EXCELLENT SURFACES AND THEIR TAUT RESOLUTION

Herwig Hauser

1. Introduction.

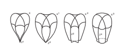
Purpose of the present paper is to reveal part of the beauty and delicacy of resolution of singularities in the case of excellent two-dimensional schemes embedded in three-space and defined over an algebraically closed field of arbitrary characteristic. The proof of strong embedded resolution we describe here combines arguments and techniques of O. Zariski, H. Hironaka, S. Abhyankar and the author.

Theorem 1. *Let W be an excellent regular three-dimensional ambient scheme over an algebraically closed field K of arbitrary characteristic. Consider a reduced hypersurface X in W. There exists a sequence of blowups*

$$W^n \to \ldots \to W^0 = W$$

of closed centers Z^i inside the singular locus of the $(i-1)$-st strict transform X^{i-1} of X such that the last strict transform X^n is smooth and has normal crossings with the exceptional divisor E^n.

Figure A: Blowups



The centers Z^i will be chosen inside the equimultiple locus of X^i. This is the subscheme of points where X has maximal multiplicity. We allow singular and non reduced centers as long as the intermediate ambient spaces W^i remain smooth. By scheme we understand a scheme of finite type over K, locally noetherian and quasi-compact. The surface X is a closed reduced subscheme of W of codimension one.

The proof of the theorem splits into two parts: First, at each stage of the process the next center to be blown up has to be chosen suitably. Then it has to be shown that when passing to the strict transform X' of X the situation improves. This is done by exhibiting certain local invariants of singularities which have dropped.

For different proofs we refer to the articles of Abhyankar and Lipman [Ab 2, Lp], as well as to the contributions of Cossart and Lê in this volume [Co 3, Le].

Definition of center. The equimultiple locus S of a surface consists of a certain number of irreducible curves and isolated points. This simplifies the definition of the centers substantially compared to higher dimensions: as long as there are singular curves in S, blow up their singular points as to make them smooth. Using resolution of curves and determining the equimultiple locus S' of the blown up scheme X' it is seen that this process yields in a finite number of steps a scheme whose equimultiple locus consists of isolated points and smooth curves having at most normal crossings. Thus it suffices to consider this situation. Here we blow up the whole equimultiple locus equipped with a suitable non reduced structure at the intersection points of its components. This choice of the center and the resulting blowup will be called *taut*. It would also be sufficient, though less canonical w.r.t. extensions to higher dimensions, to blow up S with the reduced structure, since the only singularities appearing on the ambient scheme W' are ordinary double points which lie outside the strict transform X' of X and which can be resolved easily by one point blowup.

It turns out that under the chosen blowup the highest multiplicity occurring on X either drops or remains constant when passing to the strict transform X' of X. At any point a' of X' with the same multiplicity, it will be shown that the equimultiple locus S' has again at most normal crossings, thus allowing to repeat the process. Note that S' need not equal the strict transform of S since new components can appear. These components are smooth.

Improvement of invariants: The second part of the proof consists in showing that the preceding algorithm terminates, i.e. that after a finite number of steps the obtained scheme is smooth. This is proven by induction. At each closed point a of X one defines an invariant i_a. It belongs to a well ordered subset Γ of $\mathbb{Q}^4$. Its first component is the multiplicity, i.e. the order of the power series f defining locally X at a in W. The other three components are orders of certain coefficient ideals associated to f which can be expressed through the Newton Polyhedron of f. It will be shown that if X is singular, i_a drops when passing from a point a of S to a point a' over it under the blowup as defined above. An additional argument shows that this local improvement implies a drop of the global multiplicity of X after finitely many blowups. By induction, a smooth surface is achieved.

Structure of the paper. After recalling basic properties of blowups we define in section 3 the centers of blowup selected at each stage of the resolution process. In difference to the classical treatment we allow singular centers of mild type, namely normal crossing curves with embedded components at the intersection points. These components are chosen so that the blown up ambient scheme remains regular. The choice of such centers is natural because it preserves possibly existing local or global symmetries of the scheme which may permute the two components of the normal crossing, and moreover reduces the number of required induction invariants in comparison to the treatment of Bierstone-Milman and Encinas-Villamayor [B-M, E-V 1, E-V 2]. Taking into account the history of the resolution process by distinguishing old and young exceptional divisors becomes superfluous.

As a variation, we indicate what happens when changing the structure of the embedded components at the intersection points of the center. For certain choices, the blown up ambient scheme is again smooth, but the induction invariants do not necessarily improve.

Section 4 follows Zariski's exposition [Za] in showing how to reduce the equimultiple locus to a normal crossings situation, and proving that this situation persists under taut blowup (Theorem 2). This is a prerequisite to make the induction work.

In section 5 we introduce flags. By this we understand full flags of local regular schemes centered at each point of the equimultiple locus. In three dimensions, a flag at a point a consists of a smooth curve inside a smooth surface, both passing through a. Flags are very useful to put some ordering on the coordinates and to reduce the number of allowed coordinate choices. Local coordinates subordinate to the chosen flag are needed to define the induction invariants from the Newton Polyhedron of the locally defining equation by imposing in $\mathbb{N}^3$ a hierarchy among the vertices of the polyhedron. This idea appears implicitly already in resolution of plane curves when using the Weierstrass form of the defining equation, and presents the latent basis of Hironaka's argument in [Hi 1] for surfaces. A key property of flags is their compatibility with blowup (in contrast to e.g. a collection of coordinate hyperplanes appearing as components of the exceptional divisor). A purely geometric argument allows to construct canonically from any flag at a and transversal to the center an induced flag at *any* point a' over a of the exceptional divisor (Theorem 3). Therefore subordinate coordinates are preserved under blowup and the induction invariants can be defined again in W' and compared with those in W. On the way, we will have to show that transversality of the flag w.r.t. the equimultiple locus is preserved under blowup (Theorem 4). For this, and for the control of the invariants under blowup, it is shown that for any choice of a' above a, one can perform a subordinate local coordinate choice at a which makes the local blowup *monomial* in the resulting coordinates. In this sense, flags are sufficiently restrictive to prohibit permutations of the coordinates – these present one of the main difficulties of the topic –, and sufficiently flexible to render blowups combinatorial without quitting the local setting when passing from a to a'.

Section 6 is devoted to the construction of the induction invariants. We follow the suggestion of Hironaka [Hi 1], being aware that this choice of invariants is very specific to dimension three and has no evident extension to higher dimensions. The construction is done by introducing first a vector of numbers which belongs to a certain ordered set and which depends on the choice of the coordinates subordinate to the chosen flag. To make them to genuine invariants, i.e., independent of the subordinate coordinate choice (though dependent on the flag), it is natural and appropriate to define the invariant as the maximal value of the vector over all subordinate coordinate choices. This has been done by many authors in different contexts, e.g. [Ab 1, Hi 1, Mo], and reflects the observation that the finest information on the singularity can be extracted in most specific coordinates. And these turn out – according to the setting – to be maximizing coordinates. It has to be shown that the maximum exists, at least within the set of formal coordinate choices. This is done either directly or using the general argument of [Ha 2] based on the Artin Approximation Theorem. Obviously the resulting invariant does not depend on the coordinates. In characteristic zero, maximality is usually achieved by the existence of a hypersurface of maximal contact, and it is known that this hypersurface accompanies the resolution process along any sequence of points where the multiplicity remains constant. Moreover it allows to prove semicontinuity properties for the invariant. In arbitrary characteristic, this reasoning breaks down, and one has to show that maximality of the vector persists under blowup. Actually it suffices to realize the maximum on the blown up scheme, see section 8 for more details. It seems that semi-continuity properties can be dispensed with in the special case of surfaces.

The induction invariants proposed in [B-M, E-V 1, E-V 2] in characteristic zero are inspired by Hironaka's paper on idealistic exponents [Hi 3] and are more conceptional then the ones described here. They cannot be used directly in positive characteristic, even for surfaces, due to the failure of maximal contact. There is some perspective to adapt them (simplifying them at the same time by discarding their memorative aspect on the history) to the present setting and to make them work for surfaces of arbitrary characteristic. There arises the need of complicated though probably straight forward combinatorial identities, for which we have computer evidence but no explicit proof. Due to lack of time this discussion could not be included in this article.

Section 7 establishes the induction step in the combinatorial situation. It is proven that for taut monomial blowup (and fixed coordinates) the vector of invariants drops in the lexicographic order (Theorem 5). This is done by explicit case by case calculations. Experimentation shows that there is not much freedom in changing the invariant and still having it drop.

The following section shows how to reduce an arbitrary local blowup to the combinatorial situation (Theorem 6). For any point a' of the exceptional divisor and sitting over a local subordinate coordinates are chosen at a moving a' to the origin of one chart and making the blowup monomial. Moreover this can be done so that the vector realizes in the induced coordinates at a' the maximal value, i.e., equals the actual invariant.

Section 9 combines Theorems 5 and 6 for proving the existence of resolution for surfaces (Theorem 1). Since the invariant is not obviously semi-continuous, the argument has to make a small detour to show that after finitely many blowups the resulting surface has smaller global multiplicity.

Many of the concepts presented in this article have analogues in higher dimensions, viz flags, coefficient ideals, maximality of invariants, see [Ha 1] and [E-V 2]. They lend themselves for approaching resolution in arbitrary dimension. Our exposition is occasionally more explicit than necessary in order to stress this aspect. Others like the invariants themselves or the reduction of the equimultiple locus to normal crossings have fatal drawbacks already for threefolds. Observe that in the algorithms of [B-M, E-V 1, E-V 2] the stratification used is much finer than the one given by the multiplicity and that the smallest stratum defining the center is automatically regular.

Problem 1. Extend the present proof to fields which are not necessarily algebraically closed or to schemes defined over Z. The assumption algebraically closed is only used in the lemma of section 4 and in the proofs of Theorems 1, 5 and 6 where we neglect residually algebraic irrational points in the exceptional divisor. You may consult [Bn], [Co 1], Theorem 1, p. 218, of [Hi 2], Theorem 8 of [Ha 1] and the introduction of [E-V 2].

Problem 2. Extend the present proof to the case of surfaces embedded in a regular scheme W of arbitrary dimension (non-hypersurface case).

Problem 3. Given a reduced hypersurface X in a regular four dimensional ambient scheme W, assume that its equimultiple locus S consists of isolated points, smooth curves and possibly smooth surfaces, all of them meeting with normal crossings. Define a non reduced structure Z on S and local invariants of X such that blowing up Z in W gives a regular scheme W' and a strict transform X' all of whose invariants have dropped. Observe that the normal crossing structure of the equimultiple locus may get lost under blowup, see [Ha 2, ex. 10].

In a first reading, it might to be desirable to proceed as follows: Start with the definition of the center of blowup given in section 3 omitting the propositions given there. Taking into account Theorems 2 and 3 pass directly to the construction of the invariants in section 6, followed by the study of their behaviour in section 7 and 8. Conclude by section 9 proving Theorem 1.

The author has profited from discussions with many people, among them O. Villamayor, V. Cossart, S. Encinas, A. Quirós and M. Spivakovsky. It should be understood that many of the ideas and concepts presented here have their source in the existing literature, especially in the papers [Za, Hi 1, Hi 2, Ab 1, B-M, CGO, E-V 1, E-V 2, Mo, Sp 2]. The work on this article has been supported in part by the scientific exchange program "Acciones Integradas".

2. Preliminairies

We collect several basic properties of blowups and multiplicities. For proofs and more details, see [Bn, Gi, Hi 1, Hi 2, Ha 1, Ha 2]. In this section X and W may have arbitrary dimension, where X is reduced and closed in W regular. Let $\mathcal{I}$ be the defining ideal sheaf of X in W with stalks $\mathcal{I}_a$ and local rings $\mathcal{O}_{X,a} = \mathcal{O}_{W,a}/\mathcal{I}_a$. For a a closed point of X, let m_a denote the maximal ideal of $\mathcal{O}_{W,a}$ with residue field $\mathcal{O}_{W,a}/m_a = K$. Let

$$o_a = \max\{k \in \mathbb{N},\ \mathcal{I}_a \subseteq m_a^k\}$$

denote the order of X at a. It is invariant under completion of the local rings, upper-semicontinuous w.r.t. deformation and localization and takes only finitely many values (since X is noetherian). For a proof of this in characteristic zero, see [Hi 1], p. 106. In arbitrary characteristic, we refer to [Hi 2], Thm. 1, chap. III 3, p. 218, [Bn] and [E-V 2]. In particular, the maximum $o_X = \max_{a\in X} o_a$ exists and the equimultiple locus $S = \{a \in X,\ o_a = o_X\}$ is a closed reduced subscheme of X, strictly contained and non-empty in X if X is singular (by excellence). It does not depend on the embedding of X in W. For non-hypersurfaces, the stratification given by the order can be (but not necessarily) refined by the Hilbert-Samuel stratification [Bn].

For Z a closed subscheme of X, let $\pi : W' \to W$ denote the blowup of W in Z, and denote by X^*, respectively X^{st}, the total and the strict transform of X in W' under π. Then X^{st} equals the blowup X' of X with center Z. Objects associated to X' in analogy with X will be marked by a prime. Thus a' will denote a point in X', S' the equimultiple locus of X', etc. Let $E = \pi^{-1}Z$ be the (reduced) exceptional divisor in W' and let E_a denote the fibre $\pi^{-1}(a)$ of a point a of X. When a and a' are fixed, we let $R = \mathcal{O}_{W,a}$ and $R' = \mathcal{O}_{W',a'}$ denote the local rings with completions $\overline{R}$ and $\overline{R'}$ w.r.t. the maximal ideals $M = m_a$ and $M' = m_{a'}$. Let P in R be the stalk at a of the ideal sheaf defining Z in W. For $a \in X$ and $a' \in E_a$ we call the induced map $R \to R'$ the local blowup of R with center P [Ha 2]. For $a \in Z$, $a' \in E_a$ and $Z \subseteq S$ one has $o_{a'} \leq o_a$, see e.g. the appendix to [Hi 1]. This implies that S' is contained in the total transform S^* of S if $o_{X'} = o_X$.

Assume that Z is smooth and let $a \in Z$. For any closed point a' over a there exist a regular system of parameters $x_1, \ldots, x_n$ of R, a subset J of $\{1, \ldots, n\}$ and an element $j \in J$ such that

(a) x_i, $i \in J$, generate P.
(b) W' is covered locally along E by the affine charts Spec $R[\frac{1}{x_i}]$ with $i \in J$.
(c) $y_1, \ldots, y_n$ defined by $y_i = x_i/x_j$ for $i \in J \setminus j$ and $y_i = x_i$ for $i \notin J \setminus j$ form a regular system of parameters of R'.
(d) E is defined in W' locally at a' by $y_j = 0$.

See [Hi 2, chap. III], [Ha 2] or [Bn] for more details and a description of the situation when K is not algebraically closed. We then say that a' is the origin of the x_j-chart of the blowup w.r.t. the coordinates $x_1, \ldots, x_n$ and that $R \to R'$ is monomial w.r.t. $x_1, \ldots, x_n$. Note that a regular system of parameters of R is also one for its completion. We often write x for short and speak of local coordinates of W at a. As the affine charts of W and W' at a and a' are isomorphic we shall write again x for the coordinates y at a' defined above.

Passing to completion is compatible with local blowup, i.e. the diagram below commutes.

$$\begin{array}{ccc} R' & \to & \overline{R'} \\ \uparrow & & \uparrow \\ R & \to & \overline{R} \end{array}$$

Let $Z_1,\ Z_2$ be two disjoint centers in W and denote by W'_{12} and W'_{21} the schemes obtained from W by blowing up first Z_1 and then the strict (= total) transform of Z_2, respectively inversely. Let W' be the scheme obtained by blowing up $Z_1 \cup Z_2$. Then W', W'_{12} and W'_{21} are canonically isomorphic.

Exercise 1. Let $a \in Z$ be a point and let X be a hypersurface in W defined locally at a by $f \in R$. Let $\mathrm{hom}_a f$ be the homogeneous polynomial of lowest degree appearing in the Taylor expansion of f at a. Then the points of E_a where the multiplicity has remained constant are the intersection of E_a with the strict transform of the zero set of $\mathrm{hom}_a f$.

Exercise 2. (Resolution of plane curves) Show that blowing up the singular points of a plane curve resolves the curve in a finite number of steps. To prove this, show that the pair

$i_a = (o_a, s_a)$ drops in the lexicographic order when passing from singular points $a \in X$ to points $a' \in E_a$ of the strict transform X' of the curve. Here, s_a denotes the maximum over all coordinate choices of the slope of the first segment (from the left) of the Newton Polygon of f (see section 6 for how to prove that s_a is well-defined). You may use section 7 and 8 to find a proof with slightly different induction invariants.

3. Definition of the centers of blowup

From now on X denotes a singular reduced surface in a regular three-dimensional ambient scheme W. We shall determine a convenient center to be blown up such that the invariant defined later decreases when passing to the strict transform of X. The equimultiple locus S of X consists of finitely many points and of finitely many irreducible curves. We say that S has at most normal crossings at $a \in S$, if a is either an isolated point of S, or a smooth point of a curve of S, or a normal crossing point of two or three components of S. The last condition means that S looks locally at a like two or three coordinate axes in three-space.

Exercise 3. The set T of points where S does not have at most normal crossings is finite. In a normal crossing point there cannot pass three components of S (cf. with the proof of Proposition 1, section 4).

Define the center Z in X as follows.

(a) If T is not empty, let $Z = T$.

(b) If T is empty, let Z be the closed subscheme of X supported by S with embedded components at intersection points of S given in suitable local coordinates by the ideal $P = (x, yz)(x, y)(x, z) = (x, y)^2 \cap (x, z)^2 \cap (x, y, z)^3$ with generators $x^3, x^2y, x^2z, xyz, y^2z^2$. Clearly Z is reduced if there are no intersection points.

If the center is chosen in this way we call $\pi : W' \to W$ the taut blowup of X in Z ($\tau\grave{o}$ $\alpha\grave{u}\tau\acute{o}\nu$, the same). We do not require in (b) that $\mathrm{ord}_P X = \mathrm{ord}_a X$ for all $a \in Z$. As a variant we shall also discuss the case where the embedded components equal the cube of the maximal ideal of the intersection points. Both centers yield smooth ambient blown up schemes, but the first choice makes the chosen invariant drop, whereas the second does not.

Exercise 4. Blowing up the reduced ideal (x, yz) in W gives a threefold W' with precisely one singular point which is an ordinary double point defined locally in four-space by the equation $xw - yz = 0$.

In the algorithms of [B-M] and [E-V 1, E-V 2], when applied to surfaces, the centers are almost always points, and only at the very end of the process when the situation has become almost combinatorial, also smooth curves are blown up. In occurrence of normal crossing curves the ambiguity which component to choose is solved by blowing up the intersection point, thus creating two old components, the strict transforms of the original components, and a new component, the new exceptional divisor. There then appear in W'

two normal crossings, but the symmetry can now be untied by the "age" of the components, and the two old curves are blown up. This, of course, requires additional invariants as bookholders of the history. Compare this with Propositions 2' and 3' below, where the chosen non-reduced center encapsulates in one blowup this composition of blowups.

Zariski shows that if one chooses for the center instead of normal crossings one of the smooth components and then the strict transform of the other component, the resulting strict transform of X does not depend on the choice *provided that the multiplicity remains constant* [Za] . This fails in higher dimension [Ha 2, Sp 2].

There is another possibility to get rid of normal crossings by localizing X along one component and applying resolution of curves to make the localization smooth. This allows to reduce the equimultple locus to a finite set of isolated points, but is not canonical since there is no natural indication which component to choose (imagine that there is a local but no global symmetry of X permuting the two components).

Proposition 1. *Let $S = S_y \cup S_z$ be a normal crossing at a with two smooth components S_y and S_z. Choose local coordinates x, y, z at a such that S_y and S_z are defined by $x = z = 0$ and $x = y = 0$ respectively. Let Z be the center with ideal $P = (x, yz)(x, y)(x, z)$. The blowup W' of W in Z is smooth with five affine charts. For a' a closed point over a the local blowup $R \to R'$ is given as follows.*

chart x^3: $R' = K[x, \frac{y}{x}, \frac{z}{x}]$, $(x, y, z) \to (x, xy, xz)$ $E = (x)$ $E_a = (x)$

chart x^2y: $R' = K[\frac{x}{y}, y, \frac{z}{x}]$, $(x, y, z) \to (xy, y, xyz)$ $E = (xy)$ $E_a = (y)$

chart x^2z: $R' = K[\frac{x}{z}, \frac{y}{x}, z]$, $(x, y, z) \to (xz, xyz, z)$ $E = (xz)$ $E_a = (z)$

chart xyz: $R' = K[\frac{yz}{x}, \frac{x}{z}, \frac{x}{y}]$, $(x, y, z) \to (xyz, xy, xz)$ $E = (xyz)$ $E_a = (xy, xz)$

chart y^2z^2: $R' = K[\frac{x}{yz}, y, z]$, $(x, y, z) \to (xyz, y, z)$ $E = (yz)$ $E_a = (y, z)$

Proof. The assertions are checked by computation, using the fact that W' is covered locally along E by the charts Spec $R[\frac{1}{a_i}]$ with a_i generators of P. ○

Exercise 5. Show that under taut blowup the order of the strict transform X' of X at a point a' over an intersection point a of S does not increase.

Proposition 2. *The taut blowup of W over an intersection point a of $S = S_y \cup S_z$ is the composition of the blowup of W in the reduced ideal (x, yz) of S followed by the blowup of the unique singular point s obtained by this blowup. The exceptional divisor E has three components E_y, E_z and E_s, where E_y and E_z are the closures of $\pi^{-1}(S_y \setminus a)$ and $\pi^{-1}(S_z \setminus a)$ in W' and where E_s is the fiber over s under the second blowup. The fiber E_a over $a = S_y \cap S_z$ consists of a one-dimensional component $E_t = E_y \cap E_z \cong \mathbb{P}^1$ and a two-dimensional component $E_s \cong \mathbb{P}^2$ intersecting in the origin c of the xyz-chart.*

Proof. The blowup $\pi_1 : W^1 \to W$ of S with reduced ideal (x, yz) in W has two charts:

chart yz: $R^1 = K[x, y, z, \frac{x}{yz}] = K[\frac{x}{yz}, y, z]$ smooth, $\qquad (x, y, z) \to (xyz, y, z)$.

chart x: $R^1 = K[x, y, z, \frac{yz}{x}] \cong K[x, y, z, w]/xw - yz$ singular.

To resolve the singularity of W^1, we embed W^1 into a four dimensional ambient space V with local coordinates x, y, z, w and blow up the singular point s, getting $\pi_2 : W^2 \to W^1$. Let $I^1 = (xw - yz)$ be the defining ideal of W^1 in V and I^2 its strict transform in R^2. In addition to the smooth yz-chart we get four charts with smooth strict transforms $I^2 = (w - yz)$, $(xw - z)$, $(xw - y)$ and $(x - yz)$ and substitutions as follows.

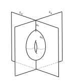
chart x: $R^2 = K[x, \frac{y}{x}, \frac{z}{x}, \frac{w}{x}]/\frac{w}{x} - \frac{y}{x}\frac{z}{x} = K[x, \frac{y}{x}, \frac{z}{x}] \qquad (x, y, z) \to (x, xy, xz)$

chart y: $R^2 = K[\frac{x}{y}, y, \frac{z}{y}, \frac{w}{y}]/\frac{x}{y}\frac{w}{y} - \frac{z}{y} = K[\frac{x}{y}, y, \frac{z}{x}] \qquad (x, y, z) \to (xy, y, xyz)$

chart z: $R^2 = K[\frac{x}{z}, \frac{y}{z}, z, \frac{w}{z}]/\frac{x}{z}\frac{w}{z} - \frac{y}{z} = K[\frac{x}{z}, \frac{y}{x}, z] \qquad (x, y, z) \to (xz, xyz, z)$

chart w: $R^2 = K[\frac{x}{w}, \frac{y}{w}, \frac{z}{w}, w]/\frac{x}{w} - \frac{y}{w}\frac{z}{w} = K[\frac{x}{z}, \frac{x}{y}, \frac{yz}{x}] \ (x, y, z) \to (xyz, xy, xz)$.

Figure B: Normal crossing blowup



Exercise 6. Show that the resulting 5 charts glue in a way which gives the blowup W' of W in the ideal $P = (x, yz)(x, y)(x, z)$.

Blowing up (x, yz) in W equals outside the origin the blowing up of a smooth curve. Hence the inverse images $\pi_1^{-1}(S_y \setminus a)$ and $\pi_1^{-1}(S_z \setminus a)$ are isomorphic to $(S_y \setminus a) \times \mathbb{P}^1$ and $(S_z \setminus a) \times \mathbb{P}^1$. Let $E_y^1 \cup E_z^1$ be the closure in W^1 of $\pi_1^{-1}(S \setminus a)$. The singular point s of W^1 lies in the intersection $E_y^1 \cap E_z^1$ and is blown up via π_2. The fiber $E_s = \pi_2^{-1}(s)$ is isomorphic to $\mathbb{P}^2$. It follows that the exceptional divisor E of the blowup π has three components E_y, E_z and E_s, where the first two are the strict transforms under π_2 of E_y^1 and E_z^1. The fiber $E_a = (\pi_2\pi_1)^{-1}(a)$ has two components, namely $E_y \cap E_z$ and E_s. Four of the charts cover E_s, whereas the yz-chart covers $(E_y \cap E_z) \setminus c$. It is checked that $E_y \cap E_z$ and E_s intersect in the origin of the xyz-chart. We shall see in section 4 that if S is the equimultiple locus of X the strict transform X' meets E_a only in the yz-chart obtained already by the blowup π_1. $\circlearrowleft$

Variation. There is an alternate non-reduced structure on normal crossings center which yields under blowup a regular ambient scheme. It is less appropriate for resolution purposes than the taut structure since the invariants we use may increase and additional invariants are necessary to show that the situation improves, cf. with [B-M] and [E-V 1]. We include a description thereof because this blowup is the composition of the blowup of the intersection point of the normal crossing followed by the blowup of the strict transforms of the two components. This is similar to the procedure used in loc. cit.

Proposition 1'. *Let $Z = Z_y \cup Z_z$ be a normal crossing in W with embedded component the cube of the maximal ideals at the intersection point a of the two smooth curves. The blowup W' of W in Z is smooth with charts and substitutions as follows:*

chart x^3: $R' = K[x, \frac{y}{x}, \frac{z}{x}],\ (x, y, z) \to (x, xy, xz),\ E = (x), E_a = (x)$.

chart xy^2: $R' = K[\frac{x}{y}, y, \frac{z}{x}],\ (x, y, z) \to (xy, y, xyz),\ E = (xy), E_a = (y)$.

chart y^2z: $R' = K[\frac{x}{z}, y, \frac{z}{y}],\ (x, y, z) \to (xyz, y, yz),\ E = (yz), E_a = (y)$.

The charts xz^2 and yz^2 are symmetric to the charts xy^2 and y^2z respectively. The remaining charts are open subsets of the preceding ones.

chart x^2y: $R' = K[\frac{x}{y}, (\frac{y}{x})^{-1}, y, \frac{z}{x}]$ *contained in chart* xy^2.

chart x^2z: *symmetric to the chart* x^2y.

chart xyz: $R' = K[\frac{x}{z}, (\frac{x}{z})^{-1}, \frac{y}{x}, (\frac{y}{y})^{-1}, z]$ *contained in chart* xz^2.

$Proof$. This is verified by computation. ⟲

Proposition 2'. *The blowup $\pi : W' \to W$ defined in the preceding proposition is the composition of the point blowup of the intersection point a followed by the blowup of the strict transforms of the two curves S_y and S_z. The exceptional divisor E of π has three components E_a, E_y and E_z, where $E_a \cong \mathbb{P}^2$ and where $E_y \cong S_y \times \mathbb{P}^1$, $E_z \cong S_z \times \mathbb{P}^1$ are the closures in W' of $\pi^{-1}(S_y \setminus a)$ and $\pi^{-1}(S_z \setminus a)$ respectively.*

$Proof$. The blow up of $P = (x, y, z)$ has three charts. The strict transforms of the two curves S_y and S_z lie in two of them.

chart x: $R^1 = K[x, \frac{y}{x}, \frac{z}{x}]$ with substitution $(x, y, z) \to (x, xy, xz)$. The strict transforms of S_y and S_z do not meet this chart.
chart y: $R^1 = K[\frac{x}{y}, y, \frac{z}{y}]$ with substitution $(x, y, z) \to (xy, y, yz)$. Only the strict transforms of S_y lies in this chart.
chart z: symmetric to preceding chart.

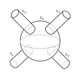
We next blow up in the y-chart the strict transform S_y^1 of S_y and get two charts. Denote by $x' = \frac{x}{y}$, $y' = y$, and $z' = \frac{z}{y}$ the induced coordinates in this chart so that S_y^1 is defined by $x' = z' = 0$.

chart x': $R^2 = K[x', y', z', \frac{z'}{x'}] = K[x', y', \frac{z'}{x'}] = K[\frac{x}{y}, y, \frac{z}{x}]$ with substitution (x', y', z') $\to (x', y', x'z')$. The map $R \to R^2$ is given by $(x, y, z) \to (xy, y, xyz)$.

chart z': $R^2 = K[x', y', z', \frac{x'}{z'}] = K[\frac{x'}{z'}, y', z'] = K[\frac{x}{z}, y, \frac{z}{y}]$ with substitution (x', y', z') $\to (x'z', y', z')$. The map $R \to R^2$ is given by $(x, y, z) \to (xyz, y, yz)$.

As a result we have five charts, namely x, (y,x'), (z,x'), (y,z'), (z,y'), with substitutions $(x,y,z) \to (x,xy,xz)$, (xy,y,xyz), (xz,xyz,z), (xyz,y,yz), (xyz,yz,z). These coincide with the ones obtained by blowing up $P=(x,yz)\cap(x,y,z)^3$.

Figure C: Variation normal crossing blowup



Exercise 7. Show that the charts glue as the charts from Proposition 1' and that the exceptional divisor decomposes as asserted. $\circlearrowleft$

4. Transformation of equimultiple locus under blowup

Let $\pi : W \to W'$ be the taut blowup with center Z. In this section, S need not have normal crossings. Assume that the maximal multiplicity has remained constant, $o_{X'} = o_X$. We describe possible configurations of the equimultiple locus S' of X' in terms of the strict and total transforms S^{st} and $S^* = \pi^{-1}S$ of S and the exceptional divisor $E = \pi^{-1}Z$.

We shall need an invariant τ which is used by various authors as induction invariant. Here it only plays an auxiliary role. Let $a \in X$ and $f \in R = \mathcal{O}_{W,a}$ be a defining equation of X locally at a. Let $f = \sum_{ijk} c_{ijk}x^iy^jz^k$ be the expansion of f in the completion $\overline{R} \cong K[[x,y,z]]$ of R. The order $o = o_a$ of f at a will be the minimal value $i+j+k$ with non zero coefficient c_{ijk}. Let $\hom_a f = \sum c_{ijk}x^iy^jz^k$ be the homogeneous form of lowest degree of f, where the sum ranges over those indices for which $i+j+k=o_a$. It defines the tangent cone TC of X at a. Let τ_a be the minimal number of variables necessary to write $\hom_a f$ over all coordinate choices.

Exercise 8. Show that it suffices to consider only linear coordinate changes to realize τ_a. More explicitly, show the following: Let ε be the graded reverse monomial order on $\mathbb{N}^3$ and $\mathrm{in}_{\varepsilon x} f$ the initial monomial of f w.r.t. ε and $x=(x,y,z)$. Set $\max_\varepsilon f = \max_x \mathrm{in}_{\varepsilon x} f$. Then τ_a equals the index of the last variable appearing in $\max_\varepsilon f$, counting x,y,z in this order [Ha 1].

Lemma. *Let a be a point of S and assume that S has at most normal crossings at a. If $\tau_a = 3$ then a is an isolated point of S. If a is a smooth point of S, then $\tau_a \le 2$, and $\tau_a = 2$ implies that there are coordinates in which S is given by the ideal (x,y). If a is an intersection point of two smooth components of S with normal crossing then $\tau_a = 1$ and there exist local coordinates such that $\hom_a f = x^o$ and S is defined by the ideal (x,yz).*

$Proof$. Assume that S contains a smooth curve C through a. Choose coordinates such that C is defined by the ideal (x,y). For $b \in C$ a closed point, local coordinates at b are obtained from the coordinates x, y, z at a by translations $(x, y, z + s)$ with $s \in K$. The Taylor expansion $\sum_{ijk} c_{ijk} x^i y^j (z + s)^k$ of f at b has order $\geq o$ for all b in C if and only if the sum

$$\sum_{ijkl} c_{ijk} \binom{k}{l} x^i y^j z^{k-l} s^l$$

over all i, j, k, l with $i + j + k - l < o$ and $0 \leq l \leq k$ is identically zero for all $s \in K$. As K is algebraically closed and hence infinite, this holds if and only if for all l

$$\sum_{ijk} c_{ijk} \binom{k}{l} x^i y^j z^{k-l} = 0$$

where the sum ranges over all i, j, k with $i + j + k - l < o$ and $k \geq l$. This implies $c_{ijk} = 0$ for all i, j, k with $i + j + k = o$ and $k > 0$, hence $\tau_a \leq 2$. If there is a second component of S passing through a the coordinates can be chosen so that it is defined by the ideal (x, z). The same argument yields $c_{ijk} = 0$ for all i, j, k with $i + j + k = o$ and $j > 0$, hence $\tau_a = 1$. ⥀

Example. Take $f = x^2 + y^2 + yz(1 + z) + z^2(1 + z)^2$ at $a = 0$ over the field with two elements. Here $\tau_a = 3$ but the equimultiple locus S is the curve defined by the ideal (x, y).

Theorem 2. *Let $W' \to W$ be the taut blowup of X in Z. Assume that the multiplicity has remained constant.*
(a) *If S has singular irreducible components or smooth components not intersecting transversally, S' is the strict transform of S augmented possibly by a curve isomorphic to $\mathbb{P}^1$. If this curve meets another smooth component of S' the intersection is transversal.*
(b) *If the equimultiple locus has at most normal crossings singularities S' has again at most normal crossings singularities.*

It follows from (a) and resolution of curves in three-space that by a finite sequence of point blowups the equimultiple locus can be transformed into a normal crossings curve as above. Then its components will be blown up until the highest occurring multiplicity drops (Theorem of Beppo Levi, see section 6 and 7 and [Za, p. 522]).

The theorem will be proven through three propositions which describe the possible transformations of the equimultiple locus in each case. We suppose throughout to be in the situation of the theorem.

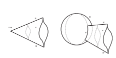
Proposition 1. *Assume that the center $Z = a$ is a point of S, S being an isolated point or an arbitrary curve. Let a' be a point over a where the multiplicity has remained constant. Then, locally at a', either $S' = S^{st}$ or $S' = S^{st} \cup (E \cap X')$ with $E \cap X' \cong \mathbb{P}^1$. In the second case, and if S has only smooth components, S^{st} and $E \cap X'$ meet transversally.*

The assertion does not hold in higher dimensions, see [Ha 2, ex. 10].

$Proof$. [Za, Thm. 1 and Lemma 3.2, p. 479]. As we blow up a, $\pi : W \to W'$ is an isomorphism outside a and hence $X \setminus a \cong X' \setminus E$ and $S \setminus a \cong S^* \setminus E$. This implies that $S^* \setminus E \subseteq S'$. As S' is closed, the strict transform S^{st} of S is formed by certain components of S'. Therefore S' is contained in the union of S^{st} and $E \cap X'$. Observe that $E \cong \mathbb{P}^2$ and that $E \cap X'$ could a priori be a reducible and singular curve. We distinguish two cases. If $\tau_a \geq 2$ only finitely many points of $X' \cap E$ can have order o, namely, by exercise 1, the intersections with E of the strict transforms of the zero set of $\mathrm{hom}_a f$ (if $\tau_a = 3$, the multiplicity drops in all points of E.) This implies that $S' = S^{st}$.

If $\tau_a = 1$ we have for suitable local coordinates x, y, z in W at a that $f \equiv x^o$ modulo M^{o+1}. At the origin of the x-chart the multiplicity drops [and also at all points, by an argument as in the proof of the next proposition]. Similarly, in the y-chart it drops at all points of E except those where $x = 0$, since $X' \cap E$ is given by $(y^{-o} f(xy, y, zy), y) = (x^o, y)$. The situation in the z-chart is symmmetric to the preceding one. Hence $S' = S^{st} \cup (X' \cap E) \cong S^{st} \cup \mathbb{P}^1$. Assume that S has only smooth components. Choose one. It is given in suitable coordinates of R by $x = z = 0$. The strict transform S^{st} only appears in the y-chart and has the same equations $x = z = 0$. Therefore it is transversal to $X' \cap E$. $\circlearrowleft$

Figure D: Equimultiple locus and point blowup

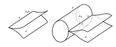


Proposition 2. *Asumme that $Z = S$ is a smooth curve. Let a' be a point over a where the multiplicity has remained constant. Then $S' \subseteq E \cap X'$. If S' has a one dimensional component, $S' = E \cap X' \cong S$ under π.*

For reduced but possibly singular centers see [Za], Thm. 2, p. 484, and its Corollary, p. 485. See also [Hi 1, p. 109].

$Proof$. The components of S different Z (and hence disjoint to Z) will transform to components of S' since π is an isomorphism outside Z and since S' is closed. Let $a \in Z$. Observe that $\tau_a \leq 2$ by the lemma. If $\tau_a = 2$ the multiplicity drops at all points of E. So we may assume $\tau_a = 1$, say $f = x^o + g(x, y, z)$ with $g \in M^{o+1}$. In the x-chart the total transform f^* of f is of form $f^* = x^o \cdot (1 + \sum c_{ijk} x^{i+j+k-o} y^j z^k) = x^o \cdot f'$ with $i+j+k > o$. It follows that f' does not vanish at the origin. The translation $x, y+s, z+t$ preserves the unit since the x-exponents in the sum remain ≥ 1. Hence f' does not vanish in the x-chart. In the y-chart $X' \cap E$ is given by the ideal $(y^{-o} f(xy, y, z), y) = (x^o + y^{-o} g(xy, y, z), y)$ where $f = x^o + g(x, y, z)$. This intersection is only o-fold if the variety defined by $x^o + y^{-o} g(xy, y, z)$ in $y = 0$ is o-fold, i.e. an o-th power $(x + h(x, z))^o$ for some h. The curve in W' given by the ideal $(x + h(x, z), y)$ is isomorphic to Z given by (x, y) in W. $\circlearrowleft$

Figure E: Equimultiple locus and curve blowup



Proposition 3. *Let S have normal crossing at a. Then $X' \cap E$ lies inside the y^2z^2-chart. If the multiplicity has remained constant every one-dimensional component of S' is isomorphic to a component of S under π.*

$Proof$. The lemma implies $\tau_a = 1$. Choose local coordinates x, y, z such that Z is defined by the ideal $P = (x, yz)(x, y)(x, z)$ locally at a. We write

$$f = x^o + \sum c_{ijk} x^i y^j z^k$$

where the sum runs over all triples i, j, k satisfying $i + j \geq o$, $i + k \geq o$, $i + j + k > o$ and $0 \leq i < o$. We consider the total transform f^* of f at a' in the various charts covering E.

chart x^3: We have $f' = 1 + \sum c_{ijk} x^{i+j+k-o} y^j z^k$ at the origin of this chart, and the sum remains in the ideal generated by x under the translations $(x, y+s, z+t)$ of this chart. Hence f' is invertible everywhere and X' does not meet this chart.

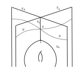
chart x^2y: We have $f' = 1 + \sum c_{ijk} x^{i+k-o} y^{i+j+k-o} z^k$ at the origin of this chart, and the sum remains in the ideal generated by y under the translations $(x, y+s, z+t)$ of this chart. Hence f' is invertible everywhere and X' does not meet this chart.

chart x^2z: symmetric to preceding chart.

chart xyz: We have $f' = 1 + \sum c_{ijk} x^{i+j+k-o} y^{j+i-o} z^{k+i-o}$ at the origin of this chart, and the sum remains in the ideal generated by x under the translations $(x, y+s, z+t)$ of this chart. The other translation $(x+s, y, z)$ requires the following argument. The sum lies in the ideal generated by y and z except possibly if $j + i = o$ and $k + i = o$, hence $j = k = o - i$. This implies $f' = 1 + \sum c_{ijk} x^{o-i}$ and X' may intersect E in this chart in a point of the x-axis off the origin. Such points lie in the y^2z^2-chart and will be treated there.

chart y^2z^2: We have $f' = x^o + \sum c_{ijk} x^i y^{j+i-o} z^{k+i-o}$ at the origin of this chart. Translations are $(x+s, y, z)$. It follows that X' intersects the x-axis in isolated points given by $x^o + \sum_{i=0}^{o-1} c_{i,o-i,o-i} x^i = 0$. The component E_y of E is given by $z = 0$, which gives $f'_{|E_y} = x^o \cdot \sum_{ij} c_{i,j,o-i} x^i y^{j+i-o}$. To have there an o-fold curve requires that $f'_{|E_y} = (x + a(y))^o$ for some series $a(y)$. Therefore this curve is isomorphic to S_y. This shows that each component of S' is isomorphic to a component of S, and two components can only meet on the x-axis of this chart. $\circlearrowleft$

Figure F: Equimultiple locus and crossing bloup

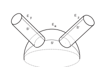


The first proposition proves (a) of the theorem. The other two show that if the multiplicity remains constant, S' will consist of smooth irreducible curves intersecting transversally and isomorphic to certain components of S, isolated points and possibly a $\mathbb{P}^1$. If C' is a one dimensional component of S' then either C' is the strict transform of some component C of S, or $C' \cong S$ and S is smooth or $C' = E \cap X' \cong \mathbb{P}^1$. This concludes the proof of the theorem. $\circlearrowleft$

Variation. We describe the transformation of the equimultiple locus when we change the embedded components of the normal crossing center Z at the intersection points.

Proposition 3'. *Let Z be a normal crossing center with embedded components the cubes of the maximal ideal at the intersection points. Assume that $o_{X'} = o_X$. If a one dimensional component of S' lies inside $E \cap X'$ then $E \cap X'$ is irreducible, equal to this component and isomorphic to a component of S under π. Moreover, S' meets E only in the y^2z- or yz^2-chart.*

Figure G: Variation equimultiple locus and crossing blowup



$Proof$. Let us place at an intersection point $a \in Z$. Choosing local coordinates x, y, z we may assume $a = 0$ and Z defined by the ideal $P = (x, yz) \cap (x, y, z)^3$ locally at a. By the lemma, $\tau_a = 1$, and we can write

$$f = x^o + \sum c_{ijk} x^i y^j z^k$$

where the sum runs over all triples i, j, k satisfying $i + j \geq o$, $i + k \geq o$ and $0 \leq i < o$. We consider the total transform f^* in the various charts.

chart x^3: The substitution is (x, xy, xz), hence $f^* = x^o + \sum c_{ijk} x^{i+j+k} y^j z^k = x^o \cdot (1 + \sum c_{ijk} x^{i+j+k-o} y^j z^k) = x^o \cdot f'$. Possible translations are $(x, y + s, z + t)$. We may restrict by symmetry to translations $y + s$. This changes only those monomials of the expansion of f' where $j \geq 1$. As $i + k \geq o$ for all i, j, k these monomials have x-exponent $i + j + k - o \geq 1$. As the x-exponent remains unchanged under the translation, the resulting f' is again a unit. Hence X' does not meet this chart.

chart xy^2: The substitution is (xy, y, xyz), hence $f^* = x^o y^o + \sum x^{i+k} y^{i+j+k} z^k = x^o y^o \cdot (1 + \sum c_{ijk} x^{i+k-o} y^{i+j+k-o} z^k) = x^o y^o \cdot f'$. Possible translations are $(x, y+s, z+t)$ or $(x+s, y, z+t)$. Translations $y+s$ affect only those monomials where $i+j+k-o \geq 1$. If $i+k-o \geq 1$ or $k \geq 1$ the resulting series f' remains a unit after the translation. So assume $i+k-o = 0$ and $k = 0$ so that these monomials are of form y^j with $j \geq 1$. This implies that in the expansion of f there appeared monomials $x^o y^j$. This case can be excluded by supposing f in Weierstrass form. Translations $z+t$ affect only those monomials where $k \geq 1$ and hence y-exponent $i+j+k-o \geq 1$, i.e. f' has no zeroes in this chart of E. Translations $x+s$ affect only those monomials where $i+k-o \geq 1$ and hence y-exponent $i+j+k-o \geq 1$, i.e. f' has no zeroes in this chart of E.

chart y^2z: The substitution is (xyz, y, yz), hence $f^* = x^o y^o z^o + \sum c_{ijk} x^i y^{i+j+k} z^{i+k} = y^o z^o \cdot (x^o + \sum c_{ijk} x^i y^{i+j+k-o} z^{i+k-o}) = y^o z^o \cdot f'$. Here E is given by $yz = 0$ and $X' \cap E$ by the ideal $(f', yz) = (x^o + \sum c_{ijk} x^i y^{i+j+k-o} z^{i+k-o}, yz) = (x^o + \sum_{ij} x^i y^j, yz)$. This is a curve in E. If it belongs to the equimultiple locus S' of X', the series $x^o + \sum_{ij} x^i y^j$ must have order o along this curve, which implies that $x^o + \sum_{ij} x^i y^j = (x + h(x,y))^o$ is an o-th power. Then $S' = S^{st} \cup (E \cap X')$ and $E \cap X'$ is a subscheme of X' locally isomorphic to S under π.

The charts xz^2 and yz^2 are symmetric to the preceding ones, the charts x^2z, x^2y and xyz are contained in the preceding charts as open subsets.

Summarizing, S' can only lie in the y^2z-chart with equation $x = y = 0$ or in the yz^2-chart with equation $x = z = 0$ or inside E_y or E_z. As we restrict to $a \in S$ the intersection point and hence $a' \in E_a$, only the first two cases will be relevant. $\circlearrowleft$

The assertion of the last proposition can also be proven by interpreting the blowup as a composition of blowups of points and smooth curves and applying Propositions 1 and 2.

5. Transformation of flags under blowup

A flag in a regular ambient scheme W at a point a is a full chain of local regular subvarieties F_i of dimension i of W at a. Flags will be needed to define the induction invariant. Let X be a surface in three-space W and let a be a closed point of the equimultiple locus S of X. We assume that S has at most normal crossings at a. The flag $\mathcal{F}$ at a consists of a smooth curve F_1 contained in a smooth surface F_2 in W, both passing through a. It is called transversal to S if one of the following cases occurs.

If a is an isolated point of S, F_1 and F_2 can be arbitrary. If a is on precisely one smooth curve of S, either F_1 and F_2 are both transversal to this curve, or F_1 is transversal to S and F_2 contains S. If a is the intersection point of two smooth curves of S meeting transversally at a, F_1 is transveral to both curves and F_2 contains one curve and is transversal to the other. We don't intend the choice of $\mathcal{F}$ to be canonical.

Local coordinates x, y, z at a are called subordinate to the flag $\mathcal{F}$ if F_1 and F_2 are defined by $y = z = 0$ and $z = 0$ respectively. In the construction of the invariants we will work in the completion $\overline{R}$ and choose coordinates there. Subordinate coordinate changes are of type $(x+a, y+b, z+c)$ where $a \in \overline{R}$ is arbitrary, b belongs to the ideal (y,z) and c to the ideal (z) of $\overline{R}$. Without loss of generality it will suffice to consider only coordinate changes where $a \in K[[y,z]]$, $b \in K[[z]]$ and $c = 0$, compare with the Gauss-Bruhat decomposition of Aut $\overline{R}$ from Theorem 2 of [Ha 1].

The flag is called partially transversal to the tangent cone TC of X if F_1 is transversal to the maximal linear space along which TC is a product [Hi 4]. This is a void condition if $\tau_a = 3$. In subordinate coordinates, it signifies that the variable x appears in at least one monomial of $\mathrm{hom}_a f$.

Theorem 3. *Assume that S has at most normal crossing at a. Let $\pi : W' \to W$ be the taut blowup of X with center Z and let a' be a point over a. Assume that either S is smooth at a or, in case where S has normal crossing at a, that a' belongs to the one dimensional component $E_y \cap E_z$ of E_a. For any flag $\mathcal{F}$ in W at a transversal to S and partially transversal to TC there exist formal subordinate local coordinates x of W at a such that π is monomial at a' w.r.t. x and Z is defined locally at a by the ideals (x,y,z), (x,y), (x,z) or $(x,yz)(x,y)(x,z)$.*

For normal crossing centers, it seems impossible to achieve monomiality at all points $a' \in E_a$. Variation: If Z has embedded components of form $(x,yz) \cap (x,y,z)^3$ at the intersection points, this is possible, see the proof below, but destroys the form of $P = (x,yz) \cap (x,y,z)^3$.

$Proof$. For blowups of smooth centers in regular schemes of arbitrary dimension this has been proven in Theorem 5(a) of [Ha 1]. We adapt the proof to dimension three, complementing it by the case of normal crossing centers.

Start with any formal subordinate coordinates x in $\overline{R}$. We first simplify by subordinate coordinate changes the ideal P defining Z. If S is a point, $P = (x,y,z)$. If S is a smooth curve, partial transversality of $\mathcal{F}$ w.r.t. TC implies that P can be written $(x + b(y,z), y + cz + d(y,z))$ or $(x + b(y,z), z + d(y))$ with series b and d of order ≥ 2 and a constant $c \in K$. The obvious subordinate coordinate change transforms P to (x,y) or $(x, z + d(y))$. Transversality of $\mathcal{F}$ with S forces in the second case $d = 0$. If S is a normal crossing, at least one component of S is defined by an ideal of form $(x + b(y,z), y + cz + d(y,z))$, hence, after subordinate coordinate change, of form (x,y). The other component is defined by an ideal of form $(x, cy + z + d(y,z))$, and transversality of $\mathcal{F}$ with S implies $c = d = 0$. The asserted form of P follows in all cases.

(1) *Point blowup.* Decompose $E_a \cong \mathbb{P}^2$ into the origin of the x-chart, the x-axis of the y-chart and the affine xy-plane of the z-chart. If a' is the origin of the x-chart, the local blowup is already monomial at a'. If a' lies in the x-axis of the y-chart, apply a translation $v = (x - s, y, z)$ in E_a to move a' to the origin of the y-chart. This translation is induced by the local subordinate coordinate change $u = (x - sy, y, z)$ of W at a. Clearly, u preserves the ideal P, and π has become monomial at a'. For a' in the z-chart, apply a translation $v = (x - s, y - t, z)$ in E_a to move a' to the origin of the z-chart. This

translation is induced by the local subordinate coordinate change $u = (x - sz, y - tz, z)$ of W at a. Clearly, u preserves the ideal P, and π has become monomial at a'.

(2) *Curve blowup.* If S is a smooth curve defined by (x, y) or (x, z), a similar argument applies. Here, $E_a \cong \mathbb{P}^1$ decomposes into the origin of the x-chart and the x-axis in the y-, respectively z-chart. We leave the details as exercise.

(3) *Crossing blowup.* Let S be defined by the ideal (x, yz). We have seen earlier that E_a has two components $E_y \cap E_z \cong \mathbb{P}^1$ and $E_b \cong \mathbb{P}^2$ intersecting at the origin of the xyz-chart. Let $a' \in E_y \cap E_z$. If it equals this origin, π is monomial at a'. Else a' lies in the y^2z^2 chart. As $E_y \cap E_z$ is the x-axis in this chart, there is a translation in W' of type $(x - s, y, z)$ which moves a' to the origin. This translation is induced from the local subordinate coordinate change $(x - syz, y, z)$ in W at a. It preserves P. This proves the assertion. $\circlearrowleft$

Variation: Let $P = (x, yz) \cap (x, y, z)^3$. Here, $E_a \cong \mathbb{P}^2$ will be decomposed into the affine plane $z = 0$ in the yz^2-chart (in which E_a is given by $z = 0$), the x-axis in the xz^2-chart (which is the projective line at infinity of $\mathbb{P}^2$ of the preceding chart) and the origin of the x^3-chart (which is the point at infinity of the preceding curve). In the first two charts translations $(x - s, y - t, z)$ and $(x + s, y, z)$ in E_a move a' to the origin of the chart. They correspond, by the substitution formulas for π, to local subordinate coordinate changes $(x - sy, y - sz, z)$ and $(x - sz, y, z)$ in W at a. The local blowup has become monomial. Note that the coordinate changes in W at a affect the equations of Z, yielding the ideals $(x - sy, yz - sz^2) \cap (x, y, z)^3$, respectively $(x - sz, yz) \cap (x, y, z)^3$.

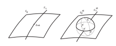
Theorem 4. *Assume that S has at most normal crossings and let $\pi : W' \to W$ be the taut blowup of X with center Z. Let $\mathcal{F}$ be a flag at a transversal to S and partially transversal to the tangent cone TC of X. Let a' be a closed point over a. If a' is an intersection point of S we suppose that a' lies in the one dimensional component $E_y \cap E_z$ of E_a, or equals one of the origins of the remaining charts. There exists a canonically defined induced flag $\mathcal{F}'$ at a', and subordinate coordinates at a induce canonically subordinates coordinates at a'. For $a' \in S'$ with $o_{a'} = o_a$, the flag $\mathcal{F}'$ is transversal to the equimultiple locus S' and partially transversal to the tangent cone TC' of X' in a'.*

Canonical means that the flag $\mathcal{F}'$ is invariant under local isomorphisms of W' induced by automorphisms of W at a preserving $\mathcal{F}$. The assertion of the theorem holds for blowups of smooth centers in any dimension [Ha 1].

Proof. (1) *Point blowup.* Define $\mathcal{F}'$ at a' as follows. Let F_1^{st} and F_2^{st} be the strict transforms of F_1 and F_2 under π. They intersect the exceptional divisor E transversally in a point $F_1^{st} \cap E$ and a smooth curve $F_2^{st} \cap E$. If $a' = F_1^{st} \cap E$ is the intersection point (which is the origin of the x-chart), let $F_1' = F_1^{st}$ and $F_2' = F_2^{st}$. If a' lies on $F_2^{st} \cap E$ but not on F_1^{st} (i.e., in the y-chart), let $F_1' = F_2^{st} \cap E$ and $F_2' = F_2^{st}$. If $a' \notin F_2^{st}$ (i.e., in the z-chart), set $F_2' = F_2^{st}$ and let F_1' be the projective line in $E = \mathbb{P}^2$ connecting a' with $F_1^{st} \cap E$.

Exercise 9. In each case, letting x, y, z denote the coordinates at a' induced by the usual formulas, show that F_1' and F_2' will be defined by $y = z = 0$ and $z = 0$.

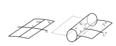
Figure H: Flag and point blowup



By Proposition 1 of section 4, we know that S' equals the strict transform of S, possibly augmented by $E \cap X' \cong \mathbb{P}^1$. In the latter case, by the same proposition, S' does not meet the x-chart, and has equation $x = z = 0$ in the y-chart, respectively $x = y = 0$ in the z-chart. Hence F_1' and S' are always transversal, and F_2' and S' are either transversal or $S' \subseteq F_2'$.

(2) *Curve blowup.* (a) If F_1 and F_2 are both transversal to S, choose coordinates x, y, z at a such that S is defined by $x = y = 0$. Define $\mathcal{F}'$ as follows.

Figure I: Flag and curve blowup



If $a' = F_1^{st} \cap E_a$ (which is the origin of the x-chart) let $F_1' = F_1^{st}$ and $F_2' = F_2^{st}$. If $a' \in E_a \setminus F_1^{st} \cap E_a$ (i.e., a' lies in the y-chart), set $F_1' = E_a$ and $F_2' = F_2^{st}$. Observe that E_a is a curve. In both cases F_1' and F_2' are defined by $y = z = 0$ and $z = 0$ respectively.

Consider now S'. If S' is a point, nothing is to show. If S' is a curve it must lie inside $E \cap X'$ and by Proposition 2 of section 4, $S' = E \cap X'$ and $S \cong S'$. Moreover S' lies in the y-chart and is given there by $x + h(z) = y = 0$ for some h. As the flag $\mathcal{F}'$ is given by $y = z = 0$ and $z = 0$ it is transversal to $F_1' = E_a$ and F_2'.

(b) If F_1 is transversal to S and $S \subseteq F_2$ choose coordinates such that S is defined by $x = z = 0$. If $a' \in F_1^{st} \cap E_a$ (which is the origin of the x-chart) let $F_1' = F_1^{st}$ and $F_2' = F_2^{st}$. If $a' \in E_a \setminus F_1^{st} \cap E_a$ (i.e., a' lies in the y-chart), set $F_1' = E_a$ and $F_2' = E$. Again E_a is a curve. In both cases F_1' and F_2' are defined by $y = z = 0$ and $z = 0$.

By Proposition 2 of section 4, $S' = E \cap X' \cong S$ lies in the z-chart and is defined there by $x + h(y) = z = 0$. The flag $\mathcal{F}'$ being given by $y = z = 0$ and $z = 0$ we have S' transversal to F_1' and contained in F_2'.

(3) *Crossing blowup*. Let $S = S_y \cup S_z$ at a. Let $a' \in E_a$ be a point of $E_y \cap E_z$ or an origin of the other charts. By Theorem 3 there is a subordinate coordinate change in W at a such that S is defined by the ideal (x, yz) and such that the local blowup $R \to R'$ is monomial. In particular, a' moves by the induced translation in $E_a \subseteq W'$ to the origin of one of the charts.

With this prior choice of coordinates it suffices to define $\mathcal{F}'$ at the origins of the charts. In the x^3-chart, set $F_1' = F_1^{st}$ and $F_2' = F_2^{st}$.

Exercise 10. Check that $F_1^{st} \cap E_a$ is the origin of this chart.

In the x^2y-chart, set $F_1' = E_a \cap F_2^{st}$ and $F_2' = F_2^{st}$.

Exercise 11. Check that F_2^{st} contains the origin of that chart.

In the x^2z-chart, the component of E_a containing the origin is isomorphic to $\mathbb{P}^2$. Let F_1' be the line in $\mathbb{P}^2$ through the origin of this chart and the origin of the x^3-chart, and set $F_2' = E_a$. In the xyz-chart and in the y^2z^2-chart, set $F_1' = E_y \cap E_z$ and $F_2' = E_y$.

Exercise 12. Check that the substitutions of the coordinates associated to the blowup and described in Proposition 2 of section 3 define coordinates at a' which are subordinate to $\mathcal{F}'$.

Exercise 13. Prove that if the multiplicity remains constant, $\mathcal{F}'$ is partially transversal to the tangent cone TC' of X'. $\circlearrowleft$

Variation: (3') Let $P = (x, yz) \cap (x, y, z)^3$ define Z where S is a normal crossing of two smooth curves S_y and S_z at a. For any a' over a, the flag $\mathcal{F}'$ is defined as follows. Choose subordinate coordinates such that S is given by (x, yz). If $a' = F_1^{st} \cap E_a$ (which is the origin of the x^3-chart), set $F_1' = F_1^{st}$ and $F_2' = F_2^{st}$. If $a' \in E_a \cap F_2^{st} \setminus F_1^{st}$ (i.e., a' lies in the xy^2-chart) set $F_1' = F_2^{st} \cap E_a$ and $F_2' = F_2^{st}$.

If $a' \in E_a \setminus F_2^{st}$ we distinguish several cases. Let $\tau : W^1 \to W$ denote the blowup of W in the intersection point a of S, and denote by S_z^1 the strict transform of S_z. There is a unique line C^1 in the exceptional divisor $E_a^1 = \tau^{-1}a \cong \mathbb{P}^2$ going through $F_1^1 \cap E_a^1$ and $S_z^1 \cap E_a^1$. Let C be the inverse image of C^1 under the blowup of S_z^1 in W^1. This is a curve in E_a which goes through the origins of the x^3-chart and the xz^2-chart.

(i) If $a' \in E_a \cap E_z \setminus C$ (i.e., a' lies in the yz^2-chart) set $F_1' = E_a \cap E_z$ and $F_2' = E_a$. (ii) If $a' \in E_a \setminus (E_z \cup C)$ (i.e., a' lies in the x^3-chart) let F_1' be the unique line in $E_a \setminus (E_y \cup E_z) \cong E_a^1 \setminus (S_y^1 \cup S_z^1) \cong \mathbb{P}^2 \setminus \{\text{two points}\}$ going through a' and $F_1^{st} \cap E_a$ and set $F_2' = E_a$. (iii) If $a' \in C \cap E_z$ (i.e., a' is the origin of the xz^2-chart) set $F_1' = C$ and $F_2' = E_a$.

It is checked that in all cases F_1' and F_2' are defined in the induced coordinates by $y = z = 0$ and $z = 0$.

The equimultiple locus S' does not appear in the charts x^3, xy^2 and xz^2, since at any point there the multiplicity has dropped, see Proposition 3' of section 4. In the two remaining charts y^2z and yz^2, we may assume that S' is a curve. By the same proposition, S' is defined in both charts by $x + h(y) = yz = 0$ for some series $h(y)$. Hence it meets $\mathcal{F}'$ as prescribed. $\circlearrowleft$

Exercise 14. Replace the construction of the flag $\mathcal{F}'$ for normal crossings of the variation by interpreting the blowup as a composition of blowups in smooth centers.

6. Construction of the induction invariant

This section is largely inspired by Hironaka's definition of the invariants used in [Hi 1]. We fix again a hypersurface X in a regular three dimensional scheme W whose equimultiple locus S has at most normal crossings. Let Z be the corresponding non reduced center with support S and embedded components of type $(x, yz)(x, y)(x, z)$ at the intersection points of S (cubes of the maximal ideal for the variation).

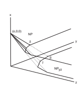
The local invariant i_a we shall associate to a closed point $a \in X$ will be defined by first constructing a quadruple $i_{ax} \in \mathbb{Q}^4$ which depends on the choice of coordinates x at a and by then specifying a set of formal coordinates for which i_{ax} takes the same value. We will then set $i_a = i_{ax}$ for x in this set of coordinates. All points considered will be closed.

Coordinate dependent definition: Let $f \in R = \mathcal{O}_{W,a}$ be a local equation of X in W at a. For coordinates $x = (x, y, z)$ in the completion $\overline{R}$ of R let $f = \sum c_{ijk} x^i y^j z^k$ be the expansion of f w.r.t. x at a. We set

$$i_{ax} = (o_a, \beta_y, s_{\beta\gamma}, |\beta|)$$

with $o_a = \mathrm{ord}_a f$ the order of f at a. The components β_y, $s_{\beta\gamma}$ and $|\beta|$ are defined as follows.

Figure J: Projection of Newton Polyhedron



Let $NP_{yz} \subseteq \mathbb{Q}^2_+$ be the projection from $(o_a, 0, 0)$ of the Newton Polyhedron NP of f, neglecting the portion of NP in $(o_a, 0, 0) + \mathbb{Q}^3_+$. A point $(\delta_x, \delta_y, \delta_z)$ with $\delta_x < o_a$ is sent to $(\frac{o_a \delta_y}{o_a - \delta_x}, \frac{o_a \delta_z}{o_a - \delta_x})$. Let β be the vertex of NP_{yz} which is closest to the y-axis, i.e., with minimal second component β_z (i.e. the point of NP_{yz} which is minimal w.r.t. the inverse lexicographic order in $\mathbb{Q}^2_+$). If β does not exist, f equals in the given coordinates the monomial x^{o_a} times a unit of $\overline{R}$ and nothing is to prove. So we may discard this case.

Let γ be the vertex of the segment of NP_{yz} adjacent to β, i.e., the vertex of NP_{yz} with second component γ_z minimal among the vertices of $NP_{yz} \setminus \beta$ (γ exists if and only if NP_{yz} is not a quadrant).

With these choices, β_y will denote the first component of β and $s_{\beta\gamma}$ the slope of the segment from β to γ, i.e., $s_{\beta\gamma} = \frac{\beta_y - \gamma_y}{\beta_z - \gamma_z} \in \mathbb{Q}_-$ (we draw the y-axis vertically). We set $s_{\beta\gamma} = -\infty$ if γ does not exist. We may equally take instead of β_y the slope of the segment in $\mathbb{Q}^2_+$ between $(o_a, 0)$ and $(0, \beta_y)$. Observe that $\beta_y + \beta_z \geq o$ and $0 > s_{\beta\gamma} \geq -\infty$. We anticipate that $s_{\beta\gamma}$ and $|\beta|$ are only used when a is an isolated point of S.

This ad-hoc definition of the invariant i_a may seem little natural. It depends highly on the ordering chosen among x, y, z and leaves rather unclear how to extend it to higher dimensions. However, without taking into account exceptional divisors when defining the invariant for the strict transforms, experimentation shows that there is not too much flexibility how to choose i_a such that it drops under taut blowup. Observe that β and $s_{\beta\gamma}$ may also serve as invariants for resolution of plane curves, see exercise 2. Projecting the Newton Polyhedron from $(o_a, 0, 0)$ to the yz-plane explains why in many expositions the case where f has form $f = x^{o_a} + g(y, z)$ is considered as representative for the problem.

As the slopes $s_{\beta\gamma}$ are negative and may become arbitrarily small it is not immediate that the invariant belongs to a well ordered set. Let Γ be the set of quadruples (a, b, c, d) with $a, b, d \in \mathbb{N}$ and c of form $c = -\frac{p}{q}$ with p, q in $\mathbb{N}$ and $p \leq b$ (allowing $c = -\infty$). By construction, $i_{ax} \in \Gamma$.

Exercise 15. Show that Γ is well ordered.

Example. Let $f = x^3 + y^4 z + y^2 z^2 + y z^7 + z^{14}$. Then $i_{ax} = (3, 4, -\frac{1}{2}, 5)$ and $\beta = (4, 1)$, $\gamma = (2, 2)$.

A more conceptual definition of the pseudo-invariant through coefficient ideals goes as follows [E-V 2, Ha 3]. Set $o = o_a$. For any ideal I of R, let

$$I_{yz} = \sum_i (a_i(y, z),\ f \in I)^{\frac{o!}{o-i}}$$

be the coefficient ideal of I w.r.t. x and o. Here $a_i(y, z)$ denote the coefficients of elements $f \in I$ given by the expansion $f \equiv \sum_{i=0}^{o-1} a_i(y, z) x^{o-i}$ modulo $x^o \cdot M$. Observe that I_{yz} is compatible with coordinate changes in y, z but not in x. For a principal ideal $I = (f)$ with f expanded as above it is in general not true that $I_{yz} = (a_i(y, z)^{\frac{o!}{o-i}}, i = 0, \ldots, o-1)$. This can be seen in the example of S. Encinas where $f = x^3 + xy^4 + z^5$ with $I_{yz} = (y^{12}, z^{10})$. Multiplying f by the unit $1 - x$ of R gives $I_{yz} = (y^{12} + y^8 z^5, z^{10})$. Consider now the Newton Polygon of I_{yz}

$$NP_{yz} = \bigcup_{g \in I_{yz}} \bigcup_{\alpha \in \mathrm{supp}\ g} \alpha + \mathbb{Q}^2_+.$$

Then β equals the inverse lexicographically minimal vertex of $\frac{1}{(o-1)!} \cdot NP_{yz}$. In [E-V 1, E-V 2] the second component of the invariant is defined as the order of I_{yz}. This order generally increases under blowup. It is necessary to factor suitable powers of the exceptional divisor from the transform $(I')_{yz}$ to get the weak transform of I_{yz} with non-increasing order. There arise, however, serious problems when making the reasoning coordinate independent, since the exceptional divisors loose their monomial form when changing coordinates.

Example. Let $f = x^2 + y^{10} + y^3z^3 + z^{10}$ with isolated singular point at zero. Hence $S = 0$ and $I_y = (y^{10} + y^3z^3 + z^{10})$ has order 6. Blowing up the origin yields in the y-chart a strict transform $f' = x^2 + y^8 + y^4z^3 + y^8z^{10}$ with $(I')_y = (y^8 + y^4z^3 + y^8z^{10})$ of order 7. However $\beta = (10, 0)$ has transformed into $\beta' = (8, 0)$ and the first component has dropped.

Coordinate free definition: We specify the coordinates which we shall select to define i_a. The coordinates will have to be subordinate to a well chosen flag, and, secondly, have to maximize i_{ax} lexicographically over all formal subordinate coordinate choices at a (restricted slightly by maximizing also β_z). Hironaka calls such coordinates well and very well prepared [Hi 1].

We first choose and then fix forever a flag $\mathcal{F}$ in W at a, transversal to S and partially transversal to the tangent cone TC of X (section 5). Given such a flag, we will only consider coordinates x, y, z in the completion $\overline{R}$ which are subordinate to $\mathcal{F}$. Hence F_1 and F_2 are defined in $\overline{R}$ by $y = z = 0$ and $z = 0$ respectively. Among all subordinate coordinates, consider those for which the vector $(\beta_z, \beta_y, s_{\beta\gamma}, |\beta|) \in \mathbb{Q}^4$ becomes maximal w.r.t. the lexicographic order on $\mathbb{Q}^4$ [instead of $s_{\beta\gamma}$ we may also maximize the projection of γ to $\mathbb{Q}_+$, and $|\beta|$ is maximized automatically through (β_z, β_y)]. Observe that we first maximize β_z which does not appear as a component of i_{ax}. It can be seen, either directly or using arguments similar to the ones used to prove Theorem 3 of [Ha 1], that the maximum actually exists and is achieved by some formal coordinates. In the algebraic setting it is convenient to work in the completion $\overline{R}$. For analytic spaces the maximum also exists in the analytic category, see [Ha 1]. In all cases, $\beta = (\beta_y, \beta_z)$ and $s_{\beta\gamma}$ are the same for such coordinates. Then set

$$i_a = i_{ax} = (o_a, \beta_y, s_{\beta\gamma}, |\beta|)$$

where x are coordinates in $\overline{R}$ subordinate to $\mathcal{F}$ and maximizing $(\beta_z, \beta_y, s_{\beta\gamma}, |\beta|)$ lexicographically. Note that instead of the slope $s_{\beta\gamma}$ one could as well project N_{yz} from β to the z-axis $\mathbb{Q}_+$ and take the slope of this line. However, the (unique) vertex of the projection may increase under blowup if one does not factor an appropriate power of the exceptional divisor.

7. Transformation of i_{ax} under monomial blowup.

In the situation of the last paragraph, let $\pi : W' \to W$ be the taut blowup associated to the singular surface X with center Z. Let $a \in X$ be a closed point inside S. We shall show in this and the next section that for all closed points $a' \in X'$ above a one has $i_{a'} < i_a$

in the lexicographic order on $\mathbb{Q}^4$. As the set Γ where i_a can vary is well ordered, there cannot occur an infinite sequence of points a', a'', ... over a along which the invariant does not stabilize. The stage at which it stabilizes depends on the choice of a, and a priori moving a in S may produce strictly decreasing sequences of arbitrary length. This does not guarantee resolution of X and we will have to show in section 9 that all sequences stabilize uniformly at a certain stage independent of a. Observe that it is not clear whether i_a takes only finitely many values on S.

The proof that i_a decreases under blowup splits into a combinatorial argument on monomial blowups and a reduction argument which reduces the general case to the combinatorial situation.

Theorem 5. *Let $R \to R'$ be a monomial blowup w.r.t. formal coordinates x of $\overline{R}$ subordinate to a chosen flag $\mathcal{F}$ at a. Assume that $\mathcal{F}$ is transversal to S and partially transversal to TC. Denote by x also the induced coordinates at a'. Then $i_{a'x} < i_{ax}$.*

Proof. The assumption implies that the point a' sits in one of the origins of the various affine charts induced in W' by the coordinates x. We denote by S', β', γ', $\mathcal{F}'$ etc. all objects associated to the strict transform X' of X as was done before for X. An asterisque * will denote objects in W' obtained from below by applying the monomial substitution of the variables and factoring the exceptional divisor o_a-times. For instance, β^* will equal in the z-chart of a point blowup the vector $(\beta_y, \beta_z + \beta_y - o_a)$ which may equal β' but may as well have moved to the interior of the projected Newton Polygon NP'_{yz} of f'. If the latter occurs, a new vertex of NP'_{yz} assumes the role of β', i.e, is the inverse lexicographically minimal vertex of NP'_{yz}. In each case we will precise in the concrete setting what is meant. The proof is purely computational and goes case by case:

(1) *Point blowup.* This only occurs when S has an isolated point at a. The argument has inductive character w.r.t. the components of i_{ax} and the three charts. We shall show that in all charts $i_{a'x} < i_{ax}$ lexicographically.

(a) Behaviour of o_a. First consider the x-chart. A vertex $\alpha = (\alpha_x, \alpha_y, \alpha_z)$ of the Newton Polygon NP of f transforms into the point $\alpha^* = (\alpha_x + \alpha_y + \alpha_z - o_a, \alpha_y, \alpha_z)$ which may be a vertex of NP' or not. Take for α an exponent of the tangent cone $\mathrm{hom}_a f$ of f, viz satisfying $|\alpha| = \alpha_x + \alpha_y + \alpha_z = o_a$. Since the coordinates are subordinate to the flag $\mathcal{F}$ which is partially transversal to TC, it follows that $\alpha_x > 0$ for at least one such α. This implies $\alpha_y + \alpha_z < o_a$ for this α and hence $|\alpha^*| < o_a$, say $o_{a'} < o_a$ and $i_{a'x} < i_{ax}$ in this chart.

Exercise 16. Show that $o_{a'} \leq o_a$ in the y-chart and z-chart. Determine when equality occurs.

(b) Behaviour of β_y. We may assume $o_{a'} = o_a$, which reduces by (a) to the y- and z-chart. In the y-chart we argue as follows. As S has an isolated point at a there exists a vertex δ of NP satisfying $\delta_x + \delta_z < o_a$ (else the curve defined by $x = z = 0$ would lie in S). Projecting δ from $(o_a, 0, 0)$ to $\mathbb{Q}^2_+$ yields the point $(\frac{o_a \delta_y}{o_a - \delta_x}, \frac{o_a \delta_z}{o_a - \delta_x})$ in NP_{yz}. This shows that $\beta = (\beta_y, \beta_z)$ has second component $< o_a$. A short computation gives

$\beta^* = (\beta_y + \beta_z - o_a, \beta_z)$, hence β_z remains constant and β^* has again minimal second component among all vertices of the projected polyhedron NP'_{yz}. Therefore $\beta^* = \beta'$. Moreover, since $\beta_z < o_a$, we get $\beta'_y = \beta_y + \beta_z - o_a < \beta_y$. We have shown $i_{a'x} < i_{ax}$ in this chart.

Now consider the z-chart. Here, $\beta^* = (\beta_y, \beta_z - o_a)$. If β^* lies in the interior of the projected polyhedron, it is replaced by a vertex β' with smaller first component because β_y was maximal among all vertices of the projected polyhedron (recall that β_z was minimal). In this case $\beta'_y < \beta_y$. If β^* remains a vertex, it equals necessarily β', and its first component β'_y has remained constant, say $\beta'_y = \beta_y$. In both cases we have $\beta'_y \leq \beta_y$.

(c) Behaviour of $s_{\beta\gamma}$ and $|\beta|$. We may assume that $o_{a'} = o_a$ by (a) and that $\beta^* = \beta'$ by (b). Hence we are left with the z-chart. By definition, the vertex $\gamma = (\gamma_y, \gamma_z)$ satisfies, if it exists, that $\beta_y > \gamma_y$ and $\beta_z < \gamma_z$. The slope $s_{\beta\gamma}$ equals $\frac{\beta_y - \gamma_y}{\beta_z - \gamma_z}$. After blowing up it becomes $\frac{\beta_y - \gamma_y}{\beta_z - \gamma_z + \beta_y - \gamma_y}$. If the denominator is negative the slope has decreased since $\beta_y - \gamma_y > 0$. If it is ≥ 0 the vertex $\gamma^* = (\gamma_y, \gamma_z + \gamma_y - o_a)$ has second component less or equal $\beta^* = (\beta_y, \beta_z + \beta_y - o_a)$ which contradicts $\beta^* = \beta'$. Therefore the slope has dropped. If γ does not exist, $NP_{yz} = \beta + \mathbb{Q}^2_+$ is a quadrant and S can only be an isolated point if both β_z and β_y are $< o_a$. Clearly, NP'_{yz} is then again a quadrant and $|\beta'| = |\beta^*| = \beta_y + \beta^*_z = \beta_y + \beta_z + \beta_y - o_a < |\beta|$. This gives $i_{a'x} < i_{ax}$ also in this chart.

(2) *Curve blowup.* The lemma of section 4 implies $\tau_a = 1$. If P equals (x, y), the x-chart is irrelevant since X' does not pass there. In the y-chart a vertex $(\delta_x, \delta_y, \delta_z)$ transforms into $(\delta_x, \delta_y + \delta_x - o_a, \delta_z)$ which shows that $o_{a'} \leq o_a$. If $o_{a'} = o_a$, the vertex $\beta = (\beta_y, \beta_z)$ moves to $\beta^* = (\beta_y - o_a, \beta_z)$. As the second component remains constant, $\beta^* = \beta'$ and β_y has dropped. If P equals (x, z) we may restrict, similarly as before, to the z-chart and $o_{a'} = o_a$. Here β moves to $\beta^* = (\beta_y, \beta_z - o_a)$. If $\beta^* = \beta'$ we get $\beta'_y = \beta_y$ and $s_{\beta\gamma} < s_{\beta\gamma}$, provided γ exists. If it does not exist we have $|\beta'| < |\beta|$. If $\beta^* \neq \beta'$, then $\beta'_y < \beta^*_y = \beta_y$. In all cases we conclude by $i_{a'x} < i_{ax}$. Only the first two components of i_{ax} were used.

(3) *Crossing blowup.* Let $P = (x, yz)(x, y)(x, z)$. Again we have by the lemma of section 4 that $\tau_a = 1$. In Proposition 3 of section 4 it was shown that X' meets E_a only in the chart y^2z^2. There the blowup is given by $(x, y, z) \to (xyz, y, z)$ with E and E_a defined by $yz = 0$, respectively $y = z = 0$. Therefore the strict transform of $f = x^{o_a} + \sum c_{ijk} x^i y^j z^k$ is given by $f' = x^{o_a} + \sum c_{ijk} x^i y^{j+i-o_a} z^{k+i-o_a}$. Hence $o_{a'} \leq o_a$. Let $(\delta_x, \delta_y, \delta_z)$ project to β. By definition of NP_{yz} we have $\delta_x < o$. Then $(\delta_x, \delta_y + \delta_x - o_a, \delta_z + \delta_x - o_a)$ projects to β', hence $\beta'_y = \frac{o_a}{o_a - \delta_x} \cdot (\delta_y + \delta_x - o_a) < \frac{o_a}{o_a - \delta_x} \cdot \delta_y = \beta_y$. This gives $i_{a'x} < i_{ax}$. Again only the first two components of i_{ax} were used.

Variation: (3') Let $P = (x, yz) \cap (x, y, z)^3$. By Proposition 3' of section 4, only the y^2z- and yz^2-charts are relevant. We show that i_{ax} may increase. In the y^2z-chart the vertices of NP move according to $(\delta_x, \delta_y, \delta_z) \to (\delta_x, \delta_y + \delta_x + \delta_z - o_a, \delta_z + \delta_x - o_a)$

and if β_z remains minimal, the component β_y may increase to $\beta'_y = \beta_y + \delta_x + \delta_z - o_a$. Observe here that $\delta_x + \delta_z \geq o_a$ since S contains the curve defined by $x = z = 0$. Hence i_a may increase in this chart.

In the yz^2-chart, nevertheless, the invariant decreases. Here a vertex $(\delta_x, \delta_y, \delta_z)$ moves to $(\delta_x, \delta_y + \delta_x - o_a, \delta_z + \delta_x + \delta_y - o_a)$. Consider $\beta = (\beta_y, \beta_z)$ moving to $(\beta_y, \beta_z + \beta_y - o_a)$. If β survives, say $\beta^* = \beta'$, then $\beta_z + \beta_y - o_a$ must again be minimal, and we have $\beta'_y = \beta^*_y = \beta_y - o_a < \beta_y$, hence β_y drops. If β moves to the interior of NP'_{yz} and the new vertex is $\beta' = \delta^*$ for some vertex δ of NP_{yz}, then $\beta^*_z = \beta_y + \beta_z - o_a > \delta^*_z = \delta_y + \delta_z - o_a$ and $\beta_z < \delta_z$. Hence $\beta'_y = \delta^*_y = \delta_y - o_a < \beta_y - o_a < \beta_y$ and β_y drops. $\circlearrowleft$

8. Reduction to monomial blowup.

We show that for local blowups $R \to R'$ the improvement $i_{a'} < i_a$ follows from the improvement of i_{ax} under monomial blowup. We work as in section 6 with the completions of the local rings.

Theorem 6. *Given $\pi : W' \to W$ and $a \in S$ a closed point, let $a' \in W'$ be a closed point over a where the multiplicity $o_{a'}$ of X' has remained constant. Assume chosen a flag $\mathcal{F}$ at a transversal to S and partially transversal to the tangent cone TC of X. There exist formal subordinate coordinates x at $a \in W$ such that $\overline{R} \to \overline{R'}$ is monomial w.r.t. x, β_z is maximal and such that the induced coordinates in $\overline{R'}$ maximize β'_z and realize $i_{a'}$ up to the relevant component.*

The various cases of the proof of Theorem 5 show that it suffices for the induction on i_a - according to the blowup and the chart considered - to maximize $i_{a'x}$ only up to the component which drops under the corresponding monomial blowup. This is the meaning of the relevant component in Theorem 6. If $o_{a'}$ has dropped, a new flag transversal to the S' and TC' has to be chosen at a' in W'.

$Proof$. The argument is inspired by the proof of Theorem 6 in [Ha 1]. Start with any subordinate coordinates in $\overline{R}$. By Theorem 3 of section 5, we may apply a subordinate coordinate change in $\overline{R}$ which makes the blowup monomial and hence moves a' to the origin of one of the charts. For crossing blowup, we have seen already that $X' \cap E$ lies in the y^2z^2-chart, so Theorem 3 does apply in this case as well. Denote by x also the induced coordinates in $\overline{R'}$.

By Theorem 3 of section 5 and since $o_a = o_{a'}$, the induced flag $\mathcal{F}'$ in $\overline{R'}$ is again transversal to S'.

We now maximize $(\beta'_z, \beta'_y, s'_{\beta\gamma}, |\beta'|)$ by subordinate coordinate changes in $\overline{R'}$. By the Gauss-Bruhat decomposition, see Theorem 2 of [Ha 1], we only have to consider subordinate coordinate changes in $\overline{R'}$ of type $(x + a(y,z), y + b(z), z)$ to maximize β'_z and $i_{a'x}$. We shall show that there exists such a change in $\overline{R'}$ which is induced from subordinate coordinate changes in $\overline{R}$ and leaves $\overline{R} \to \overline{R'}$ monomial. Simultaneously, β_z will be maximized in $\overline{R}$.

Exercise 17. Show that in the z-chart of point blowup the change $v = (x + y^k z^{k-2}, y, z)$ with $k \geq 2$ in R' is not induced by a change in R, though v fixes the exceptional divisor in this chart.

(1) *Point blowup.* It suffices to consider the y- and z-chart which cover all of $E = E_a$ except the origin of the x-chart where we already know that the multiplicity drops. In the y-chart we will maximize (β'_z, β'_y) lexicographically and can discard the remaining components by the proof of Theorem 5. It is immediate from the definition that only changes $(x + a(y,z), y, z)$ can alter β'. Now observe that in the induced coordinates in R', NP'_{yz} is contained in the cone $\{(j,k) \in \mathbb{Q}^2_+,\ j \geq k - o_a\}$ because points (j,k) of NP_{yz} move under monomial blowup in this chart to $(j + k - o_a, k)$. Set $a(y,z) = \sum a_{jk} y^j z^k$. If $v = (x + a(y,z), y, z)$ maximizes (β'_z, β'_y) the sum in $a(y,z)$ can be chosen over pairs (j,k) for which $j \geq k - 1$, since we wish to eliminate in f' only monomials whose projection lies in NP'_{yz}. This can be seen inductively, first eliminating all monomials in the expansion of f' whose projection gives β', and then starting again with the new β'.

Exercise 18. Write down this argument in all details and compare it with Lemma 3(b) of [Ha 1]. Note that it is characteristic independent.

Once we can restrict to changes of this form, it is immediate to see that v is induced from the subordinate change $u = (x + \sum_{j \geq k-1} a_{jk} y^{j-(k-1)} z^k, y, z)$ in $\overline{R}$ and that this change preserves the monomiality of the blowup (use the constancy of multiplicity and $j + k \geq 1$. If a change $(x + y, y, z)$ would be needed to maximize β_z, o_a would have dropped at a').

Exercise 19. Show that since v maximizes β'_z, u maximizes β_z.

We next treat the z-chart, where all components of i_a are relevant. To maximize (β'_z, β'_y) in $\overline{R'}$ and β_z in $\overline{R}$ the argument is the same as before with the role of y and z interchanged. To maximize $s'_{\beta\gamma}$ in $\overline{R'}$ we will need more general changes of type $v = (x + a(y,z), y + b(z), z)$. These are products of changes of type $(x + a(y,z), y, z)$ and $(x, y + b(z), z)$. For the first, the preceding reasoning applies again, since we may require that v does not create new monomials outside NP'_{yz}. For the second, it suffices to observe that changes $(x, y + b(z), z)$ are induced under point blowup in the z-chart from changes in $\overline{R}$ which have the same coordinate expression.

(2) *Curve blowup.* For $P = (x,y)$ and $P = (x,z)$ it suffices to consider the y- and z-chart respectively, and to restrict to maximizing (β'_z, β'_y). It is then straightforward to see that any change of type $(x + a(y,z), y, z)$ in $\overline{R'}$ is induced from a subordinate change in $\overline{R}$ which preserves monomiality of the blowup and that β_z is maximal if β'_z is maximal.

(3) *Crossing blowup.* Only the y^2z^2-chart and β' are relevant. The monomial substitution $\overline{R} \to \overline{R'}$ is given by $(x, y, z) \to (xyz, y, z)$. Therefore any change of type $(x + a(y,z), y, z)$ in $\overline{R'}$ is induced from a subordinate change in $\overline{R}$ which preserves monomiality

of the blowup. Check again that β_z is maximal if β'_z is maximal. This concludes the proof of the theorem. $\circlearrowleft$

9. Proof of Theorem 1.

We start with a singular reduced surface X in a regular three-dimensional ambient scheme W. By Proposition 1 of section 4 we may assume that after a finite number of point blowups the equimultiple locus S of X consists of finitely many isolated points and finitely many smooth curves which have at most normal crossings. Let $\pi : W' \to W$ be the taut blowup of X with center Z as defined in section 3. It yields the strict transform X' of X and an exceptional divisor E in W'. Choose at each point a of S a flag $\mathcal{F}$ transversal to S and partially transversal to the tangent cone TC of X.

Since it is not clear whether i_a takes finitely many values on S we cannot argue by stratifying S according to the invariant. Stratifications only work well for invariants which are semi-continuous.

Instead, we show first that any smooth component C of S dissolves after a finite number of taut blowups into finitely many isolated points (possibly none) with the same multiplicity. By Propositions 2 and 3 of section 4 we know that if the component persists after one blowp with the same multiplicity, no new component has appeared and the components C and C' are isomorphic under π. Assume that this happens infinitely many times, producing an infinite sequence of components C, C', C'', ... of the equimultiple loci of the strict transforms of X. Choose in each component $C^{(k)}$ a point $a^{(k)}$ over a where the invariant takes its minimal value on $C^{(k)}$. Such points exist because i_a varies in a well ordered set. By Theorem 5 and 6 the sequence $i_{a^{(k)}}$ strictly decreases yielding the contradiction.

Consider now an isolated point a of S. Its fibre $E_a \cap X'$ under π may be a component of S', necessarily isomorphic to $\mathbb{P}^1$. It is disjoint from the other components, cf. Proposition 1 of section 4. By the preceding observation, finitely many further blowups decompose it into finitely many points of the same multiplicity as a. Using again Theorem 5 and 6 we know that for each of these points a' we have $i_{a'} < i_a$.

It follows that after finitely many blowups, the multiplicity must drop at each point a' over any point a of S. Hence the maximal multiplicity occuring on the surface has dropped and finitely many taut blowups yield a smooth surface.

Exercise 20. Show that further blowups make the intersection of the surface and the exceptional divisor transversal.

This proves Theorem 1 and concludes the paper. $\circlearrowleft$

References.

[Ab 1] Abhyankar, S.: Desingularization of plane curves. Proc. Symp. Pure Appl. Math. **40**, Amer. Math. Soc. 1983.

[Ab 2] Abhyankar, S.: Resolution of singularities of embedded algebraic surfaces. Acad. Press 1966.

[Bn] Bennett, B.-M.: On the characteristic function of a local ring. Ann. Math. **91** (1970), 25-87.

[B-M 1] Bierstone, E., Milman, P.: Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant. Invent. Math. **128** (1997), 207-302.

[CGO] Cossart, V., Giraud, J., Orbanz, U.: Resolution of surface singularities. Springer Lecture Notes in Math. vol 1101.

[Co 1] Cossart, V.: Desingularization of embedded excellent surfaces. Tôhoku Math. J. **33** (1981), 25-33.

[Co 2] Cossart, V.: Valuations II. This volume.

[E-V 1] Encinas, S., Villamayor, O.: Good points and constructive resolution of singularities. To appear.

[E-V 2] Encinas, S., Villamayor, O.: Constructive resolution and equivariance. This volume.

[Gi] Giraud, J.: Etude locale des singularités. Cours de 3^{e} Cycle, Orsay 1971/72.

[Hi 1] Hironaka, H.: Desingularization of excellent surfaces. Notes by B. Bennett at the Conference on Algebraic Geometry, Bowdoin 1967. Reprinted in: Cossart, V., Giraud, J., Orbanz, U.: Resolution of surface singularities. Lecture Notes in Math. 1101, Springer 1984.

[Hi 2] Hironaka, H.: Resolution of singularities of an algebraic variety over a field of characteristic zero. Ann. of Math. **79** (1964), 109-326.

[Hi 3] Hironaka, H.: Idealistic exponents of singularity. In: Algebraic Geometry, the Johns Hopkins Centennial Lectures. Johns Hopkins University Press 1977.

[Hi 4] Hironaka, H.: Additive groups associated with points of a projective space. Ann. Math. **92** (1970), 327-334.

[Ha 1] Hauser, H.: Resolution techniques. Preprint 1997.

[Ha 2] Hauser, H.: Seventeen obstacles for resolution of singularities. In: Singularities, The Brieskorn Anniversary Volume (eds: V. I. Arnold, G.-M. Greuel, J. Steenbrink), Birkhäuser 1998.

[Ha 3] Hauser, H.: Tetrahedra, prismas and triangles. Preprint 1996.

[Le] Lê, D.T.: Les singularités sandwich. This volume.

[Lp] Lipman, J.: Desingularization of 2-dimensional schemes. Ann. Math. **107** (1978), 151-207.

[Mo] Moh, T.T.: On a Newton polygon approach to the uniformization of singularities in characteristic p. In: Algebraic Geometry and Singularities (eds.: A. Campillo, L. Narváez). Proc. Conf. on Singularities La Rábida. Birkhäuser 1996.

[Sp 1] Spivakovsky, M.: A counterexample to the theorem of Beppo Levi in three dimensions. Invent. Math. **96** (1989), 181-183.

[Sp 2] Spivakovsky, M.: Resolution of singularities. Preprint 1997.

[Za] Zariski, O.: Reduction of singularities of algebraic three dimensional varieties. Ann. Math. **45** (1944), 472-542.

Institut für Mathematik
Universität Innsbruck
A-6020 Austria

herwig.hauser@uibk.ac.at

February 2000 - 2