A NORMAL FORM THEOREM FOR FUCHSIAN DIFFERENTIAL EQUATIONS OVER FIELDS OF CHARACTERISTIC ZERO

Herwig HAUSER

It is shown that linear differential operators in one variable with holomorphic coefficients and regular singularities are equivalent, up to a linear automorphism of the function space on which they act, to their initial Euler operator, which thus becomes a normal form of it. As Euler equations are easy to solve, one obtains from this directly the classical theorems of Fuchs, Thomé and Frobenius about the solutions of an ordinary linear differential equation at its regular singular points. The normal form also works for irregular singularities, at the cost of losing the convergence of the solutions, while getting only part of a basis. Various applications of the normal form theorem are given, together with its prospective use for the p-curvature conjecture.

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1. Introduction

In two famous papers from 1866 and 1868, published in Crelle's Journal, Lazarus Fuchs introduced the concept of a *regular singular point* of an ordinary linear differential equations with holomorphic or meromorphic coefficients [Fuchs1, p. 146, Fuchs2, p. 360]: they are those points of $\mathbb{P}^1$ where the equation admits a basis of *moderate* local solutions; this means that their growth is at most polynomial as one approaches the singular point. Fuchs characterized regular singularities by giving explicit algebraic bounds for the order of vanishing of the coefficients of the differential equation at the given point, nowadays known as Fuchs' criterion. He then constructed a basis of local solutions as linear combinations of holomorphic functions and powers of logarithms, see e.g. formula (7) in [Fuchs2, p. 364]. Alternative constructions of the solutions were provided shortly later in a series of papers by Thomé and Frobenius in the same journal [Thom1, formula (5), p. 195, Thom2, Thom3, Frob1, formula (12), p. 222, Frob2]. Frobenius, in particular, developed a method, now named after him, of differentiating prospective solutions with respect to their local exponents, thus getting in an elegant way a full set of solutions. Subsequent references as e.g. [Fabry, Ince, Poole, Hefft, Was] reproduce the classical formulas, often restricting to the simpler case of non-resonance (i.e., the local exponents having non-integer differences) or working in the context of the associated first order system. In fact, the precise description of the solutions at regular singularities in the general case is involved and requires a careful handling of the multiplicities of the local exponents.

To illustrate: Ince [Ince, p. 396] reproduces quite accurately Frobenius' ideas, see also section 4.3 in [Mezz]. Mezzarobba presents another method to construct a basis of solutions, apparently developed by Heffter in 1894 and exposed in the book of Poole from 1936, see [Mezz, section 4.4, Poole, V.16, p. 62, and V.19, p. 70]. Haraoka and Wasow consider systems of first order equations and use gauge transformations to reduce them to a certain normal form [Hara, Thm. 2.4, p. 30, Was, Thm. 5.2, p. 21]. This allows them to describe the solutions of the normalized system, the case of eigenvalues with non-integer differences being substantially simpler [Hara, Thm. 2.5, Was, Thm. 5.5]. Of course, further expositions of the solutions of Fuchsian differential equations are numerous, at different levels of accuracy and generality.

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There also exist descriptions of the local solutions of differential equations at irregular singular points, apparently studied already by Thomé and later by Fabry [Thom3, section 5, p. 287, Thom4, Thom5, Fabr]. Here, the solutions may involve aside from logarithms also exponential functions evaluated in Laurent polynomials, and the divergence of the occurring power series is more frequent.

In the present note, the focus switches from the mere construction and description of the solutions of differential equations to the study of the involved differential operator itself. That is to say, the objective is to establish a normal form for operators $L = p_0\partial^n + \ldots + p_{n-1}\partial + p_n \in \mathcal{O}[\partial]$, with $\mathcal{O}$ the ring of germs of holomorphic functions in one variable, up to the composition from the right with an automorphism v of the space of functions $\mathcal{F}$ on which L acts,

$$L \circ v = L_0,$$

where L_0 is the *initial form* of L at the singularity. An appropriate space $\mathcal{F}$ where the solutions of $Ly = 0$ are expected to live is constructed from the local exponents of the equation and their respective multiplicities. Such a normalizing automorphism v can be given for instance as the geometric (or von Neumann) series $v = \sum_{k=0}^{\infty}(S \circ T)^k$ in a product $S \circ T$, where T is the *tail* of L (say, the terms of L beyond L_0), and S a right inverse of L_0. This shows in particular that v is the inverse $v = u^{-1}$ of the automorphism $u = \mathrm{Id}_{\mathcal{F}} - S \circ T$ of $\mathcal{F}$.

For the operator S one can choose an integration operator inverting the action of L_0. Every choice of a direct complement $\mathcal{H}$ of the kernel $\mathrm{Ker}(L_0)$ of L_0 in $\mathcal{F}$ provides such an inverse. The initial form L_0 collects the terms of smallest shift of L (say, is the initial form with respect to the V-filtration [Bud]) and is thus an Euler operator. The normal form $L \circ v = L_0$ immediately yields the classical results about the solutions $y(x)$ of $Ly = 0$: they are then given by $v(z(x)) = u^{-1}(z(x))$, where $z(x)$ runs through the solutions of the Euler equation $L_0 z = 0$. As the latter are easily described as products of (generalized) monomials x^ρ, $\rho \in \mathbb{C}$ a local exponent, with powers $\log(x)^i$ of the logarithm, $i < m_\rho$, the multiplicity of ρ, one gets an explicit description of the solutions of $Ly = 0$. This yields a precise control on the involved powers of logarithms and on the occurence of a basis of holomorphic solutions, say, the case where a regular singularity is in fact an *apparent* singularity.

It turns out that the (infinite) algorithm producing the solutions $y(x)$ of $Ly = 0$ from applying the geometric series v to the solutions $z(x)$ of the Euler equation $L_0 z = 0$ is nothing else than the algorithm computing the coefficient sequence of $y(x)$ from the linear recursion defined by L. The tricky part here is to choose the correct function space $\mathcal{F}$ involving powers of logarithms and to prove the convergence of the geometric series defining the automorphism v. The subtleties one encounters are illustrated by the study of a concrete example in an expository article of the author [Hau]. They motivate and explain the origin of the general constructions developed in the present article.

As a by-product, one also gets one implication of Fuchs' criterion for regular singularities [Fuchs1, p. 146, Fuchs2, p. 360]: the order conditions on the coefficients ensure the existence of a full set of moderate solutions. The inverse implication requires an extra argument. In our context, regularity is equivalent to saying that L_0 is an operator of the same order n as L itself. As such, the respective spaces of solutions have the same dimension, and one gets therefore all solutions of $Ly = 0$ from the Euler equation $L_0 y = 0$ by composition of the solutions of the latter equation with u^{-1}. For differential equations with irregular singularities, more refined constructions are required to describe all solutions [Thom3, Fabr]. In ongoing work of N. Merkl, an appropriate normal form theorem for irregular singularities will be formulated and proven [Merk]. Finally, the case of ground fields of characteristic $p > 0$ (for differential operators with formal power series coefficients) presents extra difficulties due to the failure of various integration operators and the absence of logarithms. This is subject of a forthcoming article of F. Fürnsinn [Fürn]. Normal form problems in the case of holomorphic differential equations in several variables have been investigated in [GaHa].

2. Constructions with differential operators

Our starting point is a simple but useful result from functional analysis.

Perturbation lemma. *If $\ell : F \to G$ is a continuous linear map between complete metric vector spaces which decomposes into $\ell = \ell_0 - t$ with $\mathrm{Im}(t) \subseteq \mathrm{Im}(\ell_0)$ and satisfies $|s(t(f))| \leq C \cdot |f|$, $0 < C < 1$, for a right inverse $s : \mathrm{Im}(\ell_0) \to F$ of $\ell_0 : F \to \mathrm{Im}(\ell_0)$ and all $f \in F$, then $u = \mathrm{Id}_F - st$ is a continuous linear automorphism of F which transforms ℓ into ℓ_0 via $\ell u^{-1} = \ell_0$.*

Proof. The prospective inverse of u is $v = \sum_{k=0}^{\infty}(st)^k$. It is well defined and continuous because of the estimate for $st(f)$ and the completeness of F. Hence u is an automorphism of F. From $\ell_0 s = \mathrm{Id}_{\mathrm{Im}(\ell_0)}$ it follows that $\ell_0 s \ell_0 = \ell_0$. From $\mathrm{Im}(t) \subseteq \mathrm{Im}(\ell_0)$ one gets that the compositions st and $s\ell$ are well defined and that $\ell_0 s \ell = \ell$ holds. Then

$$\begin{aligned}\ell_0 u &= \ell_0(\mathrm{Id}_F - st)\\ &= \ell_0(\mathrm{Id}_F - s(\ell_0 - \ell))\\ &= \ell_0(\mathrm{Id}_F - s\ell_0 + s\ell)\\ &= \ell_0 - \ell_0 s\ell_0 + \ell_0 s\ell\\ &= \ell_0 s\ell\\ &= \ell\end{aligned}$$

as required. This proves the result.

Fuchsian differential equations. Let be given a linear ordinary differential equation

$$Ly = p_0(x)y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y' + p_n(x)y = 0,$$

where

$$L = p_0\partial^n + p_1\partial^{n-1} + \ldots + p_{n-1}\partial + p_n \in \mathcal{O}[\partial]$$

is a differential operator with holomorphic coefficients in a neighborhood of a chosen singular point of L, say, the origin 0 of $\mathbb{C}$. Here, $\mathcal{O} = \mathbb{C}\{x\}$ denotes the ring of germs of holomorphic functions in one variable x at 0 and $\partial = \frac{d}{dx}$ the usual derivative with respect to x. Writing $L = \sum_{j=0}^{n}\sum_{i=0}^{\infty} c_{ij}x^i\partial^j$, the operator decomposes into a sum

$$L = L_0 + L_1 + \ldots + L_m + \ldots$$

of *homogeneous* or *Euler operators* $L_k = \sum_{i-j=\tau_k} c_{ij}x^i\partial^j$, where the *shifts* $\tau_0 < \tau_1 < \ldots$ of the operators L_k are ordered increasingly and all L_k are assumed to be non-zero. The term L_0 of smallest shift constitutes the *initial form* of L at 0, and $\tau := \tau_0$ is called the shift of L at 0. Up to multiplying L with the monomial $x^{-\tau}$ we may assume (as we will do throughout) that L has shift $\tau = 0$; thus $L_0 = \sum_{i=0}^{n} c_{ii}x^i\partial^i$. The point $x = 0$ is singular for L if at least one quotient p_i/p_0 has a pole at 0 (otherwise, 0 is called non-singular or ordinary). It is a *regular singularity* (in the sense of Fuchs) if L_0 has again order n, i.e., if $c_{nn} \neq 0$. An operator in $\mathbb{P}^1_{\mathbb{C}}$ with at most regular singularities is called *Fuchsian.* The *indicial polynomial* of L at 0 is defined as

$$\chi_L(t) = \sum_{i=0}^n c_{ii} t^{\underline{i}} = \sum_{i=0}^n c_{ii} t(t-1)\cdots(t-i+1).$$

Here, $t^{\underline{i}}$ denotes the falling factorial or Pochhammer symbol. Clearly, $\chi_L = \chi_{L_0}$, which we simply denote by χ_0. Its roots $\rho \in \mathbb{C}$ are the *local exponents* of L at 0, and $m_\rho \in \mathbb{N}$ will denote their multiplicity.

Euler equations. The solutions of Euler equations $L_0 y = 0$ are easy to find. They are of the form

$$y_{\rho,i} = x^\rho \log(x)^i,$$

where $\rho \in \mathbb{C}$ is a local exponent and i varies between 0 and $m_\rho - 1$. Here, $x^\rho = \exp(\rho \log(x))$ and $\log(x)$ may be considered either as symbols subject to the differentiation rule $\partial x^\rho = \rho x^{\rho-1}$ and $\partial \log(x) = 1/x$, or as holomorphic functions on $\mathbb{C}_{slit} = \mathbb{C} \backslash \mathbb{R}_{\geq 0}$ or on arbitrary simply connected open subsets of $\mathbb{C}^* = \mathbb{C} \backslash \{0\}$. One objective of the present paper is to lift these obvious solutions of $L_0 y = 0$ to solutions of the original equation $Ly = 0$.

Extensions of differential operators. The consideration of logarithms is best formalized by introducing a new variable z for $\log(x)$ [Honda, Mezz]. To this end, equip the the polynomial ring $\mathcal{K}[z]$ over the field $\mathcal{K} = \mathrm{Quot}(\mathcal{O})$ of meromorphic functions at 0 with the $\mathbb{C}$-derivation

$$\boldsymbol{\partial} : \mathcal{K}[z] \to \mathcal{K}[z],$$

$$\boldsymbol{\partial} x = \partial x = 1, \qquad \boldsymbol{\partial} z = x^{-1},$$

$$\boldsymbol{\partial}(x^i z^k) = (iz + k) x^{i-1} z^{k-1}.$$

This turns $\mathcal{K}[z]$ into a differential ring. It carries in addition the usual derivative ∂_z with respect to z. The same definition applies to $\mathcal{O}x^\rho[z]$ for any $\rho \in \mathbb{C}$, taking $\boldsymbol{\partial} x^\rho = \rho x^{\rho-1}$.

The j-fold composition $\boldsymbol{\partial} \circ \cdots \circ \boldsymbol{\partial}$ will be denoted by $\boldsymbol{\partial}^j$. For a differential operator $L = p_0 \partial^n + p_1 \partial^{n-1} + \ldots + p_{n-1}\partial + p_n \in \mathcal{O}[\partial]$ define its *extension* as

$$\mathbb{L} = p_0 \boldsymbol{\partial}^n + p_1 \boldsymbol{\partial}^{n-1} + \ldots + p_{n-1} \boldsymbol{\partial} + p_n.$$

Even though this is not a differential operator in the strict sense, we will speak of $\mathbb{L}$ again as an operator. If $\rho \in \mathbb{C}$ is a local exponent of L, we will likewise associate to $\mathbb{L}$ the $\mathbb{C}$-linear map

$$\mathbb{L} : \mathcal{K}x^\rho[z] \to \mathcal{K}x^\rho[z],\ x^\rho h(x) z^i \to \mathbb{L}(x^\rho h(x) z^i),$$

called again the extension of L to $\mathcal{K}x^\rho[z]$. Whenever L has shift $\tau \geq 0$ – as we will assume in the sequel – its extension $\mathbb{L}$ sends $\mathcal{O}x^\rho[z]$ to $\mathcal{O}x^\rho[z]$ and thus defines a $\mathbb{C}$-linear map

$$\mathbb{L} : \mathcal{O}x^\rho[z] \to \mathcal{O}x^\rho[z],\ x^\rho h(x) z^i \to \mathbb{L}(x^\rho h(x) z^i).$$

The Leibniz rule gives

Lemma 1. *Let L have shift $\tau \geq 0$ with extension $\mathbb{L}$ to $\mathcal{O}x^\rho[z]$. Then, for $\rho \in \mathbb{C}$, $h \in \mathcal{O}$, and $i \geq 0$,*

$$\mathbb{L}(x^\rho h(x) z^i)_{|z=\log(x)} = L(x^\rho h(x) \log(x)^i).$$

In particular, the map $\mathcal{O}x^\rho[z] \to \mathcal{O}x^\rho[\log(x)]$ given by the evaluation $z \to \log(x)$ sends solutions of $\mathbb{L}y = 0$ to solutions of $Ly = 0$.

Example. The equation $x^2y'' + 3xy' + 1 = 0$ with Euler operator $L_0 = x^2\partial^2 + 3x\partial + 1$ has indicial polynomial $\chi_0 = \rho^{\underline{2}} + 3\rho^{\underline{1}} + 1 = (\rho+1)^2$ with double root $\rho = -1$. The solutions of $L_0 y = 0$ are $y_1 = x^{-1}$ and $y_2 = x^{-1}\log(x)$. The operator $\mathbb{L}_0 = x^2\boldsymbol{\partial}^2 + 3x\boldsymbol{\partial} + 1$ on $\mathcal{O}x^{-1}[z]$ therefore has, as it should be, solutions x^{-1} and $x^{-1}z$. Indeed, $\mathbb{L}_0(x^{-1}) = L_0(x^{-1}) = 0$, whereas $\boldsymbol{\partial}(x^{-1}z) = x^{-2}(-z+1)$ and

$$\boldsymbol{\partial}^2(x^{-1}z) = \boldsymbol{\partial}(x^{-2}(-z+1)) = -2x^{-3}(-z+1) - x^{-3} = x^{-3}(2z-3)$$

give

$$\mathbb{L}_0(x^{-1}z) = x^{-1}(2z-3) + 3x^{-1}(-z+1) + x^{-1}z = 0.$$

Function spaces. If L_0 is an Euler operator with exponents set $\Omega \subseteq \mathbb{C}$ and if m_ρ denotes the multiplicity of $\rho \in \Omega$, the $\mathbb{C}$-vector space

$$\mathcal{F}_0 = \bigoplus_{\rho\in\Omega} \mathcal{O}x^\rho[z]_{<m_\rho}$$

of polynomials in z of degree $< m_\rho$ and with coefficients in $\mathcal{O}x^\rho$ is the correct space to look at for finding the solutions of the extended Euler equation $\mathbb{L}_0 y = 0$, since these are of the form $x^\rho z^i$, for $\rho \in \Omega$ and $0 \le i < m_\rho$. The space $\mathcal{F}_0$ is, however, in general too small to contain the solutions of the extension $\mathbb{L}y = 0$ if $Ly = 0$ is a general Fuchsian differential equation with initial form L_0. A suitable enlargement of $\mathcal{F}_0$ is necessary. The method how to do this goes back to Fuchs, Frobenius, Thomé; it requires some preparation.

Differentiating differential operators. If t is another variable, write the j-th derivative of $x^t = \exp(t\log(x))$ as $\partial^j x^t = t^{\underline{j}} x^{t-j}$. Define then, for $\ell \ge 1$, the *ℓ-th derivative* $(\partial^j)^{(\ell)}$ of ∂^j as

$$(\partial^j)^{(\ell)} x^t = (t^{\underline{j}})^{(\ell)} x^{t-j},$$

where $(t^{\underline{j}})^{(\ell)}$ denotes the ℓ-th derivative of $t^{\underline{j}}$ with respect to t. Clearly, $(\partial^j)^{(\ell)} = 0$ for $\ell > j$. Then, for a differential operator $L = p_0\partial^n + p_1\partial^{n-1} + \ldots + p_{n-1}\partial + p_n$ of order n, we get its ℓ-th derivative $L^{(\ell)}$ as

$$L^{(\ell)} = p_0 \cdot (\partial^n)^{(\ell)} + p_1 \cdot (\partial^{n-1})^{(\ell)} + \ldots + p_{n-1} \cdot (\partial)^{(\ell)}.$$

This is no longer a differential operator; it is just a $\mathbb{C}$-linear map $\mathcal{O}x^\rho \to \mathcal{O}x^{\rho+\tau}$, where τ is the shift of L.

The following facts are readily verified. Let L always be a differential operator of order n and shift $\tau \ge 0$. Let $\rho \in \mathbb{C}$ be arbitrary.

Lemma 2. *The extension $\mathbb{L}$ of L to $\mathcal{O}x^\rho[z]$ has expansion*

$$\mathbb{L} = L + L'\partial_z + \tfrac{1}{2!}L''\partial_z^2 + \ldots + \tfrac{1}{n!}L^{(n)}\partial_z^n,$$

where the $\mathbb{C}$-linear maps $L^{(\ell)}$ act on $\mathcal{O}x^\rho$ while leaving all z^i invariant, and ∂_z is the usual differentiation with respect to z.

Lemma 3. *If L_0 is an Euler operator of order n with shift 0, indicial polynomial $\chi_0(t)$, and extension $\mathbb{L}_0$ to $\mathcal{O}x^\rho[z]$, then*

$$\mathbb{L}_0(x^\rho z^i) = x^\rho \cdot [\chi_0(\rho)z^i + \chi_0'(\rho)iz^{i-1} + \tfrac{1}{2!}\chi_0''(\rho)i^{\underline{2}}z^{i-2} + \ldots + \tfrac{1}{n!}\chi_0^{(n)}(\rho)i^{\underline{n}}z^{i-n}].$$

Lemma 4. *The kernel of the extension $\mathbb{L}_0$ to $\mathcal{F}_0 = \bigoplus_{\rho\in\Omega}\mathcal{O}x^\rho[z]$ of an Euler operator L_0 with exponents $\rho \in \Omega \subseteq \mathbb{C}$ of multiplicity m_ρ equals*

$$\mathrm{Ker}(\mathbb{L}_0) = \bigoplus_{\rho\in\Omega}\bigoplus_{i=0}^{m_\rho-1} \mathbb{C}x^\rho z^i.$$

Lemma 5. *A $\mathbb{C}$-basis of solutions of an Euler equation $L_0 y = 0$ is given by*

$$x^\rho \log(x)^i,$$

where ρ ranges over all local exponents of L_0 at 0 and $0 \leq i < m_\rho$, with m_ρ the multiplicity of ρ.

Examples. (a) For the Euler operator $L_0 = x^2\partial^2 - 3x\partial + 3$ from before, with indicial polynomial $\chi_0(t) = (t+1)^2$ and exponent $\rho = -1$ of multiplicity $m_\rho = 2$, the extension $\mathbb{L}_0 = x^2 \boldsymbol{\partial}^2 + 3x\boldsymbol{\partial} + 1$ to $\mathcal{O}x^{-1}[z]$ has expansion

$$\mathbb{L}_0(x^\rho z^i) = x^\rho[(\rho+1)^2 z^i + 2(\rho+1)iz^{i-1} + 2i(i-1)z^{i-2}]$$

and kernel

$$\mathrm{Ker}(\mathbb{L}_0) = \mathbb{C}x^{-1} \oplus \mathbb{C}x^{-1}z.$$

(b) For the Euler operator $L_0 = x^3\partial^3 - 4x^2\partial^2 + 9x\partial - 9$ with indicial polynomial $\chi_0(t) = (t-1)(t-3)^2$ and exponents 1 and 3 of multiplicity one and two, respectively, the extension $\mathbb{L}_0 = x^3\boldsymbol{\partial}^3 - 4x^2\boldsymbol{\partial}^2 + 9x\boldsymbol{\partial} - 9$ to $\mathcal{O}x[z] \oplus \mathcal{O}x^3[z]$ has expansion

$$\mathbb{L}_0(x^\rho z^i) = x^\rho[(\rho-1)(\rho-3)^2 z^i + (3\rho-5)(\rho-3)iz^{i-1} + (6\rho-14)i^{\underline{2}}z^{i-2} + 6i^{\underline{3}}z^{i-3}]$$

and kernel

$$\mathrm{Ker}(\mathbb{L}_0) = \mathbb{C}x \oplus \mathbb{C}x^3 \oplus \mathbb{C}x^3 z.$$

Image of Euler operators. In order to apply the perturbation lemma to the extension $\mathbb{L}$ of operators L to the space $\mathcal{F}_0 = \bigoplus_{\rho\in\Omega} \mathcal{O}x^\rho[z]_{<m_\rho}$ one has to determine the image of the initial form $\mathbb{L}_0$ of $\mathbb{L}$. Write $L = L_0 - T$ and $\mathbb{L} = \mathbb{L}_0 - \mathbb{T}$. Assuming that L_0 has shift 0, it follows that T is an operator with shift > 0, that is, it increases the order in x of elements of $\mathcal{F}_0$. Therefore, $\mathbb{T}$ sends $\mathcal{F}_0$ to $\mathcal{F}_0 x = \bigoplus_{\rho\in\Omega} \mathcal{O}x^{\rho+1}[z]_{<m_\rho}$. One has no control about the precise image of $\mathbb{T}$: it can be equal to whole $\mathcal{F}_0 x$ but it can also be much smaller. The perturbation lemma requires in any case the inclusion $\mathrm{Im}(\mathbb{T}) \subseteq \mathrm{Im}(\mathbb{L}_0)$ of images. This would trivially hold if $\mathbb{L}_0$ were surjective onto $\mathcal{F}_0 x$. But this is not the case in general: it suffices to take $L_0 = x^2\partial^2 - x\partial$ with local exponents $\sigma = 0$ and $\rho = 2$, both of multiplicity one. Then $\mathcal{F}_0 = \mathcal{O} + \mathcal{O}x^2 = \mathcal{O}$ and $\mathbb{L}_0 = L_0$. The image of $\mathcal{F}_0$ under L_0 is $L_0(\mathcal{F}_0) = \mathbb{C}x + \mathcal{O}x^3 \subsetneq \mathcal{O}x = \mathcal{F}_0 x$, with a gap at x^2. However, if $L = x^2\partial^2 - x\partial - x = L_0 - T$, the operator $T = x$ sends $x \in \mathcal{F}_0$ to $x^2 \notin L_0(\mathcal{F}_0)$. So the perturbation lemma does not apply to this situation. The way out of this dilemma is a further enlargement of $\mathcal{F}_0$ to a carefully chosen function space $\mathcal{F}$ containing $\mathcal{F}_0$. This enlargement will be explained in the next section.

3. The normal form of Fuchsian differential operators

When trying to lift, for an arbitrary operator L, the solutions $x^\rho \log(x)^k$ of $L_0 y = 0$ to solutions of $Ly = 0$, two obstructions occur. First, ρ might be a multiple root of the indicial polynomial and logarithms already appear in the solutions of $L_0 y = 0$. Secondly, if ρ is not a maximal exponent of L modulo $\mathbb{Z}$, that is, if $\rho + k$ is again an exponent of L for some $k > 0$, the lifting poses additional problems since higher powers of logarithms will occur among the solutions. We will approach and solve both problems simultaneously by using the extensions $\mathbb{L}$ of operators L as defined above to appropriately chosen spaces $\mathcal{F}$ on which the image of $\mathbb{L}_0$ equals $\mathcal{F}x$. In this situation, the perturbation lemma will apply to reduce $\mathbb{L} : \mathcal{F} \to \mathcal{F}$ via a linear automorphism of $\mathcal{F}$ to $\mathbb{L}_0$.

Enlargements of function spaces. As was done already classically [Fuchs1, p. 136 and 157, Fuchs2, p. 362 and 364, Thom1, p. 193, Frob1, p. 221] it is appropriate to partition the set of exponents of a linear differential operator L into sets $\Omega \subseteq \mathbb{C}$ of exponents whose differences are integers and such that no exponent outside Ω has integer difference with an element of Ω. We list the elements of each Ω increasingly,

$$\rho_1 < \rho_2 < \cdots < \rho_r,$$

where $\rho_k < \rho_{k+1}$ stands for $\rho_{k+1} - \rho_k \in \mathbb{N}_{>0}$, and denote by $m_k \geq 1$ the respective multiplicity of ρ_k as a root of the indicial polynomial χ_0 of L at 0. Set $n_k = m_1 + \cdots + m_k$ and $n_0 = 0$. To easen the notation, we omit in each ρ_k the reference to the respective set $\Omega = \{\rho_1, ..., \rho_r\}$. Instead of $\mathcal{F}_0^\Omega = \sum_{k=1}^r \mathcal{O}x^{\rho_k}[z]_{<m_k}$ we will now allow polynomials in z of degree $< n_k$ and take the larger module

$$\mathcal{F}^\Omega = \sum_{k=1}^r \mathcal{O}x^{\rho_k}[z]_{<n_k} = \bigoplus_{k=1}^r \bigoplus_{i=n_{k-1}}^{n_k-1} \mathcal{O}x^{\rho_k}z^i = \bigoplus_{k=1}^{r-1} \bigoplus_{i=0}^{n_k-1} \bigoplus_{\sigma=\rho_k}^{\rho_{k+1}-1} \mathbb{C}x^\sigma z^i \oplus \bigoplus_{i=0}^{n_r-1} \mathcal{O}x^{\rho_r}z^i,$$

equipped with the derivation $\partial\!\!\!\partial$ from before (see Fig. 1). The two different direct sum decompositions of $\mathcal{F}$ will become relevant in a moment. Then set

$$\mathcal{F} = \bigoplus_\Omega \mathcal{F}^\Omega,$$

the sum varying over all sets Ω of exponents with integer difference. As each summand $\bigoplus_{i=n_{k-1}}^{n_k-1} \mathcal{O}x^{\rho_k}z^i$ of $\mathcal{F}^\Omega$ has rank m_k, it follows that $\mathcal{F}$ is free of rank n over $\mathcal{O}$.

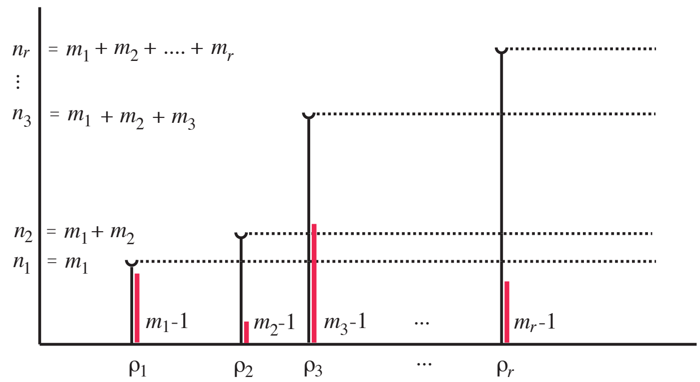

Figure 1. The sets of exponents (σ, i) of monomials $x^\sigma z^i$ in $\mathcal{F}^\Omega$; in red monomials in $\mathrm{Ker}(\mathbb{L}_0)$.

Example. Assume that the Euler operator L_0 has just two local exponents σ and ρ of multiplicities m_σ and m_ρ, respectively, say $\Omega = \{\sigma, \rho\}$. If $\rho - \sigma \notin \mathbb{Z}$, then

$$\mathcal{F} = \mathcal{O}x^\sigma[z]_{<m_\sigma} \oplus \mathcal{O}x^\rho[z]_{<m_\rho};$$

if $\rho - \sigma \in \mathbb{N}$, then

$$\mathcal{F} = \mathcal{O}x^\sigma[z]_{<m_\sigma} + \mathcal{O}x^\rho[z]_{<m_\sigma+m_\rho} = \mathcal{O}x^\sigma[z]_{<m_\sigma} \oplus \mathcal{O}x^\rho z^{m_\sigma}[z]_{<m_\rho}.$$

The extension $\mathbb{L}_0$ of L_0 to $\mathcal{F}$ has kernel $\mathbb{C}x^\sigma[z]_{<m_\sigma} \oplus \mathbb{C}x^\rho[z]_{<m_\rho}$ in the first case, and $\mathbb{C}x^\sigma[z]_{<m_\sigma} \oplus \mathbb{C}x^\rho z^{m_\sigma}[z]_{<m_\rho}$ in the second case. The respective images of $\mathbb{L}_0$ are

$$\mathcal{O}x^{\sigma+1}[z]_{<m_\sigma} \oplus \mathcal{O}x^{\rho+1}[z]_{<m_\rho}$$

and

$$\mathcal{O}x^{\sigma+1}[z]_{<m_\sigma} \oplus \mathcal{O}x^{\rho+1}z^{m_\sigma}[z]_{<m_\rho},$$

so they equal $\mathcal{F}x$ in both cases.

Normal Form Theorem. *Let $L \in \mathcal{O}[\partial]$ be a linear differential operator with holomorphic coefficients at 0, initial form L_0 and shift $\tau = 0$. Denote by $\Omega = \{\rho_1, ..., \rho_r\}$ a set of increasingly ordered local exponents ρ_k of L with integer differences and multiplicities m_k. Set $n_k = m_1 + \ldots + m_k$ and $\mathcal{F} = \mathcal{F}^\Omega = \sum_{k=1}^r \mathcal{O}x^{\rho_k}[z]_{<n_k}$. Let $\mathbb{L}, \mathbb{L}_0 : \mathcal{F} \to \mathcal{F}$ be the extensions of L and L_0 to $\mathcal{F}$ defined via $\boldsymbol{\partial} x = 1$ and $\boldsymbol{\partial} z = x^{-1}$ as in section 2. Assume in assertions (c) and (d) that L has a regular singularity at 0.*

(a) The map $\mathbb{L}$ sends $\mathcal{F}$ into

$$\mathcal{F}x = \sum_{k=1}^r \mathcal{O}x^{\rho_k+1}[z]_{<n_k}.$$

(b) The map $\mathbb{L}_0$ has image $\mathrm{Im}(\mathbb{L}_0) = \mathcal{F}x$. Its kernel $\mathrm{Ker}(\mathbb{L}_0) = \bigoplus_{k=1}^r \mathbb{C}x^{\rho_k}[z]_{<m_k}$ has direct complement

$$\mathcal{H} = \bigoplus_{k=2}^{r} \bigoplus_{i=m_k}^{n_k-1} \mathbb{C}x^{\rho_k} z^i \oplus \bigoplus_{k=1}^{r-1} \bigoplus_{e=1}^{\rho_{k+1}-\rho_k-1} \bigoplus_{i=0}^{n_k-1} \mathbb{C}x^{\rho_k+e} z^i \oplus \bigoplus_{i=0}^{n_r-1} \mathcal{O}x^{\rho_r+1} z^i,$$

in $\mathcal{F}$. Thus the restriction $\mathbb{L}_{0|\mathcal{H}}$ defines a linear isomorphism between $\mathcal{H}$ and $\mathcal{F}x$.

(c) The composition of the inverse $(\mathbb{L}_{0|\mathcal{H}})^{-1} : \mathcal{F}x \to \mathcal{H}$ of $\mathbb{L}_{0|\mathcal{H}}$ with the inclusion $\mathcal{H} \subseteq \mathcal{F}$ defines a right inverse $\mathbb{S}_0 : \mathcal{F}x \to \mathcal{F}$ of $\mathbb{L}_0$, again denoted by $(\mathbb{L}_{0|\mathcal{H}})^{-1}$. Let $\mathbb{T} : \mathcal{F} \to \mathcal{F}x$ be the extension of $T = L_0 - L$ to $\mathcal{F}$. The map

$$u = \mathrm{Id}_{\mathcal{F}} - \mathbb{S}_0 \circ \mathbb{T} : \mathcal{F} \to \mathcal{F}$$

is a linear automorphism of $\mathcal{F}$, with inverse $v = u^{-1} = \sum_{k=0}^\infty (\mathbb{S} \circ \mathbb{T})^k : \mathcal{F} \to \mathcal{F}$.

(d) The automorphism v of $\mathcal{F}$ transforms $\mathbb{L}$ into $\mathbb{L}_0$,

$$\mathbb{L} \circ v = \mathbb{L}_0.$$

(e) If 0 is an arbitrary (i.e., regular or irregular) singularity of L, statements (a) to (d) hold true with $\mathcal{O}$ replaced by the ring $\widehat{\mathcal{O}}$ of formal power series over $\mathbb{C}$ or over an arbitrary field K of characteristic 0.

Proof. (a) Recall that $\mathbb{L} = \mathbb{L}_0 - \mathbb{T}$ and that $\mathbb{T}$ sends $\mathcal{F}$ into $\mathcal{F}x$ since T has shift > 0. It therefore suffices to show that $\mathbb{L}_0$ sends $\mathcal{F}$ into $\mathcal{F}x$. But recall from Lemma 3 that

$$\mathbb{L}_0(x^\rho z^i) = x^\rho \cdot [\chi_0(\rho) z^i + \chi_0'(\rho) i z^{i-1} + \tfrac{1}{2!}\chi_0''(\rho) i^{\underline{2}} z^{i-2} + \ldots + \tfrac{1}{n!}\chi_0^{(n)}(\rho) i^{\underline{n}} z^{i-n}].$$

Therefore, as $\chi_0^{(\ell)}(\rho_k) = 0$ for $0 \leq \ell < m_k$, and using that $n_k - m_k = n_{k-1}$ for $k \geq 2$, it follows that $\mathbb{L}_0$ sends $\mathcal{F}$ into

$$\sum_{k=1}^r \mathcal{O}x^{\rho_k}[z]_{<n_k-m_k} = \sum_{k=2}^r \mathcal{O}x^{\rho_k}[z]_{<n_{k-1}} \subseteq \sum_{k=2}^r \mathcal{O}x^{\rho_{k-1}+1}[z]_{<n_{k-1}} \subseteq \mathcal{F}x.$$

Here, we use that $\rho_k - \rho_{k-1} \in \mathbb{N}_{>0}$. This proves that $\mathbb{L}_0(\mathcal{F}) \subseteq \mathcal{F}x$.

(b) From the shape of $\mathcal{F}$ and $\mathrm{Ker}(\mathbb{L}_0)$ as depicted in Fig. 1 one sees that $\mathcal{H}$ is a direct complement of $\mathrm{Ker}(\mathbb{L}_0)$ in $\mathcal{F}$. Hence $\mathbb{L}_{0|\mathcal{H}}$ is automatically injective and $\mathbb{L}_0(\mathcal{F}) = \mathbb{L}_0(\mathcal{H})$. We will show that $\mathbb{L}_0(\mathcal{F}) = \mathcal{F}x$. It suffices to check that all monomials $x^\sigma z^i \in \mathcal{F}x$ lie in the image, where $\sigma = \rho_k + e$ for some $k = 1, ..., r$ and $e \geq 1$, and where $i < n_k$. This is actually the trickiest part of the proof. We distinguish two cases.

(i) If $\sigma \notin \Omega$, proceed by induction on i. Let $i = 0$. By Lemma 2,

$$\mathbb{L}_0(x^\sigma) = L_0(x^\sigma) + \sum_{j=1}^{n} \frac{1}{j!} L^{(j)} \partial_z^j (x^\sigma) = L_0(x^\sigma) = \chi_0(\sigma) x^\sigma \neq 0,$$

since σ is not a root of χ_0. So $x^\sigma \in \mathbb{L}_0(\mathcal{F})$. Let now $i > 0$. Lemmata 2 and 3 yield

$$\mathbb{L}_0(x^\sigma z^i) = L_0(x^\sigma z^i) + \sum_{j=1}^{n} \frac{1}{j!} L^{(j)} \partial_z^j (x^\sigma z^i) = \chi_0(\sigma) x^\sigma z^i + \chi_0^{(j)}(\sigma) x^\sigma \sum_{j=1}^{n} \frac{i^{\underline{j}}}{j!} z^{i-j}.$$

By the inductive hypothesis and using again that $\chi_0(\sigma) \neq 0$, we end up with $x^\sigma z^i \in \mathbb{L}_0(\mathcal{F})$.

(ii) If $\sigma \in \Omega$, write $\sigma = \rho_k$ for some $1 \leq k \leq r$. As $x^\sigma z^i = x^{\rho_k} z^i \in \mathcal{F}x$ and $\rho_1 < \rho_2 < \cdots < \rho_r$, we know that $k \geq 2$ and

$$x^{\rho_k} z^i \notin x \cdot \sum_{\ell=k}^{r} \mathcal{O} x^{\rho_\ell} [z]_{<n_\ell}.$$

Hence

$$x^{\rho_k} z^i \in x \cdot \sum_{\ell=1}^{k-1} \mathcal{O} x^{\rho_\ell} [z]_{<n_\ell}.$$

This implies in particular that $0 \leq i < n_{k-1}$, which will be used later on. We proceed by induction on i. Let $i = 0$. By Lemma 2,

$$\begin{aligned}
\mathbb{L}_0(x^{\rho_k} z^{m_k}) &= \sum_{j=0}^{m_k-1} \frac{1}{j!} L^{(j)} \partial_z^j (x^{\rho_k} z^{m_k}) + \frac{1}{m_k!} L_0^{(m_k)} \partial_z^{m_k} (x^{\rho_k} z^{m_k}) + \sum_{j=m_k+1}^{n} \frac{1}{j!} L_0^{(j)} \partial_z^j (x^{\rho_k} z^{m_k}) \\
&= \sum_{j=0}^{m_k-1} \frac{(m_k)^{\underline{j}}}{j!} \chi_0^{(j)}(\rho_k) x^{\rho_k} z^{m_k-j} + \chi_0^{(m_k)}(\rho_k) x^{\rho_k} \\
&= \chi_0^{(m_k)}(\rho_k) x^{\rho_k}.
\end{aligned}$$

Here, the sum in the first summand in the last but one line is 0 since ρ_k is a root of χ_0 of multiplicity m_k, and for the same reason, the second summand $\chi_0^{(m_k)}(\rho_k) x^{\rho_k}$ is non-zero. So $x^\sigma = x^{\rho_k} \in \mathbb{L}_0(\mathcal{F})$. Let now $i > 0$ and consider $x^\sigma z^i = x^{\rho_k} z^i \in \mathcal{F}x$. We will use that $i < n_{k-1}$ as observed above. Namely, this implies that $m_k + i < m_k + n_{k-1} = n_k$, so that $x^{\rho_k} z^{m_k+i}$ is an element of $\mathcal{F}$. Let us apply $\mathbb{L}_0$ to it. Similarly as in the case $i = 0$ we get

$$\begin{aligned}
\mathbb{L}_0(x^{\rho_k} z^{m_k+i}) &= \sum_{j=0}^{m_k-1} \frac{1}{j!} L^{(j)} \partial_z^j (x^{\rho_k} z^{m_k+i}) + \frac{1}{m_k!} L_0^{(m_k)} \partial_z^{m_k} (x^{\rho_k} z^{m_k+i}) + \\
&\qquad + \sum_{j=m_k+1}^{n} \frac{1}{j!} L_0^{(j)} \partial_z^j (x^{\rho_k} z^{m_k+i}) \\
&= \frac{(m_k+i)^{\underline{m_k}}}{m_k!} \chi_0^{(m_k)}(\rho_k) x^{\rho_k} z^i + \sum_{j=m_k+1}^{n} \frac{(m_k+i)^{\underline{j}}}{j!} \chi_0^{(j)}(\rho_k) x^{\rho_k} z^{m_k+i-j}.
\end{aligned}$$

The sum appearing in the second summand of the last line belongs to $\mathbb{L}_0(\mathcal{F})$ by the induction hypothesis since $m_k + i - j < i$. As $\chi_0^{(m_k)}(\rho_k) \neq 0$, we end up with $x^\sigma z^i = x^{\rho_k} z^i \in \mathbb{L}_0(\mathcal{F})$. This proves that $\mathbb{L}_0(\mathcal{F}) = \mathcal{F}x$ and assertion (b).

(c) & (d) & (e) Once we show that $|\mathbb{S}_0(\mathbb{T}(f))| \leq C|f|$ holds for some $0 < C < 1$ and all $f \in \mathcal{F}$, the perturbation lemma implies that $u = \mathrm{Id}_\mathcal{F} - \mathbb{S}_0 \circ \mathbb{T}$ is a linear automorphism of $\mathcal{F}$ with $\mathbb{L} \circ u^{-1} = \mathbb{L}_0$, proving assertions (c) to (e) of the theorem. The proof of the estimate is split into two parts, first for formal power series and then for convergent ones, and uses a different metric in each case.

(i) Denote by $\widehat{\mathcal{O}} = K[[x]]$ the formal power series ring over an arbitrary field K of characteristic 0, equipped with the metric $d(f,g) = 2^{-\mathrm{ord}_0(f-g)}$, where ord_0 denotes the order of vanishing at 0. Let $\widehat{\mathcal{F}}$ denote the induced $\widehat{\mathcal{O}}$-modules $\widehat{\mathcal{F}} = \mathcal{F} \otimes_K \widehat{\mathcal{O}}$ and write again $\mathbb{L}$ for the extension $\widehat{\mathbb{L}}$ to $\widehat{\mathcal{F}}$. As $\mathbb{T}$ increases the order of series in $\widehat{\mathcal{O}}$, while $\mathbb{L}_0$ and $\mathbb{S}_0$ do not decrease the order, it follows that also $\mathbb{S}_0 \circ \mathbb{T}$ increases the order. It thus satisfies the inequality $|\mathbb{S}_0(\mathbb{T}(f))| \le C \cdot |f|$ from the beginning, for some $0 < C < 1$, having set $|f| = d(f,0) = 2^{-\mathrm{ord} f}$. Therefore the von Neumann series

$$v = \textstyle\sum_{j=0}^{\infty} (\mathbb{S}_0 \circ \mathbb{T})^j$$

defines a $\mathbb{C}$-linear map $v : \widehat{\mathcal{F}} \to \widehat{\mathcal{F}}$. Then it is clear that $v = u^{-1} = (\mathrm{Id}_{\widehat{\mathcal{F}}} - \mathbb{S} \circ \mathbb{T})^{-1}$. So u and v are automorphisms, and $\mathbb{L} \circ v = \mathbb{L}_0$ by the perturbation lemma. This proves assertion (e) of the theorem.

(ii) To prove the same thing inside $\mathcal{O}$, denote by $\mathcal{O}_s$ the subring of germs of holomorphic functions h such that $|h|_s < \infty$. Here, $s > 0$ and $|\sum_{k=0}^{\infty} a_k x^k|_s := \sum_{k=0}^{\infty} |a_k| s^k$. It is well known that the rings $\mathcal{O}_s$ are Banach spaces, and that $\mathcal{O} = \bigcup_{s>0} \mathcal{O}_s$ [GrRe]. For $s > 0$ sufficiently small, u restricts to a linear map u_s on the induced Banach space $\mathcal{F}_s$. For the convergence of v_s it therefore suffices to prove that $||\mathbb{S}_0 \circ \mathbb{T}||_s < 1$, where $|| - ||_s$ denotes the operator norm of bounded linear maps $\mathcal{F}_s \to \mathcal{F}_s$. Once this is proven, $v_s = u_s^{-1}$ holds as before and shows that u_s and hence also u are linear isomorphisms. This argument provides a compact reformulation of Frobenius' proof for the convergence of solutions [Frob1, p. 218].

The inequality $||\mathbb{S}_0 \circ \mathbb{T}||_s < 1$ is equivalent to the existence of a constant $0 < C < 1$ such that

$$|\mathbb{S}_0(\mathbb{T}(x^\rho h(x) z^i))|_s \le C \cdot |x^\rho h(x) z^i|_s$$

for all $x^\rho h(x) z^i \in \mathcal{F}_s$. This will imply in particular that $(\mathbb{S}_0 \circ \mathbb{T})(\mathcal{F}_s) \subseteq \mathcal{F}_s$.

We will treat the case where ρ is a maximal local exponent of L modulo $\mathbb{Z}$ and a simple root of χ_0. In this case, no extensions of operators are required, and we can work directly with L, S and T and $\mathcal{F} = \mathcal{O} x^\rho$. For non-maximal exponents there occur notational complications which present, however, no substantially new difficulty. So we shall omit the general case. For $h = \sum_{k=0}^{\infty} a_k x^k \in \mathcal{O}$ and writing $L = \sum_{j=0}^{n} p_j(x) \partial^j$ with $p_j = \sum_{i=0}^{\infty} c_{ij} x^i$ we have

$$T(x^\rho h) = -\sum_{i-j>0} \sum_{k=0}^{\infty} (\rho + k)^{\underline{j}} c_{ij} a_k x^{\rho+k+i-j},$$

and, recalling that L_0 is assumed to have shift 0,

$$S(T(x^\rho h)) = -\sum_{i-j>0} \sum_{k=0}^{\infty} \frac{(\rho+k)^{\underline{j}}}{\chi_L(\rho+k+i-j)} c_{ij} a_k x^{\rho+k+i-j}.$$

As $i - j > 0$, $k \ge 0$, and ρ is maximal, no $\rho + k + i - j$ appearing in the denominator is a root of χ_L. Hence the ratio

$$\frac{(\rho+k)^{\underline{j}}}{\chi_L(\rho+k+i-j)} = \frac{(\rho+k)^{\underline{j}}}{\sum_{\ell=0}^{n} c_{\ell,\ell} (\rho+k+i-j)^{\underline{\ell}}}$$

is well defined. But $c_{n,n} \neq 0$ since 0 is a regular singularity of L, and hence $(\rho + k + i - j)^{\underline{n}}$ appears in the denominator with non-zero coefficient. As $j \le n$ this ensures that the ratio remains bounded in absolute value, say $\le c$, as k tends to ∞. Taking norms on both sides of the above expression for $S(T(x^\rho h))$ yields, for $s \le 1$ and $h \in \mathcal{O}_s$, the estimate

$$|S(T(x^\rho h))|_s \le c \sum_{i-j>0} \sum_{k=0}^{\infty} |c_{ij}||a_k| s^{\rho+k+i-j} = c \sum_{i-j>0} |c_{ij}| s^{i-j} \sum_{k=0}^{\infty} |a_k| s^{\rho+k}.$$

But by assumption, $p_j = \sum_{i=0}^{\infty} c_{ij}x^i \in \mathcal{O}_s$ for all $0 < s \leq s_0$ and all $j = 0, ..., n$. This implies in particular $\sum_{i>j}^{\infty} c_{ij}x^i \in \mathcal{O}_s$ and then, after division by x^{j+1} and since $i > j$, that

$$\sum_{i>j}^{\infty} c_{ij}x^{i-(j+1)} \in \mathcal{O}_s.$$

We get for all $s \leq r$ that

$$\sum_{i-j>0} |c_{ij}|s^{i-j} = s \cdot \sum_{i-j>0} |c_{ij}|s^{i-(j+1)} \leq c's$$

for some $c' > 0$ independent of s. This inequality allows us to bound $|S(T(x^\rho h))|_s$ from above by

$$|S(T(x^\rho h))|_s \leq cc's \sum_{k=0}^{\infty} |a_k|s^{\rho+k} = cc's|x^\rho h|_s.$$

Take now $s_0 > 0$ sufficiently small, say $s_0 \leq \min(1, r)$ and $s_0 < \frac{1}{cc'}$, and get a constant $0 < C < 1$ such that for $0 < s \leq s_0$ one has

$$|S(T(x^\rho h))|_s \leq C \cdot |x^\rho h|_s.$$

This establishes $||S \circ T||_s < 1$ on $\mathcal{F}_s$ for $0 < s \leq s_0$. By the perturbation lemma, $u_s = \mathrm{Id}_{\mathcal{F}_s} - S \circ T$ is an automorphism of $\mathcal{F}_s$ with inverse $v_s = \sum_k (S \circ T)^k$. This completes the proof of the theorem. $\circlearrowleft$

4. Applications of the normal form theorem

As a first consequence of the normal form theorem we recover the classical theorem of Fuchs from 1866 and 1868 about the local solutions of differential equations at regular singular points [Fuchs1, Fuchs2]. The statement was reorganized and further detailed by Thomé and Frobenius in a series of papers between 1872 and 1875 [Thom1, Thom2, Thom3, Frob1, Frob2]. See also [Fabr, formula (9), p. 19].

Theorem of local solutions. *Let $L \in \mathcal{O}[\partial]$ be a linear differential operator with holomorphic coefficients and regular singularity at 0. For each set Ω of local exponents of L with integer differences, let $u_\Omega : \mathcal{F}^\Omega \to \mathcal{F}^\Omega$ be the automorphism of assertion (d) of the normal form theorem.*
(a) Varying Ω, a $\mathbb{C}$-basis of local solutions of $Ly = 0$ at 0 is given by

$$y_{\rho,i}(x) = u_\Omega^{-1}(x^\rho z^i)_{|z=\log(x)},$$

for $\rho \in \Omega$ a local exponent of L of multiplicity m_ρ, and $0 \leq i < m_\rho$.
(b) Order the exponents in a chosen set Ω as $\rho_1 < \ldots < \rho_r$ and write m_k for m_{ρ_k}. Set $n_k = m_1 + \ldots + m_k$. Each solution related to Ω is of the form, for $1 \leq k \leq r$ and $0 \leq i < m_k$,

$$y_{\rho_k,i}(x) = x^{\rho_k}[f_{k,i} + \ldots + f_{k,0}\log(x)^i] + \sum_{\ell=k+1}^{r} x^{\rho_\ell} \sum_{j=n_{\ell-1}}^{n_\ell - 1} h_{k,i,j}(x)\log(x)^j,$$

with holomorphic $f_{k,i}$ and $h_{k,i,j}$ in $\mathcal{O}$ with non-zero constant term.

Proof. Let Ω be a set of local exponents of L at 0 with integer differences and consider the space $\mathcal{F}^\Omega = \sum_{k=1}^{r} \mathcal{O}x^{\rho_k}[z]_{<n_k}$ as in the statement of the normal form theorem. Extend L and L_0 to operators $\mathbb{L}$ and $\mathbb{L}_0$ on $\mathcal{F} = \bigoplus_\Omega \mathcal{F}^\Omega$. By Lemma 4, a $\mathbb{C}$-basis of solutions of $\mathbb{L}_0$ is given by the monomials $x^\rho z^i$, $0 \leq i \leq m_\rho - 1$, where ρ is a local exponent of multiplicity m_ρ. By assertion (d) of the normal form theorem and since L and L_0 have the same order n, the pull-backs $u^{-1}(x^\rho z^i)$ form a $\mathbb{C}$-basis of solutions of $\mathbb{L}y = 0$. Now Lemma 1 gives the result. $\circlearrowleft$

Remark. The coefficient functions $f_{k,i}$ and $h_{\rho,i,j} \in \mathcal{O}$ of the solutions in assertion (b) of the theorem are related to each other. For instance, if ρ is a maximal exponent in Ω of multiplicity m_ρ, then

$$y_{\rho,0} = x^\rho \cdot g_0,$$

$$y_{\rho,1} = x^\rho \cdot [g_1 + g_0 \log(x)],$$

...

$$y_{\rho,m_\rho-1} = x^\rho \cdot [g_{m_\rho-1} + g_{m_\rho-2} \log(x) + \ldots + g_1 \log(x)^{m_\rho-2} + g_0 \log(x)^{m_\rho-1}],$$

with holomorphic $g_0, ..., g_{m_\rho-1}$ having non-zero constant term.

Example. If L has exactly two exponents σ and ρ, with $\rho - \sigma \in \mathbb{N}_{>0}$ and of multiplicities m_σ and m_ρ, respectively, we get accordingly

$$\mathcal{F} = x^\sigma[\mathcal{O} \oplus \cdots \oplus \mathcal{O}z^{m_\sigma-1}] + x^\rho[\mathcal{O} \oplus \cdots \oplus \mathcal{O}z^{m_\sigma+m_\rho-1}].$$

which we rewrite as

$$\mathcal{F} = x^\sigma[\mathcal{O} \oplus \cdots \oplus \mathcal{O}z^{m_\sigma-1}] \oplus x^\rho[\mathcal{O}z^{m_\sigma} \oplus \cdots \oplus \mathcal{O}z^{m_\sigma+m_\rho-1}].$$

A basis of solutions of $Ly = 0$ are $\mathcal{O}$-linear combinations

$$y_{\sigma,0} = x^\sigma \cdot h_0 + x^\rho g_0 \log(x)^{m_\sigma},$$

$$y_{\sigma,1} = x^\sigma \cdot [h_1 + h_0 \log(x)] + x^\rho \log(x)^{m_\sigma}[g_1 + g_0 \log(x)],$$

...

$$y_{\sigma,m_\sigma-1} = x^\sigma \cdot [h_{m_\sigma-1} + h_{m_\sigma-2} \log(x) + \cdots + h_1 \log(x)^{m_\sigma-2} + h_0 \log(x)^{m_\sigma-1}] +$$

$$+ x^\rho \log(x)^{m_\sigma} \cdot [g_{m_\rho-1} + \cdots + g_0 \log(x)^{m_\rho-1}],$$

$$y_{\rho,0} = x^\rho \cdot f_0,$$

$$y_{\rho,1} = x^\rho \cdot [f_1 + f_0 \log(x)],$$

...

$$y_{\rho,m_\rho-1} = x^\rho \cdot [f_{m_\rho-1} + f_{m_\rho-2} \log(x) + \ldots + f_1 \log(x)^{m_\rho-2} + f_0 \log(x)^{m_\rho-1}],$$

with holomorphic $f_0, ..., f_{m_\rho-1}, g_0, ..., g_{m_\rho-1}, h_0, ..., h_{m_\sigma-1}$ having non zero constant term.

Apparent singularities. The formulas for the solutions of $Ly = 0$ are somewhat complicated whenever the sets Ω of local exponents are not single valued. But if $\Omega = \{\rho\}$ has just one element ρ, i.e., no other local exponent of L is congruent to ρ modulo $\mathbb{Z}$, and if ρ has multiplicity m_ρ, the respective solutions are simpler, of the form, for $0 \leq i < m_\rho$,

$$y_{\rho,i}(x) = x^\rho[f_i + \ldots + f_i \log(x)^i].$$

If some local exponents have multiplicity ≥ 2 logarithms are forced to appear. If all local exponents are simple roots of the indicial polynomial, it may happen that no logarithms appear in the solutions. This situation is known as the presence of *apparent singularities.*

Theorem apparent singularities. *Let $L \in \mathcal{O}[\partial]$ be a differential operator with holomorphic coefficients and regular singularity at 0. Assume that all local exponents are integers and simple roots of the indicial polynomial of L at 0, and write $L = L_0 - T$ with initial form L_0 of L. If $\mathrm{Im}(T) \subseteq \mathrm{Im}(L_0)$ in $\mathcal{O}$, the local solutions of $Ly = 0$ at 0 are holomomorphic functions.*

Proof. This is an immediate consequence of the proof of the normal form theorem, since in case $\mathrm{Im}(T) \subseteq \mathrm{Im}(L_0)$ no extensions of the differential operators to larger function spaces involving logarithms are needed. As the local exponents are integral, the assertion follows from the description of the solutions. $\circlearrowleft$

Irregular singularities. According to Fabry it was Thomé who called Laurent series solutions of linear differential equations *regular integrals* [Fabr, pp. 29, 65]. Fabry and Forsythe mention that *normal integrals* were for Thomé solutions which involve $\exp(r(x))$, where r is a rational function [Fabr, p. 65, Fors, p. 262, Thom4, p. 75]. It seems that Thomé [Thom3, p. 292-302] only described conditions for the differential equation to allow a solution at 0 of the form

$$y(x) = \exp(r(x)) \cdot x^\rho \cdot \Big[h_0(x) + h_1(x)\log(x) + \ldots + h_k(x)\log^k(x)\Big]$$

where $r \in \mathbb{C}(x)$ is a rational function, ρ a local exponent, and h_i are holomorphic (actually, one can even take $r(x) = p(\frac{1}{x^q})$ for a polynomial $p \in \mathbb{C}[x]$ and an integer $q \geq 1$) [Salv, Thm. 3, Ince, Chap. XVII, p. 417]. Forsythe [Fors, p. 262] mentions that Thomé did not prove the existence of such solutions but that he rather assumed having such a solution in order to draw conclusions on the differential equation. Salvy refers to the thesis of Fabry as the place where the existence of a basis of solutions of this form has been proven for any (regular singular or irregular) differential equation [Salv, Thm. 3, p. 1079, Fabr, p. 28]. In chapter IV, Section 25 of [Fabr, p. 65], Fabry follows quite accurately the presentation of Thomé [Thom3, p. 293]. It is not easy to make out where Fabry actually proves that every linear differential equation has a basis of solutions of the displayed shape. Fabry himself says that he finds by his method *at most* as many solutions as the order of the differential equation indicates, but he does not claim to get a whole basis, contrary to what is alluded to by Salvy [Fabr, p. 75, Salv, Thm. 3].

Part (e) of the normal form theorem provides in the case of irregular singularities as many combinations of powers of logarithms with formal power series solutions as the order of the initial form L_0 of L indicates.

Theorem irregular singularities. *Let L in $\mathcal{O}[\partial]$ or $\widehat{\mathcal{O}}[\partial]$ be a linear differential operator with holomorphic or formal power series coefficients. For each set Ω of local exponents of L with integer differences, order the exponents in a chosen set Ω as $\rho_1 < \ldots < \rho_r$ and write m_k for m_{ρ_k}. Set $n_k = m_1 + \ldots + m_k$. Each solution of $Ly = 0$ related to Ω is of the form, for $1 \leq k \leq r$ and $0 \leq i < m_k - 1$,*

$$y_{\rho_k,i}(x) = x^{\rho_k}[f_{k,i} + \ldots + f_{k,0}\log(x)^i] + \sum_{\ell=k+1}^{r} x^{\rho_\ell} \sum_{j=n_{\ell-1}}^{n_\ell - 1} h_{k,i,j}(x)\log(x)^j,$$

with formal power series $f_{k,i}, h_{k,i,j} \in \widehat{\mathcal{O}}$ with non-zero constant term. The total number of such solutions equals the order n' of the initial form L_0 of L. $\circlearrowleft$

Remark. As mentioned before, the remaining $n - n'$ solutions may involve exponential functions in Laurent polynomials and are thus more complicated to construct [Thom4, Thom5, Fabr, Ince, p. 417]. The underlying more comprehensive normal form theorem will be treated in the forthcoming article of N. Merkl [Merk]. There, also the possible application of the normal form theorem to the proof of the index theorem of Komatsu and Malgrange will be explained [Kom, Malg].

Example. The divergent series $y(x) = \sum_{k=0}^{\infty} k! x^{k+1}$ satisfies the second order equation

$$Ly = x^3 y'' + (x^2 - x)y' + y = 0.$$

The initial form of L at 0 is given by the first order operator $L_0 = -x\partial + 1$. Hence 0 is an irregular singularity of L. The function $z(x) = \exp(-\frac{1}{x})$ is a second solution of $Ly = 0$; it is no longer a formal power series.

Gevrey series. By a theorem of Maillet, every power series solution $y(x)$ of an equation $Ly = 0$ with holomorphic coefficients is a *Gevrey-series*, i.e., there exists an $m \in \mathbb{N}$ such that the m-th Borel transform

$$y(x) = \sum_{k=0}^{\infty} a_k x^k \to \widetilde{y}(x) = \sum_{k=0}^{\infty} \frac{a_k}{(k!)^m} x^k$$

of $y(x)$ converges [Maill]. This result can also be seen as a consequence of the normal form theorem: It suffices to apply the norm estimates in part (ii) of the convergence proof to the series $\widetilde{h}(x) = \sum_{k=0}^{\infty} \frac{a_k}{(k!)^m} x^k$ with $m = n - n'$, where n' denotes again the order of the initial form L_0 of L at 0. Exploiting this one proves that the composition of the automorphism $v = u^{-1}$ of $\widehat{\mathcal{O}}$ with the m-th Borel transform sends the solutions x^ρ of $L_0 y = 0$, for $\rho \in \mathbb{Z}$ a local integer exponent of L, to a convergent power series $x^\rho \widetilde{h}(x)$. The key step is to see that the ratio

$$\frac{(\rho + k)^{\underline{j}}}{\chi_L(\rho + k + i - j)} = \frac{(\rho + k)^{\underline{j}}}{\sum_{\ell=0}^{n} c_{\ell,\ell}(\rho + k + i - j)^{\underline{\ell}}}$$

will be replaced by

$$\frac{(\rho + k)^{\underline{j}}}{\chi_L(\rho + k + i - j)} = \frac{(\rho + k)^{\underline{j}}}{\sum_{\ell=0}^{n'} c_{\ell,\ell}(\rho + k + i - j)^{\underline{\ell}}(k!)^m}$$

to obtain the required convergence. We omit the details. ◌

5. The Grothendieck-Katz p-curvature conjecture

One of the principal objectives of the normal form theorem is to prepare the ground for a characteristic p version and to study with it the conjectures of Grothendieck-Katz, André, Bézivin, Christol, the Chudnovsky brothers, Matzat and van der Put about the algebraicity of solutions of linear differential equations with polynomial coefficients defined over $\mathbb{Q}$ [Katz1, Katz2, Katz3, Andr, Béz, Chris, Chud, Matz, vdP]. We briefly comment on a prospective approach to them via the normal form theorem.

It is a classical result, already known to Abel, that algebraic power series satisfy a linear differential equation with polynomial coefficients. The intriguing and meanwhile notorious problem is to characterize those differential equations which arise in this way, a question which appears over and over again in the literature (Abel, Riemann, Autonne, Fuchs, Frobenius, Schwarz, Beukers-Heckman, ...).

Recall in this perspective the statement of (one version of) the conjecture of Grothendieck-Katz: Let $L \in \mathbb{Q}[x][\partial]$ be a differential operator with polynomial coefficients defined over $\mathbb{Q}$. The differential equation $Ly = 0$ has a $\mathbb{C}$-basis of algebraic power series solutions if (and only if) its reduction $L_p y = 0$ modulo p has for almost all primes p an $\mathbb{F}_p((x^p))$-basis of polynomial (or: power series) solutions.

The only if implication is easy to see. For the converse, one knows from the characteristic p assumption that the characteristic zero equation $Ly = 0$ will have regular singularities with pairwise distinct rational exponents [Katz3, Honda, Prop. 5.2, p. 189]. The case of order one equations is equivalent to a special case of a theorem of Kronecker (which, in turn, is a special case of Chebotarev's density theorem) [Honda].

Katz has proven spectacularly the conjecture for Picard-Fuchs equations [Katz3]. There have been recent and quite technical advances in the conjecture by various people, but the general case (even for order two equations) seems to still resist. Bost has established a striking variant of the conjecture for algebraic foliations and subgroups of Lie-groups [Bost, Chamb, Thme. 2.4]. And Bostan-Caruso-Schost have described fast algorithms to compute the p-curvature [BCS].

The conjecture of Bézivin does not resort to reduction modulo p. It reads as follows [Béz]: Let $L \in \mathbb{Z}[x][\partial]$ be a differential operator with polynomial coefficients defined over $\mathbb{Z}$. If $Ly = 0$ has a $\mathbb{Q}$-basis of power solutions with integer coefficients, these solutions are already algebraic power series.

The validity of the Grothendieck-Katz conjecture implies the validity of the Bézivin conjecture, which is, nevertheless, suspected to be strictly weaker. Neither the equivalence of the two conjectures nor their non-equivalence seem to be known. Also, Bézivin's conjecture remains unsolved.

All this suggests to extend the normal form theorem also to positive characteristic and to apply both, the characteristic 0 and the characteristic p version, to the above situation. As it turns out, extra obstructions have to be overcome in characteristic $p > 0$. Let K be a field of characteristic p, and consider a differential operator $L \in K(x)[\partial]$. The field of constants inside the field of formal Laurent series $K((x)) = \mathrm{Quot}(K[[x]])$ is now much larger than K, namely the field $K((x^p))$ of Laurent series in x^p. As such, the kernel of the initial form L_0 of L becomes larger, and its image will be smaller (in a sense that can be made precise). The main problem is to define the correct function space $\mathcal{F}$ on which a suitable extension $\mathbb{L}$ of L shall act. Whereas in characteristic 0 (and for regular singularities) it is sufficient to adjoin one variable z to $\mathcal{O}$, with differentiation $\partial z = \frac{1}{x}$, one has to consider here a countable set of new variables $z_1, z_2, ...$, mimicking each the role of an *iterated logarithm*, $z_1 = \log(x)$, $z_{i+1} = \log(z_i)$. The appropriate differentiation rule is

$$\partial z_{i+1} = \frac{1}{x} \cdot \frac{1}{z_1 \cdots z_i}.$$

This produces a differential ring $\mathcal{R} = K((x))[z_1, z_2, ...]$ which involves polynomials in infinitely many variables. In contrast to characteristic zero, the variables z_i have no counterpart as actual logarithms. They are just considered as elements of the differential ring $\mathcal{R}$. One then finds again a suitable function space $\mathcal{F}$ over $\mathcal{R}$ and an automorphism u of $\mathcal{F}$ which brings the extension $\mathbb{L}$ of L to $\mathcal{F}$ into its normal form, given again by its initial form. It is shown that the solutions of $\mathbb{L}y = 0$ will be polynomials in only *finitely many* of the variables z_i, with coefficients formal power series in x. However, the resubstitution of the variables z_i by logarithms is no longer possible, due to the characteristic $p > 0$. So one has to stick to solutions in $\mathcal{F}$. The characteristic p normal form theorem will be the content of a forthcoming article of F. Fürnsinn [Fürn]. Its formulation and proof rely on a genuine refinement of the statement and arguments given in the present paper.

Accepting the mentioned technicalities, one may now, starting with a differential equation $Ly = 0$ over $\mathbb{Q}$, look for the genuine power series solutions of its reduction $L_p y = 0$ modulo p as in the assumption in the Grothendieck-Katz conjecture. The algorithm of the normal form theorem in positive characteristic to construct solutions applies to this situation. The problem which arises lies in the observation that this algorithm does not entirely coincide with the reduction modulo p of the algorithm in zero characteristic. Very subtle disparities appear, and this makes it hard to deduce properties of the characteristic zero solutions from the characteristic p solutions, in particular, to prove their algebraicity. One hope is, however, to be able to compare the Grothendieck-Katz conjecture with the apparently weaker Bézivin conjecture.

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Faculty of Mathematics
University of Vienna, Austria
e-mail: herwig.hauser@univie.ac.at