“Platonic stars”
Construction of algebraic curves and surfaces with prescribed symmetries and singularities.

diploma thesis in mathematics

by

Alexandra Fritz

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supervisor: V.-Prof. Dr. Herwig Hauser, Faculty of Mathematics, University Vienna

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Preface

And so I raise my glass to symmetry
To the second hand and its accuracy
To the actual size of everything
The desert is the sand
You can't hold it in your hand
It won't bow to your demands
There's no difference you can make
(Bright Eyes "I Believe in Symmetry"[1])

Actually the title of the thesis already says it all: the aim of our work was to construct algebraic curves and surfaces with previously fixed properties, namely symmetries and singularities. The most prominent examples are what we called "Platonic stars" - algebraic surfaces with the symmetries of a Platonic solid and cusps in the vertices of the same solid. The idea was to produce pictures that anyone would immediately identify as "stars". Such a figures apparently must be bounded and connected, have some symmetries, and most important have a certain number of cusps. Even though in our construction we do not consider the first two properties we were able to reach this goal. Still, as we will see in the examples, it was only possible by making some "good" choices: Very often we were left with a number of free parameters, that can be chosen almost arbitrarily. If chosen right we obtain the objects we want, if not, it might still have the desired properties but becomes unbound or additional components appear. Therefore it might be interesting to include these properties while constructing the surface.
The construction is based on some results on simples singularities and invariant theory. We will present these results together with some basics from algebraic geometry and commutative algebra, as well as a small excursion on group theory and polytopes in the first chapter. The second one is dedicated to explaining the construction. First we describe all the steps in theory and give arguments why it works, then we demonstrate it with two detailed examples, and motivate three possible generalizations. The third and last chapter is basically a list of examples that should give an idea of the possibilities, but maybe also disadvantages of our construction. Besides the examples that illustrate our construction, some plane curves, the Platonic and Archimedean stars, we also motivate the generalizations with some extra examples. In the appendix one finds some technical details and recapitulatory tables.

All figures that appear through the following text are generated either with *Wolfram Mathematica*

[1] "Digital Ash in a Digital Urn", Saddle Creek Records, 2005

6 for Students[2] or the free ray-tracing software *POV-Ray*[3]. The interested reader may download the program at www.povray.org.

I want to express my gratitude to the people that made this thesis possible. Especially I want to thank my parents for giving me all opportunities and my supervisor for his patience and encouragement and Clemens, Dennis, Dominique, Eleonore, Josef Schicho and Manfred Kuhnkies for valuable disscussions, proof-reading and motivation. I am also thankful for the support by Project 21461 of the Austrian Science Fund FWF.

Love goes out to all my friends, without you I'm nothing. ♡
Innsbruck, December 11, 2009,

Alexandra Fritz.

[2]See www.wolfram.com/products/student/mathforstudents/index.html.

[3]Actually Figures 1.3, 1.4a, 2.1, 3.1, 3.3, 3.4, 3.5, 3.20a, 3.22, A.1 were produced with Mathematica, the remaining ones with POV-Ray.

Chapter 1

Preparation

1.1 Basic notations and results from algebraic geometry and commutative algebra

In the following K will always denote some *field*. A field is called *algebraically closed* if every polynomial of positive degree in one variable with coefficients in K has at least one zero in K. Note that we do not demand that the field K is algebraically closed if it is not stated explicitly. Most of the time K will be the field of real numbers $\mathbb{R}$, that is not algebraically closed. Only sometimes we will work with the algebraic closure of $\mathbb{R}$, the complex numbers $\mathbb{C}$.
The letter R will, if not stated differently, denote a *commutative ring*. Most of the time the ring will be *graded*, i.e., it permits a decomposition into a direct sum $R = \bigoplus_{i \geq 0} R_i$ such that $R_i R_j \subset R_{i+j}$ for all $i, j \geq 0$. The elements of R_i are called *homogeneous elements of degree i*.
A ring R is called *Noetherian*[1] if it satisfies the following three[2], equivalent, conditions. See for example [AM69, p. $74 - 76$] for a proof of the equivalence.

1. Every increasing sequence of ideals $p_1 \subseteq p_2 \subseteq \ldots$ of R is stationary, (i.e., there exists a $n \in \mathbb{N}$ such that $p_n = p_{n+1} = \ldots$).

2. Every non empty set of ideals of R has a maximal element.

3. Every ideal $I \subset R$ is finitely generated.

A *polynomial* in n variables with coefficients in a ring R (or a field K) is the finite sum[3]

$$f(x_1, \ldots, x_n) = f(\boldsymbol{x}) = \sum_{\alpha_1 + \ldots + \alpha_n \leq d} a_{\boldsymbol{\alpha}} x_1^{\alpha_1} \ldots x_n^{\alpha_n} = \sum_{|\boldsymbol{\alpha}| \leq d} a_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}, \quad a_{\boldsymbol{\alpha}} \in R.$$

The integer d is called the (total) degree of $f(\boldsymbol{x})$. All such polynomials form a ring, the *ring of polynomials* in n variables with coefficients in R or *polynomial ring*. It is denoted by $R[x_1, \ldots, x_n]$. We quote the important

Theorem 1 (Hilbert's Basis Theorem)**.** *Let R be a Noetherian ring, then $R[x_1, \ldots, x_n]$ is also Noetherian.*

[1]Named after the German mathematician Emmy Noether, $1882 - \mathbf{1935}$.
[2]Mostly only the first two are used for the definition and the third one is a theorem.
[3]Note that we use multiindices to write such terms more compact.

Proof. See [AM69, p. 81], Theorem 7.5 and Corollary 7.6. ★

Any field K is Notherian, therefore, by Hilbert's Basis Theorem, the polynomial ring $K[x_1, \dots, x_n]$ is also Noetherian. Hence any ideal I of $K[x_1, \dots, x_n]$ is generated by some finite set of polynomials $\{f_1, \dots, f_m\} \subset K[x_1, \dots, x_n]$.
Let $R \subset S$ be commutative rings. An element $s \in S$ is called *integral over* R if $R[s]$ is finitely graded as an R-module. The ring S is called *integral* over R if all $s \in S$ are integral over R. An element $s \in S$ is integral over R if and only if s is a rood of a monic polynomial with coefficients in R, i.e., there exists a $P(t) = t^d + a_{d-1}t^{d-1} + \ldots + a_1 t + a_0 \in R[t]$ such that $P(s) = 0$. A proof of this statement can be found in any book on commutative algebra, see for example [BIV89, p. 78].
Let $K \subset L$ be fields. Elements $a_1, \dots, a_m \in L$ are called algebraically independent if there exists no polynomial $0 \neq P(x_1, \dots, x_m) \in K[x_1, \dots, x_m]$ such that $P(s_1, \dots, s_m) = 0$.

The polynomial ring $K[x_1, \dots, x_n]$ has even more structure, besides being a ring, it is a K-algebra: A ring A is called an K-*algebra* if it has a scalar multiplication that is compatible with its ring structure. A K-algebra is said to be *finitely generated* if there exist elements $a_1, \dots, a_m \in A$ such that $A = K[a_1, \dots, a_m]$. The a_i need not be algebraic independent, i.e., the map $K[x_1, \dots . x_m] \to A : \boldsymbol{x} \mapsto \boldsymbol{a}$ is surjective but not necessarily injective. A *graded* K-algebra R is an K-algebra with a decomposition $R = R_0 \oplus R_1 \oplus R_2 \oplus \ldots$, where $R_0 = K$ and $R_i R_j \subset R_{i+j}$. Evidently the polynomial ring is naturally graded by the total degree. A homogeneous polynomial of degree i can be written as $f(\boldsymbol{x}) = \sum_{|\boldsymbol{\alpha}|=i} a_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}$.

The *affine space* over a field K, $\mathbb{A}^n_K$, is the vector space formed by all n-tubles of elements of K. When it is evident over which field we work we just write $\mathbb{A}^n$. An element $\boldsymbol{p} = (p_1, \dots, p_n)^4$ of $\mathbb{A}^n_K$ is called a *point*.
Given a set of polynomials $\{f_1, \dots, f_k\}$ in $K[x_1, \dots, x_n]$ one calls the set of all point of the affine space at which the polynomials vanish,

$$V(f_1, \dots, f_k) := \{\boldsymbol{p} = (p_1, \dots, p_n) \in \mathbb{A}^n_K,\ f_j(p_1, \dots, p_n) = 0,\ j = 1, \dots, k\},$$

the *zero set* of $f_1, \dots, f_k$. Clearly if $f_1, \dots, f_k$ vanish at a point $\boldsymbol{p}$ then all other polynomials in the ideal generated by these polynomials, $I := (f_1, \dots, f_k)$, also vanish at $\boldsymbol{p}$, therefore we can write $V(f_1, \dots, f_k) = V(I)$. Note that the affine space $\mathbb{A}^n_K$ can be viewed as the zero set of the empty set $\emptyset \subset K[x_1, \dots, x_n]$.
The zero set of some ideal, $V(I)$, is also called an *(affine) algebraic set*. Given a subset $X \subset \mathbb{A}^n_K$ then one defines the *ideal of* X as the ideal in $K[x_1, \dots, x_n]$ of all polynomials that vanish at all points of X:

$$I(X) := \{f \in K[x_1, \dots, x_n],\ f(\boldsymbol{p}) = 0 \text{ for all } \boldsymbol{p} \in X\}.$$

We already mentioned that $K[x_1, \dots, x_n]$ is Noetherian. Therefore for all algebraic sets X there exist polynomials $f_1, \dots, f_k \in K[x_1, \dots, x_n]$ such that $I(X) = (f_1, \dots, f_k)$. For algebraically closed fields Hilbert's Nullstellensatz holds. It claims that if K is algebraically closed and $X = V(J)$ is the zero set of an ideal $J \subset K[x_1, \dots, x_n]$ then the ideal of X is equal to the radical[5] of J: $I(X) = \sqrt{J}$.

[4] For the sake of readability we write p as a row vector here, but when it comes to calculations all vectors should be viewed as columns.

[5] The *radical of an ideal* J of some ring R is defined as $\sqrt{J} := \{f \in R,\ f^k \in J \text{ for some } k \in \mathbb{N}\}$.

See [Lan02, p. 380] for a proof.

A subset $U \subset \mathbb{A}^n_K$ is called *Zariski-open* if it is the complement of an algebraic set: $U = \mathbb{A}^n_K \backslash V(I)$, with I some ideal in the polynomial ring. The topology defined by these open sets is called the *Zariski-topology* of the affine space. A subset A of a topological space B is called *irreducible* if there exists no decomposition $A = A_1 \cup A_2$, where $A_1 \neq A_2$ are nonempty, proper subsets of A and both are closed in B. An *(affine) algebraic variety* is an irreducible algebraic set.

There are various ways of defining the dimension of an algebraic variety. Following [Har77] we start by defining the *dimension of an topological space* X as the supremum of the length of chains of distinct, irreducible, closed subsets of X:

$$dim(X) = sup \{r \in \mathbb{N},\ Z_0 \subset Z_1 \subset \ldots \subset Z_r,\ Z_i \neq Z_j \subset X \text{ irreducible and closed}\}.$$

The *dimension of an affine algebraic variety* is its dimension as a topological space (with the induced topology from the Zariski-topology of the affine space.). One can also define the dimension of algebraic varieties via the dimension of their (affine) coordinate rings, this will be convenient to calculate the dimension. We need some new definitions: The *(affine) coordinate ring* of an affine algebraic set $X \subset \mathbb{A}^n_K$, denoted by $K[X]$, is defined as the polynomial ring modulo the ideal of X: $K[X] := K[x_1, \ldots, x_n]/I(X)$. The coordinate ring of the affine space $\mathbb{A}^n_K$ itself is evidently the polynomial ring $K[x_1, \ldots, x_n]$. The *height of a prime ideal* of a ring R, $p \subset R$, is defined as

$$ht(p) = sup \{n \in \mathbb{N},\ p_0 \subset \ldots \subset p_n = p,\ p_i \neq p_j \subset R,\ \text{prime ideals}\}.$$

The *(Krull) dimension of a ring* R is the supremum of the heights of all prime ideals of R.
For K algebraically closed the dimension of an affine algebraic set $X \subset A^n_K$ is equal to the (Krull) dimension of its coordinate ring $K(X)$. See [Har77, p. 6] for a proof[6]. By the statement above, still for algebraically closed fields, the dimension of the affine space $\mathbb{A}^n_K$ is equal to the dimension of its coordinate ring $K[x_1, \ldots, x_n]$ which is equal[7] to n.
One calls an algebraic variety $X \subset \mathbb{A}^n_K$ of dimension $n-1$ a *hypersurface*. If K is an algebraically closed field then a variety $X \subset \mathbb{A}^n_K$ is a hypersurface if and only if it is the zero set of just one, nonconstant, irreducible polynomial $f \in K[x_1, \ldots, x_n]$, [Har77, p.]. If K is not algebraically closed this is not true: [Ful89, p. 17, Problem 1-26] $f(x, y) = y^2 + x^2(x-1)^2$ is irreducible over $\mathbb{R}$ but its zero set is not irreducible.
We will call algebraic sets of dimension one and two *algebraic curves* and *surfaces* respectively.

Let $X \in \mathbb{A}^n_K$ be an affine algebraic variety of dimension r and $I(X) = (f_1, \ldots, f_k)$ then we say X is *singular* at a point $\boldsymbol{p} \in X$ if the *Jacobean matrix*[8] at $\boldsymbol{p}$, $J_f(\boldsymbol{p}) := (\partial f_i/\partial x_j)(\boldsymbol{p})$ has rank smaller than $n-r$. A singular point is also called a *singularity*. A point is called *nonsingular* or *smooth* if the Jacobean matrix at the point has rank equal to $n-r$. Note that this definition is independent from the choice of the generators of $I(X)$. See for example [Har77]

[6]Sketch of the proof: The prime ideals of the coordinate ring $K[X] = K[x_1, \ldots, x_n]/I(X)$ are in a one to one correspondence with the prime ideal in $K[x_1, \ldots, x_n]$ that contain $I(X)$ For K algebraically closed the irreducible closed subsets of X also correspond to the prime ideals in $K[x_1, \ldots, x_n]$ that contain $I(X)$.

[7]For a proof of this statement see for example [Eis95, p. 281].

[8]The Jacobean matrix J_f of $f = (f_1, \ldots, f_k)$, is a $k \times n$ matrix with entries polynomials in $K[x_1, \ldots, x_n]$. We also write $\partial_{x_j} f_i = \partial f_i/\partial x_j$ for the j-th partial derivative of f_i.

The set of singular points of a variety X is called the *singular locus* of X and denoted by

$$\mathrm{Sing}(X) := \{\boldsymbol{p} \in X, rank(J_f(\boldsymbol{p})) < n - r\}.$$

It can be shown that it is a proper, closed subset of X.
A singularity is called *isolated* if there exists a neighborhood of it in which there are no other singular points.
For hypersurfaces of the form $X = V(f)$, $f \in \mathbb{R}[x_1, \ldots, x_n]$ the singular locus is $\mathrm{Sing}(X) = V(f, \partial_{x_1} f, \ldots, \partial_{x_n} f)$. We proof this statement:

$$\begin{aligned} Sing(X) &= \{\boldsymbol{p} \in X, rank(J_f(p)) < n - dim(X)\} = \\ &= \{\boldsymbol{p} \in \mathbb{R}^n, \boldsymbol{p} \in V(f), rank(J_f(\boldsymbol{p})) < 1\} = \\ &= \{\boldsymbol{p} \in \mathbb{R}^n, \boldsymbol{p} \in V(f), rank((\partial_{x_1} f, \ldots, \partial_{x_n} f)) < 1\} = \\ &= \{\boldsymbol{p} \in \mathbb{R}^n, \boldsymbol{p} \in V(f), (\partial_{x_1} f, \ldots, \partial_{x_n} f) = (0, \ldots, 0)\} = V(f, \partial_{x_1} f, \ldots, \partial_{x_n} f). \qquad \bigstar \end{aligned}$$

In Chapter 2 we want to construct algebraic hypersurfaces in the real space $\mathbb{R}^n$, with certain properties. To do that we construct a polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$. Because $\mathbb{R}$ is not algebraically closed the zero set $V(f) \subset \mathbb{R}^n$ is not necessarily a hypersurface and it does not need to be irreducible. In general we will only gain algebraic sets of dimension smaller or equal to $n - 1$.

1.2 Some basics from group theory

As a preparation for the next chapter about invariant theory we want to recall some basics from group theory[9].

A *group* is a set G together with a map[10] $G \times G \to G : (x, y) \mapsto xy$ that is associative, i.e., $(xy)z = x(yz)$, has a neutral element $\mathbf{1}$ ($\mathbf{1}x = \mathbf{1}x = x$) and such that for every element $x \in G$ there exists an element $y \in G$ with $xy = yx = \mathbf{1}$. This element is called the *inverse* of x and is written x^{-1}. A group is called *commutative* or *Abelian* if $xy = yx$ for all $x, y \in G$.

From now on let G be some group, not necessarily commutative.
The group G *acts* on a set X (from the left[11]) if there exists a map (the *action*) $G \times X \to X : (g, x) \mapsto g \cdot x$ that satisfies the following two conditions:

(i) For all $x \in X$: $\mathbf{1} \cdot x = x$.

(ii) For all $g, h \in G$ and $x \in X$: $g \cdot (h \cdot x) = (gh) \cdot x$.

One says that G acts *transitively* on X if for all $x, y \in X$ there exists an element $g \in G$ such that $g \cdot x = y$.
Let G be acting on the set X, then the *orbit* of an element $x \in X$ is defined as $G \cdot x := \{g \cdot x \in X;\ \text{for all } g \in G\} \subset X$. The *stabilizer* of an element x of X is the set of all group elements $g \in G$ that leave x fixed, $G_x := \{g \in G;\ g \cdot x = x\} \subset G$. The set of all elements $x \in X$ that are *fixed* by an element $g \in G$ is denoted by $X^g := \{x \in X;\ g \cdot x = x\} \subset X$.

[9] The main sources for this section are [Arm88], [Art98], [Lan02] and [Rot95].
[10] We write the map multiplicatively. One could also write it additively: $x + y$. Then the neutral element is usually denoted by 0 and the group is often considered Abelian.
[11] There is also the notion of a right group action, defined in the obvious way.

Groups as we defined them above sometimes are also called abstract groups, because the elements of the set G stay abstract. Often it is usefull to use *matrix representations* of a group G. The n-dimensional *matrix representation* of a group G as a homomorphism from G to the *general linear group* $\mathrm{GL}(K^n) = \{M \in Mat_{n\times n}(K);\ M \text{ invertible}\}$. A representation is said to be *faithful* if the homomorphism is injective. Note that one group can be represented in various ways, and that some properties might depend on the representation and not only on the group itself. Still, when it is clear what we mean, we will often just speak of a group $G \subset \mathrm{GL}(K^n)$ and not of its n-dimensional matrix representation.

In the following we are interested in symmetries of subsets of the Euclidean space, i.e., $\mathbb{R}^n$ with the usual inner product. This means bijective linear maps in the Euclidean space that preserve distances and map any point of the subset onto another point of the subset. Actually we just consider maps that leave the origin fixed. All distance preserving maps that leave the origin fixed form a group, the *real orthogonal group*. It is isomorphic to the group of all orthogonal $n \times n$-matrices: $\mathrm{O}_n(\mathbb{R}) := \{M \in \mathrm{GL}_n(\mathbb{R}),\ MM^T = M^TM = \mathbb{I}_n\}$. [Rot95, p. 65].
Any *matrix group* $G \subset Mat_{n\times n}(K)$, i.e., a representation of a group consisting of $n \times n$-matrices, with entries in a field K, acts on the vector space K^n by (matrix) multiplication. A group acting on K^n also acts naturally on the polynomial ring $K[x_1, \dots, x_n]$:

$$G \times K[x_1, \dots, x_n] \to K[x_1, \dots, x_n] :\ (\pi, P) \mapsto \pi \cdot P = P \circ \pi,$$

where $P \circ \pi$ is defined as composition with the matrix[12]: $(\pi \cdot P)(x) = (P \circ \pi)(x) = P(x\pi^T)$.
We prove this: Obviously $\mathbb{I}_n \cdot P = P$ for all polynomials P, so we just have to show the second condition of a group action: $(\pi_1\pi_2) \cdot P = \pi_1 \cdot (\pi_2 \cdot P)$. But,

$$((\pi_1\pi_2) \cdot P)(x) = P(x(\pi_1\pi_2)^T) = P(x\pi_2^T\pi_1^T) = P((x\pi_2^T)\pi_1^T) = \pi_1 \cdot (P(x\pi_2^T)) = \pi_1 \cdot (\pi_2 \cdot P(x)). \bigstar$$

Let G be a group acting on K^n and $A \subset K^n$ be a subset, then we define the *symmetry group of A* (in G), as $\mathrm{Sym}(A) = \{M \in G,\ M(a) \in A \text{ for all } a \in A\} \subseteq G$. In particular we will be interested in the symmetry groups of object in the real Euclidean space $\mathbb{R}^n$ that are subgroups of the orthogonal group $\mathrm{O}_n(\mathbb{R})$. Therefore, if we speak of the symmetry group of a subset $A \in \mathbb{R}^n$, it will always be the symmetry group[13] in $\mathrm{O}_n(\mathbb{R})$.
In Section 1.5 about invariant theory we will need the following notation. We took this definition from [Stu08, p. 44].

Definition 1. An element π of the general complex linear group $\mathrm{GL}(\mathbb{C}^n)$ is called a *reflection* if all but one of its eigenvalues are equal to one[14]. A *reflection group* is a finite subgroup $G \subset \mathrm{GL}(\mathbb{C}^n)$ that is generated by reflections.

Being a reflection group is an example of a property that does depend on the representation of a group. For example the two-dimensional representation of the *dihedral group* D_m, i.e., the symmetry group of regular polygon (see Section 1.3 on polytopes), is a reflection group while the three-dimensional is not.

[12] Note that $x = (x_1, \dots, x_n)$ is a row vector so we have to multiply the transpose matrix to it from the right. This also guarantees that the second condition of a group operation are fulfilled.

[13] Often the symmetry group is defined as a subgroup of $\mathrm{SO}_n(\mathbb{R})$ instead of $\mathrm{O}_n(\mathbb{R})$. The subgroup of $\mathrm{O}_n(\mathbb{R})$ we consider here is referred to as a *full symmetry group*.

[14] Sometimes these reflections are called "generalized" or "pseudo-reflections". A reflection by a hyperplane in $\mathbb{R}^n$ has one eigenvalue equal to -1 and all the others equal to 1.

1.3 Polytopes

In Chapter 2 we describe the construction of "stars" with the symmetries of a Platonic solid. These solids are a special type of three-dimensional polytopes. This section gives a short introduction to polytopes in general and then focuses on the 2- and 3-dimensional case. In the very beginning we discuss some preliminaries.

A set $S \subset \mathbb{R}^n$ is called *convex* if the (line) *segment*[15] between any two points $\boldsymbol{x}, \boldsymbol{y} \in S$ lies entirely in S. Given any set $S \subset \mathbb{R}^n$ one defines its *convex hull*, denoted by $conv(S)$, as the set of all points $\boldsymbol{x} \in \mathbb{R}^n$ for which there exist points $\boldsymbol{x_1}, \ldots, \boldsymbol{x_m} \in S$ and real numbers $t_1, \ldots, t_m \geq 0$, $\sum_{i=1}^m t_i = 1$, such that $\boldsymbol{x} = \sum_{i=1}^m t_i \boldsymbol{x_i}$. Such a sum is called *convex combination* of the points $\boldsymbol{x_1}, \ldots, \boldsymbol{x_m}$. Alternatively one can define the convex hull of S as the intersection of all convex sets that contain S, see [Mat02, p. 5, 6].
A *hyperplane* in $\mathbb{R}^n$ is a set $h = \{\boldsymbol{x} \in \mathbb{R}^n,\ x_1a_1 + \ldots + x_na_n = b\}$ for some $\boldsymbol{0} \neq \boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $b \in \mathbb{R}$. Hyperlanes have dimension $n-1$. The plane h is the boundary of the *(closed) half space*[16] $h^+ = \{\boldsymbol{x} \in \mathbb{R}^n,\ x_1a_1 + \ldots + x_na_n \geq b\}$.

Now we are ready to define (real) *polytopes* in $\mathbb{R}^n$. There are two ways to do that, either as the convex hull of a finite set of points of $\mathbb{R}^n$ or as a *bounded*[17] intersection of finitely many half spaces of $\mathbb{R}^n$. To see that the two definitions are equivalent, some work has to be done, see [Mat02, p. 82f]. One defines the *dimension* of a polytope as the dimension of its affine hull[18].
A *face* of a polytope P is either the polytope itself or of the form $P \cap h$, where h is a hyperplane such that $P \subset h^+$ or $P \subset h^-$. Evidently, the faces of a convex polytope are again convex polytopes. The 0-dimensional faces are called *vertices*, the 1-dimensional ones *edges*. If the polytope has dimension m its $(m-1)$-dimensional faces are called *facets*. One says that the empty set is a face of dimension -1.

A 0-dimensional polytope is a finite set of points, a 1-dimensional one is a line segment. Polytopes of dimension two are called *polygons*. Polygons are said to be *regular* if they are equilateral and equiangular. Polytopes of dimension three are called *solids*. In the following we will only consider convex polytopes, mostly of dimension three.

Given a set $S \subset \mathbb{R}^n$ one defines its *dual set* as $S^* := \{\boldsymbol{z} \in \mathbb{R}^n,\ x_1z_1 + \ldots + x_nz_n \leq 1 \text{ for all } \boldsymbol{x} \in S\}$. It is easy to verify that for any set S its dual S^* is closed, convex and contains the origin.

Lemma 1. *Let S be any subset of $\mathbb{R}^n$. Then $(S^*)^*$ is equal to the closure*[19] *of $conv(S \cup \{\boldsymbol{0}\})$. In particular*[20]*, if S is a closed convex set containing $\boldsymbol{0}$ then $(S^*)^* = S$.*

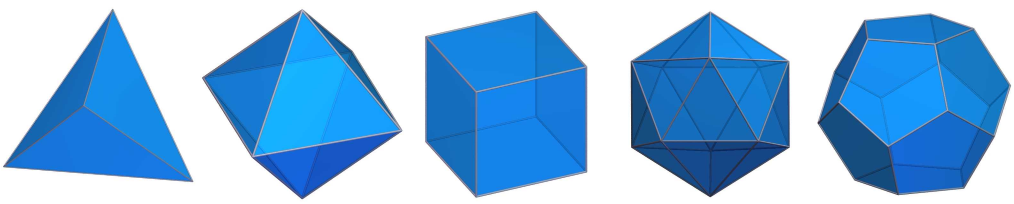
Proof. We start with the first statement and begin with proving the inclusion $cl(conv(S \cup \{\boldsymbol{0}\})) \subset (S^*)^*$. Suppose $\boldsymbol{z} \in conv(S \cup \{\boldsymbol{0}\})$, then by definition of the convex hull there exist elements

[15] The (line) segment between two points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ is defined as the set $\{\boldsymbol{z} = t\boldsymbol{x} + (1-t)\boldsymbol{y} \in \mathbb{R}^n,\ t \in [0,1]\} \subset \mathbb{R}^n$.
[16] This is one of the two half spaces determined by the hyperplane h, the other one is $h^- = \{\boldsymbol{x} \in \mathbb{R}^n,\ x_1a_1 + \ldots + x_na_n \leq b\}$.
[17] If we do not require boundedness we speak of a *polyhedron*.
[18] The *affine hull* of a set $S \subset \mathbb{R}^n$ is the set of all affine combinations of points of S. The *affine combination* of some points $\boldsymbol{x_1}, \ldots, \boldsymbol{x_m} \in \mathbb{R}^n$ is the set $\{\boldsymbol{x} = t_1\boldsymbol{x_1} + \ldots + t_m\boldsymbol{x_m} \in \mathbb{R}^n,\ \sum_{i=1}^m t_1 = 1\}$.
[19] The closure with respect to the Euclidean topology, denoted by $cl(S)$, $S \subset \mathbb{R}^n$.
[20] See [Mat02, p. 81].

$\boldsymbol{x_1}, \ldots, \boldsymbol{x_m} \in S \cup \{\boldsymbol{0}\}$ and non negative real numbers $t_1, \ldots, t_m$ with $\sum_{i=1}^m t_i = 1$, such that $\boldsymbol{z} = \sum_{i=1}^m t_i \boldsymbol{x_i}$. We have to show that $\boldsymbol{zy} = z_1 y_1 + \ldots + z_n y_n \leq 1$ for all $\boldsymbol{y} \in S^*$. But $\boldsymbol{zy} = (\sum_{i=1}^m t_i \boldsymbol{x_i})\boldsymbol{y} = \sum_{i=1}^m t_i(\boldsymbol{x_i y})$ and $\boldsymbol{x_i y} = x_{i1} + \ldots + x_{in} y_n \leq 1$ for all $1 \leq i \leq m$, because $\boldsymbol{x_i} \in S \cup \{\boldsymbol{0}\}$ and $\boldsymbol{y} \in S^*$. Therefore $\boldsymbol{zy} \leq \sum_{i=1}^m t_i = 1$ which proves that $conv(S \cup \{\boldsymbol{0}\}) \subset (S^*)^*$. But $(S^*)^*$ is closed and therefore contains the closure of $conv(S \cup \{\boldsymbol{0}\})$.
To prove the second inclusion, we assume that $C := cl(conv(S \cup \{\boldsymbol{0}\}))$ is strictly contained in $(S^*)^*$, i.e., there exists an element $\boldsymbol{z} \in (S^*)^* \setminus C$. Consider $D = \{\boldsymbol{z}\}$. Now we have two convex closed sets C and D with $C \cap D = \emptyset$ and D is bounded. By the separation theorem [Mat02, Theorem 1.2.3, p. 6] exists a hyperplane $h = \{\boldsymbol{x} \in \mathbb{R}^n,\ \boldsymbol{ax} = a_1 x_1 + \ldots + a_n x_n = 1\}$ in $\mathbb{R}^n$ such hat $C \subset h^-$, $D \subset h^+$ and $C \cap h = D \cap h = \emptyset$. The inclusion $C \subset h^-$ implies that $\boldsymbol{ax} \leq 1$ for all $x \in C = cl(conv(S \cup \{\boldsymbol{0}\}))$. Since $S \subset C$ we have $\boldsymbol{a} \in S^*$. $D \subset h^+$ means that $\boldsymbol{az} \geq 1$. But $\boldsymbol{z} \in (S^*)^*$ and therefore $\boldsymbol{yz} \leq 1$ for all $\boldsymbol{y} \in S^*$. In particular, we have $\boldsymbol{az} \leq 1$. All together we have $\boldsymbol{az} = 1$ and hence $\boldsymbol{z} \in D \cap h$, which is a contradiction.
The second statement follows directly from the first one. ★

Next consider a convex polytope P of dimension d containing the origin. By the lemma above its *dual* P^* is again a convex polytope containing the origin and $(P^*)^* = P$. Even more is true: the i-dimensional faces of P are in one-to-one correspondence with the $(d-i-1)$-dimensional faces of P^* for all $i = -1, 0, \ldots n$, see e.g. [Mat02, p. 90]. Thus for three-dimensional convex polytopes the faces correspond to the vertices of the dual solid and the edges to the edges.

A *regular solid* is a convex solid whose facets are identical regular polygons and at each of its vertices the same number of facets meet. There are exactly five regular solids. We omit a proof of this statement, see for example [Rom68, p. 24f]. The five regular solids are also named *Platonic solids*. They are called *tetrahedron, octahedron, hexahedron, icosahedron* and *dodecahedron* and are displayed in Figure 1.1.
The tetrahedron has four vertices, six edges and four facets. It is dual to itself. Its symmetry group, denoted by T_d, has 24 elements. Octahedron and hexahedron are dual to each other. The first has six vertices, twelve edges and eight facets, the second eight vertices, twelve edges and six facets. Another name for the hexahedron is cube. Their symmetry group is denoted by O_h. It is of order 48. The icosahedron has 12 vertices, 30 edges and 20 facets. It is dual to the dodecahedron, with 20 vertices, 30 edges and 12 facets. Their symmetry group consists of 120 elements and is denoted by I_h. For a list of generators of the symmetry groups of the Platonic solids, see Appendix A.2.

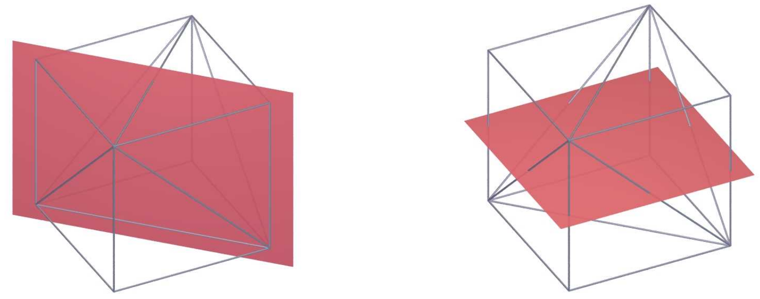


(a) Tetrahedron. (b) Octahedron. (c) Hexahedron. (d) Icosahedron. (e) Dodecahedron.

Figure 1.1: The five Platonic solids.

Remark 1. The notation for the symmetry groups of the Platonic solids has its origin in the context of crystallography and the study of symmetries of molecules. In contrast to how we defined symmetry groups in Section 1.2, in crystallography the definition is usually stated in a more heuristic way as the group of all transformations that preserve the distance between any two points of the object (i.e. the molecule or crystal) and bring it to coincide with itself. Such transformations are rotations, reflections and translations. When studying finite objects such as molecules, only rotations and reflections are possible and they have to be combined such that at least one point is left fixed under the action of the whole group. Such groups are called *point groups*. In our case the fixed point is the origin and we use certain matrix representations of the groups.

The usual approach to describe and find all these groups is to first consider only rotations. In this way one obtains the cyclic groups C_n, consiting of rotations about one axis by $2\pi/n$, the dihedral groups D_n and the rotational symmetry groups of the Platonic solids, dentoted by T, O and I, for the tetrahedral, octahedral and icosahedral group respectively. In a second step reflections are added to the rotations. For every rotational group all manners of adding a reflection such that the resulting groups is again a point group are considered. The groups that are obtained this way are denoted by the same letters as the rotational groups, attached with indices that indicate how the planes of reflections lie with respect to the axes of rotation. In the case of the tetrahedral group there are two possibilities how to add the reflections: the first one is adding a plane of reflection trough one edge and the midpoint of the opposite edge of the tetrahedron. If one considers a cube that shares four vertices with the tetrahedron this plane lies "diagonally" in it, see Figure 1.2a. Therefore the resulting group is denoted by T_d. It is the full symmetry group of the tetrahedron. The other possibility is a plane of reflection that parallel to two facets of the cube. This group is denoted T_h, the h stands for "horizontal", see Figure 1.2b. Both have 24 elements, but evidently T_h contains symmetries that the tetrahedron does not have. For the octahedral and the icosahedral groups the situation is different: there exists only one possibility to place the plane of reflection. The resulting groups are called O_h and I_h respectively. See [Ham62] for details.



(a) One of the reflection planes of T_d.

(b) One of the reflection planes of T_h.

Figure 1.2: Two possibilities of adding reflection planes to T.

After this small excursion on the notation, we introduce an important propertie of Platonic solids: They are examples of *vertex-transitive solids*, i.e., their symmetry group acts transitively on the set of their vertices. They are also *facet-transitive* and *edge-transitive*. Given a Platonic solid

P, we call the planes through the origin parallel to the planes containing a facet of the solid the *centerplanes*[21] of P.

A complex solid that has regular polygons as facets and that is vertex transitive is called *semi-regular*. The Platonic Solids, the prisms and the antiprisms[22] satisfy this condition. Besides these three families there are exactly 13 more solids that are semi-regular. These 13 solids are called *Archimedean solids* and are displayed in Figure 1.3. Note that the Archimedean solids are often defined as solids that have more than one type of regular polygons as facets but do have identical vertices in the sense that the polygons are situated around every vertex in the same way. This definition admits, besides the Platonic solids, prisms, antiprisms and the 13 Archimedean solids, an additional 14th solid called *pseudo rhomb-cub-octahedron* or *elongated square gyrobicupola.* Note its difference to the rhomb-cub-octahedron: the lower part of the first one is turned by $\pi/4$ with respect to the other one, see Figures 1.3n and 1.3e.
Some authors also include prisms and antiprisms when speaking of Archimedean solids. It is mostly due to this confusion of definitions that the existence of the pseudo rhomb-cub-octahedron has often been overseen. Also the sources we used, namely [Rom68, p. 47 − 59] and [Cro97, p. 156f and p. 367], are not very clear about it, see [Grü09][23]. We use the definition via vertex transitivity because that is a property we use in the construction described in Chapter 2.

The duals of the Archimedean solids are called *Catalan solids*[24] or *Archimedean duals.* Catalan solids are not semi-regular since they have vertices of more than one type and their facets are not regular polygons. Obviously they are still convex.
The symmetry groups of each Archimedean and Catalan solid is one of the three symmetry groups of the Platonic solids or a subgroup of them that only consists of the rotational symmetries and no reflections. With other words some of the Archimedean and Catalan solids do not have the *full* symmetry group of a Platonic solid but just the rotational symmetry group of the octahedron, $O \subset \mathrm{SO}_3(\mathbb{R})$, or the icosahedron, $I \subset \mathrm{SO}_3(\mathbb{R})$. For a table of all Archimedean and Catalan solids and their symmetry groups see Appendix A.1.
The Archimedean solids are vertex-transitive whereas the Catalan solids are not. But they are, unlike the Archimedean solids, facet-transitive[25].
The full symmetry group of a regular polygon[26] with m vertices is called the *dihedral group*, has order $2m$ and denoted by $D_m \subset \mathrm{O}_2(\mathbb{R})$. The finite subgroups of $\mathrm{O}_2(\mathbb{R})$ are precisely the cyclic and the dihedral groups, see e.g. [Arm88, p. 104]

In the 4-dimensional space $\mathbb{R}^4$ there exist exactly six convex regular polytopes. This statement is due to Ludwig Schläfli, a 19th-century mathematician from Switzerland, who first considered regular polytopes in dimension higher than three. See [Cox48, p.136 and $141f$].

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[21] We follow the notation from [Pie08].

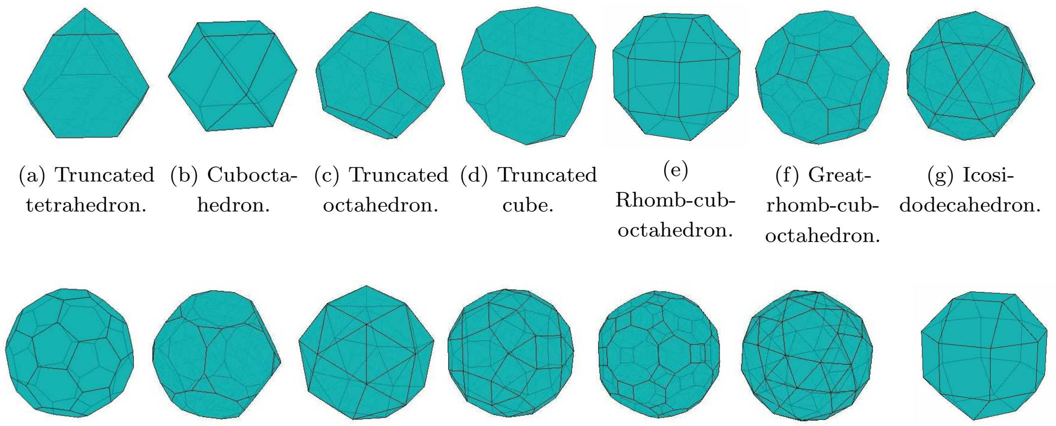
[22] For a prism or antiprism to be semi-regular, all edges must be of the same length.

[23] In [Grü09] the 13 Archimedean solids and the pseudo rhomb-cub-octahedron are called Archimedean and the 13 we called that way are called uniform or semi-regular.

[24] Named after Eugène Charles Catalan, who characterized certain semi-regular solids.

[25] This follows directly from the duality of Archimedean and Catalan solids and the vertex-transitivity.

[26] Its symmetry group in $\mathrm{SO}_2(\mathbb{R})$ is the cyclic group C_m with m elements. One obtains the dihedral groups from the cyclic groups by adding axes of reflection like described for the symmetry groups of Platonic solids in Remark 1.



(a) Truncated tetrahedron. (b) Cuboctahedron. (c) Truncated octahedron. (d) Truncated cube. (e) Rhomb-cub-octahedron. (f) Great-rhomb-cub-octahedron. (g) Icosi-dodecahedron.

(h) Truncated icosahedron. (i) Truncated dodecahedron. (j) Snub cube. (k) Rhomb-icosi-dodecahedron. (l) Great rhomb-icosi-dodecahedron. (m) Snub dodecahedron. (n) Pseudo rhomb-cub-octahedron

Figure 1.3: The 13 Archimedean solids and the pseudo rhomb-cub-octahedron.

1.4 Normal forms of simple singularities

For the construction of stars, Chapter 2, we need a tool that allows us to prescribe singularities in a previously chosen point. This section will provide us with this tool, namely a theorem that gives necessary conditions for simple singularities in a certain point. It is a result used in the classification of critical points. We will only introduce whats absolutely necessary to formulate the theorem. For more details and proofs we refer to [AGZV85, p.192ff].

Note that in [AGZV85] a more general situation is considered. In the following chapters we only need statements about polynomials $f \in \mathbb{C}[x_1, \ldots, x_n]$. The way we present the material here is therefore general enough.

A *(formal) power series* in n variables over a field K is a (possible) infinite sum,

$$f = \sum_{\boldsymbol{k} \in \mathbb{N}^n} a_{\boldsymbol{k}} x_1^{k_1} \ldots x_n^{k_n}, \quad a_{\boldsymbol{k}} \in K.$$

We say "formal" because the above sum does not necessarily need to converge. The formal power series over a field K form a ring, denoted by $K[[x_1, \ldots, x_n]]$. The polynomial ring is a subring of the ring of formal power series, $K[x_1, \ldots, x_n] \subset K[[x_1, \ldots, x_n]]$, in a canonical way.

As the polynomial ring is, the ring of formal power series is naturally graded by the degree. We denote the homogeneous parts of degree k by $K[x_1, \ldots, x_n]_k$ or $K[[x_1, \ldots, x_n]]_k$ respectively.

The *(weighted) degree* (with *weight vector* $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n) \in \mathbb{Q}^n$) of a monomial $\boldsymbol{x}^{\boldsymbol{k}} = x_1^{k_1} \ldots x_n^{k_n}$ is defined as $\boldsymbol{\omega k} = \omega_1 k_1 + \ldots + \omega_n k_n$. Obviously the "usual" degree can be viewed as a weighted degree with weight vector $\boldsymbol{\omega} = (1, \ldots, 1)$.

The *order* of a polynomial (or a power series) is the maximal integer d such that all its monomials have degree d or higher. Note that apparently the order can also be either a weighted order or the "usual" one with weights $(1, \ldots, 1)$.

We want to prescribe singularities of a certain type at a previously chosen point p. Without loss of generality we can assume that this point is $\mathbf{0} = (0, \ldots, 0)$. If not, say we want to have a singularity at $\boldsymbol{p} = (p_1, \ldots, p_n)$, we consider the polynomial $f(x_1 + p_1, \ldots, x_n + p_n) \in K[x_1, \ldots, x_n]$ instead.

We say that the function $f : \mathbb{C}^n \to \mathbb{C}$ has a *critical point* at $\mathbf{0}$ if the first derivative of f at $\mathbf{0}$ vanishes, i.e., $(\partial f/\partial x_1(\mathbf{0}), \ldots, \partial f/\partial x_n(\mathbf{0})) = \mathbf{0}$. Note that this is the same as to say that $\mathbf{0} \in \mathbb{C}^n$ is a singular point of the hypersurface $X = V(f)$ of f. The *multiplicity* of the critical point $\mathbf{0}$ of the function f is the dimension[27] below,

$$\mu := \dim_{\mathbb{C}} \mathbb{C}[[x_1, \ldots, x_n]]/(\partial f/\partial x_1, \ldots, \partial f/\partial x_n). \tag{1.1}$$

An analytic function $f : \mathbb{C}^n \to \mathbb{C}$ with $f(0, \ldots, 0) = 0$ is called *quasihomogeneous* of degree d with *weights* $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$ if for all $\lambda > 0$ the following equation holds $f(\lambda^{\omega_1} x_1, \ldots, \lambda^{\omega_n} x_n) = \lambda^d f(x_1, \ldots, x_n)$. In the following we will only allow rational weights, $\boldsymbol{\omega} \in \mathbb{Q}^n$ and consider the power series expansion of $f = \sum f_{\boldsymbol{k}} \boldsymbol{x}^{\boldsymbol{k}}$. In this case for f to be quasihomogeneous of degree d means that all indices lie in a hyperplane $\{\boldsymbol{k} = (k_1, \ldots, k_n) : \omega_1 k_1 + \ldots + \omega_n k_n = d\} \subset \mathbb{Q}^n$. If the degree d equals one we call this hyperplane the *diagonal* and denote it by Γ.
A quasihomogeneous function f is said to be *nondegenerate* if $\mathbf{0}$ is an isolated critical point, which is the same as to say that the multiplicity μ of $\mathbf{0}$ is finite. See [Dim87].
If a polynomial or power series f can be written as the sum of a nondegenerated quasihomogeneous polynomial f_0 of degree d with weights $\omega_1, \ldots, \omega_n$ and a polynomial (or power series) f' of weighted order strictly greater than d, $f = f_0 + f'$, it is called *semiquasihomogeneous* of degree d with weights $\omega_1, \ldots, \omega_n$.
One says f has a *simple singularity* at $\mathbf{0}$ if it is nondegenerate and the multiplicity μ equals one. With other words, a simple singularity is an isolated critical point of multiplicity one.

Remark 2. Every quasihomogeneous power series of degree one with weights $0 < \omega_i \leq 1/2$ is automatically a polynomial, [AGZV85, p. 192].
Suppose f is a power series, i.e., $f = \sum_{\boldsymbol{k} \in A} a_{\boldsymbol{k}} \boldsymbol{x}^{\boldsymbol{k}}$, $A \subset \mathbb{N}^n$. Being quasihomogeneous of degree 1 means that for all $\lambda > 0$ the following is true $f(\lambda^{\omega_1} x_1, \ldots, \lambda^{\omega_n} x_n) = \sum_{\boldsymbol{k} \in A} a_{\boldsymbol{k}} \lambda^{\omega_1 k_1 + \ldots + \omega_n k_n} \boldsymbol{x}^{\boldsymbol{k}} = \lambda \sum_{\boldsymbol{k} \in A} a_{\boldsymbol{k}} \boldsymbol{x}^{\boldsymbol{k}}$. Hence $\omega_1 k_1 + \ldots \omega_n k_n = 1$ for all $\boldsymbol{k} \in A$. If f is not a polynomial then A has infinite cardinality, i.e., we have an infinite system of linear equations that the weights must satisfy. Such a system of equation must generally not have solutions, hence f must be a polynomial. ★

In the following, for the sake of simplicity, we will restrict our self to the case of finite multiplicity, $\mu < \infty$, i.e., nondegenerate functions. Let f_0 be a quasihomogeneous or semiquasihomogeneous polynomial or power series of degree d with fixed weight vector $(\omega_1, \ldots, \omega_n)$. Fix a system of monomials that form a basis of $\mathbb{C}[[x_1, \ldots, x_n]]/(\partial f_0/\partial x_1, \ldots, \partial f_0/\partial x_n)$. A monomial is said to be lying *above (below or on) the diagonal* if it has (weighted) degree greater than (less than or equal to) d. Let $e_1, \ldots, e_s$ denote all monomials of the previously chosen basis that lie above the diagonal.

We say a function is *equivalent* to another, written $f \sim f'$ if there exists a biholomorphic[28] change that turns one into the other. Now we are ready to formulate the main theorem of this section.

Theorem 2. *Every semiquasihomogeneous function with quasihomogeneous part f_0 is equivalent to a function of the form $f_0 + \sum_{k=1}^{s} c_k e_k$, c_k constants.*

[27] Note that this is the dimension of $\mathbb{C}[[x_1, \ldots, x_n]]/(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ as a $\mathbb{C}$-vector space.

[28] A function is called biholomorphic, if it is holomorphic, i.e., complex-differentiable, bijective and its inverse is also holomorphic.

For the proof of this theorem the following lemma is used,

Lemma 2. *Let $g_1, \ldots, g_r$ be all basis monomials of degree $d' > d$. Every power series (polynomial) of the form $f_0 + f_1$, with the order of f_1 being strictly greater than d is equivalent to $f_0 + f_1'$, where f_1' is of the form $f_1' = (\text{monomials of } f_1 \text{ of degree } < d') + (c_1 g_1 + \ldots + c_r g_r)$.*

Proof. For a proof of this lemma we refer to [AGZV85, p. 209f]. ★

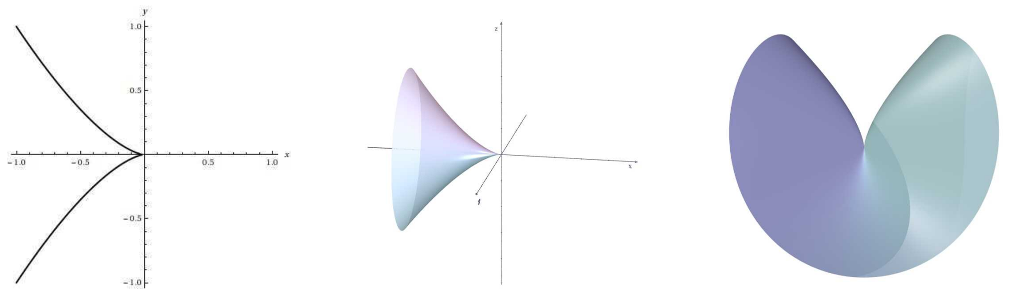
Proof of Theorem 2. Apply the lemma repeatedly. See [AGZV85, p. 209f]. ★

One part of the classification of singularities is a complete list of simple singularities, i.e., every simple singularity of a function in n variables is equivalent to one of the normal forms in Table 1.1. See [AGZV85, p. 245].

A_k	$x_1^{k+1} + x_2^2 + \ldots + x_n^2,$	$k \geq 1$
D_k	$x_1^{k-1} + x_1 x_2^2 + x_3^2 + \ldots + x_n^2,$	$k \geq 4$
E_6	$x_1^4 + x_2^3 + x_3^2 + \ldots + x_n^2,$	
E_7	$x_1^3 x_2 + x_2^3 + x_3^2 + \ldots + x_n^2,$	
E_8	$x_1^5 + x_2^3 + x_3^2 + \ldots + x_n^2,$	

Table 1.1: Normal forms of simple singularities.

Note that we want to construct *real* curves and surfaces. As in the chapter about invariant theory, also here the theory has been worked out over the algebraically closed field of the complex numbers, but again we can make use of it in the real setting as well. Theorem 2 allows to decide what kind of singularity a given polynomial has, considering complex transformations. In Chapter 2 we use it the other way around. We construct a polynomial that satisfies the conditions of Theorem 2 and then apply only real coordinate changes to it. So we still have the equivalence to the desired simple singularity, but we can not construct any polynomial equivalent to it, as we do not allow all biholomorphic transformations.



(a) Plane A_2-singularity, $x^3 + y^2 = 0$.

(b) The A_2^{++}-singularity, $x^3 + y^2 + z^2 = 0$.

(c) A_2^{+-}: $x^3 + y^2 - z^2 = 0$.

Figure 1.4: Singularities of type A_2.

The real zero-set of an A_2-singularity for $n = 2, 3$, i.e., $V(x^3 + y^2)$ respective $V(x^3 + y^2 + z^2)$ is a cusp, see Figures 1.4a and 1.4b. To construct "stars" we will only use singularities of this type. As was already mentioned, in the construction we will only admit real coordinate change and hence have to consider signs. Because of this, for $n = 3$ we will have A_2^{++} with normal form $x^3 + y^2 + z^2$ and A_2^{+-} with equation $x^3 + y^2 - z^2 = 0$ (see Figure 1.4c) instead of just A_2.

1.5 Some invariant theory

The construction of (Platonic) stars described in Chapter 2 is, among other things, based on some invariant theory. In Chapter 2 and the examples in Chapter 3 we will only be interested in polynomials with *real* coefficients (in two or three variables) that are invariant under the action of subgroups of the *real* orthogonal group. Since most sources[29] work in an algebraically closed setting we also start by giving some results on the structure of invariant rings of finite subgroups of the *complex* general linear group $\mathrm{GL}(\mathbb{C}^n)$ over $\mathbb{C}[x_1,\dots,x_n]$. While introducing them, when needed, we will also quote some results from commutative algebra. In the last part of this section we shall give an argument why the results still hold if we replace the complex numbers $\mathbb{C}$ by the real ones $\mathbb{R}$.

1.5.1 The invariant ring of finite subgroups of $GL(\mathbb{C}^n)$

For the rest of this section let $G \subset \mathrm{GL}(\mathbb{C}^n)$ be a subgroup of the complex general linear group. G acts naturally on $\mathbb{C}^n$ and hence[30] on the polynomial ring $\mathbb{C}[x_1,\dots,x_n]$. A polynomial $f \in \mathbb{C}[x_1,\dots,x_n]$ is called *invariant* under the action of the group G if it remains unchanged under this action. Evidently the set of all invariant polynomials is closed under addition and multiplication, hence it is a subring of the polynomial ring. It is called the *invariant ring of* G, denoted by $\mathbb{C}[x_1,\dots,x_n]^G$:

$$\mathbb{C}[x_1,\dots,x_n]^G := \{f \in \mathbb{C}[x_1,\dots,x_n],\ f = \pi\cdot f,\ \text{for all } \pi \in G\}. \tag{1.2}$$

Before we can present the theorems that are important for us we have to introduce an important tool: the so called *Reynolds operator*. Following [Stu08, p. 25f] we define it only for the special case of a finite group $G \subset \mathrm{GL}(\mathbb{C}^n)$ as:

$$r : \mathbb{C}[x_1,\dots,x_n] \to \mathbb{C}[x_1,\dots,x_n]^G :\ f \mapsto r(f) := \frac{1}{|G|}\sum_{\pi\in G} \pi\cdot f. \tag{1.3}$$

Lemma 3. *The Reynolds operator r is a $\mathbb{C}$-linear map, its restriction to the invariant ring $\mathbb{C}[x_1,\dots,x_n]^G$ is the identity and it is a $\mathbb{C}[x_1,\dots,x_n]^G$-module homomorphism.*

Proof. Verifying the first statement, i.e., showing that $r(a\ f + b\ g) = a\ r(f) + b\ r(f)$, $f, g \in \mathbb{C}[x_1,\dots,x_n]$ and $a, b \in \mathbb{C}$ is an easy calculation,

$$r(af+bg) = \frac{1}{|G|}\sum_{\pi\in G}\pi\cdot(af+bg) = \frac{1}{|G|}\sum_{\pi\in G}(a\pi\cdot f + b\pi\cdot g) =$$
$$= \frac{1}{|G|}a\sum_{\pi\in G}\pi f + \frac{1}{|G|}b\sum_{\pi\in G}\pi g = a\ r(f) + b\ r(g).$$

The second statement is also evident: Suppose $f \in \mathbb{C}[x_1,\dots,x_n]^G$ then $r(f) = \frac{1}{|G|}\sum_{\pi\in G}\pi\cdot f = \frac{1}{|G|}\sum_{\pi\in G} f = f = id(f)$. Finally the third statement is also easy to verify by showing that $r(fg) = fr(g)$ for all $f \in \mathbb{C}[x_1,\dots,x_n]^G$ and $g \in \mathbb{C}[x_1,\dots,x_n]$. ★

[29] For the parts on invariant theory we mostly follow [Stu08], the result on commutative algebra can be found in any book on that subject, we primarily used [AM69] and [BIV89].

[30] See Section 1.2 for details.

In general a Reynolds operator for a group is defined as a map satisfying the conditions from this lemma. Note that not any group has to admit a Reynolds operator. Groups that do admit one are called *reductive*. Now we are ready for a very important theorem in invariant theory, namely Hilbert's finiteness theorem. It can also be stated more generally than we do[31]. Namely with G being a *reductive group* and any algebraically closed field K, see [DK02, p. 46 and 49].

Theorem 3 (Hilbert's finiteness Theorem)**.** *The invariant ring $\mathbb{C}[x_1, \ldots, x_n]^G$ of a finite subgroup $G \subset \mathrm{GL}(\mathbb{C}^n)$ is finitely generated as a $\mathbb{C}$-algebra.*

Proof. Let $I_G \in \mathbb{C}[x_1, \ldots, x_n]$ be the ideal generated by all homogeneous invariants of positive degree. By the properties of the Reynolds operator from Lemma 3 it follows that any invariant of G is a linear combination of the symmetrized monomials $r(x_1^{k_1} \ldots x_n^{k_n})$. Therefore $I_G = (r(x_1^{k_1} \ldots x_n^{k_n}),\ (0, \ldots, 0) \neq (k_1, \ldots, k_n) \in \mathbb{N}^n)$. By Hilbert's Basis Theorem I_G is finitely generated, i.e., there exist finitely many homogeneous invariants $f_1, \ldots, f_m$ such that $I_G) = (f_1, \ldots, f_m)$. Now we show that every homogeneous invariant is an element of $\mathbb{C}[f_1, \ldots, f_m]$. Suppose that this is not true. Let $f \in \mathbb{C}[x_1, \ldots, x_n]^G \backslash \mathbb{C}[f_1, \ldots, f_m]$ be homogeneous and of minimal degree with these properties. As $f \in I_G$ we have $f = \sum_{j=1}^{m} a_j f_j$ where the $a_j \in \mathbb{C}[x_1, \ldots, x_n]$ are homogeneous polynomials of degree less than f. We apply the Reynolds operator:

$$f = r(f) = r(\sum_{j=1}^{m} a_j f_j) = \sum_{j=1}^{m} r(a_j) f_j.$$

The $r(a_j)$ are homogeneous invariants of degree less then the degree of f. We assumed that f has minimal degree among the polynomials in $\mathbb{C}[x_1, \ldots, x_n]^G \backslash \mathbb{C}[f_1, \ldots f_m]$, therefore the $r(a_j)$ must be contained in $C[f_1, \ldots f_m]$, but this implies that $f \in \mathbb{C}[f_1, \ldots f_m]$, which is a contradiction. ★

Hilbert's finiteness Theorem says that for any finite subgroup $G \subset \mathrm{GL}(\mathbb{C}^n)$ there exist invariants, say $g_1, \ldots, g_k \in \mathbb{C}[x_1, \ldots, x_n]^G$ such that any invariant h can be written as a polynomial in $g_1, \ldots, g_k$. With other words $\mathbb{C}[x_1, \ldots, x_n]^G = \mathbb{C}[g_1, \ldots, g_k]$. Note that the g_j need not be algebraically independent, i.e., there might exist an algebraic relation, $\mathbf{0} \not\equiv R \in \mathbb{C}[y_1, \ldots, y_k]$ such that $R(g_1, \ldots, g_k) = 0$.

Theorem 4. *The invariant ring $\mathbb{C}[x_1, \ldots, x_n]^G$ has the same (Krull) dimension as the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$:*

$$dim(\mathbb{C}[x_1, \ldots, x_n]^G) = dim(\mathbb{C}[x_1, \ldots, x_n]) = n.$$

Proof. In Section 1.1 we have allready stated that the dimension of the $\mathbb{C}[x_1, \ldots, x_n]$ is n. It remains to be shown that $dim(\mathbb{C}[x_1, \ldots, x_n]^G) = dim(\mathbb{C}[x_1, \ldots, x_n])$. By Proposition 9.2 from [Eis95] we have $dim(R) = dim(S)$ if the ring S is integral over the ring R. In our situation this means that if we only have to show that $\mathbb{C}[x_1, \ldots, x_n]$ is integral over $\mathbb{C}[x_1, \ldots, x_n]^G$.
Suppose the order of G is m. For all i we can define the polynomial

$$P_i(t) := \prod_{\pi \in G} (t - (\pi \cdot x_i)) = t^m + a_{i,1} t^m - 1 + \ldots + a_{i,m-1} t + a_{i,m} \in \mathbb{C}[x_1, \ldots, x_n][t].$$

It is invariant under the action of G and hence also its coefficients are invariants, i.e., $P_i \in \mathbb{C}[x_1, \ldots, x_n]^G[t]$ for all $i = 1, \ldots, n$. Obviously x_i is a root of P_i or with other words, all the x_i are integral over $\mathbb{C}[x_1, \ldots, x_n]^G$. Hence $\mathbb{C}[x_1, \ldots, x_n]$ is integral over $\mathbb{C}[x_1, \ldots, x_n]^G$. ★

[31] We refer to [Stu08, p. 26] for this version and proof of the theorem.

Lemma 4 (Noether's Normalization Lemma)**.** *Let R be a finitely generated $\mathbb{C}$-algebra of dimension n that does not contain zero divisors. Then there exist elements $y_1, \ldots, y_n \in R$, that are algebraically independent upon $\mathbb{C}$, such that R is a finitely generated as a $\mathbb{C}[y_1, \ldots, y_n]$-module.*
If R is additionally ***graded*** *then the y_i may be chosen* ***homogeneously****.*

Proof. For the proof we refer to [Eis95, p. 283] or [HB93, p. 37]. ★

For any graded algebra R we say that a set of homogeneous element $\{u_1, \ldots, u_n\} \subset R$, of positive degree is called a *homogeneous system of parameters*[32] if R is finitely generated as a $\mathbb{C}[u_1, \ldots, u_n]$-module and the $u_1, \ldots, u_n$ are algebraically independent.
We have just proven that there exists a homogeneous system of parameters for the invariant ring of any finite group $G \subset \mathrm{GL}(\mathbb{C}^n)$. Before we go on we resume shortly how. Hilbert's finiteness Theorem guarantees that $C[x_1, \ldots, x_n]^G$ is a finitely generated $\mathbb{C}$-algebra. By Theorem 4 it has dimension n. Hence we can apply (the graded version of) Noether's Normalization Lemma, which says that there exist n homogeneous, algebraically independent (upon $\mathbb{C}$) elements, say $u_1, \ldots, u_n$, such that the invariant ring is finitely generated as a $\mathbb{C}[u_1, \ldots, u_n]$-module, i.e., that form a homogeneous system of parameters.
Having a homogeneous system of parameter is already quite good, but even more is true: we will see in Theorem 6 that $\mathbb{C}[x_1, \ldots, x_n]^G$ is a *free* finitely generated $\mathbb{C}[u_1, \ldots, u_n]$-module. Such modules are called Cohen-Macaulay. With the help of the next theorem we make this concept precise.

Theorem 5. *Let R be a graded $\mathbb{C}$-algebra with homogeneous system of parameters $u_1, \ldots, u_n$. Then the following two conditions are equivalent.*

1. *R is a free, finitely generated $\mathbb{C}[u_1, \ldots, u_n]$-module, i.e., there exist elements $s_1, \ldots, s_t \in R$ such that $R = \bigoplus_{j=1}^{t} s_j \mathbb{C}[u_1, \ldots, u_n]$.*
2. *For all choices of homogeneous systems of parameters $v_1, \ldots, v_n$ the algebra R is a free, finitely generated $\mathbb{C}[v_1, \ldots, v_n]$-module.*

A graded $\mathbb{C}$-algebra R with homogeneous system of parameters $u_1, \ldots, u_n$ is called *Cohen-Macaulay* if the two equivalent conditions from Theorem 5 hold.
During the proof of Theorem 5 we need the notion of a *regular sequence* and the following lemmas about it. Let R be a (commutative) ring. A sequence $y_1, \ldots, y_r \in R$ is called regular if y_1 is not a zero divisor in R, y_i is not a zero divisor in $R/(y_1, \ldots, y_{i-1})$ for $2 \leq i \leq r$ and $R \neq (y_1, \ldots, y_r)$.

Lemma 5. *Let R is a graded $\mathbb{C}$-algebra of dimension n and $y_1, \ldots, y_n \in R$ homogeneous elements of positive degree and algebraically independent upon $\mathbb{C}$. Then $y_1, \ldots, y_n$ form a regular sequence if and only if R is a free module over the subring $\mathbb{C}[y_1, \ldots, y_r]$.*

Proof. We refer to [Sta79, Lemma 3.3]. ★

Lemma 6. *Let R be a graded $\mathbb{C}$-algebra.*

1. *For $a_1, \ldots, a_n \in \mathbb{N}$ positive one has: $u_1, \ldots, u_n \in R$ homogeneous of positive degree form a homogeneous system of parameters (or regular sequence) if and only if $u_1^{a_1}, \ldots, u_n^{a_n}$ are a homogeneous system of parameters (or regular sequence).*

[32] This definition is taken from [DK02, p. *p*.61], in [Stu08, p. 37] it is claimed that the algebraic independence follows from the other conditions.

2. *Let $u_1, \ldots u_n$ be a homogeneous system of parameters of R with $\deg(u_i) = \deg(u_j)$ for all $1 \leq i, j \leq n$ and $v_1, \ldots v_n$ any other homogeneous system of parameters of R. Then there exist $c_1, \ldots, c_n \in \mathbb{C}$ such that $v_1, \ldots, v_{n-i}, c_1u_1 + \ldots + c_nu_n$ is again a homogeneous system of parameters.*

Proof. See [Stu08]. ★

Now we are ready for the,

Proof of the Theorem 5. The implication from the second to the first condition is obvious. So lets assume that R a finitely generated *free* $\mathbb{C}[u_1, \ldots, u_n]$-module. By Lemma 5 this means that the homogeneous system of parameters $u_1, \ldots, u_n$ is a regular sequence. Let $v_1, \ldots, v_n$ be any homogeneous system of parameters. Again by Lemma 5 it is sufficient to show that the $v_1, \ldots, v_n$ is a regular sequence. We do that by induction on n.
$n = 1$: Let $u \in R$ be homogeneous, $\deg(u) > 0$ and regular, i.e., u is not a zero divisor and $v \in R$ a homogeneous parameter of positive degree but not regular, i.e., v is a zero divisor. Then we can choose a homogeneous element of positive degree $0 \neq f \in R$ such that $vf = 0$. This means that v is an element of the annihilator of f, $Ann(f) := \{g \in R,\ gf = 0\}$. Therefore also the ideal generated by v is contained in the annihilator: $(v) \subset Ann(f)$. But v is a parameter of the one-dimensional ring R. It follows that $R/(v)$ and hence also $R/Ann(f)$ have dimension zero. Therefore $u^m = 0$ in $R/Ann(f)$ for some $m \in N$, i.e., u^m is a zero divisor in R and hence not regular. By the first part of Lemma 6 this is a contradiction to the assumption that u is regular which proves the statement.
$(n-1) \to n$: Again by the first part of Lemma 6 we can assume that $\deg(u_i) = \deg(u_j)$ for all $i, j \in \{1, \ldots, n\}$. Now choose u as in the second part of Lemma 6, i.e., $u = c_1u_1 + \ldots + c_nu_n$, $c_i \in \mathbb{C}$ such that $v_1, \ldots, v_{n-i}, u$ is a homogeneous system of parameters. Suppose (after relabeling) that $u_1, \ldots, u_{n-1}, u$ are linear independent over $\mathbb{C}$. Then $u_1, \ldots, u_{n-1}, u$ is a regular sequence in R and hence $u_1, \ldots, u_{n-1}$ a regular sequence in $S := R/(u)$. By the choice of u the $v_1, \ldots, v_{n-1}$ form a homogeneous system of parameters of S. By induction (S is of dimension $n-1$) $v_1, \ldots, v_{n-1}$ is a regular sequence in S and $v_1, \ldots, v_{n-1}, u$ a regular sequence in R. Hence u is no zero divisor in the one-dimensional ring $R/(v_1, \ldots, v_{n-1})$. Again by induction v_n is also not a zero divisor and hence $v_1, \ldots, v_n$ a regular sequence in R. ★

The following theorem, first appeared in an article by M. Hochster and J. A. Eagon, [HE71].

Theorem 6. *If $G \subset \mathrm{GL}(\mathbb{C}^n)$ is a finite subgroup then its invariant ring $\mathbb{C}[x_1, \ldots, x_n]^G$ is Cohen-Macaulay.*

For the proof we need the following lemma.

Lemma 7. *Let R be a graded $\mathbb{C}$-algebra with homogeneous system of parameters $u_1, \ldots, u_n$ that is Cohen-Macaulay and $s_1, \ldots, s_t \in R$. Then $R = \bigoplus_{j=1}^t s_j\mathbb{C}[u_1, \ldots, u_n]$ if and only if $s_1, \ldots, s_t$ form a $\mathbb{C}$-vector space basis of $R/(u_1, \ldots, u_n)$.*

Proof. Suppose $s_1, \ldots, s_t \in R$ are as in Theorem 5, then $R = \bigoplus_{j=1}^t s_j\mathbb{C}[u_1, \ldots, u_n]$. Rewrite that as

$$R = \left(\bigoplus_{i=1}^{t} s_i\mathbb{C}\right) \oplus \left(\bigoplus_{(i_1,\ldots,i_n)\in\mathbb{N}^n\setminus\{\mathbf{0}\}} \bigoplus_{i=1}^{t} s_iu_1^{i_1}\ldots u_n^{i_n}\mathbb{C}\right).$$

The second sumand is just the ideal $(u_1, \ldots, u_n)$ so $R/(u_1, \ldots, u_n) = \bigoplus_{i=1}^{t} s_i\mathbb{C}$ which means that the s_i are a vector space basis. On the other hand if they are a vector space basis we can write $R/(u_1, \ldots, u_n) = \bigoplus_{i=1}^{t} s_i\mathbb{C}$ which implies analogously that $R = \bigoplus_{j=1}^{t} s_j\mathbb{C}[u_1, \ldots, u_n]$. ★

Proof of Theorem 6. In the proof of Theorem 4 we have already shown that $\mathbb{C}[x_1, \ldots, x_n]$ is integral over its subring $\mathbb{C}[x_1, \ldots, x_n]^G$, i.e., it is finitely generated as a $\mathbb{C}[x_1, \ldots, x_n]^G$-module.
The set of all polynomials that are mapped to zero by the Reynolds operator, $U := \{f \in \mathbb{C}[x_1, \ldots, x_n],\ r(f) = 0\}$, also form a $\mathbb{C}[x_1, \ldots, x_n]^G$-module. One can write the polynomial ring as the direct sum of modules: $\mathbb{C}[x_1, \ldots, x_n] = \mathbb{C}[x_1, \ldots, x_n]^G \oplus U$.
We allready showed that there exists a homogeneous system of parameters $u_1, \ldots, u_n$ for the invariant ring. The polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ is finitely generated as a $\mathbb{C}[x_1, \ldots, x_n]^G$-module and $\mathbb{C}[x_1, \ldots, x_n]^G$ is finitely generated as a $\mathbb{C}[u_1, \ldots, u_n]$-module. Therefore $\mathbb{C}[x_1, \ldots, x_n]$ is also finitely generated as a $\mathbb{C}[u_1, \ldots, u_n]$-module and hence $u_1, \ldots, u_n$ is a homogeneous system of parameters for $\mathbb{C}[x_1, \ldots, x_n]$ as well. Viewing $x_1, \ldots, x_n$ as a homogeneous system of parameters it becomes apparent that the polynomial ring is Cohen-Macaulay. Therefore, by Theorem 5, it is also a finitely generated, free $\mathbb{C}[u_1, \ldots, u_n]$-module.
From $\mathbb{C}[x_1, \ldots, x_n] = \mathbb{C}[x_1, \ldots, x_n]^G \oplus U$ one gets another decomposition of vector spaces,

$$\mathbb{C}[x_1, \ldots, x_n]/(u_1, \ldots, u_n) = \mathbb{C}[x_1, \ldots, x_n]^G/(u_1, \ldots, u_n) \oplus U/(u_1U + \ldots + u_nU).$$

Choose a homogeneous $\mathbb{C}$-basis $\bar{s_1}, \ldots, \bar{s_t}, \bar{s_{t+1}}, \ldots, \bar{s_r} \in \mathbb{C}[x_1, \ldots, x_n]/(u_1, \ldots, u_n)$ such that the first t elements are a basis of the first sumand above and the last $r-t$ elements a basis of the second one. Now one can choose homogeneous elements $s_1, \ldots, s_t \in \mathbb{C}[x_1, \ldots, x_n]^G$ and $s_{t+1}, \ldots s_r \in U$ such that $\bar{s_1}, \ldots, \bar{s_t}$ and $\bar{s_{t+1}}, \ldots, \bar{s_r}$ are their image under the projection $\mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n]/(u_1, \ldots, u_n)$.
By Lemma 7 we have $\mathbb{C}[x_1, \ldots, x_n] = \bigoplus_{i=1}^{r} s_i\mathbb{C}[u_1, \ldots, u_n]$ and therefore $\mathbb{C}[x_1, \ldots, x_n]^G = \bigoplus_{i=1}^{t} s_i\mathbb{C}[u_1, \ldots, u_n]$ which means that the invariant ring is Cohen-Macaulay. ★

In the situation from above theorem, i.e., when we consider an invariant ring, the direct sum from Theorem 5 is called *Hironaka decomposition*. The elements of the homogeneous system of parameters $\{u_1, \ldots, u_n\}$ are named *primary invariants* and the elements $s_1, \ldots, s_t$ *secondary invariants*. The number of secondary invariants depends on the degrees of the primary invariants and on the order of the group G, see [Stu08, p. 41]. There exist algorithms to calculate these invariants, see [Stu08]. One is implemented in the free Computer Algebra System `SINGULAR`[33].

Being Cohen-Macaulay means that each invariant polynomial f has a unique decomposition

$$f = \sum_{j=1}^{l} s_j P_j(u_1, \ldots, u_n),$$

for some polynomials $P_j \in \mathbb{C}[x_1, \ldots, x_n]$. But even better, for some groups each invariant can actually be written as just as a polynomial in the primary invariants. This is the following Theorem by Shepard, Todd and Chevalley.

[33]See `www.singular.uni-kl.de/index.html` for informations about `SINGULAR` and `www.singular.uni-kl.de/Manual/latest/sing_1189.htm#SEC1266` for the respective instruction.

Theorem 7 (Shepard-Todd-Chevalley)**.** *Let $G \subset \mathrm{GL}(\mathbb{C}^n)$ be a finite subgroup. Its invariant ring $\mathbb{C}[x_1, \ldots, x_n]^G$ is generated (as an algebra) by n algebraically independent homogeneous invariants if and only if G is a reflection group*[34]*.*

Proof. For a proof of this theorem we refer to [Stu08, p. 44ff]. ★

This means, if G is a reflection group we only need the primary invariants to generate the invariant ring, the only secondary invariant is 1. Hence we can write any invariant polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]^G$ as a (uniquely determined) polynomial in the primary invariants: $f(\boldsymbol{x}) = P(u_1, \ldots, u_n)$.

In the very beginning of this chapter we mentioned that for the construction described in Chapter 2 we need the real situation instead of the complex one presented here. Next we will show how we can deal with this problem.

1.5.2 The invariant ring $\mathbb{R}[x_1, \ldots, x_n]^G$:

The invariant ring $\mathbb{R}[x_1, \ldots, x_n]^G$: Let $G \subset \mathrm{GL}(\mathbb{R}^n)$ be a finite subgroup. Then there exist n homogeneous, algebraically independent polynomials $u_1, \ldots, u_n \in \mathbb{C}[x_1, \ldots, x_n]$ (called the primary invariants of G) and l (depending on the cardinality of G and the degrees of the u_i) polynomials $s_1, \ldots, s_l \in \mathbb{C}[x_1, \ldots, x_n]$ (the secondary invariants of G) such that the invariant ring decomposes into $\mathbb{C}[x_1, \ldots, x_n]^G = \bigoplus_{j=1}^{l} s_j \mathbb{C}[u_1, \ldots, u_n]$. There are algorithms to calculate these primary and secondary invariants, see [Stu08, p.25]. Also in [Stu08, p.1] it is claimed that if the scalars of the input for these algorithms are contained in a subfield K of C, then all the scalars in the output will also be contained in K. So in our case with $G \subset \mathrm{GL}(\mathbb{R}^n)$, the primary and secondary invariants will be real polynomials: $u_1, \ldots, u_n, s_1, \ldots, s_l \in \mathbb{R}[x_1, \ldots, x_n]$.
Now the claim is the notation above: $\mathbb{R}[x_1, \ldots, x_n]^G = \bigoplus_{j=1}^{l} s_j \mathbb{R}[u_1, \ldots, u_n]$.

Proof. The first inclusion $\mathbb{R}[x_1, \ldots, x_n]^G \supset \bigoplus_{j=1}^{l} s_j \mathbb{R}[u_1, \ldots, u_n]$ is trivial. We prove the opposite inclusion: Let $f \in \mathbb{R}[x_1, \ldots, x_n]^G \subset \mathbb{C}[x_1, \ldots, x_n]^G$ be an invariant polynomial. As $\mathbb{C}[x_1, \ldots, x_n]^G$ equals $\bigoplus_{j=1}^{l} s_j \mathbb{C}[u_1, \ldots, u_n]$, we can write f in the following, unique way:

$$f(x_1, \ldots, x_n) = \sum_{j=1}^{l} s_j \sum_{\alpha \in A} c_{j\alpha} u^{\alpha},$$

where $c_{j\alpha} = d_{j\alpha} + ie_{j\alpha}$ are complex constants, and A is some finite subset of $\mathbb{N}^n$. Then

$$\begin{aligned} f(x_1, \ldots, x_n) &= \sum_{j=1}^{l} s_j \left(\sum_{\alpha \in A} d_{j\alpha} u^{\alpha} + i \sum_{\alpha \in A} e_{j\alpha} u^{\alpha} \right) \\ &= \sum_{j=1}^{l} s_j \sum_{\alpha \in A} d_{j\alpha} u^{\alpha} + i \sum_{j=1}^{l} s_j \sum_{\alpha \in A} e_{j\alpha} u^{\alpha} \\ &= f_1(x_1, \ldots, x_n) + i f_2(x_1, \ldots, x_n). \end{aligned} \tag{1.4}$$

Here f_1 and f_2 are real polynomials. Since f is also contained in the real polynomial ring, f_2 must be equal to zero. But from $f_2(x_1, \ldots, x_n) = \sum_{j=1}^{l} s_j \sum_{\alpha \in A} e_{j\alpha} u^{\alpha} = \sum_{\alpha \in A} (\sum_{j=1}^{l} s_j e_{j\alpha}) u^{\alpha} = 0$ it

[34] See Definition 1 in Section 1.2.

would follow that for all $\alpha \in A$ the sum $\sum_{j=1}^{l} s_j e_{j\alpha}$ must be equal to zero, since the u_i are algebraically independent. Hence $f = f_1(x_1, \ldots, x_n) = \sum_{j=1}^{l} s_j \sum_{\alpha \in A} d_{j\alpha} u^{\alpha} \in \bigoplus_{j=1}^{l} s_j \mathbb{R}[u_1, \ldots, u_n]$.
★

Chapter 2

Construction of hypersurfaces with prescribed symmetries and singularities

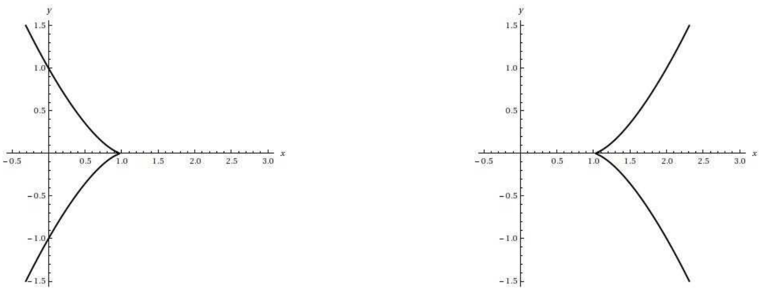
In this chapter we shall finally present a method to construct hypersurfaces with prescribed symmetries and isolated singularties of a special type[1]. We choose a group G, which is a finite subgroup of the real orthogonal group $\mathrm{O}_n(\mathbb{R})$ that is also a reflection group. Next we consider a finite set of points $P \subset A^n_{\mathbb{R}}$ on which G acts transitively. Our aim is to construct a real hypersurface whose symmetry group is equal to G and that has singularities of type A_2 exactly in the points of P.
In the examples presented in Chapter 3 we will only consider the two and three dimensional case. If $n = 2$ the group G will be a dihedral group D_m and P will be the set of vertices of a regular polygon (with m vertices). For $n = 3$, the group G will be the tetrahedral, octahedral or icosahedral group and P will denote either the set of vertices of a Platonic or of an Archimedean solid.
In Section 2.3 we shall mention two small generalizations of the construction. First the case that G is **not** a reflection group. Examples of this situation can be found in Section 3.4. Secondly we will choose a set of points P on which G does **not** act transitively. For example one can choose P to be the set of vertices of a Catalan solids, see the examples in Section 3.3.
For both $n = 2$ and $n = 3$ we demand even more than that. As we described in the introduction we want to construct "stars". For the "definition" we need one new notation. An A_2-singularity has normal form $x_1^3 + x_2^2 + \ldots + x_n^2 = 0$, see Section 1.4. For $n = 2$ the corresponding zero set is symmetric with respect to the x_1 axis, for $n = 3$ it is a rotational surface, see Figure 1.4b. Its axis of rotation is the x_1-axis. In both cases we call the x_1-axis the *tangent-line* of the cusp $Y = V(x_1^3 + x_2^2 + \ldots + x_n^2)$, $n = 2, 3$, at the origin. Apparently it is not the tangent-line in the differential geometrical sense[2]. One can also view this line as the limit of secants of Y with one point of intersection being the singular point $\mathbf{0}$ and the other point of intersection moving towards $\mathbf{0}$. Now let X be any variety with a singularity of type A_2 at a point p. Then we define the tangent-line at this point analogously. Note that it need no longer be an axis of rotation.

[1]We will only consider isolated singularities of type A_2, theoretically one could also use any other type presented in Section 1.4 about normal forms of isolated singularities.

[2]The origin is a singularity of the cusp, i.e., the surface is not a manifold there. Hence differential geometric methods fail there.

We want to emphasize that the following are not rigorous mathematical definitions.

Definition 2 (Plane stars)**.** Let P be a regular polygon with m vertices. Its symmetry group in $\mathrm{O}_2(\mathbb{R})$ is the *dihedral group* denoted by D_m. A *plane m-star* is a plane algebraic curve that is invariant under the action of the dihedral group D_m and has exactly m singularities of type A_2 in the vertices[3] of P "pointing away form the origin" (see Figure 2.1). Otherwise, i.e., if the cusps "point towards to the origin" we speak of a *plane m-anti-star*. The tangent-lines of X at p, for p being a singular point, should be the lines through the origin and p.



(a) Plane cusp "facing outside", $(x-1)^3+y^2=0$.

(b) Plane cusp "facing inside", $(x-1)^3-y^2=0$.

Figure 2.1: Plane cusps.

Definition 3 (Platonic, Archimedean, Catalan stars)**.** Let S be a Platonic (Archimedean, Catalan) solid and m the number of its vertices. Denote its symmetry group in $\mathrm{O}_3(\mathbb{R})$ by G. An algebraic surface X that is invariant under the action of G and has exactly m isolated singularities of type A_2 in the vertices of the solid, is called a *Platonic (Archimedean, Catalan) star*. We require that the cusps point outwards, otherwise we speak of an *anti-star*. In both cases for all singular points p the tangent-lines of X at p should be the lines through the origin and p.

It would be interesting to demand two more properties: boundedness[4] and connectedness. This properties would be necessary to reach the goal of actually constructing a figure that, heuristically speaking, "looks like a star". Including them during the construction would probably lead to results with less free parameters than we obtained. Nonetheless we do not consider this additional problems.

2.1 Recipe

For the rest of this section G denotes a finite subgroup of the real orthogonal group $\mathrm{O}_n(\mathbb{R})$ which additionally is a reflection group, and $P \subset A^n_{\mathbb{R}}$ a finite set of points on which G acts transitively. In Section 1.5 it has been shown that for such a group G there exists a set of primary invariants $\{u_1, \ldots, u_n\} \subset \mathbb{R}[x_1, \ldots, x_n]^G$ that generate its invariant ring as an $\mathbb{R}$-algebra: $\mathbb{R}[x_1, \ldots, x_n]^G = \mathbb{R}[u_1, \ldots, u_n]$. In the following we always assume that we have already constructed a set of homogeneous primary invariants. See A.6 for an example of the respective `SINGULAR` input and output.

[3] In the examples presented in Section 3.1 we will choose the vertices to lie in the mth roots of unity.

[4] There are approaches for constructing bounded curves or surfaces, see for example [KG99] or [TCS+94].

Our aim is to construct stars with symmetry group G and singularities in all points of P. We can reformulate this goal by saying we want to construct a polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]^G$. Note that we have to choose the degree r of the polynomial. We will discuss this choice later on.
Let d_i denote the degree of the i-th primary invariant, $d_i := \deg(u_i)$. To make the notation more compact we use multi-indices. We can write this polynomial in the following unique way,

$$f(\boldsymbol{u}) = f(u_1, \ldots, u_n) = \sum_{\boldsymbol{\alpha d} \leq r} a_{\boldsymbol{\alpha}} \boldsymbol{u}^{\boldsymbol{\alpha}}, \quad \text{where } a_{\boldsymbol{\alpha}} \in \mathbb{R}. \tag{2.1}$$

The zero set of such a polynomial has the desired symmetries, so we move on and prescribe singularities in the points of P. As the group acts transitively on P the algebraic set corresponding to the polynomial (2.1) has to have the same local geometry at each point of P. Therefore it is sufficient to choose one point and impose conditions on $f(\boldsymbol{u})$ there, in order to guarantee an A_2-singularity.
We can always suppose that P contains the point $p := (1, 0, \ldots, 0)$, otherwise we perform a coordinate change. We will use results from Section 1.4. There we assumed that the critical point is the origin. So first we have to translate, or consider the Taylor expansion of f at p, i.e., substitute $x_1 + 1$ for x_1 in $f(\boldsymbol{u}(x_1, \ldots .x_n))$ and denote it by F. We have the following necessary condition for an A_2-singularity, with $c_1, \ldots, c_n$ being real constants not equal to zero,

$$F(x_1, \ldots, x_n) := f(\boldsymbol{u}(x_1 + 1, x_2, \ldots, x_n)) = c_1 x_1^3 + c_2 x_2^2 + \ldots + c_n x_n^2 + \text{ higher order terms.} \tag{2.2}$$

To see that this is really a necessary condition we apply Theorem 2. We have to fix weights such that $f_0 = c_2 x_2^2 + \ldots + c_n x_n^2 + c_1 x_1^3$ is quasihomogeneous of degree one: $\omega = (1/3, 1/2, \ldots, 1/2)$. In (2.2) "higher order terms" refers to terms of weighted order bigger than 1. Then the polynomial F from (2.2) is a semiquasihomogeneous function with quasihomogeneous part f_0.
The theorem states that a semiquasihomogeneous function with quasihomogeneous part f_0 is equivalent to a function of the form $f_0 + \sum_{k=1}^{s} b_k e_k$, with b_k being constants and e_k all elements of a monomial basis of $\mathbb{C}[[x_1, \ldots, x_n]]/(\partial f_0/\partial x_1, \ldots, \partial f_0/\partial x_n) = \mathbb{C}[[x_1, \ldots, x_n]]/(x_1^2, x_2, \ldots, x_n)$ that lie above the diagonal, i.e., $\boldsymbol{x}^\alpha$ with $1/3\alpha_1 + 1/2\alpha_2 + \ldots + 1/2\alpha_n > 1$. But no such basis monomials exist. Therefore such a function is equivalent to f_0, i.e. has the desired singularity. ★
If we are in the plane case, i.e., $n = 2$, we will demand that c_1 and c_2 have the same sign, to guarantee that the cusps will "face outside", otherwise they will "face inside". If $n = 3$ we want c_2 and c_3 to have the same sing, or even to be equal, to prescribe an A_2^{++} and not an A_2^{+-}. If additionally c_1 has the same sign as c_2 and c_3 the cusps will "face outside", otherwise they will "face inside".
If we expand $F(x_1, \ldots, x_n)$ and compare the coefficients of $x_1, x_2, \ldots, x_n$ with the right hand side of Equation (2.2), we obtain a system of linear equations in the unknown coefficients of f from (2.1), i.e., in our notation the $a_{\boldsymbol{\alpha}}$. In general this system of equations will be under-determined. We will be left with free parameters, as we will see in the examples. By choosing the free parameters well we can achieve the additional properties mentioned in the beginning of this section, i.e., boundedness and connectedness. Note that for the visualizations of the examples the parameters are often chosen that way, but in no systematic manner but merely by "good guessing".
Evidently, in this construction we have to choose the degree r of the indetermined polynomial f. If we choose it too small the system of equation will be over-determined and we may not have a solution. We will chose r "as small as possible" in the sense that the system is still solvable. Obviously the degree r has to be greater or equal to three and depends on the degrees of the

primary invariants u_i.
Now we demonstrate the construction in one detailed example.

2.2 Explanatory example: Octahedral and hexahedral stars.

Example 1 (Octahedral and hexahedral stars). The octahedron and the cube - or hexahedron - have the same symmetry group O_h, of order 48, see Section 1.3. We choose coordinates x, y and z of $\mathbb{R}^3$ such that in these coordinates the vertices of the octahedron are $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$. Then O_h is generated by two rotations σ_1, σ_2 around the x and the y-axes by $\pi/2$ and the reflection against the x, y-plane τ, see A.2.

These matrices are the input for the algorithm implemented in `SINGULAR` that computes the primary and secondary invariants[5]. In this example the primary invariants that generate the invariant ring are the following,

$$\begin{aligned} u(x,y,z) &= x^2 + y^2 + z^2, \\ v(x,y,z) &= x^2y^2 + y^2z^2 + x^2z^2, \\ w(x,y,z) &= x^2y^2z^2. \end{aligned} \tag{2.3}$$

Octahedral stars: Clearly we need to start with an indeterminate polynomial of even degree greater than two. A degree four polynomial yields no solvable system of equations therefore we try a polynomial of degree six,

$$f(u,v,w) = 1 + a_1u + a_2u^2 + a_3u^3 + a_4uv + a_5v + a_6w.$$

We substitute $x+1$ for x and expand the resulting polynomial $F(x,y,z) = f(u(x+1,y,z), v(x+1,y,z), w(x+1,y,z))$. As described in Section 2.1 all monomials which have weighted norm (with weights $\boldsymbol{\omega} = (1/3, 1/2, 1/2)$) smaller or equal to 1, except x^3, y^2 and z^2, must not appear. All such monomials are the constants, the linear and the quadratic terms. Therefore the coefficients of these terms in the left hand side of (2.2) have to be zero. This yields the following system of linear equations[6]:

$$\begin{array}{lrcl} \text{Constant term of } F: & 1 + a_1 + a_2 + a_3 &=& 0, \\ \text{Coefficient of } x: & 2a_1 + 4a_2 + 6a_3 &=& 0, \\ \text{Coefficient of } x^2: & a_1 + 6a_2 + 15a_3 &=& 0, \\ \text{Coefficient of } y^2 \text{ and } z^2: & a_5 + a_1 + a_4 + 2a_2 + 3a_3 &=& c_1, \\ \text{Coefficient of } x^3: & 4a_2 + 20a_3 &=& c_2. \end{array} \tag{2.4}$$

Solving the first three equations from the system (2.4) yields the polynomial (2.5) with three free parameters. In addition we get an inequality from the condition that the coefficient of x^3 must have the same sign as the coefficient of y^2 and z^2 if we want to obtain a star. Substituting the solution of the first three equations yields $c_1 = a_4 + a_5$ and $c_2 = -8$. Therefore we impose $a_4 + a_5 \neq 0$ to obtain a star or an anti-star,

$$f(u,v,w) = (1-u)^3 + a_4uv + a_5v + a_6w, \quad \text{with } a_4 + a_5 \neq 0. \tag{2.5}$$

[5]See A.6 for the `SINGULAR` input in the (more involved) example of the icosahedral group I_h.
[6]The monomials y, z, xy, xz, yz do not appear, we do not obtain further equations from them.

From the construction it is clear that for $a_4 + a_5 = 0$ the zero set of (2.5) cannot have singularities of type A_2, so it has to be either smooth or have singularities of a different type. If all three parameters are equal to zero we obtain the sphere of radius one, i.e., a smooth[7] surface. Another choice for which $a_4 + a_5 = 0$ holds, $-a_5 = a_4 = 1$ and $a_6 = -10$, is displayed in Figure 2.2. The vertices of the corresponding octahedron are still isolated singularities but not of type A_2, actually they are not even simple. In the other examples similar behavior may appear. If we choose $a_4 = c$,

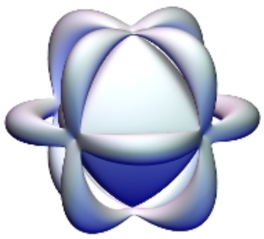
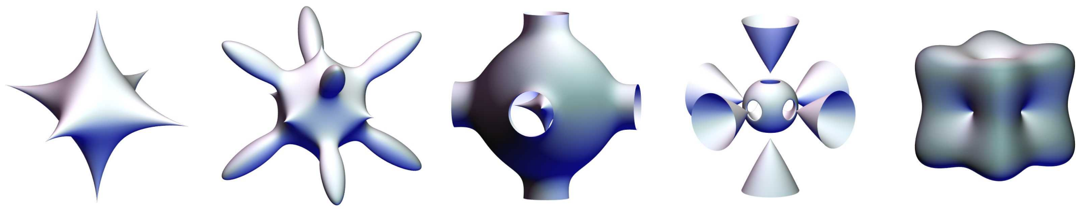


Figure 2.2: $V(f)$ with $a_4 = 1$, $a_5 = -1$ and $a_6 = -10$.

$a_5 = 0$ and $a_6 = -9c$, $c \neq 0$, the corresponding zero set is neither an octahedral star since it has too many singularities (we will describe this case more detailed in Section 3.3, Example 12). For the other choices of parameters the corresponding zero sets are octahedral stars for $a_4 + a_5 < 0$ (Figures 2.3a, 2.3b and 2.3c), or anti-stars for $a_4 + a_5 > 0$ (Figures 2.3d and 2.3e). Sometimes additional components appear and the stars or anti-stars become unbounded.



(a) Octahedral star, $a_4 = 0$, $a_5 = -100$, $a_6 = 0$.
(b) Octahedral star, $a_4 = -10$, $a_5 = 0$, $a_6 = 100$.
(c) Octahedral star, $a_4 = 20$, $a_5 = -100$, $a_6 = 0$.
(d) Octahedral anti-star, $a_4 = 8$, $a_5 = -6$, $a_6 = 0$.
(e) Octahedral anti-star, $a_4 = 2$, $a_5 = 1$, $a_6 = 0$.

Figure 2.3: Octahedral stars and anti-stars.

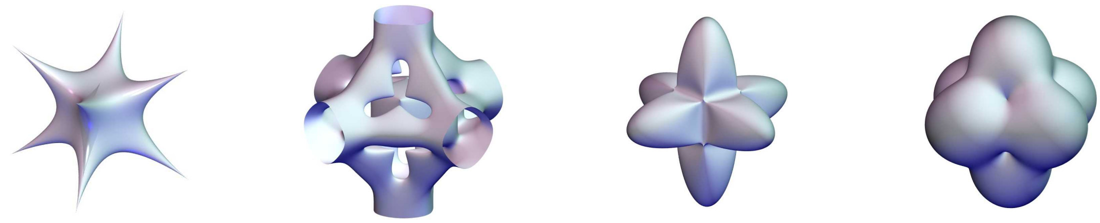
Hexahedral stars: Now we present the case of the Platonic solid dual to the octahedron, namely the cube, or hexahedron. If we use the same coordinates as before, it has vertices in $(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}})$. For the construction we want one vertex to lie in $(1, 0, 0)$. We need to perform a coordinate change after which the hexahedron has one vertex in $(1, 0, 0)$. This is the same as rotating the hexahedron. After this coordinate change we remain with new invariants in the new coordinates. With these invariants we can proceed as in the example of the octahedron. In the

[7] Actually if we choose all three parameters to be zero the resulting polynomial is $(1-u)^3$. Its zero set is a sphere, but it is taken three times. Therefore, if we stick to our definition of a singular point from Section 1.1, actually all of its points are singular.

Appendix A.4 we discuss the coordinated change explicitly. Again we need a polynomial of degree six, since degree four yields no solution. After solving the system of equations we perform the converse coordinate change and obtain the following polynomials (2.6) as candidates for hexahedral stars or anti-stars,

$$f(u,v,w) = 1 - 3u + a_2u^2 + a_3u^3 + a_4uv + (9 - 3a_2)v - 9(3 + a_4 + 3a_3)w, \tag{2.6}$$

with $3a_2 + 9a_3 + 2a_4 \neq 0$. For $a_2 = 3$, $a_3 = -1$ and $a_4 = 0$ we obtain the sphere. If we choose the parameters of the polynomial in (2.6) such that $3a_2 + 9a_3 + 2a_4 = 0$ we can not have A_2-singularities. Again there exists one choice of parameters, namely $a_2 = 3$, $a_3 = -1$ and $a_4 = c \neq 0$, for which the surface has too many singularities. We obtain the same object as in the example of the octahedral star, see Example 12 for details. In the other cases we obtain a hexahedral star for $3a_2 + 9a_3 + 2a_4 < 0$ (Figures 2.4a, 2.4b and 2.4c), or anti-star for $3a_2 + 9a_3 + 2a_4 > 0$ (Figure 2.4d), even though, as in the example of the octahedral stars, additional components may appear.



(a) Hexahedral star, $a_2 = -100$, $a_3 = 0$, $a_4 = 0$.

(b) Hexahedral star, $a_2 = 1$, $a_3 = 0$, $a_4 = -2$.

(c) Hexahedral star, $a_2 = 6$, $a_3 = -1$, $a_4 = -6$.

(d) Hexahedral anti-star, $a_2 = 6$, $a_3 = -1$, $a_4 = 0$.

Figure 2.4: Hexahedral stars and anti-stars.

2.3 Three generalizations

2.3.1 G is not a reflection group

Let G be a finite subgroup of $O_n(\mathbb{R})$ but not a reflection group. Then by Theorem 6 and the remark on the real situation from Section 1.5 the invariant ring admits the Hironaka decomposition $\mathbb{R}[x_1,\ldots,x_n]^G = \bigoplus_{j=1}^t s_j\mathbb{R}[u_1,\ldots,u_n]$. We assume that we have already constructed a set of primary and secondary invariants, $u_1,\ldots,u_n$ and $s_1,\ldots,s_t$. Then a polynomial $f \in \mathbb{R}[x_1,\ldots,x_n]^G$ of degree r can uniquely be written in the form, (with $f_j \in \mathbb{R}[u_1,\ldots,u_n]$),

$$f(x_1,\ldots,x_n) = \sum_{j=1}^t s_jf_j = \sum_{j=1}^t s_j \sum_{\boldsymbol{\alpha_j d} \leq r-e_j} a_{\boldsymbol{\alpha_j}}\boldsymbol{u}^{\boldsymbol{\alpha_j}}, \quad \text{where } a_{\boldsymbol{\alpha_j}} \in \mathbb{R}, \tag{2.7}$$

where $e_j = \deg s_j$, $j = 1,\ldots,t$, $\boldsymbol{d} = (d_1,\ldots,d_n)$ and $d_i = \deg u_i$. Then we can proceed as in the case of G being a reflection group, i.e. consider the Equation (2.2). See Section 3.4 for examples of this construction.

2.3.2 G does not act transitively

Let G be a finite subgroup of $O_n(\mathbb{R})$ that is a reflection group (or not) as before. But now let $P \subset \mathbb{R}^n$ be a finite set of points on which G does not act transitively. This situation occurs for example if we want to construct a Catalan star, since Catalan solids are not vertex-transitive. See Section 3.3 for examples.

Assume that P consists of k orbits under the action of G. Then we have to choose k points $p_1, \ldots, p_k \in P$ and prescribe singularities there. We probably will have to perform a coordinate change for every point p_j after which it is of the form $(b_j, 0, \ldots, 0)$, $b_j \neq 0 \in \mathbb{R}$. Note that, as the points probably have different Euclidean norm we can not just assume that after the coordinate change they are equal to $(1, 0, \ldots, 0)$. Using the general invariant polynomial (2.1) (or (2.7) if G is not a reflection group) we obtain k equations of the form,

$$F_j(x_1, \ldots, x_n) := f(\boldsymbol{u}(x_1 + b_j, x_2, \ldots, x_n)) = c_1^j x_1^3 + c_2^j x_2^2 + \ldots + c_n^j x_n^2 + \text{ higher order terms.} \quad (2.8)$$

Here "higher order terms" also refers to terms of weighted[8] order greater than one. Again we want $c_2^j, \ldots, c_n^j$ all to have the same sign to obtain A_2^{++} singularities. If $b_j > 0$ then c_1^j and $c_2^j, \ldots, c_n^j$ have to have the same sign to obtain stars, if $b_j < 0$ they have to have different signs.

Singularities of different types than A_2: Applying Theorem 2 one could theoretically prescribe any simple singularity in any point. Still there might emerge some problems. We will discuss this with the help of some examples, see Section 3.5.

[8] With weights $\omega = (1/3, 1/2, \ldots, 1/2)$.

Chapter 3

Examples

3.1 Plane dihedral stars

In this section we present examples of plane stars, that we defined in the previous chapter, Definition 2. We shall compare the results from our construction with another one, namely the *hypocycloids*, that also produces plane stars in our sense. So before we go on to presenting the examples we shortly introduce hypocycloids. For a list of all primary invariants used in the examples see Appendix A.3.

3.1.1 Hypocycloids and the implicitization of trigonometric curves

A *hypocycloid* is the trace of a point P on a circle of radius r, rolling within a bigger circle of radius R. This situation is illustrated with Figure 3.1, for $R = 1$ and $r = 1/4$. To find the parameterization

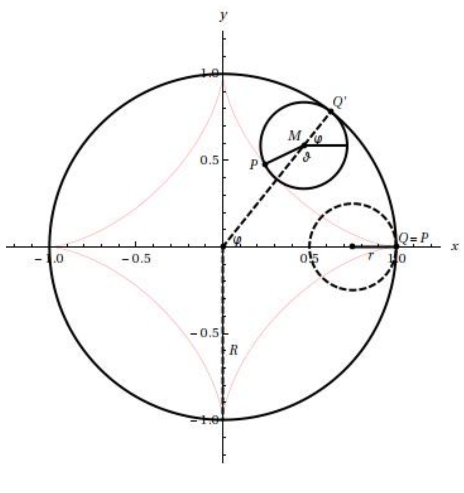


Figure 3.1: An Astroid with generating circles.

of a hypocycloid we have to give the coordinates of the point P depending on the angle φ. The center M of the smaller circle has coordinates $((R - r)\cos\varphi, (R - r)\sin\varphi)$. The coordinates of P with respect to the center M are $(r\cos(2\pi - \vartheta), r\sin(2\pi - \vartheta)) = (r\cos\vartheta, -r\sin\vartheta)$. Therefore the coordinates of P with respect to the origin are $((R - r)\cos\varphi + r\cos\vartheta, (R - r)\sin\varphi - r\sin\vartheta)$. For

$\varphi = 0$ the point P coincides with the point Q. Because the smaller circle rolls on the bigger one, the length of the arc $Q'P$ on the small circle has to be the same as the length of the arc QQ' on the big one. Hence $R\varphi = r(\vartheta + \varphi)$. Subsituting ϑ according to this equation yields

$$P(\varphi) = \begin{pmatrix} (R-r)\cos\varphi + r\cos\frac{R-r}{r}\varphi \\ (R-r)\sin\varphi - r\sin\frac{R-r}{r}\varphi \end{pmatrix}, \quad \varphi \in [0, 2\pi]. \tag{3.1}$$

If the ratio of the radii is an integer, $R : r = k$, the curve is closed and has exactly k "cusps" but no self-intersections. The parameterization of such a hypocycloid is given in (3.2).

$$\begin{pmatrix} x(\varphi) \\ y(\varphi) \end{pmatrix} = \begin{pmatrix} (k-1)r\cos\varphi + r\cos{(k-1)\varphi} \\ (k-1)r\sin\varphi - r\sin{(k-1)\varphi} \end{pmatrix}, \quad \varphi \in [0, 2\pi]. \tag{3.2}$$

Hypocycloids are only one example of curves generated by tracing a point related to a circle. Others are *cycloids* - the trace of a point of a circle rolling on a line - or *epicycloids*, where the small circle rolls on the outside of the bigger circle. The resulting curves of the latter ones could be compared with the plane anti-stars like we will compare the hypocycloids with the plane stars. There are a lot of sources on cycloids and related curves, see for example [Law72].

Hypocycloids with parameterization (3.2) evidently "look like stars" but we have to show that they really are plane stars in the sense of Definition 2. First of all we have to see if they are algebraic after all, i.e., if there exists a polynomial whose zero set coincides with the parameterized curve. Such a polynomial is also called implicitization of the curve. In [HS98] it is shown when there exists an implicitization for trigonometric curves such as the hypocycloids. We do not want to go into detail, but we sketch how one can find the implicitization if it exists. A *trigonometric curve* is a curve that can be parameterized in the following way,

$$[0, 2\pi] \to \mathbb{R}^2 : \ \varphi \mapsto \begin{pmatrix} \sum_{k=0}^{m} a_k \cos k\varphi + b_k \sin k\varphi \\ \sum_{k=0}^{n} c_k \cos k\varphi + d_k \sin k\varphi \end{pmatrix}, \quad a_k, b_k, c_k, d_k \in \mathbb{R}.$$

Parameterizations are called *simple* if the curve (possibly except finitely many points) is traced only once. A *simplification* of a parameterized curve is a simple parameterization of the same curve. A trigonometric curve admits an implicitization if and only if it has a trigonometric simplification. Therefore lets assume that a simple trigonometric parameterization[1] $[0, 2\pi] \to \mathbb{C}$. By substituting $\cos{(k\varphi)}$ by $\frac{z^k + z^{-k}}{2}$ and $\sin{(k\varphi)}$ by $\frac{z^k - z^{-k}}{2i}$, with z lying on the unit circle $S^1 \subset \mathbb{C}$, one obtains a parameterization $S^1 \to \mathbb{C}$. It can be written in the following way, called the *complex form* of the trigonometric parameterization, $z \to (P(z)/z^m, Q(z)/z^n)$. Here P and Q are complex polynomials in one variable of degree $2m$ and $2n$ respectively. Given a simple trigonometric parameterization, its implicitization is given by

$$f(x, y) = resultant_z(P(z) - z^m x, Q(z) - z^n y).$$

This is Theorem 3.2 in [HS98]. For a proof and more details on simplifications and implicitization we refere to the same paper.

Recalling the way hypocycloids are constructed it becomes apparent that, for $k > 2$, the parameterization we give is simple and hence admits an implicitization[2]. It is also evident that the

[1] Note that now, for the sake of convenience, we consider the curve to be in the complex plane.

[2] The hypocycloid for $k = 2$ is parameterized by $(2r\cos\varphi, 0)$ where φ is in $[0, 2\pi]$. So it is not an algebraic curve but an interval on the x-axis.

symmetry group of a hypocycloid with parameterization (3.2) is the dihedral group D_k. It remains to be shown that the "cusps" indeed are singularities of type A_2 to show that they are plane stars. We checked this, using Theorem 2 on simple singularities, for each example presented separately.

In the construction of stars via primary invariants we always try to find a polynomial of minimal degree that satisfies these properties. We will see that sometimes the hypocycloids coincide with the stars we obtain that way. In one of the examples presented here, namely the 5-star, the degree of the implicitization of the hypocycloid is higher than the degree of the polynomial our construction yields.

3.1.2 Examples

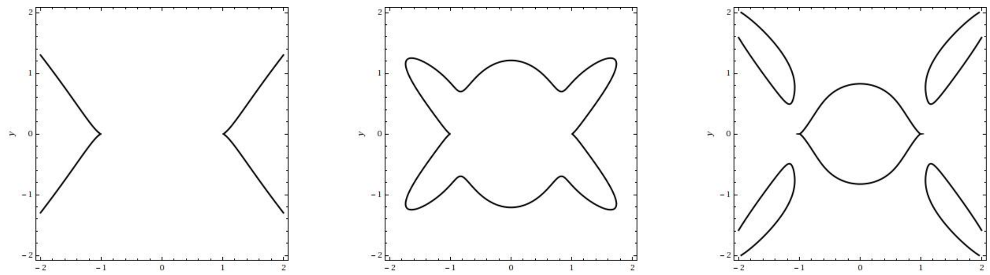
Example 2 (2-stars). The group D_2 has primary invariants

$$\begin{aligned} u(x,y) &= x^2, \\ v(x,y) &= y^2. \end{aligned} \tag{3.3}$$

Our constructions yields the degree six polynomial (3.4) with six free parameters,

$$f(u,v) = (1-u)^3 + a_1 v + a_2 uv + a_3 v^2 + a_4 uv^2 + a_5 u^2 v + a_6 v^3, \tag{3.4}$$

with $a_1 + a_2 + a_5 \neq 0$. The corresponding curves are stars for $a_1 + a_2 + a_5 < 0$ an'd anti-stars otherwise. Note that it can look a lot different then we might expect or desire, see Figure 3.2. The



(a) $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $a_4 = 0$, $a_5 = 1$, $a_6 = 0$.

(b) $a_1 = 0$, $a_2 = -1$, $a_3 = 1$, $a_4 = -1$, $a_5 = 1.5$, $a_6 = -1$.

(c) $a_1 = -1$, $a_2 = 0$, $a_3 = 0$, $a_4 = 1$, $a_5 = 0.5$, $a_6 = -1$.

Figure 3.2: Some plane 2-stars and anti-stars.

choice $a_1 \neq 0$ and the remaining parameters equal to zero yields the simple equation,

$$f(u,v) = (1-u)^3 + a_1 v, \quad \text{with } a_1 \neq 0. \tag{3.5}$$

For $a_1 < 0$ we obtain a 2-star, for $a_1 > 0$ anti-stars. The corresponding curve runs through the points $(0, \pm\frac{1}{\sqrt{-a_1}})$ and is bounded. See Figure 3.3a. For $a_1 > 0$ it is an unbounded anti-star. In both cases it has two singularities in $(\pm 1, 0)$.

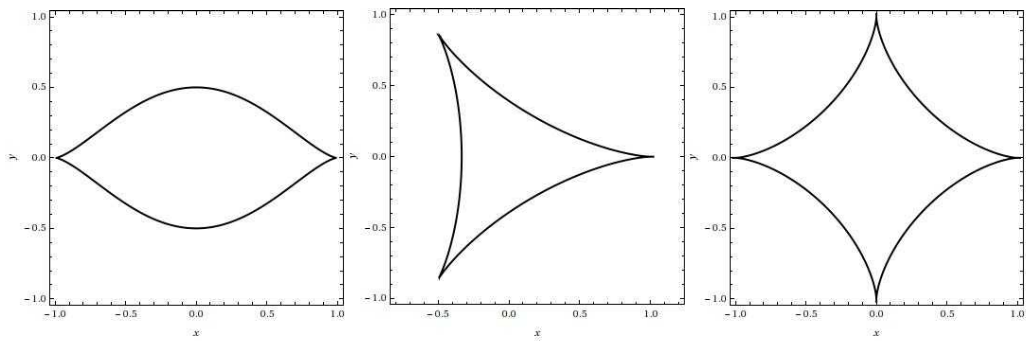
Note that we managed to construct 2-stars, while the other possible construction, via hypocycloids does not work here.

Example 3 (3-stars). The primary invariants of D_3 are

$$\begin{aligned} u(x,y) &= x^2 + y^2, \\ v(x,y) &= x^3 - 3xy^2. \end{aligned} \tag{3.6}$$

In this case a degree four polynomial suffices to generate a star, see Figure 3.3b. The polynomial (3.7) is completely determined, we have no free parameters. It coincides with the hypocycloid for $k = 3$, which is also called *deltoid*,

$$f(u, v) = 1 - 6u - 3u^2 + 8v. \tag{3.7}$$



(a) A 2-star (3.5), $c_1 = -4$. (b) The deltoid. (c) The astroid.

Figure 3.3: Some plane dihedral stars.

Example 4 (4-stars). The dihedral group of order eight, D_4, has primary invariants,

$$\begin{aligned} u(x, y) &= x^2 + y^2, \\ v(x, y) &= x^2 y^2. \end{aligned} \tag{3.8}$$

The construction described in Chapter 2 yields the following polynomial of degree six with two free parameters,

$$f(u, v) = (1 - u)^3 + a_1 v + a_2 uv, \qquad \text{with} \;\; a_1 + a_2 \neq 0. \tag{3.9}$$

For $a_1 + a_2 < 0$ we obtain stars, for $a_1 + a_2 > 0$ anti-stars. In both cases additional components may appear. The curves become unbounded for $a_2 > 4$.
The hypocycloid with four cusps is also called *astroid*. Its implicit equation is $(1 - u)^3 - 27v = 0$. So if we choose $a_1 = -27$ and $a_2 = 0$ in (3.9) we obtain the same curve. It is displayed in Figure 3.3c.

Example 5 (5-stars). The primary invariants of D_5 are

$$\begin{aligned} u(x, y) &= x^2 + y^2, \\ v(x, y) &= x^5 - 10x^3y^2 + 5xy^4. \end{aligned} \tag{3.10}$$

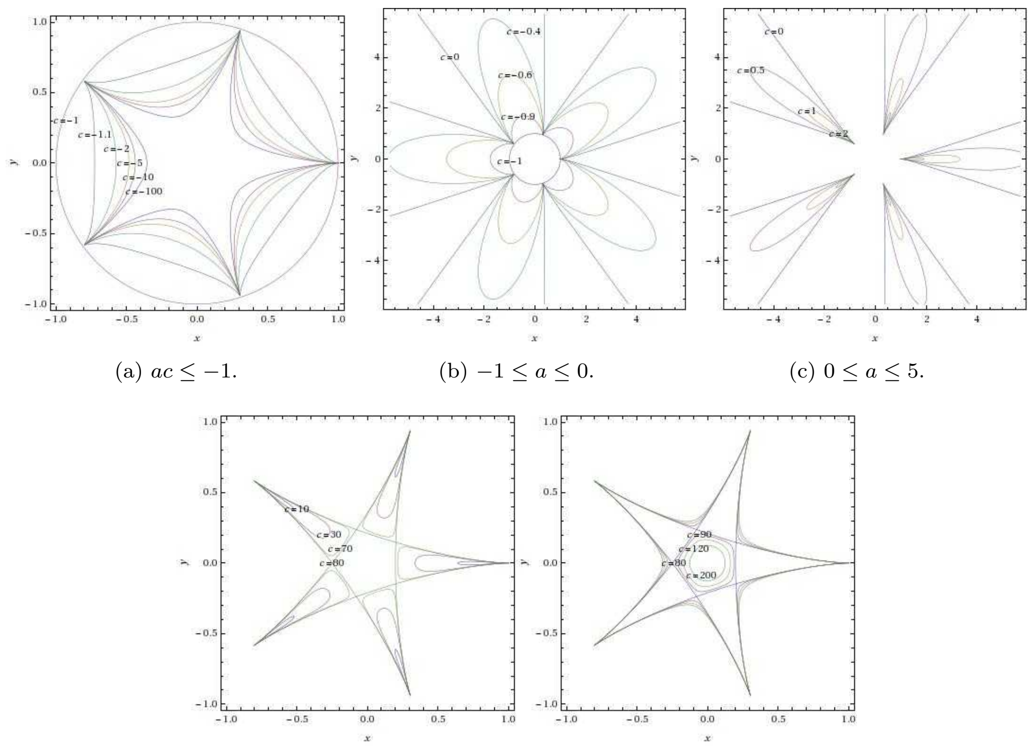
A polynomial of degree four yields no solution. If we try a degree five polynomial, we obtain polynomial (3.11) with no free parameters. It only permits anti-stars.

$$f(u, v) = 1 - \frac{10}{3}u + 5u^2 - \frac{8}{3}v. \tag{3.11}$$

Note that it is just a special case, choosing the free parameter $a = 0$, of the following polynomial for plane 5-stars and anti-stars of degree six,

$$f(u, v) = 1 - \frac{a + 10}{3}u + (2a + 5)u^2 - \frac{8}{3}(1 + a)v + au^3, \quad \text{with } a \neq -1,\ 5. \tag{3.12}$$

If we look closer at this equation and let the parameter value a vary we observe a quite interesting behavior. For $a < -1$ one obtains a star, the smaller a gets, the smaller gets its "inner radius", see Figure 3.4a. The choice $a = -1$ yields a circle with radius one. Note that the five singularities of the curves (3.12) always lie on this circle. For $-1 < a < 5$ the cusps of (3.12) point inwards, i.e., we have anti-stars. For $-1 < a < 0$ the curve has one bounded component, for $a = 0$, it is unbounded with five components, Figure 3.4b. For $0 < a < 5$ the curve is again bounded, but has five components, like drops falling away from the center, Figure 3.4c. For $a = 5$ we only have finitely many real solutions, the five points that are singular in the other cases. If we choose $a > 5$ we obtain stars again, i.e., the cusps point outwards, even though for $5 < a < 80$ the curve also has five components, like drops falling towards the origin, Figure 3.4d. The curve we obtain for $a = 80$ is special since it has self intersections, i.e., five additional singularities. They lie on a circle with radius one quarter, on a regular pentagon. These "extra singularities" are of type[3] A_1. One could call this curve an *algebraic pentagram.* For $a > 80$ the curve has two components, see Figure 3.4e.



(a) $ac \leq -1$. (b) $-1 \leq a \leq 0$. (c) $0 \leq a \leq 5$.

(d) $5 \leq a \leq 80$. (e) $a \geq 80$.

Figure 3.4: 5-stars and anti-stars with varing parameter value a.

The implicit equation of the hypocycloid (3.13) with five cusps is already of degree eight, while the polynomial we found with our construction has degree six. The two cannot coincide for any choice of the free parameter a.

[3]Plane curve A_1-singularities, see Table 1.1 in Section 1.4.

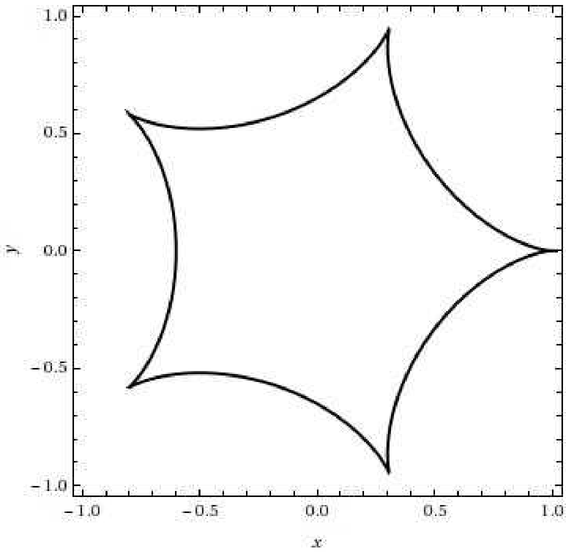


Figure 3.5: Zero set of the implicit equation of the hypocycloid with five cusps (3.13).

$$f(x,y) = -\frac{243}{125} + \frac{108}{25}(x^2+y^2) + \frac{6}{25}(x^2+y^2)^2 + \frac{12}{25}(x^2+y^2)^3 + (x^2+y^2)^4 - \\ - \frac{512}{125}(x^5 + 5y^4x - 10x^3y^2). \quad (3.13)$$

3.2 Platonic and Archimedean stars

We already gave the examples of two Platonic stars[4], namely the octahedral and hexahedral stars, in the previous chapter. In this section we will present the remaining ones, i.e., tetrahedral, icosahedral and dodecahedral stars. Then we give examples of stars corresponding to selected Archimedean solids.

For details on Platonic and Archimedean solids see Section 1.3, for a table of the generators and primary invariants of the symmetry groups used, T_d, O_h and I_h, see Appendix A.2 and A.3.

3.2.1 The remaining Platonic stars

Example 6 (Tetrahedral stars). If we choose coordinates x, y, z such that the tetrahedron has vertices $p_1 = (1,1,1)$, $p_2 = (-1,-1,1)$, $p_3 = (1,-1,-1)$ and $p_4 = (-1,1,-1)$ the invariant ring of the tetrahedral group T_d is generated by the primary invariants displayed in (3.14). One could also choose $(1,0,0)$ as a vertex to avoid a coordinate change during the construction, but then the invariants would be more complicated. Note the difference to the invariants (2.3) of the octahedron and the hexahedron,

$$\begin{aligned} u(x,y,z) &= x^2+y^2+z^2, \\ v(x,y,z) &= xyz, \\ w(x,y,z) &= x^2y^2+y^2z^2+z^2x^2. \end{aligned} \quad (3.14)$$

[4]See Definition 3 in Section 2.1.

A degree three polynomial yields no solution but a degree four polynomial already suffices,

$$f(u, v, w) = 1 - 2u + au^2 + 8v - (3a + 1)w, \quad \text{with } a \neq 1. \tag{3.15}$$

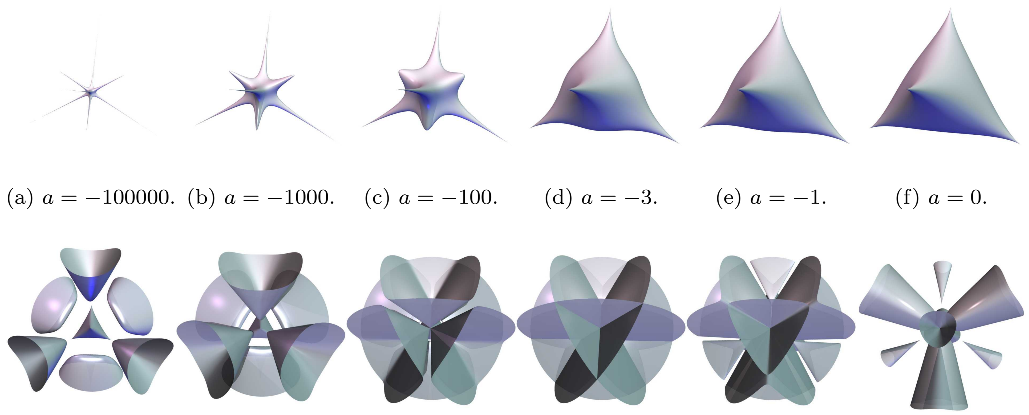
For $a < 1$ we obtain a star, for $a > 1$ an anti-star (with singular points exactly in the vertices given above). If we choose $a = 1$ in (3.15) the polynomial has four linear factors, see Figure 3.6j:

$$(x - 1 + z - y)(x - 1 - z + y)(x + 1 - z - y)(x + 1 + z + y).$$

The planes corresponding to this linear factors intersect each other in the six lines $\{x = 1, y - z\}$, $\{x = -1, y + z\}$, $\{y = 1, x - z\}$, $\{y = -1, x + z\}$, $\{z = 1, x - y\}$ and $\{z = -1, x + y\}$, that contain the edges of our tetrahedron. This lines are evidently the singular locus of the surfaces.
For very small a values there seem to appear four additional cusps in the vertices of a tetrahedron that would be dual to the first one. But these points stay smooth for all $a \in \mathbb{R}$.

Remark 3. We proof this statement: The vertices of a tetrahedron dual to the first one - with vertices $p_1, \ldots .p_4$ - are just $-p_i$, $i = 1, \ldots 4$. Because of the symmetry of the surface it is sufficient to consider just one vertex of the dual tetrahedron, say $q = -p_1 = (-1, -1, -1)$. We subsitute $dp_1 = (d, d, d)$ in $(f, \partial_x f, \partial_y f, \partial_z f)$ from (3.15) and get the system of equations $(1 - 6d^2 + 8d^3 - 3d^4, -4d + 8d^2 - 4d^3, -4d + 8d^2 - 4d^3, -4d + 8d^2 - 4d^3)$. Its only zero is $d = 1$ which is just the singularities in the vertices of the tetrahedron we prescribed. This system of equations is independent from the parameter a, so there is no other singularity in this direction for all values of a. ★
Substituting (d, d, d) just into f yields another zero, $d = -1/3$. This means for all $a \in \mathbb{R}$ the surface $V(f)$ passes through the (smooth) points $-1/3\ p_i$.

For $0 < a < 1$ the zero set of our polynomial has additional components, besides the desired "star shape". For $a > 1$ we get anti-stars, see Figure 3.6.



(a) $a = -100000$. (b) $a = -1000$. (c) $a = -100$. (d) $a = -3$. (e) $a = -1$. (f) $a = 0$.

(g) $a = 0.3$. (h) $a = 0.6$. (i) $a = 0.98$. (j) $a = 1$. (k) $a = 1.02$. (l) $a = 3$.

Figure 3.6: Tetrahedral star (and anti-star) with varying parameter value a, for $a > 0$ the images are clipped by a sphere with radius 5.

Example 7 (Icosahedral and dodecahedral stars). As was mentionend in the first chapter the icosahedron and the dodecahedron share the same symmetry group I_h. Its invariant ring is generated by the following polynomials[5],

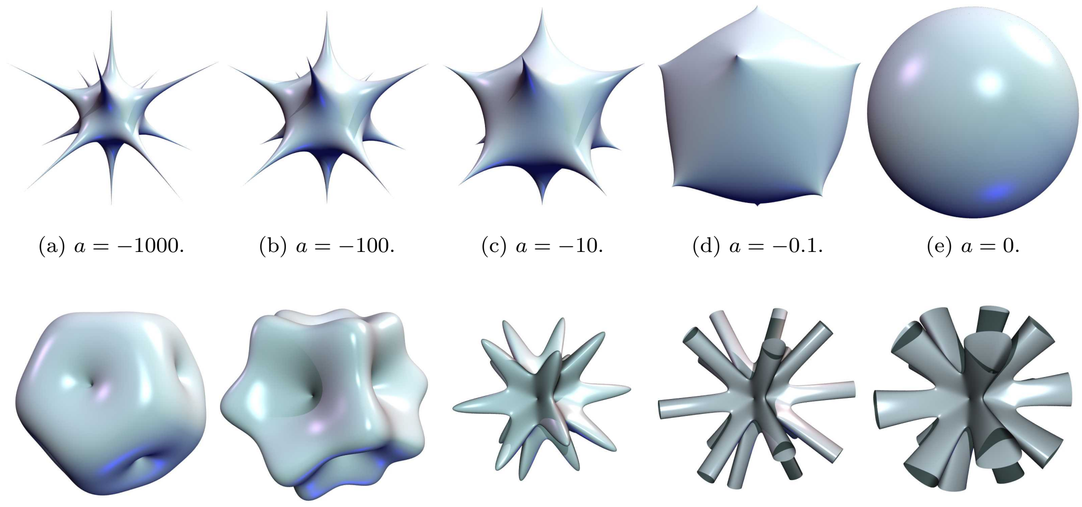
$$\begin{aligned}
u(x,y,z) &= x^2+y^2+z^2,\\
v(x,y,z) &= -z(2x+z)(x^4-x^2z^2+z^4+2(x^3z-xz^3)+5(y^4-y^2z^2)+10(xy^2z-x^2y^2)),\\
w(x,y,z) &= \left(4x^2+z^2-6xz\right)\\
&\qquad \left(z^4-2z^3x-x^2z^2+2zx^3+x^4-25y^2z^2-30xy^2z-10x^2y^2+5y^4\right)\\
&\qquad \left(z^4+8z^3x+14x^2z^2-8zx^3+x^4-10y^2z^2-10x^2y^2+5y^4\right).
\end{aligned} \tag{3.16}$$

Note that both v and w factor into linear polynomials. In Appendix A.5 we give this factorization explicitly.
In both of the following examples, the icosahedral and dodecahedral stars, the "smallest possible degree" is six. The third invariant w has degree ten so we do not use it in both cases.

Icosahedral stars. The equation for the icosahedral star is the following,

$$f(u,v,w) = (1-u)^3 + au^3 + av, \quad \text{with } a \neq 0. \tag{3.17}$$

Figure 3.7 shows icosahedral stars ($a < 0$) and anti-stars ($a > 0$) for various a values. For $a = 0$ we get a sphere of radius one. For all $a \neq 0$ the 12 singularities of the surface lie on this sphere. For $a = 27/32$ the surface has points at infinity in the direction of normals to the facets of the corresponding icosahedron. Note that this is just the negative value of a for which the dodecahedral stars are unbounded. The illustrations suggest that for $a > 27/32$ the surfaces become unbounded while they are bounded for $a < 27/32$.



(a) $a = -1000$. (b) $a = -100$. (c) $a = -10$. (d) $a = -0.1$. (e) $a = 0$.

(f) $a = 0.1$. (g) $a = 0.5$. (h) $a = 0.8$. (i) $a = 27/32$. (j) $a = 0.9$.

Figure 3.7: Icosahedral star and anti-star, with varying parameter a, for $a > 27/32$ the surfaces are clipped by a sphere with radius 11.

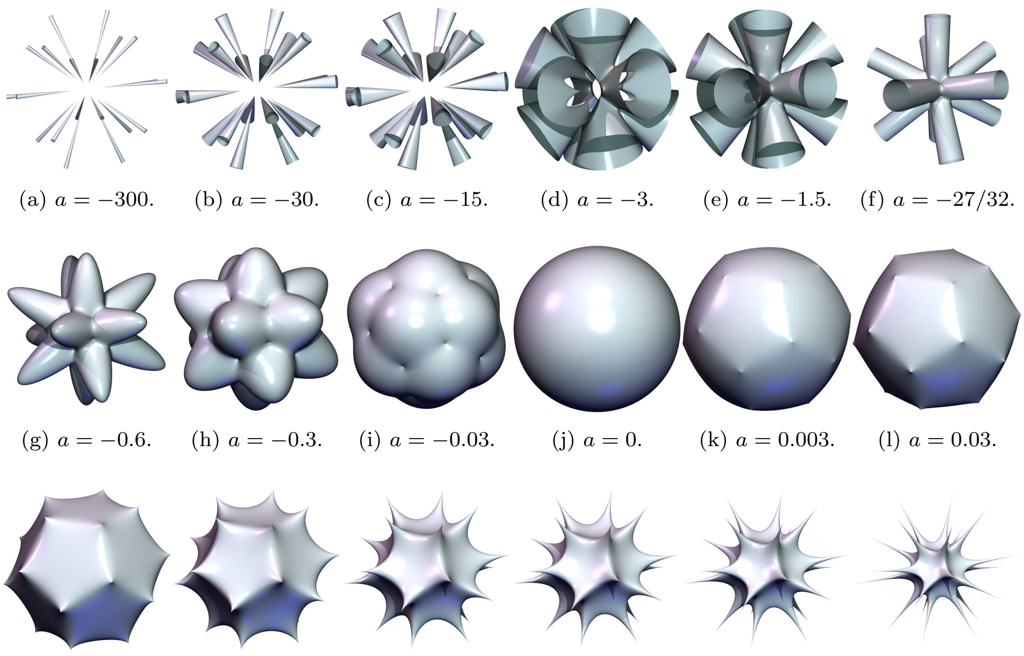
[5]See Appendix A.6 for the `SINGULAR` input and output for calculating them.

Dodecahedral stars Our construction yields the following polynomial of degree six in the invariants (3.16),

$$f(u,v,w) = (1-u)^3 - \frac{5}{27}au^3 + av, \quad \text{with } a \neq 0, \tag{3.18}$$

For $a > 0$ each of the surfaces of this family is a Platonic star. If we choose $a < 0$ we get anti-stars. The choice $a = 0$ yields an ordinary sphere of radius one. See Figure 3.8 for the effect of varying the parameter a. Note that the singularities stay fixed at a sphere of radius one for all parameter values, so for $a < 0$ we have to zoom out to be able to show the whole picture. For $a = -27/32$ the anti-star has a point at infinity in the direction of the z-axis, which is one of the normals of the facets of the dodecahedron. By symmetry it will also have points at infinity in the direction of the normals of the remaining facets. As in the example of icosahedral anti-stars the pictures suggest that for $a > -27/32$ the dodecahedral anti-stars and stars are bounded while they remain unbounded for $a < -27/32$.

Remark 4. On the (un)boundedness of dodecahedral (anti-) stars: f depends on a free parameter c, our aim is to find out for which values of c the zero set of f is bounded, i.e., has no zeros "at infinity". Regard the homogenization of (3.18) $F(x,y,z,t) = t^d f(\frac{x}{t}, \frac{y}{t}, \frac{z}{t}) \in \mathbb{R}[x,y,z,t]$, where d is the degree of f. The zero set of f is unbounded (has zeros at infinity) if $F(x,y,z,0)$ has (real) solutions.
$F(0,0,1,0)$ is a linear equation in c with solution $c = -27/32$, so we know that $V(f)$ is unbounded at the point $(0,0,1)$, which is a normal to a facet of the dodecahedron corresponding to $V(f)$. Because of the symmetry it will also be unbounded along the normals of the other facets.



(a) $a = -300$. (b) $a = -30$. (c) $a = -15$. (d) $a = -3$. (e) $a = -1.5$. (f) $a = -27/32$.

(g) $a = -0.6$. (h) $a = -0.3$. (i) $a = -0.03$. (j) $a = 0$. (k) $a = 0.003$. (l) $a = 0.03$.

(m) $a = 0.3$. (n) $a = 3$. (o) $a = 30$. (p) $a = 81$. (q) $a = 300$. (r) $a = 3000$.

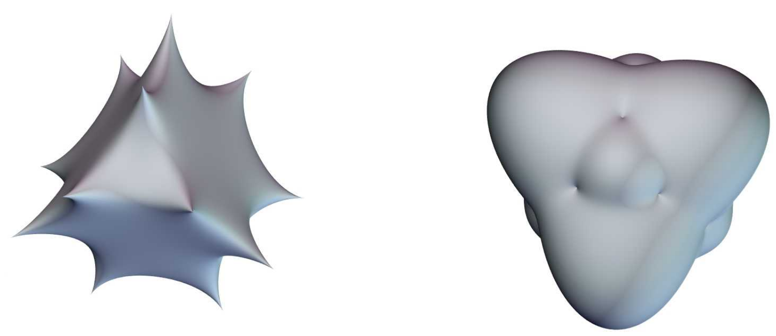
Figure 3.8: Dodecahedral star with varying parameter value a, for $a \leq -27/32$ the surfaces are clipped by a sphere of radius 4.5.

3.2.2 Some Archimedean stars

Example 8 (Truncated tetrahedral stars). One obtains the truncated tetrahedron by cutting of the vertices of a tetrahedron, such that one remains with a solid with 12 vertices[6], 18 edges and 8 facets - 4 regular triangles and 4 regular hexagons. See Figure 1.3a. Its dual is the Triakis tetrahedron that will be presented in Example 11. Evidently it shares its symmetry group with the tetrahedron, hence we use the same invariants (3.14). We obtain the following polynomial of degree six,

$$\begin{aligned} f(u,v,w) = (1-u)^3 + \frac{3}{88\sqrt{11}}(a_2 - 3a_1)u + a_1 v + (\frac{5}{22\sqrt{11}}a_1 - \frac{2}{33\sqrt{11}}a_2)u^2 - \\ -\frac{\sqrt{11}}{4}(a_1 + \frac{1}{3}a_2)w + a_2 uv - \frac{1}{264\sqrt{11}}(53a_1 + 13a_2)u^3 - \frac{11\sqrt{11}}{10}(a_1+a_2)v^2 + \\ + \frac{\sqrt{11}}{60}(11a_1 + a_2)uw, \end{aligned} \tag{3.19}$$

with $a_1 + a_2 \neq 0$ and $3(a_1 - a_2) - 176\sqrt{11} \neq 0$. If this two terms have different signs we obtain stars, see Figure 3.9a, if they are both either positive or negative anti-stars, Figure 3.9b. For $a_1 = a_2 = 0$ the zero set of equation (3.19) is a sphere.



(a) Truncated tetrahedral star, $a_1 = 0$, $a_2 = 100$.

(b) Truncated tetrahedral anti-star, $a_1 = 0$, $a_2 = -1$.

Figure 3.9: Truncated tetrahedral stars and anti-stars.

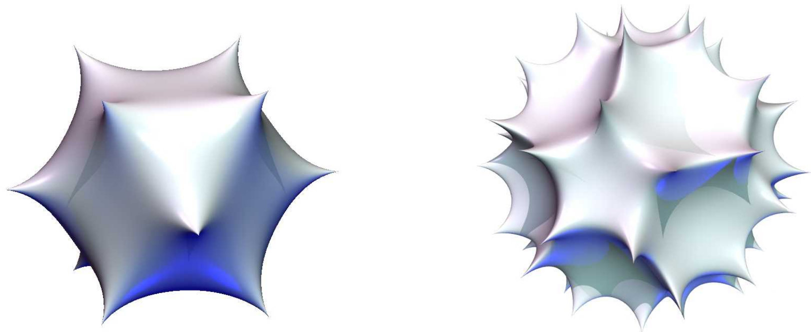
Example 9 (Cub-octahedral stars). The cub-octahedron is the Archimedean solid with 14 facets (6 squares and 8 equilateral triangles), 24 edges and 12 vertices. See Figure 1.3b. Its symmetry group is the one of the octahedron and cube. Therefore we use the invariants (2.3). Our construction yields the following polynomial of degree six, with three free parameters,

$$f(u,v,w) = 1 - 3u + a_1 u^2 + (12 - 4a_1)v + a_2 u^3 - (4 + 4a_2)uv + a_3 w, \tag{3.20}$$

with $a_1 + a_2 \neq 2$ and $8(a_1 + a_2) - a_3 \neq 16$. For $a_1 = 3$, $a_2 = -1$ and $a_3 = 0$ we obtain a sphere. In this example we have a new kind of behavior. So far the coefficients of y^2 and z^2 in the Taylor

[6] We choose them to lie on a sphere of radius one, therefore the vertices of "our" truncated tetrahedron and hence the singularities of the corresponding stars are $(3/\sqrt{11}, 1/\sqrt{11}, 1/\sqrt{11})$, $(3/\sqrt{11}, -1/\sqrt{11}, -1/\sqrt{11})$, $(1/\sqrt{11}, 1/\sqrt{11}, 3/\sqrt{11})$, $(-1/\sqrt{11}, -1/\sqrt{11}, 3/\sqrt{11})$, $(-1/\sqrt{11}, -3/\sqrt{11}, 1/\sqrt{11})$, $(1/\sqrt{11}, -3/\sqrt{11}, -1/\sqrt{11})$, $(1/\sqrt{11}, 3/\sqrt{11}, 1/\sqrt{11})$, $(-1/\sqrt{11}, 3/\sqrt{11}, -1/\sqrt{11})$, $(-3/\sqrt{11}, -1/\sqrt{11}, 1/\sqrt{11})$, $(-3/\sqrt{11}, 1/\sqrt{11}, -1/\sqrt{11})$, $(1/\sqrt{11}, -1/\sqrt{11}, -3/\sqrt{11})$, $(-1/\sqrt{11}, 1/\sqrt{11}, -3/\sqrt{11})$.

expansion of f in $(1,0,0)$ were the same and we got inequalities from the conditions that it should have the same sign as the one of x^3. In this case the coefficient of x^3 is -8, but y^2 and z^2 have different coefficients, namely $a_1 + a_2 - 2$ and $16 - 8(a_1 + a_2) + a_3$ respectively. So if both are negative we obtain stars, see Figure 3.10a, if both are positive anti-stars, but if they have different signs, we will no longer have a A_2^{++}- or A_2^{--}-singularity, as we want, but a A_2^{+-}-singularity[7]. The singularities always lie on a sphere of radius one.



(a) Cuboctahedral star, $a_1 = 0$, $a_2 = 0$, $a_3 = -100$.

(b) Soccer star, $a_1 = -100$, $a_2 = -100$, $a_3 = -100$, $a_4 = -100$.

Figure 3.10: Two Archimedean stars.

Example 10 (Soccer star). The truncated icosahedron is the Archimedean solid which is obtained by "cutting off the vertices" of a icosahedron, dividing the edges into three segments of the same length. It is known as the shape of a soccer ball. It has 32 facets (12 regular pentagons and 20 regular hexagons), 60 vertices and 90 edges. See Figure 1.3h. Its symmetry group is the icosahedral group I_h. For this example we finally need the third of the invariants (3.16), since the first polynomial that yields a solvable system of equations is of degree 10. We obtain the following equation with four free parameters,

$$\begin{aligned} f(u,v,w) = 1 &+ \left(\frac{128565 + 115200\sqrt{5}}{1295029} a_3 + \frac{49231296000\sqrt{5} - 93078919125}{15386239549} a_4 - a_1 - 3a_2 - 3\right) u + \\ &+ \left(\frac{-230400\sqrt{5} - 257130}{1295029} a_3 + \frac{238926989250 - 126373248000\sqrt{5}}{15386239549} a_4 + 3a_1 + 8a_2 + 3\right) u^2 + \\ &+ \left(\frac{115200\sqrt{5} + 128565}{1295029} a_3 + \frac{91097280000\sqrt{5} - 172232645625}{15386239549} a_4 - 3a_1 - 6a_2 - 1\right) u^3 + \\ &+ \left(a_3 + \frac{121075 - 51200\sqrt{5}}{11881} a_4\right) v + \left(\frac{102400\sqrt{5} - 242150}{11881} - 2a_3\right) uv + a_1 u^4 + a_2 u^5 + a_3 u^2 v + a_4 w, \end{aligned} \tag{3.21}$$

with

$$\begin{aligned} & a_4 \neq 0, \\ & b(a_1,a_2,a_3,a_4) := (991604250 - 419328000\sqrt{5})a_4 + 20316510 a_3 + (135776068 - 121661440\sqrt{5})a_2 \\ & \quad + (33944017 - 30415360\sqrt{5})a_1 + 30415360\sqrt{5} - 33944017 \neq 0. \end{aligned} \tag{3.22}$$

We obtain stars if we choose a_1, a_2, a_3 and a_4 such that a_4 and $b(a_1,a_2,a_3,a_4)$ have the same sign. Otherwise we obtain anti-stars. See Figure 3.10b.

[7] With normalform $x^3 + y^2 - z^2 = 0$. See Figure 1.4c in Section 1.4.

3.3 Catalan stars and "relatives"

So far we only constructed surfaces with a set of isolated singularities on which its symmetry group acts transitively, now we want to see what happens if it does not act transitively. How this works theoretically has been explained in Section 2.3.2. An example for this are Catalan stars as "defined" in the beginning of Chapter 2. The vertices of a Catalan solid do not all have the same Euclidean norm. Sometimes we also give examples of stars with singularities in the direction of the vertices of a Catalan solid that all have the same norm. This is want we mean with "relatives". Such a surface appeared by chance as a special case in Example 1 of hexahedral and octahedral stars, we describe the resulting surface in Example 12.

Example 11 (Triakis tetrahedral stars). The triakis tetrahedron is the Catalan solid dual to the truncated tetrahedron. Its symmetry group is the tetrahedral group T_d, so we use its invariants (3.14). It has 8 vertices[8], 18 edges and 12 facets, which are isosceles triangles. We obtain the following polynomial of degree seven with four free parameters,

$$\begin{aligned} f(u,v,w) = 1 - \frac{23}{9}u + a_1u^2 + \frac{26}{27}v + (\frac{509}{81} - 3a_1)w - \frac{20}{9}uv + a_2u^3 - \\ - (\frac{425}{27} + 9a_4 + 27a_2)v^2 + a_4uw + a_3u^2v + (\frac{250}{81} - 3a_3)vw, \end{aligned} \tag{3.23}$$

with

$$\begin{aligned} b_1(a_1,a_2,a_3,a_4) &:= 6a_1 + 54a_2 + 6a_3 + 12a_4 + \frac{248}{27} \neq 0, \\ b_2(a_1,a_2,a_3,a_4) &:= \frac{54}{25}a_1 + \frac{4374}{625}a_2 - \frac{1458}{3125}a_3 + \frac{972}{625}a_4 - \frac{8}{75} \neq 0. \end{aligned} \tag{3.24}$$

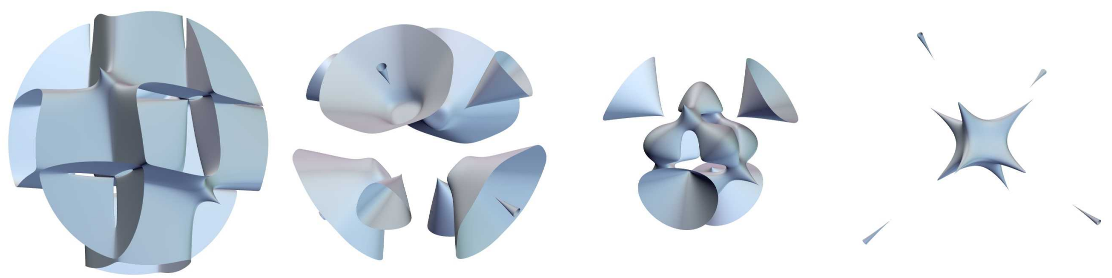
In this examples we have to fix two vertices, one in each orbit of the group action on the vertices. The term b_1 is the coefficient of y^2 and z^2 in the expansion (2.8) at the first point, b_2 the one at the second point. At both points the coefficient of x^3 does not depend on the free parameters, at the first point it is $512/243\sqrt{3}$, at the second one $5632/1215\sqrt{3}$. Therefore the appearance of the surface depends on b_1 and b_2 in the following way[9]: If $b_1 > 0$ and $b_2 < 0$ we have stars (Figure 3.11a), if $b_1 < 0$ and $b_2 > 0$ anti-stars (Figure 3.11b). If both are either positive (Figure 3.11c) or negative (Figure 3.11d) at the same time we have cusps pointing outwards at four vertices forming one orbit and cusps pointing inwards at the remaining four. We did not find "a good choice" of parameters that would produce "nice" pictures as in the other examples, but we did not search systematically for it so this does not necessarily mean that there exists none.

Remark 5. Note that the "relative" of a triakis tetrahedral star would be one with cusps in the vertices of a tetrahedron and the tetrahedron dual to it with the same length. This is the same as a hexahedral star. If we try to construct it with the invariants (3.14) of the tetrahedron, fixing two vertices, we yield the same polynomial (2.6) as with the invariants (2.3) of the octahedral group. Note that the primary invariants of the tetrahedral group T_d and of the octahedral group O_h correspond to each other in the following way:

$$\begin{aligned} u_{O_h} &= u_{T_d}, \\ v_{O_h} &= w_{T_d}, \\ w_{O_h} &= v_{T_d}^2. \end{aligned}$$

[8]The vertices of a tetrahedron and its dual, which have differnet lengths, namely in an aspect ratio of 5/3. We used $p_1 = (1,1,1)$, $p_2 = (-1,-1,1)$, $p_3 = (-1,1,-1)$, $p_4 = (1,-1,-1)$, $q_1 = (-3/5,-3/5,-3/5)$, $q_2 = (3/5,3/5,-3/5)$, $q_3 = (3/5,-3/5,3/5)$ and $q_4 = (-3/5,3/5,3/5)$.

[9]See Section 2.3 for details.



(a) A star, $a_1 = a_2 = a_3 = a_4 = 0$. (b) An antistar, $a_1 = 23/9$, $a_2 = 1$, $a_3 = -10$, $a_4 = -5$. (c) $a_1 = 1$, $a_2 = a_3 = a_4 = 0$. (d) $a_1 = -100$, $a_2 = a_3 = a_4 = 0$.

Figure 3.11: Triakis tetrahedral stars.

Example 12 (Rhombic dodecahedral and 14-stars).

14-stars. In Section 2.2 we mentioned that in both the example of the octahedral and the hexahedral star, there exits a choice of the free parameters[10], for which a special behavior appears: The surfaces have "too many" singularities, namely 14 instead of the 8 respective 6 we prescribed. The singularities are of type A_2 and lie exactly in the vertices[11] of the octahedron and the hexahedron dual to it. This object has the following defining polynomial, with u, v and w the primary invariants of the octahedral group (2.3),

$$f(u, v, w) = (1 - u)^3 + auv - 9aw, \quad \text{with } a \neq 0. \tag{3.25}$$

We will call this object a 14-*star* or 14-*anti-star* for $a < 0$ or $a > 0$ respectively. The parameter value $a = 0$ yields obviously a sphere of radius one. See Figure 3.12 for an illustration of the effect of varying the parameter.

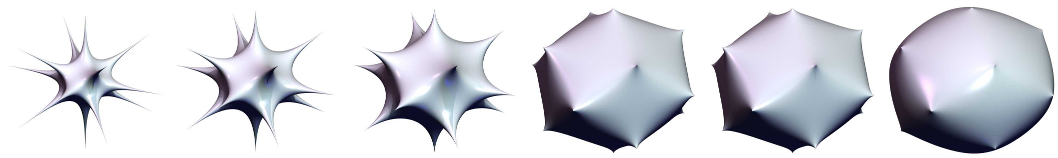
This star does not correspond to a Platonic or Archimedean solid, but to the solid S that is the convex hull both of the vertices of a hexahedron and an octahedron that lie on one sphere. It has 14 vertices, 36 edges and 24 facets, which are isosceles triangles. See Figure 3.13a. It is remarkable that it appears as a special case of hexahedral and octahedral stars since the symmetry group O_h does not act transitively on its vertices. Evidently its vertices lie in two orbits of the action of O_h on $\mathbb{R}^3$. Note that, because of this, if we would construct such a star, we would have to choose two points and prescribe singularities there, like it was described in Section 2.3. Performing this construction yields the same polynomial (3.25) as we obtained by chance.

Note that if the vertices of the octahedron and the cube do not to lie on one sphere but have have different Euclidean norms of a certain ration, namely $2/\sqrt{3}$, this solid is a Catalan solid, called *rhombic dodecahedron* (14 vertices, 12 facets, every two triangles form a rhombus, 24 edges) which is the dual of the Archimedean solid called cub-octahedron. See Section 3.2 for cub-octahedral stars. In the next paragraph we will present the rhombic dodecahedral stars.

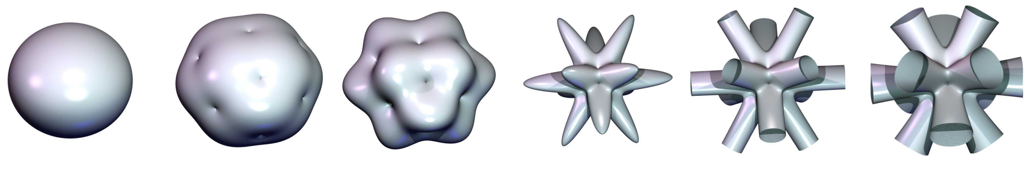
Rhombic dodecahedral stars. To construct a rhombic dodecahedral star, that is, an algebraic surface with symmetry group O_h and singularities in the vertices of the rhombic dodecahedron

[10]We obtain the polynomial (3.25) by substituting $a_4 = a$, $a_5 = 0$ and $a_6 = -9a$, $a \neq 0$ in the polynomial of the octahedral stars (2.5) or $a_2 = 3$, $a_3 = -1$ and $a_4 = a \neq 0$ in the one of the hexahedral stars (2.6).

[11]This is $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$ and $(\pm 1/\sqrt{3}, \pm 1/\sqrt{3}, \pm 1/\sqrt{3})$.

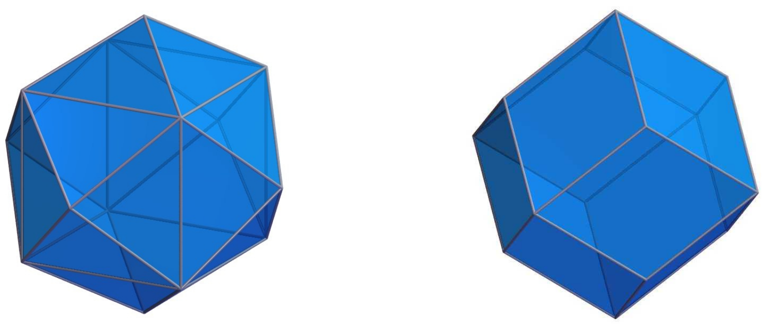


(a) $a = -10000$. (b) $a = -1000$. (c) $a = -100$. (d) $a = -10$. (e) $a = -1$. (f) $a = -0.1$.



(g) $a = 0$. (h) $a = 0.1$. (i) $a = 1$. (j) $a = 3.5$. (k) $a = 4$. (l) $a = 5$.

Figure 3.12: 14-stars and anti-stars, for $a \geq 4$ the surfaces are clipped by a sphere with radius 5.



(a) Solid S corresponding to the 14-star. (b) The rhombic dodecahedron.

Figure 3.13: 14-solid and the rhombic dodecahedron.

(see Appendix A.1), we need a polynomial of degree eight. With the invariants (2.3) we get,

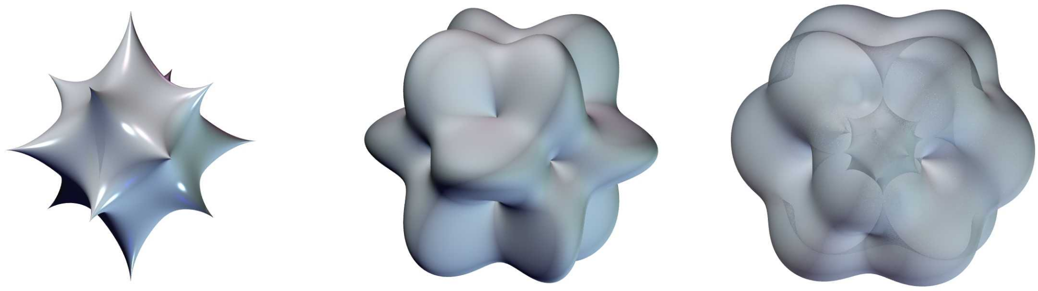
$$\begin{aligned} f(u,v,w) = 1 - (\frac{9}{4} + \frac{64}{27}a_1)u + (\frac{27}{16} + \frac{16}{3}a_1)u^2 + (\frac{16}{3}a_1 - \frac{45}{16})v - (\frac{27}{64} + 4a_1)u^3 + \\ + (\frac{321}{64} - \frac{28}{3}a_1 - \frac{1}{9}a_3)uv + a_3w + a_1u^4 + a_4v^2 + a_2u^2v + (37a_1 - \frac{81}{4} - 3a_4 - 9a_2)uw, \end{aligned} \tag{3.26}$$

with

$$\begin{aligned} b_1 &= \frac{837}{2} - 768a_1 - 16a_3 + 192a_2) \neq 0, \\ b_2 &= 256a_1 - 81 \neq 0, \\ b_3 &= \frac{2619}{32} - 150a_1 + 36a_2 - 4a_3 + 6a_4 \neq 0, \\ b_4 &= 256a_1 - 189 \neq 0. \end{aligned} \tag{3.27}$$

It is a star if each of the pairs b_1, b_2 and b_3, b_4 have the same sign, see Figure 3.14a. If b_1 and b_2 have different signs and b_3 and b_4 also have different signs we obtain anti-stars, Figure 3.14b. If b_1

has the same sign as b_2 but b_3 and b_4 have different signs, the cusps point outwards in the vertices the rhombic dodecahedron shares with an octahedron, and inwards in the vertices that come from a hexahedron, Figure 3.14c. If b_1 and b_2 have the same sign but b_3 and b_4 have different ones it is the other way around.



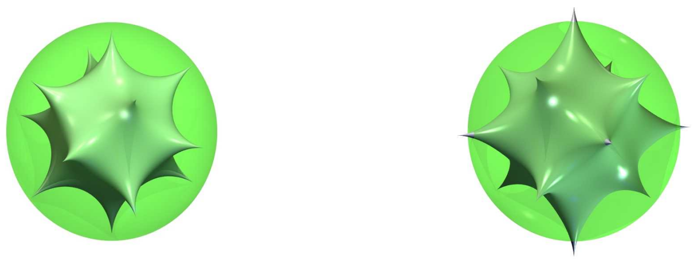
(a) Rhombic dodecahedral star, $a_1 = 3$, $a_2 = 100$, $a_3 = -100$, $a_4 = 0$.

(b) Rhombic dodecahedral anit-star, $a_1 = 3$, $a_2 = 100$, $a_3 = -100$, $a_4 = 0$.

(c) $a_1 = 100/256$, $a_2 = 0$, $a_3 = 100$, $a_4 = 0$.

Figure 3.14: Rhombic dodecahedral stars and anti-stars.

Figures 3.15 should emphasize the difference between the 14- and the rhombic dodecahedral star, with the help of a sphere of radius one.

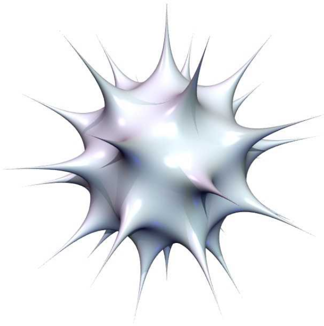


(a) 14-star, $c = -50$, with sphere of radius one.

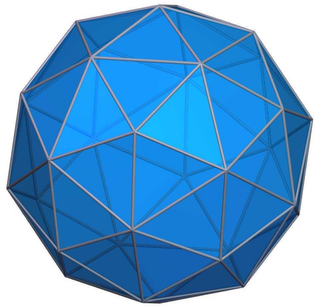
(b) Rhombic dodecahedral star with sphere of radius one.

Figure 3.15: 14-star and rhombic dodecahedral star.

Example 13 (32-stars). In this example we try to construct a star with 32 singularities. Twelve of them should lie in the vertices of an icosahedron and the remaining 20 in the vertices of the dodecahedron dual to the icosahedron. They should all lie on a sphere or radius one. The convex hull of this vertices is a solid with 32 vertices, 90 edges and 60 facets (isosceles triangles). It is displayed in Figure 3.16b. Such a solid obviously has the symmetry group of the icosahedron and the dodecahedron, so we use their primary invariants (3.16) for the construction of its corresponding stars. Since it can not be vertex transitive, in this case we have to “fix” two vertices, one vertex of the icosahedron and one of the dodecahedron. The smallest degree for which our construction



(a) 32-star, $a_1 = 3$, $a_2 = -1$, $a_3 = 10$.



(b) Solid with 32 vertices.

Figure 3.16: 32-star and corresponding solid.

yields the a solution is ten,

$$f(u,v,w) = 6 - (3a_1 + a_2 + 10)\,u + 6a_1u^2 + 6a_2u^3 + (-6a_1 - 8a_2 + 10)\,u^4 + \\ + (3a_1 + 3a_2 - 124a_3 - 6)\,u^5 - 124a_3u^2v + 6a_3w, \tag{3.28}$$

with $a_3 \neq 0$ and $10 - 2a_2 - 3a_1 \neq 0$. To be more precise, we obtains a star ifa_3 and $10 - 2a_2 - 3a_1$ have the same sign and an anti-star otherwise.

3.4 Dihedral stars

In this section we give examples for the construction of algebraic surfaces with symmetry groups that are not reflection groups. The general concept is described in Section 2.3.
We consider the dihedral groups D_m as subgroups of $O_3(\mathbb{R})$. Represented as 2×2-matrices the dihedral groups are reflection groups, therefore in the examples from Section 3.1 we only had to consider primary invariants[12]. In the representation as 3×3-matrices they are not reflection groups. In the examples we give here the number of secondary invariants is always two. The first one is obviously always one, so we do not mention it in every example but just give the second one, s_2. For a table with all primary and secondary invariants of the groups we considered see the Appendix A.3.
Our aim is to construct surfaces that are invariant under the action of D_m with singularities in the m-th roots of unity, in the xy-plane. Additionally we want them to pass through the points $(0, 0, \pm c)$ with $0 \neq c \in \mathbb{R}$. Like in the previous examples we speak of *stars* if the cusps "point inwards" and *anti-stars* if they "point outwards". The surfaces we obtain all have a high number of free parameters. The choice of parameters we made for the illustration are based on purely aesthetic reasons. Namely to fit the aim that the resulting surface should "look like a pillow". To reach this intuition without arbitrary choices more conditions, such as boundedness and connectedness, that we do not consider, would be necessary.

[12]Note that two of the primary invariants in all examples of this section coincide with the primary invariants of the respective examples in Section 3.1.

Example 14 (D_2). The dihedral group $D_2 \subset \mathrm{O}_3(\mathbb{R})$ has primary invariants

$$\begin{aligned} u(x,y,z) &= z^2, \\ v(x,y,z) &= x^2, \\ w(x,y,z) &= y^2. \end{aligned} \tag{3.29}$$

Its secondary invariant is

$$s_2(x,y,z) = xyz. \tag{3.30}$$

To obtain a solvable system of equations we need a polynomial of degree six,

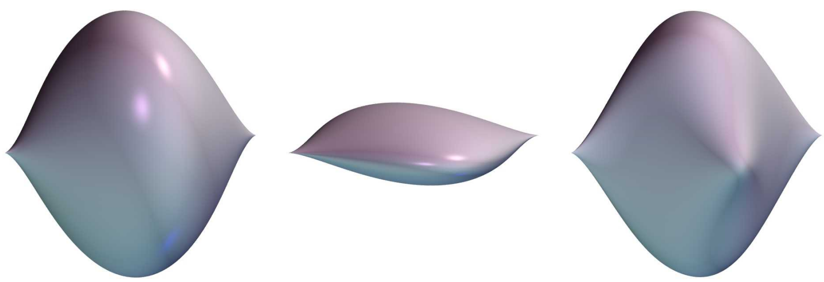
$$\begin{aligned} f(u,v,w,s_2) = 1 - \frac{1+a_2c^4+a_7c^6}{c^2}u - 3v + a_1w + a_2u^2 + 3v^2 + a_3w^2 + a_4uv + a_5uw + \\ + a_6vw + a_7u^3 - v^3 + a_8w^3 + a_9u^2w + a_{10}u^2v + a_{11}v^2w + a_{12}uv^2 + a_{13}uw^2 + \\ + a_{14}vw^2 + a_{15}uvw + (a_{16}u + a_{17}(v-1) + a_{18}w)s_2. \end{aligned} \tag{3.31}$$

with

$$\begin{aligned} b_1 &:= a_1 + a_6 + a_{11} \neq 0, \\ b_2 &:= -\frac{1+a_2c^4+a_7c^6}{c^2} + a_4 + a_{12} \neq 0. \end{aligned} \tag{3.32}$$

For b_1 and b_2 negative we obtain stars, if both are positive anti-stars. If b_1 and b_2 have different signs the singularities are of type A_2^{+-}.
The choice $a_1 = -1$ and all the remaining parameters equal to zero yields a "nice" surface. For $c = 1$ it has infinite rotational symmetry, and therefore more symmetry than we demanded. We call this surface "Zitrus" (Figure 3.17a).

$$f(u,v,w) = (1-v)^3 - \frac{1}{c^2}u - w. \tag{3.33}$$



(a) Zitrus, $c = 1$, $a_1 = -1$, rest equal to 0.
(b) $c = 1/4$, $a_1 = -1$, rest equal to 0.
(c) $c = 1$, $a_1 = -1$, $a_6 = -10$, $a_13 = -10$, rest equal to 0.

Figure 3.17: D_2 stars.

Example 15 (D_3). The primary invariants of $D_3 \subset \mathrm{O}_3(\mathbb{R})$ are

$$\begin{aligned} u(x,y,z) &= z^2, \\ v(x,y,z) &= x^2 + y^2, \\ w(x,y,z) &= x^3 - 3xy^2, \end{aligned} \tag{3.34}$$

its secondary invariant is

$$s_2(x, y, z) = 3x^2yz - y^3z. \tag{3.35}$$

A polynomial of degree three yields no solution. The general equation of a degree four polynomial in the invariant ring of D_3 is $f_1(u, v, w) + bs_2$, where $f_1(u, v, w)$ is an indeterminate polynomial of degree four[13] in $\mathbb{R}[u, v, w]$ and $b \in \mathbb{R}$ is a constant. A degree four polynomial suffices to obtain a solvable system of equation. We obtain the following polynomial. It has three free parameters,

$$f(u, v, w) = 1 - \frac{1 + a_1c^4}{c^2}u + a_1u^2 + a_2uv - 6v - 3v^2 + 8w, \tag{3.36}$$

with $-(1+a_1c^4)+c^2a_2 \neq 0$. For $-(1+a_1c^4)+c^2a_2 < 0$ we obtain stars, for $-(1+a_1c^4)+c^2a_2 > 0$ anti-stars. Note that the secondary invariant s_2 does not appear in the above polynomial, its coefficient b is zero. We obtain a nice result for $a_1 = a_2 = 0$, $c = 1/3$. It is displayed in Figure 3.18a.

Example 16 (D_4). The group D_4 has the following primary and secondary invariants,

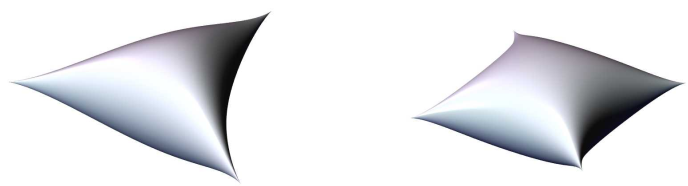
$$\begin{aligned} u(x, y, z) &= z^2, \\ v(x, y, z) &= x^2 + y^2, \\ w(x, y, z) &= x^2y^2, \end{aligned} \tag{3.37}$$

$$s_2(x, y, z) = x^3yz - xy^3z. \tag{3.38}$$

Our construction yields a degree six polynomial. As in the previous example the secondary invariant s_2 does not appear,

$$\begin{aligned} f(u, v, w) = 1 - \frac{1 + a_1c^4 + a_4c^6}{c^2}u - 3v + a_1u^2 + a_2uv + 3v^2 + a_3w + a_4u^3 - v^3 + a_5uw+ \\ + a_6vw + a_7uv^2 + a_8u^2v, \end{aligned} \tag{3.39}$$

with $a_3 + a_6 \neq 0$ and $-(1 + a_1c^4 + a_4c^6) + c^2(a_2 + a_7) \neq 0$. If both of this terms are negative we obtain stars, if both are positive anti-stars. If they have different signs the singularities are of type A_2^{+-}. See Figure 3.18b for the resulting surface if we choose $c = 1/3$, $a_3 = -27$ and all the other parameters are set equal to zero.



(a) A D_3 star. (b) A D_4 star.

Figure 3.18: Dihedral stars.

[13] The usual total degree in the variables x, y, z.

Example 17 (D_5). The primary invariants of D_5 are

$$\begin{aligned} u(x,y,z) &= z^2, \\ v(x,y,z) &= x^2 + y^2, \\ w(x,y,z) &= x^5 - 10x^3y^2 + 5xy^4. \end{aligned} \tag{3.40}$$

Its secondary invariant is

$$s_2(x,y,z) = 5x^4yz - 10x^2y^3z + y^5z. \tag{3.41}$$

A degree five polynomial already produces a solvable system of equations. We get,

$$f(u,v,w) = 1 - \frac{1+a_1c^4}{c^2}u - \frac{10}{3}v + a_1u^2 + a_2uv + 5v^2 - \frac{8}{3}w, \tag{3.42}$$

with $a_2c^2 - (1+a_1c^4) \neq 0$. This polynomial only permits anti-stars because the coefficient of x_3 in (2.2) is $-\frac{20}{3}$, while the one of y^2 is $\frac{100}{3}$ and the one of z^2 is $a_2 - (1+a_1c^4)/c^2$, i.e., we have anti-stars for $a_1 + a_2 > 0$ and singularities of type A_2^{+-} otherwise. See Figure 3.42 for an illustration of above polynomial with $c = 1$, $a_1 = -3$, $a_2 = 5$. Note that equation (3.42) is just a special case of the following degree six polynomial (3.43) with $a_3 = a_4 = a_5 = a_6 = 0$. Again s_2 does not appear,

$$\begin{aligned} f(u,v,w) = 1 - \frac{1+a_1c^4+a_3c^6}{c^2}u - \frac{10+a_4}{3}v + a_1u^2 + a_2uv + (5+2a_4)v^2 - \frac{8}{3}(1+a_4)w + \\ + a_3u^3 + a_4v^3 + a_5uv^2 + a_6u^2v, \end{aligned} \tag{3.43}$$

with

$$\begin{aligned} b_1 &:= a_4 + 1 \neq 0 \\ b_2 &:= -(1 + a_1c^4 + a_3c^6) + c^2(a_2 + a_5) \neq 0 \\ b_3 &:= a_4 - 5 \neq 0. \end{aligned} \tag{3.44}$$

The zero sets of these polynomials are stars if all three terms from above are either negative or positive. If b_1 and b_2 have the same sign but b_3 has a different one we obtain anti-stars. Otherwise, i.e., if b_1 and b_2 also have different signs, we have singularities of type A_2^{+-}.
A nice choice for the free parameters is $c = 1/3$, $a_4 = -3$ and all the other parameters equal to zero. See Figure 3.19b.

(a) A D_5 anti-star (3.42), $c = 1$, $a_1 = -3$, $a_2 = 5$.

(b) A D_5 star, $c = 1/3$, $a_4 = -3$ and all the others equal to 0.

Figure 3.19: Surfaces with symmetry group D_5, clipped by a sphere of radius 6.

3.5 Some examples of curves and surfaces with simple singularities of a type different to A_2

In Chapter 2 we mentioned that in theory the same construction could be performed with any other type of simple singularities, not just A_2. In this section we actually do the calculation for one type, namely E_6. The zero sets of its normal forms in 2- and 3-dimensional real space are displayed in Figure 3.20.
Examples 18 and 19 are a curve and a surface both with singularities of type E_6 and dihedral symmetry. Analogosly to the previous examples we will call them "stars" (or "anti-stars" depending on wether the "peaks" of the singularities point "outside" or "inside"). Trying to construct surfaces

(a) $V(x^3 + y^4)$. (b) $V(x^3 + y^4 + z^2)$. (c) $V(x^3 + y^4 - z^2)$.

Figure 3.20: Singularities of type E_6.

with singularities of type E_6 one faces a problem: It is not as symmetric as, for example, A_2. This makes it impossible to construct surfaces with the symmetries of a Platonic solid: Consider for example a tetrahedron and start with an E_6-singularity in one vertex, for example the blue one in Figure 3.21. If one turns it by π about the z-axis one obtains the red E_6-singularity in a second vertex of the tetrahedron. But if we turn it by $4\pi/3$ about the axis through the vertex $(1, 1, 1)$ and the center of the opposite face we obtain the green singularity.

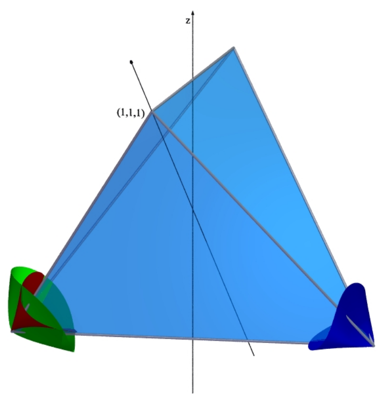


Figure 3.21: Tetrahedron with singularities of type E_6.

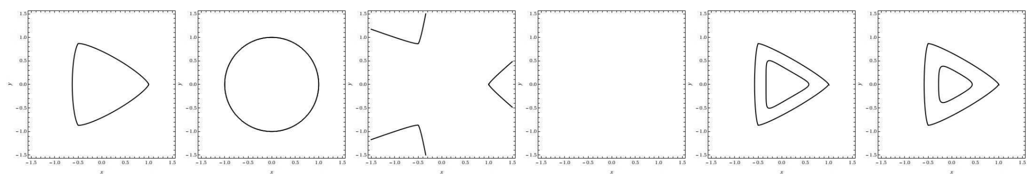
Remark 6. Again we need necessary conditions for a singularity of type E_6. To apply Theorem 2 to $f_0 = x^3+y^4+z^2$ (analogously for $n = 2$ and $f_0 = x^3+y^4$) we choose weights $\boldsymbol{\omega} = (1/3, 1/4, 1/2)$ that make f_0 into a quasihomogeneous function of degree 1. Again (as in the case of A_2-singularities) no basis monomials of $\mathbb{C}[x, y, z]/(\partial f_0/\partial x, \partial f_0/\partial y, \partial f_0/\partial z) = {}_{\mathbb{C}}< 1, x, y, y^2, xy, xy^2 >$ lying above the diagonal exist, therefore the theorem applies and we can prescribe singularities by demanding that the polynomial[14] we construct (at p) looks like,

$$c_1x^3 + c_2y^4 + c_3z^2 + \text{ higher order terms } . \tag{3.45}$$

Example 18. We want to construct a curve similar to a plane 3-star, as in Example 3, but with singularities of type E_6 instead of A_2. To do so we need the primary invariants of D_3 (3.6). Advancing analogously as in the examples before one finds the following polynomial of degree six, with one free parameter,

$$f(u,v) = 1 - 3(a+2)u + 4(a+1)(v+v^2) + 3(2a+3)u^2 - 12(a+1)uv + cu^3, \tag{3.46}$$

with $a \neq -1$ and $a \neq 0$. For $a < -1$ and $a > 0$ we have "stars", for $-1 < a < 0$ "anti-stars" and for $a = -1$ $V(f)$ is a circle. See Figure 3.22.



(a) $a = -2$. (b) $a = -1$. (c) $a = -0.5$. (d) $a = 0$. (e) $a = 1$. (f) $a = 2$.

Figure 3.22: $V(f)$ with varying parameter a.

Example 19. Next we construct a surface comparable to the "pillow-stars" from Section 3.4. To be more precise we want to find a surface with symmetry group $D_3 \subset \mathrm{O}_3(\mathbb{R})$ and singularities of type E_6 in the vertices of regular triangle lying in the xy-plane. We use the primary and secondary invariants of D_3, (3.34) and (3.35). One obtains a polynomial of degree six with 11 free parameters.

$$\begin{aligned} f = {} & 1 + a_1u - 3(a_2+2)v + 4(1+a_2)w + a_3u^2 + 3\,(2a_2+3)\,v^2 + a_4uv + a_5uw- \\ & - 12(1+a_2)vw + a_6u^3 + a_2v^3 + a_8u^2v + a_9uv^2 + 4(1+a_2)w^2 + (-b_1 + b_2u + b_1v)\,s_2. \end{aligned} \tag{3.47}$$

with $a_1 + a_4 + a_5 + a_9 \neq 0$ and $a_2 \neq 0$ and $a_2 \neq -1$. Note that the choice of the free parameters, $a_1 = -5$, $a_2 = -2$ and all the other coefficients equal to 0, we made for the visualization (Figure 3.23) are purely based on aesthetic reasons.

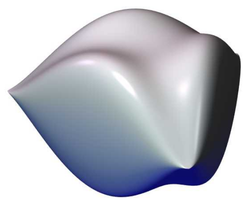


Figure 3.23: $V(f)$ with $a_1 = -5$, $a_2 = -2$ and all the other coefficients equal to 0.

[14]Compare this situation with the situation described in Section 2.1, especially Equation (2.2).

Appendix A

Technical details

A.1 The Archimedean and Catalan solids

We give two tables of the Archimedean and Catalan solids, plus the pseudo rhomb-cub-octahedron and its dual the *pseudo deltoidal icositetrahedron*. The symbol v stands for the number of vertices, e of edges and f of facets. The number of facets, that are polygons with n vertices, is denoted by p_n. In the last column, Sym, the symmetry group of the solid is displayed. The Archimedean and Catalan solids with the same number are dual to each other. The sources we used are [Cro97, p. 82f], [Rom68, p. 55] and [Grü09] for the pseudo rhomb-cub-octahedron.

No.	**Archimedean solids**	v	e	f	p_3	p_4	p_5	p_6	p_8	p_{10}	Sym
1	truncated tetrahedron	12	18	8	4			4			T_d
2	cub-octahedron	12	24	14	8	6					O_h
3	truncated octahedron	24	36	14		6		8			O_h
4	truncated cube	24	36	14	8				6		O_h
5	rhomb-cub-octahedron	24	48	26	8	18					O_h
6	great-rhomb-cub-octahedron	48	72	26		12		8	6		O_h
7	icosi-dodecahedron	30	60	32	20		12				I_h
8	truncated icosahedron	60	90	32			12	20			I_h
9	truncated dodecahedron	60	90	32	20					12	I_h
10	snub cube	24	60	38	32	6					O
11	rhomb-icosi-dodecahedron	60	120	62	20	30	12				I_h
12	great rhomb-icosi-dodecahedron	120	180	62		30		20		12	I_h
13	snub dodecahedron	60	150	92	80		12				I

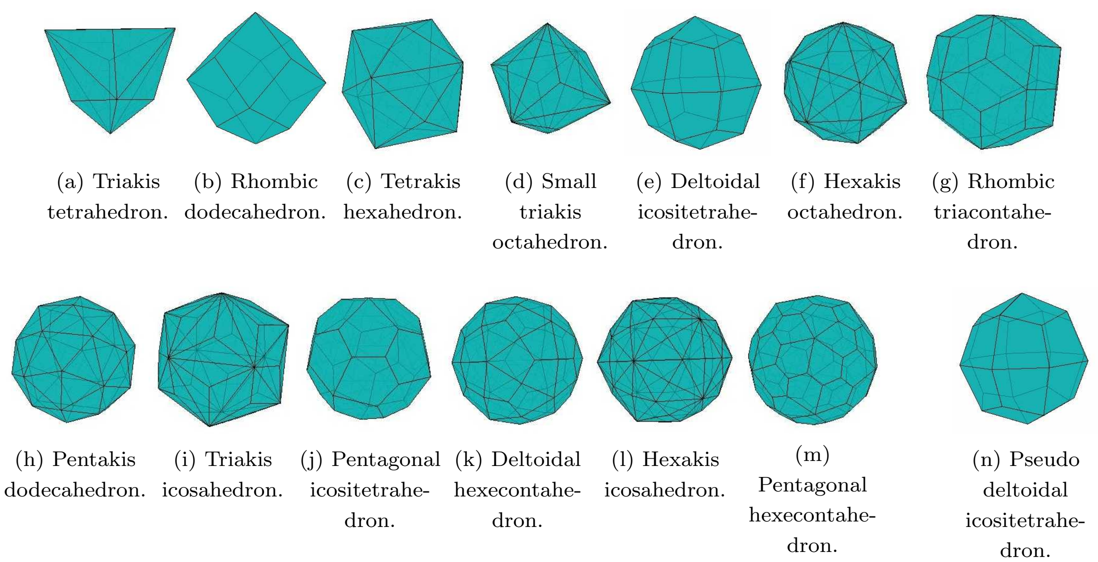
Table A.1: The Archimedean solids.

No.		v	e	f	
1	Pseudo rhomb-cub-octahedron	24	48	26	
2	Pseudo deltoidal icositetrahedron	26	48	24	

Table A.2: The pseudo rhomb-cub-octahedron and its dual.

No.	**Catalan solids**	v	e	f	Sym
1	triakis tetrahedron	8	18	12	T_d
2	rhombic dodecahedron	14	24	14	O_h
3	tetrakis hexahedron	14	36	24	O_h
4	small triakis octahedron	14	36	24	O_h
5	deltoidal icositetrahedron	26	48	24	O_h
6	hexakis octahedron	26	72	48	O_h
7	rhombic triacontahedron	32	60	30	I_h
8	pentakis dodecahedron	32	90	60	I_h
9	triakis icosahedron	32	90	60	I_h
10	pentagonal icositetrahedron	38	60	24	O
11	deltoidal hexecontahedron	62	120	60	I_h
12	hexakis icosahedron	62	180	120	I_h
13	pentagonal hexecontahedron	92	150	60	I

Table A.3: The Catalan solids.



(a) Triakis tetrahedron. (b) Rhombic dodecahedron. (c) Tetrakis hexahedron. (d) Small triakis octahedron. (e) Deltoidal icositetrahedron. (f) Hexakis octahedron. (g) Rhombic triacontahedron.

(h) Pentakis dodecahedron. (i) Triakis icosahedron. (j) Pentagonal icositetrahedron. (k) Deltoidal hexecontahedron. (l) Hexakis icosahedron. (m) Pentagonal hexecontahedron. (n) Pseudo deltoidal icositetrahedron.

Figure A.1: The 13 Catalan solids and the Pseudo Deltoidal icositetrahedron.

A.2 Generators of the symmetry groups of the Platonic solids in $\mathrm{O}_3(\mathbb{R})$

The tetrahedral group T_d: If we choose our coordinates x, y, z such that the tetrahedron has vertices $(1,1,1)$, $(-1,-1,1)$, $(-1,1,-1)$ and $(1,-1,-1)$, then its symmetry group is generated by the following three matrices, namely a rotation σ_1 around the z-axis by π, a rotation σ_2 around the axis trough $(1,1,1)$ by $2\pi/3$ and a reflection τ against the $\{x=y\}$-plane. [Art98, p. 225]

$$\sigma_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ \tau = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The octahedral group O_h: We choose coordinates x, y and z of $\mathbb{R}^3$ such that in these coordinates the vertices of the octahedron are $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$. Then O_h is generated by two rotations σ_1, σ_2 around the x and the y-axes by $\pi/2$ and the reflection against the $\{z = 0\}$-plane τ:

$$\sigma_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The icosahedral group I_h: With coordinates x, y, z such that two of its vertices are $p_1 = (0, 0, 1)$ and $p_2 = (\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}})$, then the symmetry group is generated by a rotation around the z-axis by $2\pi/5$, a rotation around the axis through p_2 by $2\pi/5$ and a reflection against the $\{y = 0\}$-plane.

$$\sigma_1 = \begin{pmatrix} \frac{-1+\sqrt{5}}{4} & -\frac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}} & 0 \\ \frac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}} & \frac{-1+\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} \frac{15+\sqrt{5}}{20} & -\frac{\sqrt{5+\sqrt{5}}}{2\sqrt{10}} & \frac{5-\sqrt{5}}{10} \\ \frac{\sqrt{5+\sqrt{5}}}{2\sqrt{10}} & \frac{-1+\sqrt{5}}{4} & -\frac{\sqrt{5+\sqrt{5}}}{\sqrt{10}} \\ \frac{5-\sqrt{5}}{10} & \frac{\sqrt{5+\sqrt{5}}}{\sqrt{10}} & \frac{1}{\sqrt{5}} \end{pmatrix}, \ \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A.3 Primary and secondary invariants

Group	Primary invariants
D_2	x^2, y^2,
D_3	$x^2 + y^2$, $x^3 - 3xy^2$,
D_4	$x^2 + y^2$, x^2y^2,
D_5	$x^2 + y^2$, $x^5 - 10x^3y^2 + 5xy^4$,
D_{42}	$x^2 + y^2$, $269128937221x^{42} + 5651707680759x^{40}y^2 + 56517076928130x^{38}y^4 +$ $357941481256814x^{36}y^6 + 1610736807291885x^{34}y^8 + 5476503272046807x^{32}y^{10} +$ $14604023707422968x^{30}y^{12} + 31294259959712520x^{28}y^{14} + 54765213944619402x^{26}y^{16} +$ $79104714819953550x^{24}y^{18} + 94926596012096940x^{22}y^{20} + 94925568428882100x^{20}y^{22} +$ $79105422214195650x^{18}y^{24} + 54764880925176198x^{16}y^{26} + 31294365680170680x^{14}y^{28} +$ $14604001591189192x^{12}y^{30} + 5476506214932753x^{10}y^{32} + 1610736571231515x^8y^{34} +$ $357941491748386x^6y^{36} + 56517076704270x^4y^{38} + 5651707682481x^2y^{40} +$ $269128937219y^{42}$.

Table A.4: Primary invariants of some reflection groups contained in $\mathrm{O}_2(\mathbb{R})$

Group	Primary invariants
T_d	$x^2+y^2+z^2$,
	xyz,
	$x^2y^2+y^2z^2+z^2x^2$,
O_h	$x^2+y^2+z^2$,
	$x^2y^2+y^2z^2+z^2x^2$,
	$x^2y^2z^2$,
I_h	$x^2+y^2+z^2$,
	$-z(2x+z)(x^4+2zx^3-10x^2y^2-x^2z^2+10xy^2z-2z^3x+5y^4-5y^2z^2+z^4)$,
	$(4x^2+z^2-6xz)(z^4-2z^3x-x^2z^2+2zx^3+x^4-25y^2z^2-30xy^2z-10x^2y^2+5y^4)(z^4+$ $8z^3x+14x^2z^2-8zx^3+x^4-10y^2z^2-10x^2y^2+5y^4)$.

Table A.5: Primary invariants of some reflection groups contained in $O_3(\mathbb{R})$

Group	Primary invariants	Secondary invariants
D_2	z^2,	1,
	x^2,	xyz,
	y^2,	
D_3	z^2,	1,
	x^2+y^2,	$3x^2yz-y^3z$,
	x^3-3xy^2,	
D_4	z^2,	1,
	x^2+y^2,	x^3yz-xy^3z,
	x^2y^2,	
D_5	z^2,	1,
	x^2+y^2,	$5x^4yz-10x^2y^3z+y^5z$,
	$x^5-10x^3y^2+5xy^4$.	

Table A.6: Primary and secondary invariants of some non reflection groups contained in $O_3(\mathbb{R})$

A.4 Coordinate change

In Chapter 2 we described the construction and mentioned that if the set P, in which we want to prescribe singularities, does not contain the point $(1,0,\dots,0)^T$ we have to perform a coordinate change. Here we present this in detail in one example, namely in the case of the hexahedron, see 1. Then we give a list of the matrices needed in the remaining examples. Note that so far we always used column vectors while here they are rows. Evidently this is no big difference.

We need to perform a coordinate change after which the hexahedron, with vertices $(\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}})^T$, has one vertex in $(1,0,0)^T$. This is the same as rotating the hexahedron. Let $M\in O_3(\mathbb{R})$ be a rotation that turns $(1,0,0)^T$ into $(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})^T$:

$$M = R.R_z.R^{-1} = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{2}(\frac{1}{\sqrt{3}}+1) & \frac{1}{2}(\frac{1}{\sqrt{3}}-1) \\ \frac{1}{\sqrt{3}} & \frac{1}{2}(\frac{1}{\sqrt{3}}-1) & \frac{1}{2}(\frac{1}{\sqrt{3}}+1) \end{pmatrix}, \tag{A.1}$$

with

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, R_z = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} & 0 \\ \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If we perform the coordinate change with the help of this matrix, that is, we substitute x, y and z by the first, second and third row of $M(\bar{x}, \bar{y}, \bar{z})^T$ in the invariants (2.3) we obtain the primary invariants corresponding to the rotated hexahedron, $\bar{u}$, $\bar{v}$, $\bar{w}$.

$$\begin{aligned} \bar{u} &= x^2 + y^2 + z^2, \\ \bar{v} &= xyz^2 + xy^2z + \frac{1}{3}(x^4 - xy^3 - xz^3) + \frac{1}{4}(y^4 + z^4) + \frac{1}{2}y^2z^2, \\ \bar{w} &= \frac{1}{3888}(x - y - z)^2(2\sqrt{3}x + 3y + \sqrt{3}y - 3z + \sqrt{3}z)^2(2\sqrt{3}x - 3y + \sqrt{3}y + 3z + \sqrt{3}z)^2. \end{aligned} \tag{A.2}$$

Using these invariants we can proceed as before. To obtain the polynomial of the hexahedral star in the original invariants (2.6) we perform another coordinate change. We substitute $\bar{x}$, $\bar{y}$ and $\bar{z}$ by the first, second and third row of $M^{-1}(x, y, z)^T$.

To calculate the tetrahedral star we need to perform a coordinate change but we do not need a new matrix: Since one of its vertices, $(1, 1, 1)^T$, is just a scalar of the vertex of the hexahedron, we can use the same matrix A.1 as in the previous example. It turns $(\sqrt{3}, 0, 0)^T$ into $(1, 1, 1)^T$.

Dodecahedron See Example 7. The matrix $M = R_x.R_z.R_x^{-1}$ turns $(1, 0, 0)^T$ to one vertex of the dodecahedron, $q = (-\sqrt{\frac{5-2\sqrt{5}}{15}}, \frac{1}{\sqrt{(3)}}, \frac{\sqrt{5-2\sqrt{5}}}{\sqrt{15(\sqrt{5}-1)}})^T$.

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{(\sqrt{5-\sqrt{5}})^3}{2\sqrt{10}(\sqrt{5}-1)} & \frac{-1}{\sqrt{5}-1} \\ 0 & \frac{1}{\sqrt{5}-1} & \frac{(\sqrt{5-\sqrt{5}})^3}{2\sqrt{10}(\sqrt{5}-1)} \end{pmatrix}, R_z = \begin{pmatrix} -\sqrt{\frac{5-2\sqrt{5}}{15}} & -\sqrt{\frac{10+2\sqrt{5}}{15}} & 0 \\ \sqrt{\frac{10+2\sqrt{5}}{15}} & -\sqrt{\frac{5-2\sqrt{5}}{15}} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Cub-octahedron See Example 9. The matrix M turns the vertex $(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})^T$ of the cub-octahedron, into $(0, 0, 1)^T$.

$$M = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Truncated icosahedron See Example 10. M turns the vertex $(\frac{2}{\sqrt{25+4\sqrt{5}}}, 0, \frac{10+\sqrt{5}}{\sqrt{5}\sqrt{25+4\sqrt{5}}})^T$ of the truncated icosahedron into $(0, 0, 1)^T$.

$$M = \begin{pmatrix} \frac{10+\sqrt{5}}{\sqrt{5}\sqrt{25+4\sqrt{5}}} & 0 & -\frac{2}{\sqrt{25+4\sqrt{5}}} \\ 0 & 1 & 0 \\ \frac{2}{\sqrt{25+4\sqrt{5}}} & 0 & \frac{10+\sqrt{5}}{\sqrt{5}\sqrt{25+4\sqrt{5}}} \end{pmatrix}.$$

For the examples from Section 3.3 on Catalan stars and their “relatives” we do not need any “new” coordinate changes since in our examples we only have vertices that coincide with vertices of some Platonic solid.

A.5 Factorization of the primary invariants of I_h

In Example 7 of the icosahedral and dodecahedral stars we claimed that two of the primary invariants of I_h (3.16) factor into linear polynomials. The zero sets of these linear polynomials correspond to the dodecahedron and icosahedron. The six linear factors of the second invariant v are the zero sets of the six centerplanes of the dodecahedron. Analogously the icosahedron has ten centerplanes, which give the linear factors of w. In the following we give this factorization explicitly:

$$
\begin{aligned}
v(x,y,z) = &-\frac{1}{16}z(2x+z)\left((\sqrt{5}+1)x-\sqrt{10-2\sqrt{5}}y-2z\right)\left((\sqrt{5}+1)x+\sqrt{10-2\sqrt{5}}y-2z\right)\\
&\left((\sqrt{5}-1)x-\sqrt{10+2\sqrt{5}}y+2z\right)\left((\sqrt{5}-1)x+\sqrt{10+2\sqrt{5}}y+2z\right),\\
w(x,y,z) = &-\frac{1}{20250000}\left(-3x+x\sqrt{5}+z\right)\left(3x+x\sqrt{5}-z\right)\\
&\left(-2x\sqrt{75+30\sqrt{5}}+x\sqrt{75+30\sqrt{5}}\sqrt{5}+5\sqrt{3}y-\sqrt{75+30\sqrt{5}}z\right)\\
&\left(-2x\sqrt{75+30\sqrt{5}}+x\sqrt{75+30\sqrt{5}}\sqrt{5}-5\sqrt{3}y-\sqrt{75+30\sqrt{5}}z\right)\\
&\left(2x\sqrt{75-30\sqrt{5}}+x\sqrt{75-30\sqrt{5}}\sqrt{5}-5\sqrt{3}y+\sqrt{75-30\sqrt{5}}z\right)\\
&\left(2x\sqrt{75-30\sqrt{5}}+x\sqrt{75-30\sqrt{5}}\sqrt{5}+5\sqrt{3}y+\sqrt{75-30\sqrt{5}}z\right)\\
&\left(-x\sqrt{75+30\sqrt{5}}+x\sqrt{75+30\sqrt{5}}\sqrt{5}-5y\sqrt{5}\sqrt{3}+5\sqrt{3}y+2\sqrt{75+30\sqrt{5}}z\right)\\
&\left(-x\sqrt{75+30\sqrt{5}}+x\sqrt{75+30\sqrt{5}}\sqrt{5}+5y\sqrt{5}\sqrt{3}-5\sqrt{3}y+2\sqrt{75+30\sqrt{5}}z\right)\\
&\left(x\sqrt{75-30\sqrt{5}}+x\sqrt{75-30\sqrt{5}}\sqrt{5}+5y\sqrt{5}\sqrt{3}+5\sqrt{3}y-2\sqrt{75-30\sqrt{5}}z\right)\\
&\left(x\sqrt{75-30\sqrt{5}}+x\sqrt{75-30\sqrt{5}}\sqrt{5}-5y\sqrt{5}\sqrt{3}-5\sqrt{3}y-2\sqrt{75-30\sqrt{5}}z\right).
\end{aligned}
\tag{A.3}
$$

A.6 SINGULAR input and output

As a selected example we give the **SINGULAR** input (and output) to calculate the primary and secondary invariants of the symmetry group of the icosahedron and the dodecahedron I_h. We need the generators of I_h (A.2). Note that in (A.2) a lot of square roots appear. To work with roots in SINGULAR we need to adjoin them to the base ring with the `splitring` command. For this we need the minimal polynomial of the roots we want to add. In this example it is sufficient to calculate the minimal polynomial of $a = \sqrt{10+2\sqrt{5}}$, the other roots can be calculated from a. We get the minimal polynomial with the help of Wolfram Mathematica 6.0, by using the command `MinimalPolynomial[a,x]`. The minimal polynomial of a is $80-20x^2+x^4$.

```
LIB "finvar.lib";
LIB "primitiv.lib";
ring r=0,(x,y,z),dp;
def r1=splitring(x4-20x2+80);
def b=(a2-10)/2;

matrix sigma1[3][3]= (-1+b)/4,-a/4,0, a/4,(-1+b)/4,0,  0,0,1;
```

```
matrix sigma2[3][3]= 3/4+b/20,-(a*b)/20,1/2-b/10,
(a*b)/20,(-1+b)/4, -(a*b)/10, 1/2-b/10, (a*b)/10,b/5;
matrix tau[3][3]= 1,0,0, 0,-1,0, 0,0,1;

matrix P,2,IS=invariant_ring(sigma1,sigma2,tau);
P[1,1]=x2+y2+z2
P[1,2]=35*x6+105*x4y2+105*x2y4+35*y6-6*x5z+60*x3y2z-30*xy4z+90*x4z2+180*x2y2z2+
90*y4z2+120*x2z4+120*y2z4+32*z6
P[1,3]=5906*x10+28755*x8y2+62160*x6y4+55650*x4y6+30150*x2y8+5875*y10-3870*x9z+
30960*x7y2z+54180*x5y4z-19350*xy8z+20475*x8z2+81900*x6y2z2+122850*x4y4z2+
81900*x2y6z2+20475*y8z2-5880*x7z3+52920*x5y2z3+29400*x3y4z3-29400*xy6z3+
48300*x6z4+144900*x4y2z4+144900*x2y4z4+48300*y6z4-5544*x5z5+55440*x3y2z5-
27720*xy4z5+68040*x4z6+136080*x2y2z6+68040*y4z6+34560*x2z8+34560*y2z8+4064*z10
S;
S[1,1]=1
IS;
IS2[1,1]=0
```

We obtain the primary invariants $P[1,1]$, $P[1,2]$ and $P[1,3]$. Since I_h is a reflection group the secondary invariants are not interesting. Please note that the invariants u, v, w (3.16) we used in the calculations differ from the ones above because we simplified them in the following manner: $u = P[1,1]$, $v = (P[1,2] - 35u^3)/3$, $w = (4P[1,3] - 23500u^5 - 7275u^2v)/31$.

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