COMPOSITIO MATHEMATICA

HERWIG HAUSER

Comparing modules of differential operators by their evaluation on polynomials

Compositio Mathematica, tome 69, n° 3 (1989), p. 295-307.

<http://www.numdam.org/item?id=CM_1989__69_3_295_0>

Compositio Mathematica 69: 295–307, 1989

Comparing modules of differential operators by their evaluation on polynomials

HERWIG HAUSER
Institut für Mathematik, Universität Innsbruck, A-6020 Innsbruck, Austria

Received 23 June 1987; accepted in revised form 27 June 1988

Introduction

Any non constant power series can be written for d sufficiently large as a linear combination of its derivatives of order less than d.

Conversely, given an integer d there always exists a power series which is not a linear combination of its derivatives of order less than d.

The first statement is obvious. The second seems obvious, too: if $d = 1$ it just asserts the existence of a non-quasihomogeneous power series. This is immediate. If d is arbitrary, one may expect that any generic polynomial with sufficiently many summands should fulfil the assertion.

It turns out that even if d is small the search for a convenient polynomial is very unpleasant: the size and the coefficients of the systems of linear equations one has to solve increase rapidly with d. As a common phenomenon, generic objects are despite their number hard to grasp.

This paper proposes a general algorithm for computing such generic polynomials. Actually we shall construct a universal family $\mathscr{P}$ of testing polynomials valuable for all finitely generated modules of differential operators: two modules will be equal if and only if their evaluations on a suitable polynomial of $\mathscr{P}$ are equal. To make this more precise let us fix some notation.

Let A denote the ring of germs of analytic functions on $\mathbb{C}^n$ at 0 and let $\mathbb{D}$ be the A-module of differential operators on $\mathbb{C}^n$ with coefficients in A. Given coordinates $x_1, \ldots, x_n$ on $\mathbb{C}^n$ we can write $A = \mathbb{C}\{x\}$ and $D = \Sigma_{\alpha\varepsilon} c_{\alpha\varepsilon} x^{\alpha+\varepsilon} \partial^\alpha \in \mathbb{D}$ with $c_{\alpha\varepsilon} \in \mathbb{C}$, $\alpha \in \mathbb{N}^n$, $\varepsilon \in \mathbb{Z}^n$, $\alpha + \varepsilon \in \mathbb{N}^n$. Set $|\varepsilon| = \varepsilon^1 + \cdots + \varepsilon^n \in \mathbb{Z}$. For a differential operator $D \in \mathbb{D}$ and a finitely generated A-submodule $\mathbb{F}$ of $\mathbb{D}$ we introduce:

$$\operatorname{supp} D = \text{support of } D = \{\alpha \in \mathbb{N}^n, \exists\, \varepsilon: c_{\alpha\varepsilon} \neq 0\} \subset \mathbb{N}^n \text{ finite},$$

$$\operatorname{carr} D = \text{carrier of } D = \{\varepsilon \in \mathbb{Z}^n, \exists\, \alpha: c_{\alpha\varepsilon} \neq 0\} \subset \mathbb{Z}^n,$$

$$\begin{aligned}
\text{ord } D &= \text{order of } D = \sup\{|\alpha|, \alpha \in \text{supp } D\} \in \mathbb{N}, \\
\text{lev } D &= \text{level of } D = \inf\{|\varepsilon|, \varepsilon \in \text{carr } D\} \in \mathbb{Z}, \\
\text{ord } 0 &= 0, \text{ lev } 0 = \infty, \\
\text{ord } \mathbb{F} &= \sup\{\text{ord } D, D \in \mathbb{F}\} \in \mathbb{N}, \\
\text{lev } \mathbb{F} &= \sup\{\text{lev } D_i;\ D_1, \ldots, D_m \text{ minimal standard base of } \mathbb{F}\} \in \mathbb{Z}
\end{aligned}$$

(cf. sec. 1).

For a power series $z \in A$ denote finally by $\mathbb{F}z$ the ideal of evaluations, $\mathbb{F}z = \{Dz, D \in \mathbb{F}\}$. We then have for all finitely generated A-submodules of $\mathbb{D}$:

THEOREM: *Assume $n \geqslant 2$. For any $d, e \in \mathbb{Z}$ there exists an explicit construction of a polynomial $z = z_{de} \in A$ with the following universal property: Two A-submodules $\mathbb{F}$ and $\mathbb{G}$ of $\mathbb{D}$ with* ord $\mathbb{F}$, ord $\mathbb{G}$ $\leqslant d$ *and* lev$(\mathbb{F} + \mathbb{G}) \leqslant e$ *are equal if and only if the ideals $\mathbb{F}z$ and $\mathbb{G}z$ are equal.*

Remarks: 1. It is equivalent to say that any two A-submodules $\mathbb{F} \subset \mathbb{G}$ of $\mathbb{D}$ of order $\leqslant d$ and level $\leqslant e$ are equal if and only if $\mathbb{F}z$ and $\mathbb{G}z$ are equal.

2. The polynomial z of A is not unique: the construction algorithm we describe provides a whole range of suitable polynomials. But no matter how z is chosen, its degree and number of summands increase very quickly with d and e.

3. In practical computations the situation is generally more specific and allows the choice of simpler testing polynomials. Typically are given a differential operator D and a sub-module $\mathbb{F}$ of $\mathbb{D}$; knowing that $D \notin \mathbb{F}$ one wants to find a $z \in A$ with $Dz \notin \mathbb{F}z$. For instance, consider the case $n = 2$, $D = 1$ and $\mathbb{F}$ the module generated by all $\partial_x^i \partial_y^j$ with $0 < i + j \leqslant 2$. A possible polynomial z satisfying $z \notin \mathbb{F}z$ is

$$z = x^{15} + x^{12}y^3 + x^9y^6 + x^6y^{10} + x^3y^{13} + y^{16}.$$

This polynomial has two characteristic properties: its exponents have componentwise distance in $\mathbb{N}^n$ strictly bigger than 2 (they are sufficiently *spare*), and the 6×6 matrix $((\gamma)_\alpha)$ has rank 6, where γ (resp. α) runs over the exponents of z (resp. D and $\mathbb{F}$), and $(\gamma)_\alpha \in \mathbb{N}$ is defined by $\partial^\alpha(x, y)^\gamma = (\gamma)_\alpha \cdot (x, y)^{\gamma-\alpha}$ (the γ's are *generic* w.r.t. the α's). These two features will form the basis of the construction of the testing polynomial z in general.

1. Division Theorem for differential operators

One ingredient for proofing the result stated in the Introduction is the Division Theorem for finitely generated modules of differential operators (cf. [B-M], [C]). We shall need a slightly different version of it and thus provide an independent presentation of the theorem.

Consider $\mathbb{Z}^n$ equipped with the following total order: $\varepsilon < \varepsilon'$ if either $|\varepsilon| < |\varepsilon'|$ or $|\varepsilon| = |\varepsilon'|$ and $\varepsilon <_{\text{lex}} \varepsilon'$, where $<_{\text{lex}}$ denotes lexicographical order. For a differential operator $D \in \mathbb{D}$, $D = \Sigma\, c_{\alpha\varepsilon} x^{\alpha+\varepsilon}\partial^\alpha$ and a finitely generated A-submodule $\mathbb{F}$ of $\mathbb{D}$ we define:

$$tcD = \text{tangent cone of } D = \sum_\alpha c_{\alpha\varepsilon_0} x^{\alpha+\varepsilon_0}\partial^\alpha \quad \text{with } \varepsilon_0 = \inf \operatorname{carr} D,$$

$$inD = \text{initial term of } D = c_{\alpha_0\varepsilon_0} x^{\alpha_0+\varepsilon_0}\partial^{\alpha_0} \quad \text{with } \alpha_0 = \inf \operatorname{supp}(tcD),$$

$$tc0 = in0 = 0,$$

$$tc\mathbb{F} = (tcD,\, D \in \mathbb{F}) \cdot A \subset \mathbb{D},$$

$$in\mathbb{F} = (inD,\, D \in \mathbb{F}) \cdot A \subset \mathbb{D},$$

$$\Delta\mathbb{F} = \left\{D = \sum_{\alpha\varepsilon} c_{\alpha\varepsilon} x^{\alpha+\varepsilon}\partial^\alpha \in \mathbb{D},\ x^{\alpha+\varepsilon}\partial^\alpha \notin in\,\mathbb{F} \text{ if } c_{\alpha\varepsilon} \neq 0\right\}.$$

Both $tc\mathbb{F}$ and $in\mathbb{F}$ are A-submodules of $\mathbb{D}$, whereas $\Delta\mathbb{F}$ is only a $\mathbb{C}$-subspace. All three depend on the chosen coordinates $x_1, \ldots, x_n$ on $\mathbb{C}^n$ (however, [G, Th.2] suggests that $in\mathbb{F}$ and $\Delta\mathbb{F}$ are constant for generic coordinates). One clearly has the direct sum decomposition $\mathbb{D} = in\mathbb{F} \oplus \Delta\mathbb{F}$; the Division Theorem asserts that actually $\mathbb{D} = \mathbb{F} \oplus \Delta\mathbb{F}$. This provides a very effective description of the vector space $\mathbb{D}/\mathbb{F}$. We start with some elementary properties of "*in*" and "*tc*".

LEMMA 1: (a) *If D and $E \in \mathbb{D}$ with $tcD + tcE \neq 0$ then $tc(D+E)$ equals either $tc\,D$, $tc\,E$ or $tcD + tcE$. The same holds for initial terms.*
(b) *If $D \in \mathbb{D}$ and $y \in A$ with $(tcD)\,(iny) \neq 0$ then $in(Dy) = (tcD)\,(iny)$.*
(c) *One has for $D \in \mathbb{D}$:* $\operatorname{lev} D = \operatorname{lev}(inD) = \operatorname{lev}(tcD)$.
(d) *If D and $E \in \mathbb{D}$ satisfy $tcD + tcE \neq 0$ then* $\operatorname{lev}(tc(D+E)) \leqslant \operatorname{lev} D$.

Proof: (a) Follows from the definitions.
(b) Write $D = tcD + \acute{D}$ and $y = iny + \acute{y}$. Then $Dy = (tcD)(iny) + (tcD)\acute{y} + \acute{D}(iny) + \acute{D}\acute{y}$ and comparison of exponents gives (b).
(c) The two equalities follow from the definition and the choice of the total order on $\mathbb{Z}^n$.
(d) Follows from (a) and (c).

DIVISION THEOREM: *Let* $\mathbb{F}$ *be a finitely generated A-submodule of* $\mathbb{D}$.
(1) $\mathbb{F} \oplus \Delta\mathbb{F} = \mathbb{D}$.
(2) *There exist generators* $D_1, \ldots, D_m$ *of* $\mathbb{F}$ *with* $in\mathbb{F} = (inD_1, \ldots, inD_m) \cdot A$ *and* $tc\mathbb{F} = (tcD_1, \ldots, tcD_m) \cdot A$.
(3) *For such generators* $D_1, \ldots, D_m$ *of* $\mathbb{F}$ *there exist for any* $D \in \mathbb{D}$ *unique* $y_1, \ldots, y_m \in A$ *and a unique* $E \in \Delta\mathbb{F}$ *such that*

$$D = \sum y_i D_i + E$$

and $\acute{y}_i \cdot inD_i \notin (inD_1, \ldots, inD_{i-1}) \cdot A$ *for all monomials* $\acute{y}_i$ *of the expansion of* y_i.
(4) *For any finitely generated* A*-submodule* $\mathbb{G}$ *of* $\mathbb{D}$ *with* $\mathbb{G} \subset \mathbb{F}$:

$$\mathbb{G} = \mathbb{F} \Leftrightarrow in\mathbb{G} = in\mathbb{F} \Leftrightarrow tc\mathbb{G} = tc\mathbb{F}.$$

Remark: Elements $D_1, \ldots, D_m$ of $\mathbb{F}$ are called a (minimal) standard base of $\mathbb{F}$ (w.r.t. the given coordinates and the total order on $\mathbb{Z}^n$) if $in\mathbb{F} = (inD_1, \ldots, inD_m) \cdot A$ (and $m \in \mathbb{N}$ is minimal for this property). A standard base is automatically a generator system and satisfies $tc\mathbb{F} = (tcD_1, \ldots, tcD_m) \cdot A$: indeed, by (4) of the Theorem, the inclusions of A-modules $(D_1, \ldots, D_m) \cdot A \subset \mathbb{F}$ and $(tcD_1, \ldots, tcD_m) \cdot A \subset tc\mathbb{F}$ are actually equalities. Note moreover that the definition of the level of $\mathbb{F}$ does not depend on the choice of the minimal standard base.

Proof: Clearly (3) $\Rightarrow$ (1) $\Rightarrow$ (4) and (2) is immediate since in$\mathbb{F}$ is finitely generated. In order to prove (3) let us first show uniqueness. If $D = \Sigma y_i D_i + E = \Sigma \bar{y}_i D_i + \bar{E}$ then $E - \bar{E} \in \Delta\mathbb{F} \cap \mathbb{F} = 0$, thus $E = \bar{E}$ and $\Sigma (y_i - \bar{y}_i)D_i = 0$. We may assume $in y_i \neq in\bar{y}_i$ for all i. From $in(\Sigma(y_i - \bar{y}_i)D_i) = 0$ follows similarly as in Lemma 1(a) that there is a set $I \subset \{1, \ldots, m\}$ such that $\Sigma_{i \in I}\, in((y_i - \bar{y}_i)D_i) = 0$ and thus $\Sigma_{i \in I}\, in(y_i - \bar{y}_i)inD_i = 0$. Let $j = \sup$ I. Then $in(y_j - \bar{y}_j)inD_j \in (inD_1, \ldots, inD_{j-1}) \cdot A$ and contradiction. Therefore $y_i = \bar{y}_i$ for all i.

The proof of existence goes in several steps. Let $d = \text{ord}\, \mathbb{F}$. It suffices to show (3) with $\mathbb{D}$ replaced by $\mathbb{D}_d = \{D \in \mathbb{D},\ \text{ord}\, D \leqslant d\}$. By abuse of notation we shall write $\mathbb{D}^{\cdot}$ for $\mathbb{D}_d$ throughout this proof. We have to show that the $\mathbb{C}$-linear map

$$w: A^m \times \Delta^{\cdot}\mathbb{F} \to \mathbb{D}^{\cdot}: (y, E) \to \sum y_i D_i + E$$

is surjective. This will be done by choosing suitable filtrations of $A^m \times \Delta^{\cdot}\mathbb{F}$ and $\mathbb{D}^{\cdot}$ by Banach spaces and proving surjectivity of the corresponding restrictions of w.

(a) Let $o, \acute{o}: \mathbb{Z}^n \to \mathbb{R}$ be injective linear forms. For $D \in \mathbb{D}^{\cdot}$, $D = \Sigma c_{\alpha\varepsilon} x^{\alpha+\varepsilon}\partial^{\alpha}$ and $0 < r \in \mathbb{R}$ define

$$\| D \|_r = \sum | c_{\alpha\varepsilon} | \cdot r^{o(\varepsilon)+\acute{o}(\alpha)}$$

and $\mathbb{D}^{\cdot}_r = \{D \in \mathbb{D}^{\cdot}, \| D \|_r < \infty\}$. The $\mathbb{D}^{\cdot}_r$ are Banach spaces and $\mathbb{D}^{\cdot} = \bigcup_{r>0} \mathbb{D}^{\cdot}_r$. Consider $A_r^m \times \Delta^{\cdot}\mathbb{F}_r$ as the Banach space with norm

$$\| (y, E) \|_r = \sum \| y_i inD_i \|_r + \| E \|_r,$$

where $A_r = A \cap \mathbb{D}^{\cdot}_r$ and $\Delta^{\cdot}\mathbb{F}_r = \Delta^{\cdot}\mathbb{F} \cap \mathbb{D}^{\cdot}_r$. Then the

$$w_r: A_r^m \times \Delta^{\cdot}\mathbb{F}_r \to \mathbb{D}^{\cdot}_r: (y, E) \to \sum y_i D_i + E$$

are well defined $\mathbb{C}$-linear maps between Banach spaces for all $r > 0$ for which $D_i \in \mathbb{D}^{\cdot}_r$. If we show that w_r is surjective for all sufficiently small $r > 0$ then w itself will be surjective.

(b) Setting $\acute{D}_i = D_i - inD_i$ the maps w_r decompose into $w_r = u_r + v_r$ where

$$u_r(y, E) = \sum y_i \cdot in\ D_i + E$$

$$v_r(y, E) = \sum y_i \cdot \acute{D}_i.$$

By definition of $\Delta^{\cdot}\mathbb{F}_r$, u_r is already surjective and it suffices to show that v_r is small enough not to destroy the surjectivity. By the criterion of [H, Lemma 1, p. 47] one has to prove that the norm of v_r is strictly smaller than the conorm of u_r: $\| v_r \| < \text{con}\ u_r$.

(c) con $u_r \geqslant 1$ for all $r > 0$: For $D \in \mathbb{D}^{\cdot}_r$ there exist unique $y_1, \ldots, y_m \in A_r$ and a unique $E \in \Delta^{\cdot}\mathbb{F}_r$ with

$$D = \sum y_i \cdot inD_i + E$$

and such that $\acute{y}_i \cdot inD_i \notin (inD_1, \ldots, inD_{i-1}) \cdot A$ for all monomials $\acute{y}_i$ of y_i. From this and the definition of the norms one obtains:

$$\| D \|_r = \left\| \sum y_i \cdot inD_i + E \right\|_r = \left\| \sum y_i \cdot inD_i \right\|_r + \| E \|_r$$

$$= \sum \| y_i \cdot inD_i \|_r + \| E \|_r = \| (y, E) \|_r.$$

This proves con $u_r \geqslant 1$.

(d) $\| v_r \| < 1$ for suitable $o, \acute{o}: \mathbb{Z}^n \to \mathbb{R}$ and sufficiently small $r > 0$: Let $D \in \mathbb{D}^{\cdot}$ and set $\acute{D} = D - inD$. The choice of the total order on $\mathbb{Z}^n$ used to define tcD and inD allows to choose $o\colon \mathbb{Z}^n \to \mathbb{R}$ such that $o(\varepsilon) - o(\varepsilon_0) > 2c$ for some constant $c > 0$ and $\varepsilon_0 = \text{carr } inD$ and all $\varepsilon \in \text{carr } \acute{D}$. Setting $\acute{o} = t \cdot o$ with $0 < t \in \mathbb{R}$ small enough one can then achieve

$$o(\varepsilon) - o(\varepsilon_0) + \acute{o}(\alpha) - \acute{o}(\alpha_0) > c$$

for $\alpha_0 = \text{supp } inD$ and all $\alpha \in \text{supp } \acute{D}$. Consider now

$$\frac{\|\acute{D}\|_r}{\|inD\|_r} = \frac{\sum |c_{\alpha\varepsilon}| \cdot r^{o(\varepsilon)+\acute{o}(\alpha)}}{|c_{\alpha_0\varepsilon_0}| \cdot r^{o(\varepsilon_0)+\acute{o}(\alpha_0)}} = \left[\sum \frac{|c_{\alpha\varepsilon}|}{|c_{\alpha_0\varepsilon_0}|} \cdot r^{o(\varepsilon)-o(\varepsilon_0)+\acute{o}(\alpha)-\acute{o}(\alpha_0)-c}\right] \cdot r^c.$$

From the above inequality follows that the term in the brackets remains bounded as $r \to 0$. Thus there exists a $0 < a < 1$ such that for $r > 0$ sufficiently small one has

$$\|\acute{D}\|_r \leqslant a \cdot \|inD\|_r.$$

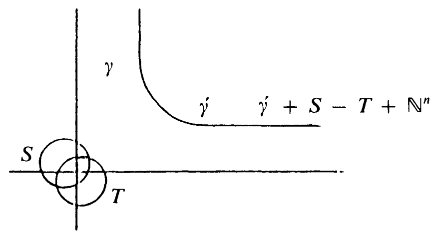
It is then clear that by suitable choices of o and $\acute{o}$ such an inequality can be achieved simultanously for finitely many D's, in particular for the generators $D_1, \ldots, D_m$ of $\mathbb{F}$. We thus get

$$\begin{aligned} \|v_r(y, E)\|_r &= \left\| \sum y_i \acute{D}_i \right\|_r \leqslant \sum \|y_i \acute{D}_i\|_r \leqslant \sum \|y_i\|_r \|\acute{D}_i\|_r \\ &\leqslant a \cdot \sum \|y_i\|_r \|inD_i\|_r = a \cdot \sum \|y_i \cdot inD_i\|_r \\ &= a \cdot \|(y, 0)\|_r \leqslant a \cdot \|(y, E)\|_r. \end{aligned}$$

This establishes $\|v_r\| < 1$ and concludes the proof of the Theorem.

2. Combinatorics

A subset Γ of $\mathbb{Z}^n$ will be called *spare* w.r.t. a couple (S, T) of subsets of $\mathbb{Z}^n$ if for all $\gamma \neq \acute{\gamma} \in \Gamma$ one has $\gamma - \acute{\gamma} \notin S - T + \mathbb{N}^n \subset \mathbb{Z}^n$:



PROPOSITION 1: *Let* $\Gamma \subset \mathbb{N}^n$ *be spare w.r.t. a couple* (S, T) *of subsets of* $\mathbb{Z}^n$. *Let* $D, E \in \mathbb{D}$ *be differential operators satisfying* $\operatorname{carr}(tcD) \subset T$ *and* $\operatorname{carr} E \subset S$. *If for some* $\gamma \in \Gamma$:

$$Dx^{\gamma} = \sum_{\acute{\gamma} \neq \gamma} Ex^{\acute{\gamma}}$$

then

$$(tcD)x^{\gamma} = 0.$$

Proof: Let $\operatorname{carr} tcD = \{\varepsilon\}$ and assume $(tcD)x^{\gamma} \neq 0$. By Lemma 1(b), $in(Dx^{\gamma}) = (tcD)x^{\gamma} \neq 0$ and therefore

$$x^{\gamma+\varepsilon} \in \sum_{\acute{\gamma} \neq \gamma} \sum_{\acute{\varepsilon} \in \operatorname{carr} E} A \cdot x^{\acute{\gamma}+\acute{\varepsilon}}.$$

This implies $\gamma \in \bigcup_{\acute{\gamma} \neq \gamma} \bigcup_{\acute{\varepsilon}} (\acute{\gamma} + \acute{\varepsilon} - \varepsilon + \mathbb{N}^n)$ and contradiction.

We next prove that there exist sufficiently many spare sets.

LEMMA 2: *Assume* $n \geqslant 2$. *Let* $T \subset \mathbb{Z}^n$ *be finite,* $\delta \in \mathbb{Z}^n$, $S \subset \delta + \mathbb{N}^n$ *and* $t \in \mathbb{N}$. *For* $\zeta \in \mathbb{Z}^{nt} = (\mathbb{Z}^n)^t$ *set* $\Gamma_\zeta = \{\gamma \in \mathbb{Z}^n, \gamma$ *is a component of* $\zeta\}$. *The set of* $\zeta \in \mathbb{N}^{nt}$ *such that* $\Gamma_\zeta \subset \mathbb{N}^n$ *is spare w.r.t.* (S, T), *contains balls of* $\mathbb{N}^{nt}$ *of arbitrary radius.*

Proof: The set T being finite we may assume that $S - T + \mathbb{N}^n \subset \delta + \mathbb{N}^n$ replacing possibly δ. Moreover we can choose $\delta \in (-\mathbb{N})^n$. Let $\bar{\delta} \in \mathbb{N}^{n-1} \times (-\mathbb{N})$ be defined by

$$\bar{\delta}^i = -\delta^i + 1 \quad 1 \leqslant i \leqslant n-1$$

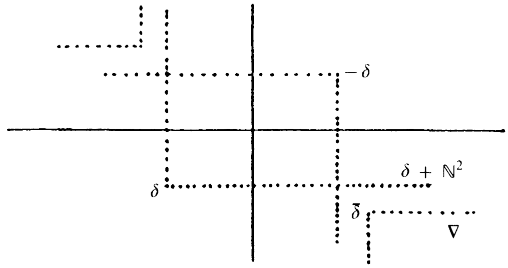
$$\bar{\delta}^n = \delta^n - 1.$$

The set $\nabla = \bar{\delta} + \mathbb{N}^{n-1} \times (-\mathbb{N})$ is closed under addition and does not intersect $\pm(\delta + \mathbb{N}^n)$: the first assertion is clear since $\bar{\delta} \in \mathbb{N}^{n-1} \times (-\mathbb{N})$. Furthermore, if α would belong to ∇ and $\pm(\delta + \mathbb{N}^n)$ then either

$$\alpha^i \in (-\delta^i + 1 + \mathbb{N}) \cap (-\delta^i - \mathbb{N}) \quad 1 \leqslant i \leqslant n-1$$

or

$$\alpha^n \in (\delta^n - 1 - \mathbb{N}) \cap (\delta^n + \mathbb{N}).$$



The linear isomorphism L: $\mathbb{Z}^{nt} \to \mathbb{Z}^{nt}, (\zeta_1, \ldots, \zeta_t) \to (\zeta_1, \zeta_1 + \zeta_2, \ldots, \zeta_1 + \ldots + \zeta_t)$, sends the t-fold cartesian product ∇^t of ∇ to some nt-dimensional cone $L(\nabla^t)$. Let Δ denote the n-dimensional diagonal in $\mathbb{N}^{nt} = (\mathbb{N}^n)^t$, $\Delta = \{(\omega_1, \ldots, \omega_t) \in \mathbb{N}^{nt}, \omega_i = \omega_j\}$. For any t-tuple $\zeta = (\gamma_1, \ldots, \gamma_t)$ of $\Delta + L(\nabla^t) \subset \mathbb{Z}^{nt}$ the differences $\gamma_i - \gamma_j$ for $i > j$ are sums of elements of ∇ by definition of L. As ∇ is closed under addition and $\nabla \cap \pm(\delta + \mathbb{N}^n) = \phi$, the $\gamma_i - \gamma_j$ do not belong to $\pm(\delta + \mathbb{N}^n) \supset S - T + \mathbb{N}^n$. This shows that $\Gamma_\zeta = \{\gamma_1, \ldots, \gamma_t\}$ is spare w.r.t. (S, T). Moreover $L(\nabla^t) \subset \mathbb{Z}^{nt}$ contains balls of $\mathbb{Z}^{nt}$ of arbitrary radius. For any such ball B there exists an $\omega \in \Delta$ such that $\acute{B} = \omega + B \subset \mathbb{N}^{nt}$ proving the Lemma.

For γ and α in $\mathbb{N}^n$ define $(\gamma)_\alpha \in \mathbb{N}$ by the formula $\partial^\alpha x^\gamma = (\gamma)_\alpha x^{\gamma-\alpha}$, say

$$(\gamma)_\alpha = \prod_{i=1}^{n} \frac{\gamma^i!}{(\gamma^i - \alpha^i)!}.$$

A set $\Gamma \subset \mathbb{N}^n$ is called *generic* w.r.t. some finite set $R \subset \mathbb{N}^n$ if the matrix

$$((\gamma)_\alpha)_{\gamma \in \Gamma, \alpha \in R}$$

has rank equal to the cardinality of R.

PROPOSITION 2: *Let* $\Gamma \subset \mathbb{N}^n$ *be generic w.r.t. to some finite* R $\subset \mathbb{N}^n$. *Let* $D \in \mathbb{D}$ *be a differential operator with* supp $(tcD) \subset R$. *If* $Dx^\gamma = 0$ *for all* $\gamma \in \Gamma$ *then* $D = 0$.

Proof: Assume $D \neq 0$. Then $tcD = \Sigma_{\alpha \in R} c_{\alpha\varepsilon_0} x^{\alpha+\varepsilon_0} \partial^\alpha \neq 0$. From $Dx^\gamma = 0$ follows by Lemma 1(b) that $(tcD)x^\gamma = \Sigma_{\alpha \in R} c_{\alpha\varepsilon_0} (\gamma)_\alpha x^{\varepsilon_0+\gamma} = 0$ for all γ. In matrices:

$$(c_{\alpha\varepsilon_0})_{\alpha \in R} \cdot ((\gamma)_\alpha)_{\gamma \in \Gamma, \alpha \in R} = 0.$$

Hence $c_{\alpha\varepsilon_0} = 0$ for all $\alpha \in R$.

LEMMA 3: *Let $R \subset \mathbb{N}^n$ be finite, $t = \operatorname{card} R$ and let Γ_ζ be defined as in Lemma 2. The set Z of $\zeta \in \mathbb{N}^{nt}$ for which Γ_ζ is generic w.r.t. R is a non-empty Zariski-open subset of $\mathbb{N}^{nt}$.*

Proof: We only have to show that Z is non-empty. This signifies that the polynomial

$$\det\, ((x_i)_\alpha)_{1 \leqslant i \leqslant t, \alpha \in R}$$

is not identically zero, where $x_i = (x_i^1, \ldots, x_i^n)$ denote variables on $\mathbb{N}^n$ for all i. But $\alpha < \acute{\alpha}$ w.r.t. the total order on $\mathbb{Z}^n$ implies that $\alpha^j < \acute{\alpha}^j$ for some components α^j, $\acute{\alpha}^j$ of α and $\acute{\alpha}$. Thus $((\alpha)_{\acute{\alpha}})_{\alpha, \acute{\alpha} \in R}$ is a triangular matrix with non-zero entries on the diagonal. It follows that $\det((x_i)_\alpha) \not\equiv 0$.

PROPOSITION 3: For any finite $R \subset \mathbb{N}^n$, $T \subset \mathbb{Z}^n$ and any $S \subset \delta + \mathbb{N}^n (\delta \in \mathbb{Z}^n)$ there exists a subset Γ of $\mathbb{N}^n$ which is spare w.r.t. (S, T) and generic w.r.t. R.

Proof: This is an immediate consequence of Prop. 1 and 2.

3. Proof of the Theorem

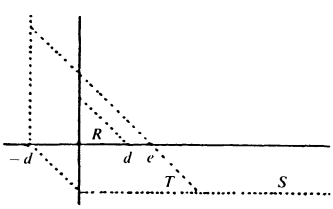
For $d, e \in \mathbb{N}$ define the following sets:

$$R = \operatorname{supp} \{D \in \mathbb{D}, \operatorname{ord} D \leqslant d\} = \{\alpha \in \mathbb{N}^n, |\alpha| \leqslant d\},$$

$$S = \operatorname{carr} \{D \in \mathbb{D}, \operatorname{ord} D \leqslant d\} = \{\varepsilon \in \mathbb{Z}^n, \exists\, \alpha \in R \text{ with } \alpha + \varepsilon \in \mathbb{N}^n\},$$

$$T = \operatorname{carr} \{tcD, D \in \mathbb{D}, \operatorname{ord} D \leqslant d, \operatorname{lev} D \leqslant e\}$$

$$= \{\varepsilon \in \mathbb{Z}^n, |\varepsilon| \leqslant e, \exists\, \alpha \in R \text{ with } \alpha + \varepsilon \in \mathbb{N}^n\}.$$



Both R and T are finite and $S \subset \delta + \mathbb{N}^n$ for some $\delta \in \mathbb{Z}^n$. By Prop. 3 there exists a finite subset Γ of $\mathbb{N}^n$ which is spare w.r.t. (S, T) and generic w.r.t. R. We define the polynomial $z = z_{de} \in A$ as:

$$z = \sum_{\gamma \in \Gamma} x^\gamma .$$

Let now $\mathbb{F}$ and $\mathbb{G}$ be submodules of $\mathbb{D}$ as in the assertion of the Theorem. Assume $\mathbb{F}z = \mathbb{G}z$. We shall deduce that $tc(\mathbb{F} + \mathbb{G}) \subset tc\mathbb{F}$. Part (4) of the Division Theorem will then imply that $\mathbb{F} + \mathbb{G} = \mathbb{F}$ and by symmetry we will obtain $\mathbb{F} = \mathbb{G}$.

Choose a minimal standard base $D_1, \ldots, D_m$ of $\mathbb{F} + \mathbb{G}$. We have $\text{lev}\, D_i \leqslant \text{lev}\,(\mathbb{F} + \mathbb{G}) \leqslant e$. As $(tcD_1, \ldots, tcD_m) \cdot A = tc(\mathbb{F} + \mathbb{G})$ the inclusion $tc(\mathbb{F} + \mathbb{G}) \subset tc\mathbb{F}$ will follow if we show that $tcD_i \in tc\mathbb{F}$ for all i. Actually we shall prove more generally that for any $D \in \mathbb{D}$ of order $\leqslant d$ and level $\leqslant e$ the inclusion $Dz \in \mathbb{F}z$ already implies $tcD \in tc\mathbb{F}$.

Let us write $Dz = Ez$ with $E \in \mathbb{F}$ and assume that $tcD \neq tcE$. By Lemma 1(d) we have $\text{lev}(tc(D - E)) \leqslant \text{lev}\, D \leqslant e$ and therefore $\text{carr}(tc(D - E)) \subset T$. Let $\gamma \in \Gamma$ and write $Dz = Ez$ as

$$(D - E)x^\gamma = \sum_{\gamma' \neq \gamma} (E - D)x^{\gamma'} .$$

As $\text{carr}(E - D) \subset S$ and Γ is spare w.r.t. (S, T) Prop. 1 implies that for all $\gamma \in \Gamma$

$$tc(D - E)x^\gamma = 0.$$

But $\text{supp}(tc(D - E)) \subset R$. As Γ is generic w.r.t. R, Prop. 2 implies that $tc(D - E) = 0$, i.e. $D = E$. This proves the Theorem.

4. Examples

In this section we compute the polynomial z of the Theorem in more specific situations and show possible simplifications. Namely we assume given a differential operator $D \in \mathbb{D}$ and a finitely generated submodule $\mathbb{F}$ of $\mathbb{D}$ such that $D \notin \mathbb{F}$. Adding to D a convenient element of $\mathbb{F}$ we may assume by the Division Theorem that $tcD \notin tc\mathbb{F}$. Our aim is to find explicitly a polynomial $z = \Sigma_{\gamma \in \Gamma} x^\gamma \in A$ such that $Dz \notin \mathbb{F}z$.

In this situation one can proceed as follows. Set:

$$R = \operatorname{supp}(tcD) \cup \operatorname{supp}\{tcE,\ E \in \mathbb{F},\ \operatorname{carr}(tcE) \leqslant \operatorname{carr}(tcD)\},$$

$$S = (\operatorname{carr} D \cup \operatorname{carr} \mathbb{F}) + \mathbb{N}^n$$

$$T = \operatorname{carr}(tcD) \cup \operatorname{carr}\{tcE,\ E \in \mathbb{F},\ \operatorname{carr}(tcE) \leqslant \operatorname{carr}(tcD)\}.$$

Choose a (finite) subset Γ of $\mathbb{N}^n$ which is spare w.r.t. (S, T) and generic w.r.t. R, and set

$$z = \sum_{\gamma \in \Gamma} x^\gamma.$$

The three sets R, S, T are generally smaller than the one defined in the proof of the Theorem. Nevertheless, the proof applies as well, for if we would have $Dz = Ez$ for some $E \in \mathbb{F}$ then

$$\operatorname{supp}(tc(D - E)) \subset R, \quad \operatorname{carr}(E - D) \subset S, \quad \operatorname{carr}(tc(D - E) \subset T.$$

And this will yield by the same arguments $tcD = tcE$ and contradiction.

Let us carry out the above procedure in three examples of modules of differential operators on $\mathbb{C}^2$:

EXAMPLE 1: Let $D = 1$ and $\mathbb{F} \subset \mathbb{D}$ be generated by $\partial_x^i \partial_y^j$ with $0 < i + j \leqslant 2$. Then

$$R = \{(i, j) \in \mathbb{N}^2,\ 0 \leqslant i + j \leqslant 2\},$$

$$S = \{(p, q) \in \mathbb{Z}^2,\ -2 \leqslant p + q \leqslant 0\} + \mathbb{N}^2,$$

$$T = \{(p, q) \in \mathbb{Z}^2,\ -2 \leqslant p + q \leqslant 0\}.$$

Note that $S - T \subset [(-2, -2) + \mathbb{N}^2] \cup [(2, 2) - \mathbb{N}^2]$ and hence $(3, -3) + \mathbb{N} \times (-\mathbb{N})$ does not intersect $\pm(S - T)$. It follows that

$$\Gamma = \{(15, 0), (12, 3), (9, 6), (6, 10), (3, 13), (0, 16)\}$$

is spare w.r.t. (S, T). One then checks by computation that the matrix $((\gamma)_\alpha)_{\gamma \in \Gamma, \alpha \in R}$ has rank 6, i.e., that Γ is generic w.r.t. R. Thus $z = x^{15} + x^{12}y^3 + x^9y^6 + x^6y^{10} + x^3y^{13} + y^{16}$ does not belong to $\mathbb{F}z$.

EXAMPLE 2: Let again $D = 1$ and $\mathbb{F}$ be now generated by $\partial_x^i \partial_y^j$ with $0 < i + j \leqslant 3$. Analogous considerations as before yield for instance

$$z = x^{36} + x^{32}y^4 + x^{28}y^8 + x^{24}y^{13} + x^{20}y^{17} + x^{16}y^{21} + x^{12}y^{25} + x^8y^{30} + x^4y^{34} + y^{38}.$$

In both examples the polynomial z is relatively complicated and not the simplest one satisfying $z \notin \mathbb{F}z$. But aside of the computation of the rank of the matrix $((\gamma)_\alpha)_{\gamma\in\Gamma,\alpha\in R}$ its construction is very easy.

EXAMPLE 3: We conclude with an example where inspite of the complicated structure of D and $\mathbb{F}$ the polynomial z is simple. Let

$$D = x^2\partial_{xx} + y^2\partial_{yy},$$

and $\mathbb{F} \subset \mathbb{D}$ be generated by $E_1, \ldots, E_6$, where:

$$E_1 = xy\partial_{xy} + y^3\partial_{yy} \qquad E_4 = xy\partial_{xx} + x^2\partial_{xy}$$

$$E_2 = x\partial_{xx} + xy\partial_{yy} \qquad E_5 = xy\partial_y + x^2y^2\partial_{xx}$$

$$E_3 = y\partial_{xy} + y^2\partial_{yy} \qquad E_6 = xy\partial_x + x^3\partial_{xy}.$$

Then $tcD = D$ and $E_2, E_3, E_5, E_6, E_7, E_8$ form a minimal standard base of $\mathbb{F}$, where:

$$E_7 = y^3\partial_{yy} - xy^2\partial_{yy} \qquad E_8 = x^2\partial_{xy} - xy^2\partial_{yy}.$$

Note that $tcD \notin tc\mathbb{F} = (x\partial_{xx}, y\partial_{xy}, xy\partial_y, xy\partial_x, y^3\partial_{yy}, x^2\partial_{xy}) \cdot \mathbb{C}\{x, y\}$. Computation gives

$$R = \{(2, 0), (1, 1), (0, 2)\},$$

$$S = \{(-1, 0), (1, -1)\} + \mathbb{N}^2,$$

$$T = \{(0, 0), (-1, 0), (-1, 1)\}$$

One observes that $S - T \subset [(-1, -2) + \mathbb{N}^2] \cup [(1, 2) - \mathbb{N}^2]$ and that $(2, -3) + \mathbb{N} \times (-\mathbb{N})$ does not intersect $\pm(S - T)$. Thus

$$\Gamma = \{(4, 0), (2, 3), (0, 6)\}$$

is spare w.r.t. to (S, T) and one checks immediately that Γ is also generic w.r.t. R. Therefore, $z = x^4 + x^2y^3 + y^6$ satisfies $Dz \notin \mathbb{F}z$ as desired.

Acknowledgements

We would like to thank A. Dür and A. Ostermann for valuable contributions to section 2, resp. section 4.

References

[B-M] J. Briançon and P. Maisonobe: Idéaux de germes d'opérateurs différentiels à une variable, *L'Ens. Math.* 30 (1984) 7–38.

[C] F. Castro: Calcul de la dimension et des multiplicités d'un $\mathscr{D}$-module monogène, *C.R. Acad. Sc. Paris* 302, sér. I, n° 14 (1986) 487–490.

[G] A. Galligo: A propos du théorème de préparation. *Lecture Notes in Mathematics*, Springer, n° 409 (1973) 543–579.

[H] H. Hauser: La construction de la déformation semi-universelle d'un germe de variété analytique complexe. *Ann. E.N.S. Paris* 18 (1985) 1–56.