the Taylor expansions of $\gamma(t)$ at 1 and -1 are

$$\gamma(t) = (2(t-1)+(t-1)^2, 2(t-1)+3(t-1)^2+(t-1)^3)$$

and

$$\gamma(t) = (-2(t+1)+(t+1)^2, 2(t+1)-3(t+1)^2+(t+1)^3).$$

Because of the terms $\pm 2(t \pm 1)$ in each component of γ we have $\mathrm{ord}_{\pm 1}(\gamma) = 1$ and so the branches of X that are parametrized by γ are regular. However the curve X itself is singular at the origin. The reason for that is the analytical reducibility of X at the origin. X has namely at the origin two branches defined by convergent power series $g_1 = x\sqrt{x+1} - y, g_2 = x\sqrt{x+1} + y \in \mathbb{C}\{x,y\}$.

3.3 Modified higher curvatures of plane algebraic curves

The goal of this section is to introduce one system of rational invariants that will then later play the key role in the resolution of singularities. We will also study the geometric invariants of plane algebraic curves corresponding to this system of rational invariants, the so-called modified higher curvatures, and the behavior of their t-adic orders.

From now on let $X \subseteq \mathbb{A}^2$ be a plane algebraic curve and $\gamma(t) = (\mathsf{x}(t), \mathsf{y}(t))$ a parametrization of X at the origin with $\gamma(0) = 0$. We denote $a = \mathrm{ord}_0(\mathsf{x}(t)), b = \mathrm{ord}_0(\mathsf{y}(t))$ and assume that $b \leq a$.

Proposition 3.3.1. *The rational function*

$$R_{\mathsf{s}} := \frac{u_1}{v_1} \in \mathbb{C}(u_0, v_0, u_1, v_1)$$

is a rational invariant. This induces the geometric invariant

$$\mathsf{s}(t) = \frac{\mathsf{x}'(t)}{\mathsf{y}'(t)}$$

of X.

Proof. See Example 3.2.3. □

Lemma 3.3.2. *Furthermore, we have in $\mathbb{C}\{\{t\}\}$ the equality*

$$\mathsf{s}(t) = \frac{\mathsf{x}'(t)}{\mathsf{y}'(t)} = -\frac{f_y}{f_x}(\mathsf{x}(t), \mathsf{y}(t)).$$

Proof. Follows directly from Lemma 3.2.5. □

Remark 3.3.3. The evaluation of the geometric invariant $\mathsf{s}(t)$ at 0 can be also interpreted as one of the affine chart expressions of the projective point $(\mathsf{y}'(0) : \mathsf{x}'(0)) \in \mathbb{P}^1_{\mathbb{C}}$ which is known from differential geometry and called the slope of the tangent vector of X at 0. The other chart expression is given by $\frac{1}{\mathsf{s}(0)}$ and stems from the geometric invariant $\frac{1}{\mathsf{s}(t)}$. Both these expressions give us equivalent geometric informations about X at the points parametrized by $(\mathsf{x}(t), \mathsf{y}(t))$ and therefore, it is enough to work with only one of them.

We will call in this thesis also the geometric invariants $\mathsf{s}(t)$ and $\frac{1}{\mathsf{s}(t)}$ the slope of the tangent vector. As for the t-adic order of s, we have the equality

$$\mathrm{ord}_0(\mathsf{s}(t)) = \mathrm{ord}_0(\mathsf{x}'(t)) - \mathrm{ord}_0(\mathsf{y}'(t)) = a - 1 - (b - 1) = a - b.$$

The task now is to find further rational invariants. We can consider for example the modified derivative of R_{s},

$$\partial R_{\mathsf{s}} = \frac{u_2 v_1 - u_1 v_2}{v_1^2}.$$

But, as we have already seen in Example 3.2.3, ∂R_{s} is not a rational invariant. It turns out that a small modification in the denominator of ∂R_{s} does yield a geometric invariant. The multiplication of ∂R_{s} with the term

$$S(u_0, v_0, u_1, v_1) = \frac{v_1^2}{(u_1 + v_1)^3}$$

is such a possible modification. The rational function we get in this way,

$$R_\kappa := \frac{u_2 v_1 - u_1 v_2}{(u_1 + v_1)^3},$$

is then a rational invariant. We namely have

$$\begin{aligned} R_\kappa \star (\mathsf{x}(\varphi(t)), \mathsf{y}(\varphi(t))) = \partial R_{\mathsf{s}} \star (\mathsf{x}(\varphi(t)), \mathsf{y}(\varphi(t))) \cdot S \star (\mathsf{x}(\varphi(t)), \mathsf{y}(\varphi(t))) = \\ = \varphi'(t) \cdot [\partial R_{\mathsf{s}} \star (\mathsf{x}(t), \mathsf{y}(t))]|_{t=\varphi(t)} \cdot \frac{\varphi'(t)^2 \cdot \mathsf{y}'(\varphi(t))^2}{\varphi'(t)^3 \cdot (\mathsf{x}'(\varphi(t)) + \mathsf{y}'(\varphi(t)))^3} = \\ = [\partial R_{\mathsf{s}} \star (\mathsf{x}(t), \mathsf{y}(t))]|_{t=\varphi(t)} \cdot [S \star (\mathsf{x}(t), \mathsf{y}(t))]|_{t=\varphi(t)} = [R_\kappa \star (\mathsf{x}(t), \mathsf{y}(t))]|_{t=\varphi(t)}. \end{aligned}$$

Proposition 3.3.4. *The rational function*

$$R_\kappa := \frac{u_2 v_1 - u_1 v_2}{(u_1 + v_1)^3} \in \mathbb{C}(u_0, \ldots, v_2)$$

is a rational invariant that induces the geometric invariant

$$\kappa(t) := \frac{\mathsf{x}''(t) \cdot \mathsf{y}'(t) - \mathsf{x}'(t) \cdot \mathsf{y}''(t)}{(\mathsf{x}'(t) + \mathsf{y}'(t))^3}$$

of X.

Lemma 3.3.5. *We have the equality*

$$\kappa(t) = \left(\frac{f_{xx} f_y^2 + 2 f_{xy} f_x f_y - f_{yy} f_x^2}{(-f_y + f_x)^3} \right) (\mathsf{x}(t), \mathsf{y}(t))$$

in $\mathbb{C}\{\{t\}\}$.

Proof. The invariance under reparametrization has been already shown.
Let us write $\mathsf{x} = \mathsf{x}(t), \mathsf{y} = \mathsf{y}(t)$ and $f_x = f_x(\mathsf{x}(t), \mathsf{y}(t)), f_y = f_y(\mathsf{x}(t), \mathsf{y}(t))$, etc. We have

$$\mathsf{s}'(t) = \left(-\frac{f_y}{f_x} \right)' = -\frac{(f_{yx} \cdot \mathsf{x}' + f_{yy} \cdot \mathsf{y}') \cdot f_x + (f_{xx} \cdot \mathsf{x}' + f_{xy} \cdot \mathsf{y}') \cdot f_y}{f_x^2}.$$

Furthermore,

$$\frac{\mathsf{y}'^2}{(\mathsf{x}'+\mathsf{y}')^3} = \frac{f_x^2}{p(t)\cdot(-f_y+f_x)^3}.$$

When we substitute $(\mathsf{x}',\mathsf{y}') = p(t)\cdot(-f_y, f_x)$ into the first equality for $\mathsf{s}'(t)$ and use that $f_{xy} = f_{yx}$ we get for κ the following equality

$$\kappa(t) = \frac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}{(-f_y+f_x)^3}.$$

$\square$

Note that we have the equality $\operatorname{ord}_0(\kappa(t)) = a - 2b$.

Remark 3.3.6. The expression

$$\kappa(t) = \frac{\mathsf{x}''(t)\mathsf{y}'(t) - \mathsf{x}'(t)\mathsf{y}''(t)}{(\mathsf{x}'(t)+\mathsf{y}'(t))^3}$$

is closely related to

$$\widetilde{\kappa}(t) := \frac{\mathsf{x}''(t)\mathsf{y}'(t) - \mathsf{x}'(t)\mathsf{y}''(t)}{\sqrt{(\mathsf{x}'(t)^2+\mathsf{y}'(t)^2)^3}}$$

which is the formula for the curvature of X of $(\mathsf{x}(t),\mathsf{y}(t))$ known from differential geometry. Hence we call $\kappa(t)$ the *modified curvature* of X.

In the same way we got a new geometric invariant of X from the modified derivative of the slope of the tangent vector, we can define further geometric invariants of X recursively.

Proposition 3.3.7. *Let $R \in \mathbb{C}(u_0, v_0, \ldots, u_k, v_k)$, for some $k \in \mathbb{N}$, be a rational invariant. Let $G(u_0, v_0, u_1, v_1) \in \mathbb{C}[u_0, v_0, u_1, v_1], G \not\equiv 0$, be a polynomial in four variables that satisfies the equality*

$$G \star (x(\varphi(t)), y(\varphi(t))) = [G \star (x(t), y(t))]|_{t=\varphi(t)} \cdot \varphi'(t)$$

for all pairs of convergent power series $(x(t), y(t)) \in \mathbb{C}\{t\}^2$ and all reparametrizations $\varphi \in Aut(\mathbb{C}\{t\})$. Then the rational function

$$\frac{\partial R}{G} \in \mathbb{C}(u_0, v_0, \ldots, u_{k+1}, v_{k+1})$$

is a rational invariant as well. Furthermore, for all pairs of convergent power series $(x(t), y(t)) \in \mathbb{C}\{t\}^2$ that satisfy $\operatorname{ord}_0(R \star (x(t), y(t))) > 0$ we have the inequality

$$\operatorname{ord}_0(R \star (x(t), y(t))) > \operatorname{ord}\left(\left(\frac{\partial R}{G}\right) \star (x(t), y(t))\right).$$

Proof. For each pair of convergent power series $(x(t), y(t))) \in \mathbb{C}\{t\}^2$ and each reparametrization $\varphi \in \mathrm{Aut}(\mathbb{C}\{t\})$ we have

$$\frac{\partial}{\partial t}(R \star (x(\varphi(t)), y(\varphi(t)))) = [\partial R \star (x(t), y(t))]|_{t=\varphi(t)} \cdot \varphi'(t).$$

With the conditions on G we get

$$\left(\frac{\partial R}{G}\right) \star (x(\varphi(t)), y(\varphi(t))) = \frac{\frac{\partial}{\partial t}(R \star (x(\varphi(t)), y(\varphi(t))))}{G \star (x(\varphi(t)), y(\varphi(t)))} =$$

$$= \frac{[\partial R \star (x(t), y(t))]|_{t=\varphi(t)} \cdot \varphi'(t)}{[G \star (x(t), y(t))]|_{t=\varphi(t)} \cdot \varphi'(t)} = \left[\left(\frac{\partial R}{G}\right) \star (x(t), y(t))\right] |_{t=\varphi(t)}.$$

Hence, the rational function $\frac{\partial R}{G}$ is a rational invariant. Regarding the order of $R \star (x(t), y(t))$, the equalities

$$\mathrm{ord}_0(\partial R \star (x(t), y(t))) = \mathrm{ord}_0(R \star (x(t), y(t))) - 1$$

and

$$\mathrm{ord}_0(G \star (x(t), y(t))) \geq 0$$

imply

$$\mathrm{ord}_0\left(\left(\frac{\partial R}{G}\right) \star (x(t), y(t))\right) \leq \mathrm{ord}_0(R \star (x(t), y(t))) - 1 < \mathrm{ord}_0(R \star (x(t), y(t))).$$

□

Thus, in the way described in Proposition 3.3.7 we can generate a sequence of geometric invariants of X of decreasing order. We can recursively construct the rational invariants

$$\begin{aligned}
R_{\mathsf{s}} &:= \frac{u_1}{v_1} \in \mathbb{C}(u_0, v_0, u_1, v_1), \\
R_{\kappa} &:= \frac{\partial R_{\mathsf{s}}}{u_1 + v_1} \in \mathbb{C}(u_0, v_0, \ldots, u_2, v_2), \\
R_{\kappa_1} &:= \frac{\partial R_{\kappa}}{u_1 + v_1} \in \mathbb{C}(u_0, v_0, \ldots, u_3, v_3), \\
&\vdots \\
R_{\kappa_n} &:= \frac{\partial R_{\kappa_{n-1}}}{u_1 + v_1} \in \mathbb{C}(u_0, v_0, \ldots, u_{n+2}, v_{n+2}), \\
&\vdots
\end{aligned}$$

that induce the following geometric invariants of X:

$$
\begin{aligned}
\mathsf{s}(t) &:= \frac{\mathsf{x}'(t)}{\mathsf{y}'(t)} \text{ with } \operatorname{ord}_0(\mathsf{s}(t)) = a - b,\\
\kappa(t) &:= \frac{\mathsf{s}'(t)}{\mathsf{x}'(t)+\mathsf{y}'(t)} \text{ with } \operatorname{ord}_0(\kappa(t)) = a - 2b,\\
\kappa_1(t) &:= \frac{\kappa'(t)}{\mathsf{x}'(t)+\mathsf{y}'(t)} \text{ with } \operatorname{ord}_0(\kappa_1(t)) = a - 3b,\\
&\vdots\\
\kappa_n(t) &:= \frac{\kappa_{n-1}'(t)}{\mathsf{x}'(t)+\mathsf{y}'(t)} \text{ with } \operatorname{ord}_0(\kappa_n(t)) = a - (2+n)b,\\
&\vdots
\end{aligned}
$$

We call each $\kappa_i(t), i \in \mathbb{N}$ a *modified higher curvature of* X.

4 Resolution of singularities of plane algebraic curves via geometric invariants

4.1 Resolution of analytically irreducible plane algebraic curves

In this section we will put into practice the knowledge about rational and geometric invariants we gained in the last chapter. We will use certain properties of geometric invariants to get an idea how to construct an algorithm that resolves with help of rational invariants singularities of analytically irreducible plane algebraic curves. More precisely, for an analytically irreducible plane algebraic curve $X \subseteq \mathbb{A}^2$ with a singularity at the origin the algorithm will construct a rational invariant $R = \frac{R_1}{R_2}$ so that the Zariski-closure of the graph Δ of the map

$$\delta : X\backslash Z \to \mathbb{P}^2$$

$$(x, y) \mapsto (R_1(x, y) : R_2(x, y))$$

is the blowup of X with center $Z = X \cap V(R_1, R_2)$ and is regular at the points lying over the origin. Repated use of this algorithm to each singular point together with an additional ingredient will then construct for an arbitrary analytically irreducible plane algebraic curve a regular blowup of the curve with a suitable center.

As already mentioned, we will use some important properties of geometric invariants for the construction of the algorithm. We will even prove with help of Puiseux parametrizations and geometric invariatns of plane algebraic curves that the algorithm really works. The main component of the proof will be Theorem 3.2.9 which uses orders of parametrizations of an algebraic curve to decide whether a point on the curve is regular or singular. To guarantee that the Theorem 3.2.9 applies in the situation described above let us prove the following proposition:

Proposition 4.1.1. *Let $X \subseteq \mathbb{A}^2$ be a plane algebraic curve parametrized by $(\mathsf{x}(t), \mathsf{y}(t))$ at the origin. Consider rational invariants $R_i \in \Lambda, i = 1, \dots k$ and the corresponding geometric invariants $\mathsf{z}_i(t) = R_i \star (\mathsf{x}(t), \mathsf{y}(t)) \in \Lambda_X$ with $\mathrm{ord}_0(\mathsf{z}_i) \geq 0$ for all i. If X is analytically irreducible at the origin, then the algebraic curve $X_{\mathsf{z}} \subseteq \mathbb{A}^{k+2}$ that is parametrized by $(\mathsf{x}(t), \mathsf{y}(t), \mathsf{z}_1(t), \dots, \mathsf{z}_k(t))$ is analytically irreducible at the points lying over $(0, 0)$.*

Proof. Let $f \in \mathbb{C}[x, y]$ be the defining polynomial of X. For all $i \in \{1, \dots, k\}$ let $P_{\mathsf{z}_i}, Q_{\mathsf{z}_i}$ be the to z_i corresponding modified differential operators. Let $c \in X_{\mathsf{z}}$ be a point lying over $(0, 0)$. If X_{z} were not analytically irreducible at c, then X_{z} would have at least two distinct branches (Y_1, c) and (Y_2, c) at this point. We distinguish two cases:
1) Assume that there exists a representative $\widetilde{Y_1}$ of (Y_1, c) and a representative $\widetilde{Y_2}$ of (Y_2, c) such that $\pi(\widetilde{Y_2}) = \pi(\widetilde{Y_1})$ for the projection

$$\pi : \mathbb{A}^{2+k} \to \mathbb{A}^2$$
$$(x, y, z_1, \dots, z_k) \mapsto (x, y).$$

This means that the only differences between the points of $\widetilde{Y}_1$ and $\widetilde{Y}_2$ are their $(z_1, \ldots, z_k)$-coordinates. But the $(z_1, \ldots, z_k)$-coordinates of almost all points lying on X_{z} are uniquely determined by $z_i = \left(\frac{P_{\mathsf{z}_i}(f)}{Q_{\mathsf{z}_i}(f)}\right)(x, y)$. Hence, $\pi|_{X_{\mathsf{z}}}$ is injective locally at c and this situation can not appear.
2) Assume that for all representatives $\widetilde{Y}_1$ of (Y_1, c) and $\widetilde{Y}_2$ of (Y_2, c) the images $\pi(\widetilde{Y}_1)$ and $\pi(\widetilde{Y}_2)$ are representatives of two distinct branches of X at the origin. But this is a contradiction to the assumption that X itself is analytically irreducible at the origin. $\square$

Remark 4.1.2. The statement of Proposition 4.1.1 is in general no longer true for algebraic space curves parametrized by $(\mathsf{x}, \mathsf{y}, \mathsf{z})$, with an arbitrary convergent power series $\mathsf{z}(t) \in \mathbb{C}\{t\}$. To illustrate the problem let us consider the following example:
The pair $(0, t^2 - 1)$ parametrizes the y-axis in $\mathbb{A}^2$ which is an anlytically irreducible curve at each point $(0, c), c \in \mathbb{C}$. We set $\mathsf{z}(t) = t^3 - t$. Then the curve X_{z} that is parametrized by $(0, t^2 - 1, t^3 - t)$ is the node in the yz-plane and is analytically reducible at the origin.

Corollary 4.1.3. *Let $X \subseteq \mathbb{A}^2$ be a plane algebraic curve and $0 \in X$. Assume that X is analytically irreducible at the origin. Let $\mathsf{z}_1, \ldots, \mathsf{z}_k \in \mathbb{C}\{t\}$ be geometric invariants of X that stem all from the same parametrization $\gamma = (\mathsf{x}, \mathsf{y})$ of X at the origin. Let X_{z} be the algebraic space curve parametrized by $(\mathsf{x}, \mathsf{y}, \mathsf{z}_1, \ldots, \mathsf{z}_k)$. Consider the map*

$$\pi : \mathbb{A}^{2+k} \to \mathbb{A}^2$$

$$(x, y, z_1, \ldots, z_k) \mapsto (x, y)$$

and the restriction of this map $\pi|_{X_{\mathsf{z}}}$. Then the fiber

$$\pi|_{X_{\mathsf{z}}}^{-1}(0, 0) = \{(0, 0, \mathsf{z}_1(0), \ldots, \mathsf{z}_k(0))\}$$

consists of only one point.

Proof. Assume that the fiber $\pi|_{X_{\mathsf{z}}}^{-1}(0, 0)$ consists of at least two different points $c_1, c_2, c_1 \neq c_2$. Let

$$\gamma_1(t) = (t^n, s_1(t), \ldots, s_{k+1}(t)), \gamma_2(t) = (t^m, r_1(t), \ldots, r_{k+1}(t))$$

be the Puiseux parametrizations of the branches (Y_1, c_1) and (Y_2, c_2) of X at c_1 and c_2, respectively. Then $(t^n, s_1(t)), (t^m, r_1(t))$ parametrize two distinct branches of X at the origin, according to the discussion in the case 1) in the proof of Proposition 4.1.1. But this is a contradiction to the assumption of the analytical irreducibility of X at 0. $\square$

We consider from now on an irreducible plane algebraic curve $X = V(f) \subseteq \mathbb{A}^2$ with a singularity at the origin. Assume that X is analytically irreducible at the origin. Let $\gamma(t) = (\mathsf{x}(t), \mathsf{y}(t))$ be a parametrization of X at the origin with $a = \operatorname{ord}_0(\mathsf{x}(t)) > 0, b = \operatorname{ord}_0(\mathsf{y}(t)) > 0$. W.l.o.g. we may assume $b \neq a$. Because if we had the equality $a = b$, then we could use the coordinate change $(x, y) \mapsto (x, x - c \cdot y)$ for a suitable constant $c \in \mathbb{C}$ to reach a parametrization with components of different t-adic orders. Let $R \in \Lambda$ be a rational invariant which induces the geometric invariant $\mathsf{z} = R \star (\mathsf{x}, \mathsf{y})$ of X with

$\mathrm{ord}_0(\mathsf{z}(t)) = 1$. We conclude then from Theorem 3.2.9, Proposition 4.1.1 and Corollary 4.1.3 that adding z as the third component to the parametrization γ gives us the triple $(\mathsf{x}, \mathsf{y}, \mathsf{z})$ that parametrizes an algebraic space curve X_{z} that is regular at the origin and that is birationally equivalent to X. Thus, the goal is to construct from γ such a geometric invariant z of X of t-adic order one.

We already know, using Theorem 3.2.7, that each triple $\gamma_i(t) = (\mathsf{x}(t), \mathsf{y}(t), \kappa_i(t)), i \in \mathbb{N} \cup \{0, -1\}$ parametrizes an algebraic space curve if $\mathrm{ord}_0(\kappa_i(t)) \geq 0$. Here $\kappa_{-1}(t) := \mathsf{s}(t), \kappa_0(t) := \kappa(t)$. Even more, we have $\mathrm{ord}_0(\gamma_i(t)) > \mathrm{ord}_0(\gamma_{i+1}(t))$ for all $i \in \mathbb{N} \cup \{0, -1\}$ which can be interpreted as a betterment of the singularity of X at the origin. But we are still missing a geometric invariant of X of order one to be able to construct a resolution of X at the origin. To do so, we proceed stepwise. Firstly, we construct by iterative use of the formulas for modified higher curvatures a geometric invariant of X of t-adic order equal to $\gcd(a, b)$.

Proposition 4.1.4. *Applying iteratively in a particular manner the formulas for modified higher curvatures to the parametrization* (x, y) *yields a geometric invariant of* X *of* t*-adic order equal to* $\gcd(a, b)$.

Proof. Let us w.l.o.g. assume that $a > b$. Let $i_1 \in \mathbb{N}$ be chosen such that $c_1 := \mathrm{ord}_0(\kappa_{i_1}(t)) > 0$ and $c_1 - b < 0$. Then the triple $(\mathsf{x}, \mathsf{y}, \kappa_{i_1})$ parametrizes an algebraic space curve X_1. Let us consider the projection map

$$\pi : \mathbb{A}^3 \to \mathbb{A}^2$$

$$(x, y, z) \mapsto (y, z).$$

Then $\overline{\pi(X_1)}$ is a plane algebraic curve according to Lemma 3.1.11 and it is parametrized by $\gamma_1(t) = (\mathsf{y}(t), \kappa_{i_1}(t))$. Here $\overline{\pi(X_1)}$ denotes the Zariski-closure of $\pi(X_1)$. Then we have the inequality

$$b = \mathrm{ord}_0(\gamma) > \mathrm{ord}_0(\gamma_1) = c_1.$$

We proceed now in the same way on $\overline{\pi(X_1)}$. We choose $i_2 \in \mathbb{N}$ such that $c_2 := \mathrm{ord}_0(\kappa_{i_2}(t)) > 0$ and $c_2 - c_1 < 0$, where $\kappa_{i_2}(t)$ denotes a modified higher curvature of $\overline{\pi(X_1)}$ that stems from the parametrization $(\mathsf{y}(t), \kappa_{i_1}(t))$. Then again the triple $(\mathsf{y}(t), \kappa_{i_1}(t), \kappa_{i_2}(t))$ parametrizes an algebraic space curve X_2 and $\overline{\pi(X_2)}$ is a plane algebraic curve parametrized by $\gamma_2(t) = (\kappa_{i_1}(t), \kappa_{i_2}(t))$ with

$$c_1 = \mathrm{ord}_0(\gamma_1) > \mathrm{ord}_0(\gamma_2) = c_2.$$

For $j \geq 3$ we define recursively κ_{i_j} as the modified higher curvature that stems from the parametrization $\gamma_{j-1}(t) = (\kappa_{i_{j-2}}(t), \kappa_{i_{j-1}}(t))$ of the plane algebraic curve $\overline{\pi(X_{j-1})}$ and that satisfies the inequalities $c_j := \mathrm{ord}_0(\kappa_{i_j}(t)) > 0$ and $c_j - c_{j-1} < 0$. The algebraic space curve X_j let be the curve parametrized by the triple $(\kappa_{i_{j-2}}(t), \kappa_{i_{j-1}}(t), \kappa_{i_j}(t))$. Notice that c_j is exactly the value we get in the j-th step of the Euclidean algorithm applied to the constants a and b. Therefore, there exists $k \in \mathbb{N}$ so that $\mathrm{ord}_0(\kappa_{i_k}(t)) = \gcd(a, b) = d$.

It can be shown by induction that $\kappa_{i_n}(t)$ is a geometric invariant of X for all $n \in \mathbb{N}$. Notice that for such a $\kappa_{i_n}(t)$ there exist polynomials in two variables $P, Q \in \mathbb{C}[x, y]$ so that the equality

$$\kappa_{i_n}(t) = \frac{P(\kappa_{i_{n-2}}(t), \kappa_{i_{n-1}}(t))}{Q(\kappa_{i_{n-2}}(t), \kappa_{i_{n-1}}(t))}$$

is fulfilled. By the induction hypothesis there exist polynomials in two variables $R_1, \ldots, R_4 \in \mathbb{C}[x, y]$ so that we have the equalities

$$\kappa_{i_{n-1}}(t) = \frac{R_1(\mathsf{x}(t), \mathsf{y}(t))}{R_2(\mathsf{x}(t), \mathsf{y}(t))}, \kappa_{i_{n-2}}(t) = \frac{R_3(\mathsf{x}(t), \mathsf{y}(t))}{R_4(\mathsf{x}(t), \mathsf{y}(t))}.$$

Thus,

$$\kappa_{i_n}(t) = \frac{P\left(\frac{R_1(\mathsf{x},\mathsf{y})}{R_2(\mathsf{x},\mathsf{y})}, \frac{R_3(\mathsf{x},\mathsf{y})}{R_4(\mathsf{x},\mathsf{y})}\right)}{Q\left(\frac{R_1(\mathsf{x},\mathsf{y})}{R_2(\mathsf{x},\mathsf{y})}, \frac{R_3(\mathsf{x},\mathsf{y})}{R_4(\mathsf{x},\mathsf{y})}\right)}$$

from which it follows that also $\kappa_{i_k}(t)$ is a geometric invariant. $\square$

We denote from now on the modified higher curvature of t-adic order $d = \gcd(a, b)$ constructed in the way described in the proof of Proposition 4.1.4 by $\kappa^*(\mathsf{x}, \mathsf{y})$.

Corollary 4.1.5. *Let $\mathsf{z}_1, \ldots, \mathsf{z}_m$ be geometric invariants of X that stem all from the parametrization (x, y) with $k_i = \operatorname{ord}_0(\mathsf{z}_i(t))$. Then a geometric invariant of X of t-adic order $\gcd(k_1, \ldots, k_m)$ can be constructed from $\mathsf{z}_1, \ldots, \mathsf{z}_m$ by repeated use of the formulas for modified higher curvatures.*

Proof. According to Corollary 3.2.8, the m-tuple $(\mathsf{z}_1, \ldots, \mathsf{z}_m)$ parametrizes an algebraic space curve $Z \subseteq \mathbb{A}^m$. Let us consider the projection

$$\pi : \mathbb{A}^m \to \mathbb{A}^2$$

$$(x_1, \ldots, x_m) \mapsto (x_{m-1}, x_m).$$

Then $\overline{\pi(Z)}$ is a plane algebraic curve and it is parametrized by the pair $(\mathsf{z}_{m-1}, \mathsf{z}_m)$. We compute the modified higher curvature $\kappa^*(\mathsf{z}_{m-1}, \mathsf{z}_m)$. As for its t-adic order we have $\operatorname{ord}_0(\kappa^*(\mathsf{z}_{m-1}, \mathsf{z}_m)) = \gcd(k_{m-1}, k_m)$. For each $i = 1, \ldots, m-1$ we can construct recursively the modified higher curvature

$$\kappa^*(\mathsf{z}_i, \kappa^*(\mathsf{z}_{i+1}, \kappa^*(\ldots, \kappa^*(\mathsf{z}_{m-1}, \mathsf{z}_m))))$$

of t-adic order equal to

$$\gcd(k_i, \gcd(k_{i+1}, \gcd(\ldots, \gcd(k_{m-1}, k_m)))) = \gcd(k_i, \ldots, k_m).$$

Hence, the modified higher curvature

$$\kappa^*(\mathsf{z}_1, \ldots, \mathsf{z}_m) := \kappa^*(\mathsf{z}_1, \kappa^*(\ldots, \kappa^*(\mathsf{z}_{m-1}, \mathsf{z}_m)))$$

is of t-adic order $\gcd(k_1, \ldots, k_m)$. Finally, it can be shown in the same way as in the proof of Proposition 4.1.4 that $\kappa^*(\mathsf{z}_1, \ldots, \mathsf{z}_m)$ is a geometric invariant of X. $\square$

For each finite set of geometric invariants $\mathsf{z}_1, \ldots, \mathsf{z}_m$ of X that stem from the same parametrization of X we denote by $\kappa^*(\mathsf{z}_1, \ldots, \mathsf{z}_m)$ a modified higher curvature of t-adic order $\gcd(\mathrm{ord}_0(\mathsf{z}_i(t)), i = 1, \ldots, k)$ that is constructed in the way we described in the proof of Corollary 4.1.5.

Lemma 4.1.6. *Let $s(t) \in \mathrm{Puiseux}_{\mathbb{C}}(t)$ be a Puiseux series and $n = \nu(s)$ its polydromy order. Assume that $s(t^n) \in \mathbb{C}\{t\}$. For a reparametrization φ let us denote $s_\varphi(t) = \varphi(s(t^n))$. Then the polydromy order of $s_\varphi(t^{\frac{1}{n}})$ equals n as well.*

Proof. We assume indirectly that the polydromy order of $s_\varphi(t^{\frac{1}{n}})$ is smaller than n. Then there is a constant $c \in \mathbb{N}, c \neq 1$ so that n and each $i \in \mathrm{supp}(s_\varphi)$ are divisible by c and we can write

$$\varphi(s(t^n)) = \sum_{i \geq 0} a_i t^{c \cdot i}$$

for some constants $a_i \in \mathbb{C}$. Hence each term of $s(t^n) = \varphi^{-1}(\varphi(s(t^n)))$ is (up to a multiplication with a constant) a product of the monomials $a_i t^{c \cdot i}$ and has a power that is divisible by c. But as n is the polydromy order of $s(t)$, there exists a subset $\{i_1, \ldots, i_r\} \subseteq \mathrm{supp}(s(t^n))$ of the support of $s(t^n)$ such that $\gcd(n, i_1, \ldots, i_r) = 1$, a contradiction. $\square$

Theorem 4.1.7. *Let $\eta(t) = (\mathsf{u}(t), \mathsf{v}(t))$ be a Puiseux parametrization of X at the origin. Then a geometric invariant $\mathsf{z}(t)$ of t-adic order one of X can be constructed from the parametrization $\eta(t)$ by iterative use of the formulas for modified higher curvatures.*

Proof. We may assume that a is the polydromy order of $\mathsf{v}(t^{\frac{1}{a}})$. Let $b = \mathrm{ord}_0(\mathsf{v}(t))$. According to Proposition 4.1.4 there exists a modified higher curvature $\mathsf{z}_1 = \kappa^*(\mathsf{u}, \mathsf{v})$ of t-adic order $d_1 = \gcd(a, b)$. Then the triple $(\mathsf{u}, \mathsf{v}, \mathsf{z}_1)$ parametrizes an algebraic space curve X_1. Furthermore, there is a reparametrization φ_1 such that $\varphi_1(\mathsf{z}_1) = t^{d_1}$. Thus, applying the reparametrization φ_1 to the triple $(\mathsf{u}, \mathsf{v}, \mathsf{z}_1)$ gives us another parametrization of X_1,

$$(\varphi_1(\mathsf{u}), \varphi_1(\mathsf{v}), t^{d_1}),$$

with $\mathrm{ord}_0(\varphi_1(\mathsf{u}(t))) = a, \mathrm{ord}_0(\varphi_1(\mathsf{v}(t))) = b$. Via a polynomial triangular coordinate change, the curve X_1 is isomorphic to the curve parametrized by

$$(\varphi_1(\mathsf{u}), \varphi_1(\mathsf{v}) - k \cdot (t^{d_1})^q, t^{d_1}) = (\mathsf{x}_1, \mathsf{y}, t^{d_1})$$

for $q = \frac{b}{d_1}$ and suitable $k \in \mathbb{C}$ with $\mathrm{ord}_0(\mathsf{y}(t)) > b$. If $\gcd(a, \mathrm{ord}_0(\mathsf{y}(t)), d_1) = d_1$, we reapply a triangular coordinate change to increase again the order of $\mathsf{y}(t)$. By Lemma 4.1.6 this process must terminate after finitely many steps when achieving y_1 so that

$$\gcd(a, \mathrm{ord}_0(\mathsf{y}_1(t)), d_1) < d_1.$$

Furthermore, by Lemma 4.1.6 we find a finite subset $\{i_1, \ldots, i_{r_1}\} \subseteq \mathrm{supp}(\mathsf{y}_1)$ of the support of y_1 so that the equality $\gcd(a, d_1, i_1, \ldots, i_{r_1}) = 1$ holds. Now, according to Corollary 4.1.5 there exists a modified higher curvature $\mathsf{z}_2 = \kappa^*(\mathsf{x}_1, \mathsf{y}_1, t^{d_1})$ of t-adic order equal to $d_2 = \gcd(a, \mathrm{ord}_0(\mathsf{y}_1(t)), d_1)$. We apply again a suitable reparametrization and a triangular coordinate change to the algebraic space curve X_2 parametrized by $(\mathsf{x}_1, \mathsf{y}_1, t^{d_1}, \mathsf{z}_2)$ and get an isomorphic algebraic space curve parametrized by $(\mathsf{x}_2, \mathsf{y}_2, \mathsf{s}_1, t^{d_2})$ for suitable $\mathsf{x}_2, \mathsf{y}_2, \mathsf{s}_1 \in \mathbb{C}\{t\}$ with $\operatorname{ord}_0(\mathsf{x}_2) = a, \operatorname{ord}_0(\mathsf{y}_2) = \operatorname{ord}_0(\mathsf{y}_1), \operatorname{ord}_0(\mathsf{s}_1) = d_1$ so that the inequality

$$\gcd(a, \operatorname{ord}_0(\mathsf{y}_2), d_1, d_2) < d_1$$

holds. By an iterative use of this procedure we achieve an m-tuple

$$(\mathsf{x}_{m-3}, \mathsf{y}_{m-3}, \mathsf{s}_1, \ldots, \mathsf{s}_{m-3}, \mathsf{z}_{m-2})$$

of geometric invariants of X that satisfy

$$\gcd(\operatorname{ord}_0(\mathsf{x}_{m-3}), \operatorname{ord}_0(\mathsf{y}_{m-3}), \operatorname{ord}_0(\mathsf{s}_1), \ldots, \operatorname{ord}_0(\mathsf{s}_{m-3}), \operatorname{ord}_0(\mathsf{z}_{m-2})) = 1.$$

Now, from Corollary 4.1.5 the rest follows. □

Let us now assume that X has more than the only one singularity at the origin. Let $s_1, \ldots, s_m = \operatorname{Sing}(X) \backslash \{0\}$ be the other finitely many singularities of X. We consider at each singularity s_i a Puiseux parametrization $\gamma_i(t) = (\mathsf{x}_i^{s_i}, \mathsf{y}_i^{s_i})$ of X with $\gamma_i(b_i) = s_i$ for some $b_i \in \mathbb{C}$ and construct then for each $i = 1, \ldots, m$ according to Theorem 4.1.7

$$\mathsf{z}_i^{s_i} = \frac{P_i(\mathsf{x}_i^{s_i}, \mathsf{y}_i^{s_i})}{Q_i(\mathsf{x}_i^{s_i}, \mathsf{y}_i^{s_i})},$$

a geometric invariant of X of $(t - b_i)$-adic equal to one. Here $P_i, Q_i \in \mathbb{C}[x, y]$ for all $i = 1, \ldots, m$. If $\mathsf{z} = \frac{P_0(\mathsf{x},\mathsf{y})}{Q_0(\mathsf{x},\mathsf{y})}$ denotes the geometric invariant of X of t-adic order one that stems from the parametrization (x, y) of X, then the $(m+3)$-tuple

$$\eta(t) = \left(\mathsf{x}, \mathsf{y}, \frac{P_0(\mathsf{x},\mathsf{y})}{Q_0(\mathsf{x},\mathsf{y})}, \frac{P_1(\mathsf{x},\mathsf{y})}{Q_1(\mathsf{x},\mathsf{y})}, \ldots, \frac{P_m(\mathsf{x},\mathsf{y})}{Q_m(\mathsf{x},\mathsf{y})}\right)$$

parametrizes one chart expression of the blowup $\widetilde{X}$ of X with the center

$$(P_0 \cdot \prod_{i \geq 1} Q_i, P_1 Q_0 \cdot \prod_{i \neq 1} Q_i, \ldots, P_m Q_0 \cdot \prod_{i \neq m} Q_i, Q_0 \cdot \prod_{i \geq 1} Q_i).$$

Even more, since for each $i = 1, \ldots, m$ there is according to Corollary 4.1.3 only one point $\widetilde{s}_i$ lying on the curve parametrized by $\eta(t)$ over s_i, and since the curve parameatrized by $\eta(t)$ is at each point $\widetilde{s}_i$ parametrized by

$$\eta_i(t) = \left(\mathsf{x}^{s_i}, \mathsf{y}^{s_i}, \frac{P_0(\mathsf{x}^{s_i},\mathsf{y}^{s_i})}{Q_0(\mathsf{x}^{s_i},\mathsf{y}^{s_i})}, \frac{P_1(\mathsf{x}^{s_i},\mathsf{y}^{s_i})}{Q_1(\mathsf{x}^{s_i},\mathsf{y}^{s_i})}, \ldots, \frac{P_m(\mathsf{x}^{s_i},\mathsf{y}^{s_i})}{Q_m(\mathsf{x}^{s_i},\mathsf{y}^{s_i})}\right)$$

with $\eta_i(b_i) = \widetilde{s}_i$ and with one component $\mathsf{z}_i^{s_i}$ satisfying $\operatorname{ord}_{b_i}(\mathsf{z}_i^{s_i}(t)) = 1$, it is regular at each point. And the regularity of $\widetilde{X}$ follows. Thus, we have already proven the following statement:

Theorem 4.1.8. *Let* $\mathrm{Sing}(X) = \{s_1, \ldots, s_m\}$. *Then by repeatedly using the formulas for modified higher curvatures to the parametrization* (x, y) *geometric invariants*

$$\kappa_{s_i}(t) = \left(\frac{\kappa_{s_i}^{(1)}(f)}{\kappa_{s_i}^{(2)}(f)}\right)(\mathsf{x}(t), \mathsf{y}(t)), i = 1, \ldots, m,$$

can be constructed so that the $(m+2)$*-tuple*

$$(\mathsf{x}, \mathsf{y}, \kappa_{s_1}, \ldots, \kappa_{s_m})$$

parametrizes one of the affine chart expressions of a regular blowup $\widetilde{X}$ *of* X *with a suitable center. Here* $\kappa_{s_i}^{(j)}(f) \in \mathbb{C}[x, y], j = 1, 2$ *denotes a modified differential operator* $\kappa_{s_i}^{(j)}$ *applied to* f.

However, there is still a little problem with the implicit equations for $\widetilde{X}$. Even though we can compute the center of the blowup and also the exceptional divisor, to find the defining equations of $\widetilde{X}$ is in general a very complicated process. The reason for that is that it is generally not straightforward to factor out the equations of the exceptional divisor from the equations of the total transform of X.

4.2 Resolution of analytically reducible plane algebraic curves

We will now use the knowledge we have from the previous section about the resolution of analytically irreducible curves to discuss how the resolution of an analytically reducible curve can be constructed with the help of rational invariants. The most important part here will play the discussion of the separation of different branches at one point via geometric invariants.

Let $X \subseteq \mathbb{A}^2$ be a plane algebraic curve with a singularity at the origin. Let us assume that X is analytically reducible at the origin and that $(Y_1, 0), \ldots, (Y_m, 0)$ are the branches of X at the origin. Note that we have already seen in the section 4.1 how to resolve each of the branches of X at the origin. Hence, we can transform the original question asking for the resolution of an arbitrary analytically reducible plane algebraic curve to the problem of searching for a resolution of a space algebraic curve with regular branches. It is clear that two regular branches of an algebraic space curve can meet at a point in different ways. We will in this section distinguish between two types of how the branches can meet.

At first notice that the concept of geometric invariants was defined only for plane aglebraic curves. Hence, for the construction of the resolution of algebraic space curves a new strategy has to be figured out.

1. Two branches with different tangent vectors at the meeting point:
In this case the separation of these two branches can be done via the Nash modification that takes the slope of the tangent vector as a new coordinate.

2. Two branches with the same tangent vectors at the meeting point:
Let $Z \subseteq \mathbb{A}^n$ be a blowup of X with a suitable center and let $(Z_1, 0)$ and $(Z_2, 0)$ be two distinct branches of Z with the same tangent vector at the origin. We consider the projection

$$\pi : \mathbb{A}^n \to \mathbb{A}^2$$
$$(x_1, \ldots, x_n) \mapsto (x_1, x_2).$$

Let $(X_1, 0), (X_2, 0)$ be the projections of the branches $(Z_1, 0)$ and $(Z_2, 0)$ of Z under π, respectively. Then $(X_1, 0), (X_2, 0)$ are two distinct branches of X at the origin. Let us consider the Puiseux parametrizations $(\mathsf{x}_1(t), \mathsf{y}_1(t))$, $(\mathsf{x}_2(t), \mathsf{y}_2(t))$ of the branches $(X_1, 0)$ and $(X_2, 0)$, respectively. Then the goal is to find a rational invariant $R \in \mathbb{C}(u_0, v_0, \ldots, u_k, v_k)$ such that

$$[R \star (\mathsf{x}_1(t), \mathsf{y}_1(t))]|_{t=0} \neq [R \star (\mathsf{x}_2(t), \mathsf{y}_2(t))]|_{t=0}.$$

This means that we must have $\mathrm{ord}_0(R \star (\mathsf{x}_i(t), \mathsf{y}_i(t))) = 0$ for at least one $i = 1, 2$ and if we had $\mathrm{ord}_0(R \star (\mathsf{x}_i(t), \mathsf{y}_i(t))) = 0$ for both $i = 1, 2$, then it would be necessary that the geometric invariants $R \star (\mathsf{x}_1(t), \mathsf{y}_1(t))$ and $R \star (\mathsf{x}_2(t), \mathsf{y}_2(t))$ have different constant terms. We denote $a_i = \mathrm{ord}_0(\mathsf{x}_i(t)), b_i = \mathrm{ord}_0(\mathsf{y}_i(t))$ for $i = 1, 2$. If we have the inequality $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$ then there exist positive integers $c, d \in \mathbb{N}$ such that

$$c \cdot a_1 - d \cdot b_1 = 0, c \cdot a_2 - d \cdot b_2 \neq 0.$$

Then we have for the rational invariant $R = \frac{u_0^c}{v_0^d} \in \mathbb{C}(u_0, v_0)$ the following:

$$\operatorname{ord}_0(R \star (\mathsf{x}_1(t), \mathsf{y}_1(t))) = 0, \operatorname{ord}_0(R \star (\mathsf{x}_2(t), \mathsf{y}_2(t))) \neq 0.$$

There is also the possibility to use the formulas for modified higher curvatures for the construction of the searched rational invariant. Remember that for the modified higher curvature κ_j that stems form the parametrization $(\mathsf{x}_1, \mathsf{y}_1)$ we have the equality $\operatorname{ord}_0(\kappa_j(t)) = a_1 - (j+2)b_1$. Thus, we can construct from the parametrization $(\mathsf{x}_1, \mathsf{y}_1)$ the modified higher curvature $\kappa_{d-2}(t)$ of t-adic order $a_1 - d \cdot b_1$ and then continue with the pair $(\frac{1}{\kappa_{d-2}}, \mathsf{x}_1)$ and construct from this the modified higher curvature κ_{c-3} of t-adic order $\operatorname{ord}_0(\kappa_{c-3}(t)) = d \cdot b_1 - c \cdot a_1 = 0$. Then κ_{c-3} is clearly a geometric invariant of X as well and the corresponding rational invariant S satisfies the required conditions. Thus, if $\eta(t) = (\mathsf{x}, \mathsf{y}, \mathsf{z}_1, \ldots, \mathsf{z}_{n-2})$ is a parametrization of Z, then the addition of the geometric invariant $S \star (\mathsf{x}, \mathsf{y})$ or $R \star (\mathsf{x}, \mathsf{y})$ as a new component to the parametrization $\eta(t)$ separate the both branches $(Z_1, 0)$, $(Z_2, 0)$ and the $(n+1)$-tuple $(\mathsf{x}, \mathsf{y}, \mathsf{z}_1, \ldots, \mathsf{z}_{n-2}, S \star (\mathsf{x}, \mathsf{y}))$ parametrizes one chart expression of a blowup of X with a suitable center. However, if there are more than two branches at the origin, this strategy does not apply. Also in the case that the equality $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ holds it is still open how to separate the branches.

5 Appendix

5.1 Puiseux parametrizations of plane algebraic curves

The aim of this section is to present the concept of Puiseux parametrizations of branches of plane algebraic curves. We will present in this section the Newton-Puiseux algorithm which constructs these parametrizations.

Let us now briefly introduce the concept of Puiseux series. The *field of Puiseux series over* $\mathbb{C}$ is defined as

$$\mathrm{Puiseux}_{\mathbb{C}}(x) := \bigcup_{n\geq 1} \mathbb{C}((x^{\frac{1}{n}})).$$

The elements are Laurent series with rational exponents with a fixed denominator $m \in \mathbb{N}$,

$$s(x) = \sum_{i=i_0}^{\infty} a_i \cdot x^{\frac{i}{m}}, i_0 \in \mathbb{Z},$$

with coefficients $a_i \in \mathbb{C}$, called *Puiseux series* over $\mathbb{C}$. The choice of m is not unique but after reduction of all fractionary exponents we may take the minimal one, i.e., m and the set of integers $\{i \in \mathbb{Z} | a_i \neq 0\}$ have no non-trivial common divisor. The minimal value m is called the *polydromy order* of $s(x)$ and denoted by $\nu(s)$.

Let us fix the coordinates x, y in $\mathbb{A}^2$ and let $f(x,y) \in \mathbb{C}[x,y]$ be a non-constant squarefree polynomial and $X = V(f) \subseteq \mathbb{A}^2$ the corresponding plane algebraic curve. We suppose from now on $0 \in X$. Let $f = f_1 \cdots f_r$, $f_i \neq f_j$ for $i \neq j$ be the factorization of f into irreducible factors $f_i \in \mathbb{C}[x,y]$. If $r \geq 2$, the curve X is reducible and in this case X cannot be parametrized by a pair of convergent power series $(\mathsf{x}(t), \mathsf{y}(t))$ (for the definition of a parametrization of an algebraic curve or its branches see Section 3.1). This can be seen indirectly in the following way:
Assume the existence of a parametrization $(\mathsf{x}(t), \mathsf{y}(t))$ of X. Then the equality $f(\mathsf{x}(t), \mathsf{y}(t)) = 0$ holds in $\mathbb{C}\{t\}$. But then $f_i(\mathsf{x}(t), \mathsf{y}(t)) = 0$ for one $i \in \{1, \ldots, r\}$. This means that $(\mathsf{x}(t), \mathsf{y}(t))$ parametrizes already the irreducible component $X_i \subsetneq X$ and, as X_i is the Zariski-closure of the image of the map

$$\gamma : D(\mathsf{x}(t)) \cap D(\mathsf{y}(t)) \to X$$

$$a \mapsto (\mathsf{x}(a), \mathsf{y}(a)),$$

it parametrizes none of the other irreducible components $X_j, j \neq i$. Therefore, the pair$(\mathsf{x}(t), \mathsf{y}(t))$ cannot parametrize the whole curve X.

We will therefore look only at irreducible plane algebraic curves for searching for a parametrization. But even in this case, there could be a similar problem with analytical irreducibility. We hence look at the decomposition of the curve X in its branches at one point. We will ask whether there exists a parametrization of each branch of X at 0. Moreover, we will look for a special type of parametrizations called Puiseux parametrizations. A *Puiseux parametrization* of a branch $(Y,0)$ of X at 0 is a parametrization of the branch of the form $(\mathsf{x}(t),\mathsf{y}(t)) = (t^m, s(t))$ or $(\mathsf{x}(t),\mathsf{y}(t)) = (s(t),t^m)$, for some $m \in \mathbb{N}$ and $s(t) \in \mathbb{C}\{t\}$. Note that for a parametrization $(t^m, s(t))$ of X we have the equality $f(t, s(t^{\frac{1}{m}})) = 0$ in $\text{Puiseux}_{\mathbb{C}}(t)$. Hence, the Puiseux parametrizations are closely related to Puiseux series. However, as required $s(t)$ to be a power series and not allowed to be a Laurent series with a negative order in t, we do not have a 1:1 correspondence between Puiseux series and Puiseux parametrizations.

Let $g(x_1,\ldots,x_n) \in \mathbb{C}[[x_1,\ldots,x_n]]$ be a power series. An n-tuple of Puiseux series $(x_1(t),\ldots,x_n(t)) \in \text{Puiseux}_{\mathbb{C}}(t)^n$ is called a *zero of* g if the equality

$$g(x_1(t),\ldots,x_n(t)) = 0$$

is fulfilled in $\text{Puiseux}_{\mathbb{C}}(t)$. A Puiseux series $s(t) \in \text{Puiseux}_{\mathbb{C}}(t)$ is said to be an *x_i-root of g* if we have after substituing $s(t)$ for the variable x_i and t for all other variables the equality

$$g(t,\ldots,t,s(t),t,\ldots,t) = 0$$

in $\text{Puiseux}_{\mathbb{C}}(t)$. Let $J \subseteq \mathbb{C}[[x_1,\ldots,x_n]]$ be an ideal. An n-tuple of Puiseux series $(x_1(t),\ldots,x_n(t)) \in \text{Puiseux}_{\mathbb{C}}(t)^n$ is said to be a *zero of* J if it is a zero of all power series $g \in J$. For an algebraic plane curve $X = V(f)$ it is clear, using the fact that the ring of Laurent series is an integral domain, that a Puiseux series $s(t)$ is an x- or y-root of f if and only if it is an x- or y-root of a power series defining one of the branches of X.

Proposition 5.1.1. *For every zero $(t^m, s(t))$ or $(s(t),t^m), m \in \mathbb{N}$ of a polynomial $f \in \mathbb{C}[x,y]$ with $s \in \mathbb{C}[[t]]$ the power series s is convergent.*

Proof. From the equalities $f(s(t),t^m) = 0$ and $f(t^m,s(t)) = 0$ we conclude that $s(t)$ is a zero of $f(x,t^m)$ or $f(t^m,x)$, respectively. As they are polynomials in x with coefficients in the polynomial ring $\mathbb{C}[t]$, the power series s is algebraic and hence convergent. □

Let $g(x,y) \in \mathbb{C}[[x,y]]$ be a formal power series without constant term such that $g(0,y) = y^n \cdot h(y)$ for some integer $n \in \mathbb{N}$ and some formal power series $h(y)$ with $h(0) \neq 0$. A power series of this form is said to be *y-regular of order n*. The x-regularity is defined in the same way.

We will show that each y-regular power series $g \in \mathbb{C}[[x,y]]$ possesses a y-root $y(x)$. Let us briefly look at one concrete example, at the polynomial $f(x,y) = x^2 - y^3$ which is y-regular of order 3. Then we see in general it is not possible $y(x)$ to be a power series. The only y-roots of f are $y(x) = \zeta^j x^{\frac{2}{3}}$, where $\zeta = e^{\frac{2\pi i}{3}}$ is a third primitive root of unity and $j = 0,1,2$. And we see that $y(x)$ is a polynomial with fractional exponents. Thus, in general there is no hope to find a y-root which would be a power series. However, there is still the possibility that $y(x) \in \mathbb{C}[[x^{\frac{1}{m}}]]$ for some fixed integer m. And this is what we will show. Even more, we will show that this root is of the form $y(x) = \phi(x^{\frac{1}{m}})$, where $\phi(x) \in \mathbb{C}\{x\}$ and m is the y-regularity order of g.

Lemma 5.1.2. *For each power series $g \in \mathbb{C}[[x, y]]$, there exists a change of coordinates $(x, y) \mapsto (x + \lambda y, y), \lambda \in \mathbb{C}$ such that g is y-regular of order $o = \mathrm{ord}_0(g)$. Here $\mathrm{ord}_0(g)$ denotes the t-adic order of g.*

Proof. Let us write the power series g as follows:

$$g = \sum_{i+j \geq o} a_{i,j} x^i y^j.$$

Let

$$\mathrm{hom}_o(g) := \sum_{i+j=o} a_{i,j} x^i y^j = \sum_{j=0}^{o} a_{j,o-j} x^j y^{o-j}$$

be the homogeneous part of g of degree o. We apply the change of coordinates λ : $(x, y) \mapsto (x + \lambda y, y)$ for a constant $\lambda \in \mathbb{C}$ and we get

$$\mathrm{hom}_o(g \circ \lambda) = \sum_{j=0}^{o} a_{j,o-j} (x + \lambda y)^j y^{o-j} = \sum_{i=0}^{o} \underbrace{\sum_{j=i}^{o} a_{j,o-j} \begin{pmatrix} j \\ i \end{pmatrix} \lambda^{j-i}}_{\tilde{a}_{i,o-i}(\lambda)} x^i y^{o-i}$$

with

$$\tilde{a}_{0,o}(t) = \sum_{j=0}^{o} a_{j,o-j} t^j.$$

Indeed $\tilde{a}_{0,o}$ is a non-zero polynomial in the variable λ, thus $\tilde{a}_{o,0}$ has only finitely many zeros in $\mathbb{C}$. For a generic $\lambda \in \mathbb{C}$ we therefore have $\tilde{a}_{0,o}(\lambda) \neq 0$ and so we can find a change of coordinates so that the term y^o appears with a non-zero coefficient in the Taylor expansion of $g(x + \lambda y, y)$. $\square$

Thus we can w.l.o.g assume that each power series $g \in \mathbb{C}[[x, y]]$ we are working with is y-regular. Let us sketch the main ideas of the Newton-Puiseux algorithm. We show the existence of a y-root $\phi(t)$ of each y-regular power series $g \in \mathbb{C}[[x, y]]$ of order n by induction on the integer n. But first let us discuss the case that $g = y^m$ for some $m \in \mathbb{N}$. Then g is parametrized by the pair $(t, 0)$ which is a Puiseux parametrization. For an arbitrary power series $g \in \mathbb{C}[[x]]$ of y-regularity order $n = 1$ we have $g_y(0, 0) \neq 0$ and according to the Implicit functions theorem there exists a unique power series $y(x) \in \mathbb{C}[[x]]$ with $g(x, y(x)) = 0, y(0) = 0$. Let us now consider $n > 1$. For a power series $g \in \mathbb{C}[[x, y]]$ we will seek a y-root $y(x)$ of the form

$$y(x) = \phi(x^{\frac{1}{m}}) = x^{\nu}(c_0 + \phi_0(x^{\frac{1}{m}})),$$

with constants $c_0 \in \mathbb{C}\backslash\{0\}, \nu \in \mathbb{Q}^+$ and $\phi_0(x) \in \mathbb{C}[[x]]$ without constant term. The substitution of the searched y-root $y(x)$ into g with $o = \mathrm{ord}_0(g)$ must satisfy

$$g(x, y(x)) = \sum_{i+j \geq o} a_{i,j} x^i y(x)^j = \sum_{i+j \geq o} a_{i,j} x^{i+\nu j} (c_0 + \phi_0(x^{\frac{1}{m}}))^j = 0.$$

Thus, we look for constants ν, c_0 and a Puiseux series $\phi_0(x^{\frac{1}{m}})$ as above such that the last equality holds. When we set $\mu = \min\{i + \nu j | a_{i,j} \neq 0\}$, we can write

$$g(x, y(x)) = x^\mu \sum_{i+\nu j=\mu} a_{i,j} c_0^j + x^\mu \sum_{i+\nu j=\mu} a_{i,j} \sum_{k=0}^{j-1} \left(\begin{array}{c} j \\ k \end{array} \right) c_0^k \phi_0(x^{\frac{1}{m}})^{j-k} +$$

$$+ x^\mu \sum_{i+\nu j>\mu} a_{i,j} x^{i+\nu j-\mu} (c_0 + \phi_0(x^{\frac{1}{m}}))^j = x^\mu \sum_{i+\nu j=\mu} a_{i,j} c_0^j + x^\mu h(x^{\frac{1}{m}})$$

for some $h(x) \in \mathbb{C}[[x]]$. Notice that $h(x)$ has no constant term as ϕ_0 has not niether. Therefore, to achieve the equality $g(x, y(x)) = 0$, we must have

$$\sum_{i+\nu j=\mu} a_{i,j} c_0^j = 0.$$

And we conclude that at least two distinct constants $a_{m,n}, a_{k,l}$ with $m + \nu n = \mu = k + \nu l$ are non-zero (by assumption $c_0 \neq 0$).

Let us now recall the concept of the Newton polygon which is the key tool of the Newton-Puiseux algorithm. The set $\mathcal{N}(g) := \{(i, j) | a_{i,j} \neq 0\} \subseteq \mathbb{R}^2_{\geq 0}$ lying in the (i, j)-plane is called the *Newton cloud* of g. The *Newton polygon* of g, $\mathcal{NP}(g)$, is defined as the boundary of the convex hull of the set $\mathcal{N}_+(g) := \mathcal{N}(g) + \mathbb{R}^2_{\geq 0}$. The bounded edges, i.e., edges of finite length, of the Newton polygon are called *segments*. We denote the segments of the Newton polygon by $s_1, s_2, \ldots, s_l$. The first segment s_1 is the closest one to the j-axis. The points on the Newton polygon where two edges meet are called *vertices*. We can note that the line defining the first segment of the Newton polygon is of the smallest slope. Furthermore each line defining one segment s_i is of smaller slope than the line defining the segment s_{i+1} lying on the right side of the segment s_i. Notice that there is a geometric interpretation of $\mu = \min\{i + \nu j | a_{i,j} \neq 0\}$. The constant μ is the minimal value $d \in \mathbb{R}$ for which the line $i + \nu j = d$ crosses the Newton polygon. And hence it is also the minimal value lying on the intersection of the i-axis and the line $i + \nu j = d$ crossing the Newton polygon. By the defintion of μ, the line $i + \nu j = \mu$ meets the Newton polygon in exactly one vertex or contains one whole segment of the Newton polygon.

In the case that the line $i + \nu j = \mu$ contains one whole segment s_k of the Newton polygon, the value $-\frac{1}{\nu}$ equals the slope of the segment s_k. We call the number ν the *inclination* of the segment s_k.

Remark 5.1.3. The first segment of the Newton polygon has the smallest inclination.

Remark 5.1.4. The polynomial

$$\sum_{i+\nu j=\mu} a_{i,j} c_0^j$$

then consists of the terms $a_{i,j}c_0^j$, where (i,j) lies on the segment of the Newton polygon with the inclination ν.

But now back to the induction step in the proof of the existence of Puiseux parametrizations. The induction step from $n-1$ to n consists of several parts. For the simplicity, we handle these parts separately. But all parts are connected and each part uses the results and definitions from the previous parts. So the individual parts cannot be understood individually without reading the previous parts. The following steps work only for a power series $g \neq y^m, m \in \mathbb{N}$. Hence, for the further procedure we will consider only y-regular power series g with at least one finite segment of the Newton polygon.

<u>1^{st} *part :*</u>

Let $\nu_0 = \frac{l}{h}$ be the inclination of the first segment of the Newton polygon $\mathcal{NP}(g)$ of the power series

$$g(x,y) = \sum_{i+j\geq o} a_{i,j}x^i y^j, o = \mathrm{ord}_0(g).$$

We chose a constant $c_0 \in \mathbb{C}$ such that $c_0 \neq 0$ and the equation

$$\sum_{i+\nu_0 j=\mu} a_{i,j} c_0^j = 0.$$

is fulfilled. As $g \neq y^m$ for all $m \in \mathbb{N}$, the first segment of $\mathcal{NP}(g)$ lies above the j-axis and this choice of c_0 is possible. We study now the effect of the change of variables

$$\begin{aligned} x &= x_1^h \\ y &= x_1^l(c_0 + y_1) \end{aligned}$$

on the power series g. The substitution of the new variables into g and using the definition of μ as before together with the fact

$$i + \nu_0 j \geq \mu \Leftrightarrow hi + lj \geq \mu h$$

yields

$$g(x_1^h, x_1^l(c_0+y_1)) = \sum_{i+j\geq o} a_{i,j}x_1^{hi+lj}(c_0+y_1)^j = \sum_{hi+lj\geq \mu h} a_{i,j}x_1^{hi+lj}(c_0+y_1)^j =$$

$$= x_1^{\mu h} \underbrace{\sum_{hi+lj\geq \mu h} a_{i,j} x_1^{hi+lj-\mu h}(c_0+y_1)^j}_{=:\tilde{g}(x_1,y_1)}.$$

So we get the following factorization

$$g(x_1^h, x_1^l(c_0+y_1)) = x_1^{\mu h}\tilde{g}(x_1,y_1)$$

with

$$\tilde{g}(0,y_1) = \sum_{i+\nu_0 j=\mu} a_{i,j}(c_0+y_1)^j.$$

Since $g(0,y) = y^n h(y), h(0) \neq 0$, we know that $a_{0,n} \neq 0$. Therefore, the point $(0,n)$ is the left-boundary point of the first segment of $\mathcal{NP}(g)$ and it is also the point lying on the line $i+\nu_0 j = \mu$. Thus, the term $a_{0,n}(c_0+y_1)^n$ must appear in the Taylor expansion of $\tilde{g}(0,y_1)$ and so the y_1-order of $\tilde{g}(0,y_1)$ is smaller or equal to n. Because of the choice of c_0 as a root of the polynomial $\sum_{i+\nu_0 j=\mu} a_{i,j}c_0^j$, the y_1-order of $\tilde{g}(0,y_1)$ is strictly greater than 0. Even more, there is a necessary and sufficient condition for y_1-order of $\tilde{g}(0,y_1)$ to be equal to n:

Proposition 5.1.5. *The y_1-order of $\tilde{g}(0,y_1)$ is equal to n if and only if c_0 is a root of multiplicity n of the polynomial*

$$\sum_{i+\nu_0 j=\mu} a_{i,j}t^j$$

.

Proof. $\Leftarrow$: Let c_0 be a root of

$$\sum_{i+\nu_0 j=\mu} a_{i,j}t^j = 0$$

of multiplicity n. Then we have the equality

$$\sum_{i+\nu_0 j=\mu} a_{i,j}t^j = b(t-c_0)^n$$

with some constant $b \neq 0$ and finally

$$\tilde{g}(0,y_1) = \sum_{i+\nu_0 j=\mu} a_{i,j}(c_0+y_1)^j = b(c_0+y_1-c_0)^n = by_1^n.$$

$\Rightarrow$: If the multiplicity m of the root c_0 is strictly smaller than n, then we have the factorization

$$\sum_{i+\nu_0 j=\mu} a_{i,j}t^j = b(t-c_0)^m(t-d_0)^{r_0}\cdots(t-d_s)^{r_s},$$

with $m + r_0 + \cdots + r_s = n$ and some constants $b \neq 0$, $d_i \neq d_j$ for $i \neq j$ and $d_i \neq c_0$ for all $i = 1, ..., s$. After substituting $t \mapsto c_0 + y_1$ into $\sum a_{i,j} t^j$ we get for $\tilde{g}$ the following

$$\tilde{g}(0, y_1) = \sum_{i+\nu_0 j=\mu} a_{i,j}(c_0 + y_1)^j = \underbrace{b(c_0 - d_0)^{r_0} \cdots (c_0 - d_s)^{r_s}}_{\neq 0} y_1^m.$$

And so $\tilde{g}(0, y_1)$ has y_1-order $m < n$. □

Let us now discuss the case that the y_1-order of the power series $\tilde{g}(0, y_1)$ equals n, i.e., $c_0 \in \mathbb{C}$ is a root of multiplicity n of the polynomial $\sum_{i+\nu_0 j=\mu} a_{i,j} t^j$. Then we have the following equality

$$\sum_{i+\nu_0 j=\mu} a_{i,j} t^j = b(t - c_0)^n = b \sum_{k=0}^{n} t^{n-k} (-1)^k c_0^k,$$

with some constant $b \neq 0$. Comparing the terms $a_{0,n} t^n$ (we already know that $(0, n)$ lies on the line $i + \nu_0 j = \mu$) and bt^n, we get $b = a_{0,n}$. Furthermore, comparing the terms $a_{\mu-\nu_0(n-1),n-1} t^{n-1}$ and $-a_{0,n} c_0 t^{n-1}$ yields the equality $a_{\mu-\nu_0(n-1),n-1} = -a_{0,n} c_0 \neq 0$. This implies the existence of a positive integer i' such that $i' + \nu_0(n-1) = \mu$. Then $i' = \mu - \underbrace{\nu_0 n}_{=\mu} - \nu_0 = \nu_0$ and we get the following proposition:

Proposition 5.1.6. *If the y_1-order of $\tilde{g}(0, y_1)$ equals n, then ν_0 is a positive integer.*

Furthermore, comparing the coefficients of t^0 in the above equality ensures that $a_{\nu_0 n,0} = (-1)^n a_{0,n} c_0^n \neq 0$. Thus, the first segment of $\mathcal{NP}(g)$ connects the points $(0, n)$ and $(\nu_0 n, 0)$ which correspond to the monomials y^n and $x^{\nu_0 n}$. Analogously, we can compute the other coefficients $a_{i,j}$ with $(i, j) \in \mathcal{NP}(g)$ satisfying the condition $i + \nu_0 j = \mu$ and we see that:

Remark 5.1.7. If the y_1-order of $\tilde{g}(0, y_1)$ equals n, then g can be written in the following way

$$g(x, y) = \sum_{i+\nu_0 j=\mu} a_{i,j} x^i y^j + \sum_{i+\nu_0 j>\mu} a_{i,j} x^i y^j = a_{0,n}(y - c_0 x^{\nu_0})^n + \sum_{i+\nu_0 j>\mu} a_{i,j} x^i y^j$$

and the first segment is therefore the only bounded segment of $\mathcal{NP}(g)$.

2^{nd} *part :*

If the y_1-order of $\tilde{g}(0, y_1)$ equals n, we apply to g the following change of variables

$$x \mapsto x$$
$$y \mapsto y + c_0 x^{\nu_0}.$$

Substituting these newly defined variables into g and using Remark 5.1.7 yields

$$g_1(x,y) := g(x, y + c_0 x^{\nu_0}) = a_{0,n}y^n + \sum_{i+\nu_0 j>\mu} a_{i,j}x^i(y + c_0 x^{\nu_0})^j =$$

$$= a_{0,n}y^n + \sum_{i+\nu_0 j>\mu} \sum_{k=0}^{j} \left(\begin{array}{c} j \\ k \end{array} \right) a_{i,j} c_0^k x^{i+\nu_0 k} y^{j-k}.$$

Notice that the point $(0, n)$ is a vertex of $\mathcal{NP}(g_1)$. Furthermore, the other terms appearing in the above sum are of the form $x^{i+\nu_0 k}y^{j-k}$ with $i+\nu_0 k+\nu_0(j-k) = i+\nu_0 j > \mu = \nu_0 n$ (because of the special choice of c_0). But this is equivalent to $\frac{i}{n-j} > \nu_0$ from which we conclude that the inclination of the first segment of $\mathcal{NP}(g_1)$ is strictly greater than ν_0. Here ν_0 is the inclination of the first segment of $\mathcal{NP}(g)$. And we obtain the following proposition from our observation:

Proposition 5.1.8. *If the y_1-order of $\tilde{g}(0, y_1)$ equals n, then g_1 is y-regular of order n. Moreover, $\mathcal{NP}(g_1)$ still contains the point $(0, n)$ and $\nu_1 > \nu_0$, where ν_1 denotes the inclination of the first segment of $\mathcal{NP}(g_1)$.*

$\underline{3^{rd}\textit{part}:}$ *induction step*

a) If $\tilde{g}(x_1, y_1) = y_1^n$, then $\phi(t) = t$ is a y-root of $\tilde{g}$. Or if the y_1-order of $\tilde{g}(0, y_1)$ is strictly smaller than n, then by the induction hypothesis there exists a positive integer m_1 and a Puiseux series $\phi_1(x_1^{\frac{1}{m_1}}) \in \mathbb{C}[[x_1^{\frac{1}{m_1}}]]$ fulfilling the equality

$$\tilde{g}(x_1, \phi_1(x_1^{\frac{1}{m_1}})) = 0.$$

For g we then have:

$$g(x_1^h, x_1^l(c_0 + \phi_1(x_1^{\frac{1}{m_1}})) = x_1^{\mu h}\tilde{g}(x_1, \phi_1(x_1^{\frac{1}{m_1}})) = 0.$$

Setting $x = x_1^h$, $m = m_1 h$ and $\phi(x^{\frac{1}{m}}) = x^{\frac{l}{h}}(c_0 + \phi_1(x^{\frac{1}{m}})) = x^{\nu_0}(c_0 + \phi_1(x^{\frac{1}{m}}))$ yields $g(x, \phi(x^{\frac{1}{m}})) = 0$ which finishes the proof in this case.

b) Let us now discuss the case that the y_1-order of $\tilde{g}(0, y_1)$ equals n and $\tilde{g}(x_1, y_1) \neq y_1^n$. Because of Proposition 5.1.8, the inclination of the first segment of $\mathcal{NP}(g_1)$, which we denote by ν_1, is strictly greater than ν_0. Now we have to apply the procedure described in parts $1-3$ again to the power series g_1. We then get the corresponding power series $\tilde{g_1}(x_2, y_2)$ and observe its behavior after substituing $x_2 = 0$. If its y_2-order is strictly smaller than n or $\tilde{g_1}(x_2, y_2) = y_2^n$, we can apply the induction hypothesis. We then find a positive integer m_2 and a Puiseux series $\phi_2(x_2^{\frac{1}{m_2}}) \in \mathbb{C}[[x_2^{\frac{1}{m_2}}]]$ fulfilling the equality

$$\tilde{g}_1(x_2, \phi_2(x_2^{\frac{1}{m_2}})) = 0.$$

For g_1 we then have

$$g_1(x_2^{h_1}, x_2^{l_1}(c_1 + \phi_2(x_2^{\frac{1}{m_2}}))) = x_2^{\mu_1 h_1}\tilde{g}_1(x_2, \phi_2(x_2^{\frac{1}{m_2}})) = 0,$$

where $\nu_1 = \frac{l_1}{h_1}$ is the inclination of the first segment of $\mathcal{NP}(g_1)$ and μ_1 the corresponding minimal value $d \in \mathbb{R}$ for which the line $i + \nu_1 j = d$ crosses $\mathcal{NP}(g_1)$. Finally for g we get

$$g(x_2^{h_1}, x_2^{l_1}(c_1 + \phi_2(x_2^{\frac{1}{m_2}})) + c_0 x_2^{h_1\nu_0}) = g_1(x_2^{h_1}, x_2^{l_1}(c_1 + \phi_2(x_2^{\frac{1}{m_2}}))) = 0.$$

And the claim follows when setting $x = x_2^{h_1}$, $m = m_2 h_1$ and $\phi(x^{\frac{1}{m}}) = x^{\frac{l_1}{h_1}}(c_1 + \phi_2(x^{\frac{1}{m}})) + c_0 x^{\nu_0} = c_0 x^{\nu_0} + x^{\nu_1}(c_1 + \phi_2(x^{\frac{1}{m}}))$.

If this is not the case and the y_2-order of $\tilde{g}_1(0, y_2)$ is equal to n again and $\tilde{g}_1(x_2, y_2) \neq y_2^n$, we have to define the power series $g_2(x, y)$ realizing the change of the variables

$$\begin{aligned} x &\mapsto x \\ y &\mapsto y + c_0 x^{\nu_0} + c_1 x^{\nu_1}. \end{aligned}$$

Because of Proposition 5.1.8, the inclination ν_2 of the first segment of $\mathcal{NP}(g_2)$ is strictly greater than ν_1 and we get the inequalities $\nu_0 < \nu_1 < \nu_2$. Now two cases have to be discussed. The first one is the situation where after finitely many repetitions of steps 1-3 we come to some $p \in \mathbb{N}$ so that the power series $\tilde{g}_{p-1}(0, y_p)$ has strictly smaller y_p-order than n or $\tilde{g}_{p-1}(x_p, y_p) = y_p^n$. Then, by the induction hypothesis, there exists a power series with fractional exponents $\phi_p(x_p^{\frac{1}{m_p}}) \in \mathbb{C}[[x_p^{\frac{1}{m_p}}]]$ which fulfills the equality $\tilde{g}_{p-1}(x_p, \phi_p(x_p^{\frac{1}{m_p}})) = 0$. And analogously as above we can show that the searched y-root of g has the form

$$y(x) = c_0 x^{\nu_0} + c_1 x^{\nu_1} + \cdots + x^{\nu_{p-1}}(c_{p-1} + \phi_p(x^{\frac{1}{m_p h_{p-1}}})).$$

But it is also possible that the y_j-order of $\tilde{g}_j(0, y_{j+1})$ remains equal to n and $\tilde{g}_j(x_{j+1}, y_{j+1}) \neq y_{j+1}^n$ for every $j \in \mathbb{N}$. This is the second case we have to discussat the moment. In this case we have an integer sequence $\nu_0 < \nu_1 < \cdots < \nu_k < \ldots$ of the inclinations of the first segments of the associated Newton polygons $\mathcal{NP}(g_j)$ and the formal power series

$$\phi_\infty(x) = c_0 x^{\nu_0} + c_1 x^{\nu_1} + \cdots + c_j x^{\nu_j} + \cdots \in \mathbb{C}[[x]].$$

Therefore, the inclination of the first segment of the Newton polygon of the limit $g_\infty(x, y)$ is equal to ∞. Here $g_\infty(x, y)$ can be obtained from $g(x, y)$ with the change of variables

$$\begin{aligned} x &\mapsto x \\ y &\mapsto y + \phi_\infty(x). \end{aligned}$$

Furthermore, we have

$$g(x, c_0 x^{\nu_0} + c_1 x^{\nu_1} + \cdots + c_j x^{\nu_j}) = g_j(x, 0)$$

which converges to $g_\infty(x, 0)$ as j tends to ∞. But because of $\nu_\infty = \infty$, the power series g_∞ must be divisible by y^n and we can write

$$g_\infty(x, y) = y^n h(x, y),$$

for some unit of the power series ring $h \in \mathbb{C}[[x, y]]^*$. Thus, we finally get the equality

$$g(x, \phi_\infty(x)) = 0$$

which finishes the proof of the existence of a y-root of a power series g.

Remark 5.1.9. The choice of the constants c_i's determines completely the Puiseux series produced by the Newton-Puiseux algorithm.

Newton-Puiseux theorem (first version) 5.1.10. *Let $g \in \mathbb{C}[[x, y]]$ be a y-regular power series. Then there exists a y-root g, namely $g(x, y(x^{\frac{1}{m}})) = 0$, with $y(x) \in \mathbb{C}[[x]]$ for and some positive integer $m \in \mathbb{N}$.*

And using Lemma 5.1.1 we see that the Newton-Puiseux algorithm even produces parametrizations of branches of plane algebraic curves.

Newton-Puiseux theorem (second version) 5.1.11. *Let $X \subseteq \mathbb{A}^2$, be a plane algebraic curve. Then each branch of X at an arbitrary point of X can be parametrized by a pair of convergent power series.*

There is an even more general theorem saying that the field of Puiseux series is algebraically closed:

Newton-Puiseux theorem 5.1.12. *The algebraic closure of the field $\mathbb{C}\{\{x\}\}$ (resp. $\mathbb{C}((x))$) is the field $\bigcup_{n\geq 1} \mathbb{C}\{\{x^{\frac{1}{n}}\}\}$ (resp. $\bigcup_{n\geq 1} \mathbb{C}((x^{\frac{1}{n}}))$).*
Here $\mathbb{C}\{\{x\}\} := \mathrm{Quot}(\mathbb{C}\{x\})$.

Now, to show that each Puiseux parametrization of a branch is indeed a parametrization of the curve itself, we need to show the Zariski density of the image of the Puiseux parametrization. To see the density of the parametrization, the following lemma can be helpful :

Lemma 5.1.13. *Let $X, Y \subseteq \mathbb{A}^n$ be two irreducible algebraic curves with $X \neq Y$ and $0 \in X \cap Y$. Let $(X_i, 0), i \in I$ be the branches of X at 0 and $(Y_j, 0), j \in J$ the branches of Y at 0, with suitable index sets I and J. Then $(X_i, 0) \not\equiv (Y_j, 0)$ for all $i \in I$ and $j \in J$.*

Proof. We assume indirectly that there exist some $i \in I$ and $j \in J$ so that $(X_i, 0) \equiv (Y_j, 0)$. Let Z be a representative of this equivalence class. Then the Zariski-closure of Z must equal X and also Y as Z is a representative of a branch of X and of a branch of Y and as X and Y are irreducible. But this is impossible as $X \neq Y$. $\square$

Lemma 5.1.14. *Let $I, J \subseteq \mathbb{C}\{x_1, \ldots, x_n\}, I \neq J$, be two prime idelas of height $n-1$. Then I and J cannot have the same zero $\gamma(t) = (x_1(t), \ldots, x_n(t)) \in \mathbb{C}\{t\}^n$, with convergent power series $x_i(t) \neq const.$ for all $i = 1, \ldots n$.*

Proof. Let us indirectly assume that $\gamma(t)$ is a zero of I and J. Let us consider the map

$$\begin{aligned} \gamma^* : \mathbb{C}\{x_1, \ldots, x_n\} &\to \mathbb{C}\{t\} \\ x_1 &\mapsto x_1(t) \\ &\vdots \\ x_n &\mapsto x_n(t). \end{aligned}$$

At first notice that γ^* is a map between two integral domains. Hence, $ker(\gamma^*)$ is a prime ideal in $\mathbb{C}\{x_1, \ldots, x_n\}$. As $x_i(t) \neq const.$ for all $i = 1, \ldots n$, the height of $ker(\gamma^*)$ is at most $n-1$. Since $(x_1(t), \ldots, x_n(t))$ is a zero of I, the height of $ker(\gamma^*)$ equals $n-1$. Then we have $I, J \subseteq ker(\gamma^*)$. But then $ker(\gamma^*) = I = J$ because I, J and $ker(\gamma^*)$ are prime ideals of the same height. $\square$

Corollary 5.1.15. *Two not associated irreducible convergent power series $g_1, g_2 \in \mathbb{C}\{x, y\}$, $g_1, g_1 \neq 0$, cannot have the same zero $\gamma(t) = (x(t), y(t)) \in \mathbb{C}\{t\}^2$, with convergent power series $x(t), y(t) \neq const.$*

Proposition 5.1.16. *Let $X = V(I) \subseteq \mathbb{A}^n$ be an algebraic curve and $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ its defining ideal. Consider the following map*

$$\begin{aligned} \gamma : \bigcap_{i=1}^{n} D(x_i(t)) &\to X \\ t &\mapsto (x_1(t), ..., x_n(t)), \end{aligned}$$

where $x_i(t) \in \mathbb{C}\{t\}$ are convergent power series for all $i = 1, ..., n$. Let

$$\begin{aligned} \gamma^* : \mathbb{C}[x_1, ..., x_n]/I &\to \mathbb{C}\{t\} \\ \overline{x}_i &\mapsto x_i(t) \\ &\vdots \\ \overline{x}_n &\mapsto x_n(t) \end{aligned}$$

be the induced map. Then the image of γ is Zariski-dense in X if and only if γ^* is injective.

Proof. ⇒: If the map γ^* was not injective, then there would exist a polynomial $h \in \mathbb{C}[x_1, \dots, x_n] \backslash I$ so that $h(x_1(t), ..., x_n(t)) = 0$ in $\mathbb{C}\{x_1, ..., x_n\}$. Then $ker(\gamma^*)$ would be a prime ideal of height at least 1. But then the Zariski-closure of the image of γ is an algebraic variety of codimension n and therefore strictly contained in X.
⇐: Assume that the image of γ is not Zariski-dense in X. Then the Zariski-closure of the image of γ is an irreducible algebraic curve $Y = V(J)$ for some ideal $J \subseteq \mathbb{C}[x_1, \dots, x_n]$ with $I \subsetneq J$. But the n-tuple $(x_1(t), ..., x_n(t))$ must be then a zero of the ideal J and consequently the map γ^* is not injective. □

Hence, for the density of the image of a non-constant map $\gamma : D(x_1(t)) \cap D(x_2(t)) \to X$, with $\gamma(t)$ a root of a convergent power series g_i defining a branch $(X_i, 0)$ of a plane algebraic curve $X = V(I) \subseteq \mathbb{A}^2$, it is sufficient to show the injectivity of the corresponding map γ^* defined as in Proposition 5.1.16. But according to Lemma 5.1.13 and 5.1.14 and Corollary 5.1.15, the map γ^* is injective. And we deduce the following lemma:

Lemma 5.1.17. *Let $X \subseteq \mathbb{A}^2$ be a plane algebraic curve. Let $(X_i, 0)$ be the branches of X at the origin and $g_i \in \mathbb{C}\{x, y\}$ the defining power series of the branches. Then we have the correspondences:*

$$\left\{ \begin{array}{c} (x_i(t), y_i(t))\ zeros\ of\ g_i \\ constructed\ by\ the \\ Newton-Puiseux\ alg. \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} (x_i(t), y_i(t))\ Puiseux\ parametrizations \\ of\ the\ branches\ (X_i, 0) \end{array} \right\}.$$

Even more:

$$\left\{ \begin{array}{c} (x_i(t), y_i(t))\ zeros\ of\ g_i \\ constructed\ by\ the \\ Newton-Puiseux\ alg. \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} (x_i(t), y_i(t))\ Puiseux \\ parametrizations\ of\ X \end{array} \right\}.$$

Proof. Notice that an irreducible convergent power series $g \in \mathbb{C}$ is either y- or x-regular. Hence, after a suitable coordinate change, the Newton-Puiseux algorithm applies to each branch of X. Let $(x, \phi(x^{\frac{1}{n}}))$ be a zero of $g \in \mathbb{C}\{x, y\}$, where g defines a branch of X. Then $x \mapsto x^n$ is the correspondence between Puiseux parametrizations of the branches of X or X itself and the zeros of the defining power series of the branches of X constructed due to the Newton-Puiseux algorithm. The rest follows from the discussion before. □

And we conclude the existence of a parametrization of an arbitrary irreducible plane algebraic curve.

Theorem 5.1.18. *Each irreducible plane algebraic curve can be parametrized by a pair of convergent power series $(\mathsf{x}(t), \mathsf{y}(t))$. In addition, each branch of a plane algebraic curve can be parametrized by a Puiseux parametrization.*

Even more, for a parametrization $(\mathsf{x}(t), \mathsf{y}(t))$ of a branch of a plane algebraic curve the orders $\mathrm{ord}_0(\mathsf{x})$ and $\mathrm{ord}_0(\mathsf{y})$ can be read off from the Newton-Puiseux algorithm.

Lemma 5.1.19. *Let $X \subseteq \mathbb{A}^2$ be a plane algebraic curve and $(Y, 0)$ one of the branches of X at the origin with defining power series $g \in \mathbb{C}\{x, y\}$. Let $\nu = \frac{l}{h}$ be the inclination of the first segment of $\mathcal{NP}(g)$. If $g \neq y^k, k \in \mathbb{N}$, then for a Puiseux parametrization $(\mathsf{x}(t), \mathsf{y}(t))$ of $(Y, 0)$ we have $\mathsf{x}(t) = t^h$ and $\mathrm{ord}_0(\mathsf{y}) = l$.*

Proof. This is a direct consequence of the construction of y-roots according to the Newton-Puiseux algorithm $\square$

Lemma 5.1.20. *If $y(t)$ is a y-root of a y-regular polynomial f, then it is one of the series the Newton-Puiseux algorithm gives rise to.*

Proof. If $y(t) = 0$ then $f = y \cdot g(x, y)$ and the parametrization of $V(y)$ was discussed at the beginning of the Newton-Puiseux algorithm.
So let $y(t) \neq 0$. We write $y(t) = t^{\frac{l}{h}}(c_0 + y_1(t))$ with $\frac{l}{h}$ the inclination of the first segment of the Newton polygon of f, some constant $c_0 \neq 0$ and power series $y_1 \in \mathbb{C}[[t]]$. Then using the definition of $\tilde{f}$ from the Newton-Puiseux algorithm we see that if $f(t, y(t)) = 0$, then $f(t^h, t^l(c_0 + y_1(t^h))) = t^{\mu h}\tilde{f}(t, y_1) = 0$ and so we get $\tilde{f}(t, y_1) = 0$. And the claim follows by iteration. $\square$

We will now discuss some more properties on Puiseux parametrizations of plane algebraic curves. Finally, we show that if a polynomial f factors into a product of formal power series $f = g_1^{s_1} \cdots g_n^{s_n}, g_i \in \mathbb{C}[[x, y]]$, then the factorization is already unique up to a multiplication with a unit and all the factors (except the unit) are already convergent power series.

For each n-th root of unity $\zeta \in \{\xi \in \mathbb{C} | \xi^n = 1\}$ we define the automorphism

$$\sigma_\zeta : \mathbb{C}((x^{\frac{1}{n}}))[[y]] \to \mathbb{C}((x^{\frac{1}{n}}))[[y]]$$

$$x^{\frac{1}{n}} \mapsto \zeta x^{\frac{1}{n}}$$

$$y \mapsto y.$$

For a Puiseux series

$$s(x) = \sum_{i \geq i_0} a_i x^{\frac{i}{n}},$$

with $i_0 \in \mathbb{Z}, n \in \mathbb{N}$ fixed, the image of s under σ_ζ is then

$$\sigma_\zeta(s) = \sum_{i \geq i_0} \zeta^i a_i x^{\frac{i}{n}},$$

and is called a *conjugate* of s. It is clear, by definition, that the image under σ_ζ of a Puiseux series of polydromy order n is again a Puiseux series of polydromy order n . The set of all conjugates of s is called the *conjugacy class* of s.

Lemma 5.1.21. *The number of different conjugates of a Puiseux series equals its polydromy order.*

Proof. Let

$$s(x)=\sum_{i\geq i_0} a_i x^{\frac{i}{n}},$$

be a Puiseux series of polydromy order $n=\nu(s)$. We select the indices $i_1,...,i_r\in\mathbb{Z}$ such that $a_{i_j}\neq 0$ for all $j=1,...,r$ and $\gcd\{n,i_1,...,i_r\}=1$. If the equality $\sigma_\zeta(s)=\sigma_\eta(s)$ holds for some $\eta^n=\zeta^n=1$, then the equalities $\eta^{i_j}a_{i_j}=\zeta^{i_j}a_{i_j}$ and so $\eta^{i_j}=\zeta^{i_j}$ are satisfied for all $j=1,...,r$. But from $\gcd\{n,i_1,...,i_r\}=1$ it follows that $\eta=\zeta$. Thus $s(x)$ has n different conjugates. $\square$

Lemma 5.1.22. *A Puiseux series $y(x^{\frac{1}{n}})\in\mathbb{C}[[x^{\frac{1}{n}}]]$ is y-root of $g\in\mathbb{C}[[x,y]]$ if and only if $g(x,y)=(y-y(x^{\frac{1}{n}}))\cdot h(x^{\frac{1}{n}},y)$ holds for some $h\in\mathbb{C}[[x^{\frac{1}{n}},y]]$.*

Proof. Consider the automorphism

$$\varphi:\mathbb{C}[[x^{\frac{1}{n}},y]]\to\mathbb{C}[[x^{\frac{1}{n}},y]]$$

$$g\mapsto\varphi(g):=g(x,y+y(x^{\frac{1}{n}})).$$

Then $\varphi(y-y(x^{\frac{1}{n}}))=y$ and $(\varphi(g))(x,0)=g(x,y(x^{\frac{1}{n}}))$. Thus we can w.l.o.g. assume that $y(x^{\frac{1}{n}})=0$. But in this case we have $g(x,y)=y\cdot h(x,y)$ for some $h\in\mathbb{C}[[x,y]]$. $\square$

Lemma 5.1.23. *If a Puiseux series $y(x^{\frac{1}{n}})\in\mathbb{C}[[x^{\frac{1}{n}}]]$ is a y-root of a power series $g\in\mathbb{C}[[x,y]]$, then all its conjugates are again y-roots of g.*

Proof. By the previous lemma we have $g(x,y)=(y-y(x^{\frac{1}{n}}))\cdot h(x^{\frac{1}{n}},y)$ for some $h\in\mathbb{C}[[x^{\frac{1}{n}},y]]$. Furthermore, we have the equality $g(x,y)=\sigma_\zeta(g(x,y))=(y-y(\zeta x^{\frac{1}{n}}))\cdot h(\zeta x^{\frac{1}{n}},y)$ for each n-th root of unity ζ. Using the previous lemma again we get the claim. $\square$

For a Puiseux series $s\in\mathbb{C}[[x^{\frac{1}{n}}]]$ of polydromy order $\nu(s)=n$, we define the following power series

$$g_s=\prod_{i=1}^{\nu(s)}(y-\sigma_{\zeta^i}(s))\in\mathbb{C}[[x]][y],$$

where ζ is a primitive n-th root of unity and so $\sigma_\zeta(s),...,\sigma_{\zeta^n}(s)$ are the different conjugates of s. The fact $g_s\in\mathbb{C}[[x]][y]$ can be seen in the following way. Note that for each $j,k=1,...,n$ there exists $i\in\{1,...,n\}$ such that $\sigma_{\zeta^k}(\sigma_{\zeta^j}(s))=\sigma_{\zeta^i}(s)\in\mathbb{C}[[x^{\frac{1}{n}}]]$ is fulfilled. Furthermore, for $i\neq j$ we have $\sigma_{\zeta^i}(s)\neq\sigma_{\zeta^j}(s)$. Thus, we conclude the equality $\sigma_{\zeta^j}(g_s)=g_s$ for all $j=1,...,n$ from which it follows that the polydromy order of g_s equals 1 and so $g_s(x)\in\mathbb{C}[[x]][y]$. Then using Lemmata 5.1.21, 5.1.22 and 5.1.23 we can show the following:

Lemma 5.1.24. *A Puiseux series $s\in\mathbb{C}[[x^{\frac{1}{n}}]]$ is a y-root of a power series $g\in\mathbb{C}[[x,y]]$ if and only if the equality $g(x,y)=g_s(x,y)\cdot h(x,y)$ is fulfilled for some $h(x,y)\in\mathbb{C}[[x,y]]$.*

Proof. $\Leftarrow$: This direction is straightforward with Lemmata 5.1.22 and 5.1.23.
$\Rightarrow$: Let the equality $g(x, s(x)) = 0$ be fulfilled. Then $g = (y - s) \cdot h_1$ for some $h_1 \in \mathbb{C}[[x^{\frac{1}{n}}, y]]$. By Lemma 5.1.23, $\sigma_\zeta(s)$, with ζ a primitive n-th root of unity, is a y-root of g and hence of h_1. Thus we get $h_1 = (y - \sigma_\zeta(s)) \cdot h_2$ for some $h_2 \in \mathbb{C}[[x^{\frac{1}{n}}, y]]$ and so $g = (y-s)\cdot(y-\sigma_\zeta(s))\cdot h_2$. Iteratively, we obtain $g = (y-s)\cdots(y-\sigma_{\zeta^{n-1}}(s))\cdot h_n = g_s h_n$ for some $h_n \in \mathbb{C}[[x^{\frac{1}{n}}, y]]$ and since $g \in \mathbb{C}[x, y]$ and $g_s \in \mathbb{C}[[x]][y]$, we have $h_n \in \mathbb{C}[[x, y]]$. $\square$

Lemma 5.1.25. *Let $s \in \mathbb{C}[[x^{\frac{1}{n}}]]$ be a Puiseux series. Then the power series g_s is irreducible in the power series ring $\mathbb{C}[[x, y]]$.*

Proof. If we had $g_s = h_1 \cdot h_2$ for some power series $h_1, h_2 \in \mathbb{C}[[x, y]]$, then s would have to be a y-root of h_1 or h_2. Let us assume that s is a y-root of h_1. But then g_s has to divide the power series h_1 and so $h_2 \equiv 1$. $\square$

With these last results we are now able to show the convergence of power series appearing in the factorization of a polynomial in the power series ring.

Corollary 5.1.26. *Let $X = V(f)$ be a plane algebraic curve with $0 \in X$. Then there exist unique Puiseux series $s_1, ..., s_k$, unit $u \in \mathbb{C}[[x, y]]^*$ and non-negative integers $l_1, l_2 \geq 0$ such that f factors into $f = u \cdot x^{l_1} y^{l_2} \cdot g_{s_1} \cdots g_{s_k}$, with $g_{s_i} \in \mathbb{C}\{x, y\}$ for all $i = 1, \ldots, k$. Especially, the branches of X at 0 are unique.*

Proof. The existence of unique non-negative integers l_1, l_2 that fulfil the equality $f = x^{l_1} y^{l_2} \tilde{f}$ with some polynomial $\tilde{f}$, which is not divisible by x and y, is clear. So let us further assume that f is not divisible by y and w.l.o.g. let f be y-regular. We proceed now by induction on the order of y-regularity of f. If f is y-regular of order one, then by the Implicit functions theorem there exists a Puiseux series s of polydromy order 1 that is a y-root of f and hence $f = (y - s) \cdot u = g_s \cdot u$ for some power series u with $u(0, 0) \neq 0$. As for the case that the y-order of f is strictly bigger than 1 we shall construct, using the Newton-Puiseux algorithm, a Puiseux series s_1 that is a y-root of f. Then f satisfies the equality $f = g_{s_1} \cdot f_1$ for some power series $f_1 \in \mathbb{C}[x, y]$. As s_1 is a convergent power series, g_{s_1} is convergent as well. Obviously the order of y-regularity of f_1 is strictly smaller than the y-regularity order of f. Hence the induction hypothesis applies to f_1 and we get the claim. The uniqueness of such a factorization is then clear. $\square$

5.2 Puiseux parametrizations of algebraic space curves

In the previous section we saw how to construct parametrizations of irreducible plane algebraic curves according to the Newton-Puiseux algorithm. We will discuss now whether for every algebraic space curve $X \subseteq \mathbb{A}^n$ and a point $a \in X$ there exists a parametrization of X at a. The key role in the answer to this question will play the knowledge about Puiseux parametrizations of plane algebraic curves from the previous section.

There is also a modification of the Newton-Puiseux algorithm for plane algebraic curves to algebraic space curves. This modified algorithm replace the inclination of the first segment of the Newton polygon by the so-called tropism of the defining ideal of an algebraic space curve. The algorithm is very similar to the Newton-Puiseux algorithm for plane algebraic curves. However, some steps are more technical and need more estimations and arguments.

Let $X \in \mathbb{A}^n$ be an algebraic space curve. Assume that $0 \in X$. We will investigate the branches of X at the origin by looking at the projections of the branches to the coordinate planes. We define for each $i = 2, \dots, n$ the map

$$\pi_i : \mathbb{A}^n \to \mathbb{A}^2$$

$$(x_1, \dots, x_n) \mapsto (x_1, x_i)$$

to be the projection of $\mathbb{A}^n$ to the (x_1, x_i)- coordinate plane. For a branch $(Y, 0)$ of X at the origin we define $\overline{\pi_i((Y,0))} := \overline{\pi_i(\widetilde{Y})}$ for a representative $\widetilde{Y}$ of $(Y, 0)$. Here $\overline{\pi_i(\widetilde{Y})}$ is the Zariski-closure of $\pi_i(\widetilde{Y})$. As two different plane algebraic curves have distinct branches and any two different branches can meet only at finitely many points, the definition of $\overline{\pi_i((Y,0))}$ does not depend on the choice of a representative $\widetilde{Y}$ of $(Y, 0)$. Hence, the definition is well-defined.

Lemma 5.2.1. *Let $(Y, 0)$ be a branch of X at the origin. If $(Y, 0) \not\equiv (V(x_1), 0)$, then $\dim \overline{\pi_i((Y,0))} = 1$ for all $i = 2, \dots, n$.*

Proof. Let $J = (g_1, \dots, g_k) \subseteq \mathbb{C}\{x_1, \dots, x_n\}$ be the defining ideal of the branch $(Y, 0)$. It is enough to show that the image of one representative $\widetilde{Y}$ of $(Y, 0)$ under each projection π_i contains infinitely many points. This implies $\dim \overline{\pi_i((Y,0))} > 0$. And as the dimension under morphisms cannot increase the claim follows. We will show this by a contradiction. Let $\widetilde{Y} = V(J)$. Let us assume that there exists some $j \in \{2, \dots, n\}$ for which the projection $\pi_j(\widetilde{Y})$ consists only of finitely many points $\{(0,0), (a_1, b_1), \dots, (a_m, b_m)\} \subseteq \mathbb{A}^2$. This means that $\widetilde{Y}$ consists only of points with the first coordinate from the set $\{0, a_1, \dots, a_m\}$. Because of $(Y, 0) \not\equiv (V(x_1), 0)$ we have $\widetilde{Y} \not\subseteq V(x_1)$ and so $a_i \neq 0$ for at least one a_i. Thus, we can write $J = (x_1(x_1 - a_1) \cdots (x_1 - a_m), g_1, \dots, g_k)$ with at least one a_i different from 0. We claim that $(x_1(x_1 - a_1) \cdots (x_1 - a_m), g_1, \dots, g_k) = (x_1, g_1, \dots, g_k) \cap (x_1 - a_1, g_1, \dots, g_k) \cap \cdots \cap (x_1 - a_m, g_1, \dots, g_k)$. The inclusion $'' \subseteq''$ is easy to see, so let us discuss the other inclusion, $'' \supseteq''$. Let $h \in (x_1, g_1, \dots, g_k) \cap$

$(x_1 - a_1, g_1, \ldots, g_k) \cap \cdots \cap (x_1 - a_m, g_1, \ldots, g_k)$. Then there are following possibilites of what this h could be. Either $h \in (g_1, \ldots, g_k)$ or $\overline{h} \neq 0$ in $((x_1, g_1, \ldots, g_k) \cap (x_1 - a_1, g_1, \ldots, g_k) \cap \cdots \cap (x_1 - a_m, g_1, \ldots, g_k))/(g_1, \ldots, g_k)$. As for the second case, as $h \in (x_1, g_1, \ldots, g_k)$, we have $h = x_1 \cdot h_0 + (g_1, \ldots, g_k)$ for some $h_0 \in \mathbb{C}\{x, y\}$. Analogously from $h \in (x_1 - a_1, g_1, \ldots, g_k)$ we get $h = x_1(x_1 - a_1) \cdot h_1 + (g_1, \ldots, g_k)$ for some $h_1 \in \mathbb{C}\{x, y\}$. Iteratively we get $h = x_1(x_1 - a_1) \cdots (x_1 - a_m) \cdot h_m + (g_1, \ldots, g_k)$ for some $h_m \in \mathbb{C}\{x, y\}$. Thus, $h \in (x_1(x_1 - a_1) \cdots (x_1 - a_m), g_1, \ldots, g_k)$ and the claim follows. But this is a contradiction to the analytical irreducibility of $\widetilde{Y}$. $\square$

Lemma 5.2.2. *Let $(Y, 0)$ be a branch of X with $(Y, 0) \not\equiv (V(x_1), 0)$. Then for each representative $\widetilde{Y}$ of $(Y, 0)$ and all $i = 2, \ldots, n$, we have $\pi_i(\widetilde{Y}) \subseteq V(g_i)$, with $g_i \in \mathbb{C}\{x_1, x_i\}$ an irreducible x_i-regular convergent power series.*

Proof. Using Lemma 5.2.1 we already know that $\pi_i(\widetilde{Y})$ defines a branch at the origin of a plane algebraic curve. Hence $\pi_i(\widetilde{Y}) = V(g_i)$ for some convergent power series $g_i \in \mathbb{C}\{x, y\}$. Clearly $V(g_i)$ contains the origin and so g_i has no constant term. Let us write $g_i = \sum c_{k,l} x^k y^l$ with $c_{0,0} = 0$. Then $g_i = x^s \cdot y^r \cdot \tilde{g}_i$ for some $s, r \in \mathbb{N}$ and some convergent power series $\tilde{g}_i \in \mathbb{C}\{x, y\}$ with $\tilde{g}_i(0, y) \neq 0$ and so $\tilde{g}_i$ is y-regular. Thus, we have three possibilities, either $(\pi_i(\widetilde{Y}), 0) \equiv (V(x), 0)$ or $(\pi_i(\widetilde{Y}), 0) \equiv (V(y), 0)$ or $(\pi_i(\widetilde{Y}), 0) \equiv (V(\tilde{g}_i), 0)$. But because of the assumption $(Y, 0) \not\equiv (V(x_1), 0)$, the case that $(\pi_i(\widetilde{Y}), 0) \equiv (V(x), 0)$ is not possible. $\square$

From these two lemmata we conclude directly a generalization of the Newton-Puiseux theorem to algebraic space curves.

Generalized Newton-Puiseux Theorem for algebraic space curves 5.2.3. *Let $(Y, 0)$ be a branch at the origin of an algebraic space curve $X \subseteq \mathbb{A}^n$. Assume that $(Y, 0) \not\equiv (V(x_1), 0)$. Then there exist convergent power series $s_2(t), \ldots, s_n(t) \in \mathbb{C}\{t\}$ and a positive integer $m \in \mathbb{N}$ such that the n-tuple $(t^m, s_2, \ldots, s_n)$ parametrizes $(Y, 0)$ at the origin.*

Proof. Let $\widetilde{Y}$ be a representative of $(Y, 0)$. As $\widetilde{Y}$ is not contained in the x_1-coordinate hyperplane, each projection $\pi_i(\widetilde{Y}), i = 2, \ldots n$, is a branch of a plane algebraic curve and is defined by a convergent power series $\tilde{g}_i \in \mathbb{C}\{x, y\}$ that is y-regular. For the branch $(\pi_i(\widetilde{Y}), 0) \equiv (V(\tilde{g}_i), 0)$ we get according to the Newton-Puiseux algorithm a Puiseux parametrization $(t^{m_i}, s_i(t)), s_i \in \mathbb{C}\{t\}$ with $s_i(0) = 0$. From the Puiseux parametrizations $(t^{m_i}, s_i(t))$ of the projections $\pi_i(\widetilde{Y})$ for all $i = 2, \ldots n$, we can easily reconstruct a parametrization of $(Y, 0)$. We set now $m = m_2 \cdots m_n$ and obtain n-tuple of convergent power series

$$(t^m, s_2(t^{\frac{m}{m_2}}), \ldots, s_n(t^{\frac{m}{m_n}}))$$

parametrizing the branch $(Y, 0)$ at the origin. $\square$

As we are always able to transform an algebraic space curve to an algebraic space curve all whose branches at the origin are not contained in the x_1-coordinate hyperplane, we are able to construct in this way a parametrization of each branch of an arbitrary algebraic space curve. Even more, using the results from the last section, we know that these are already parametrizations of the curve itself. And so the following theorem was proven:

Theorem 5.2.4. *Each branch at a point a,* (Y, a)*, of an algebraic space curve* $X \subseteq \mathbb{A}^n$ *can be parametrized by an n-tuple of convergent power series* $(s_1(t), \ldots, s_n(t))$*,* $s_i(t) \in \mathbb{C}\{t\}$ *for all* $i = 1, \ldots, n$*.*

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