ISOSINGULAR LOCI OF ALGEBRAIC VARIETIES

CHRISTOPHER CHIU AND HERWIG HAUSER

Abstract. We define the notion of isosingular loci of algebraic varieties, following the analytic case first studied by Ephraim. In particular, we give a partial extension of his main result in arbitrary characteristic and a full extension assuming characteristic 0. One of the main obstructions in the positive characteristic case is the non-separability of the orbit map associated to the contact group, as first observed by Greuel and Pham for isolated singularities.

Let X be a complex analytic space and $p \in X$. In [2] the *isosingular locus* of X at p was defined to be the subset of X with a prescribed singularity type, that is,

$$\operatorname{Iso}(X, p) = \{q \in X \mid X_q \simeq X_p\},$$

where X_p and X_q denote the analytic germs of X at p and q. The main result of [2] asserts the following:

Theorem ([2, Theorem 0.2, Observation 2.5]). *For a complex analytic space X and $p \in X$ the set $\operatorname{Iso}(X, p)$ is locally closed and smooth as a (reduced) analytic subspace of X. Furthermore, there exists a germ of an analytic space Y_y such that $X_p \simeq Y_y \times \operatorname{Iso}(X, p)_p$, with Y_y having the additional property that there does not exist an isomorphism of germs $Y_y \simeq Y'_{y'} \times \mathbb{C}_0$.*

Generalizations of this result to the relative case of morphisms between analytic spaces were obtained in [8, 9]. We call an analytic germ (X, p) *harmonic* if $\operatorname{Iso}(X, p) = \operatorname{Iso}(\operatorname{Sing} X, p)$. In [4] the classical Mather–Yau theorem (see [10]) was extended from the isolated singularity case to general harmonic singularities. Results in a very similar spirit appeared recently in the preprint [3].

In this paper, our main goal is to study isosingular loci of an algebraic variety X over an arbitrary algebraically closed field k. As we do not have the analytic topology at our disposal, we will replace it by considering the corresponding *formal neighborhoods* instead. That is, for any $x \in X(k)$ the *isosingular locus* of X at x is defined to be the set

$$\operatorname{Iso}(X, x) := \{x' \in X \mid \widehat{X}_{x'} \simeq \widehat{X}_x\},$$

where $\widehat{X}_x$ denotes the formal completion of X along the singleton $\{x\}$. Note that any isomorphism of such formal completions induces an isomorphism of residue fields and thus $\operatorname{Iso}(X, x) \subset X(k)$. Our first main result is an extension of the first part of the above theorem in [2].

Theorem A. *Let X be a variety over an algebraically closed field k.*

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(1) *For each* $x \in X(k)$ *the subset* $\mathrm{Iso}(X,x)$ *is locally closed in* $X(k)$ *(endowed with the Zariski topology).*
(2) *Denote by* $X^{(x)}$ *the unique reduced subscheme of* X *whose* k*-points agree with* $\mathrm{Iso}(X,x)$*. Then* $X^{(x)}$ *is smooth.*

The proof follows the analytic case and uses the action of the contact group $\mathcal{K}$ as well as a version of Artin's celebrated approximation result in [1]. As for remaining part of [2, Theorem 0.2], assuming characteristic 0 we obtain the following result:

Theorem B. *Let* X *be a variety over an algebraically closed field of characteristic* 0*. For each* $x \in X(k)$ *there exists a scheme* Y *of finite type over* k*, a point* $y \in Y(k)$ *and an isomorphism*

$$\widehat{X}_x \simeq \widehat{(X^{(x)})}_x \times \widehat{Y}_y,$$

such that $\widehat{Y}_y$ *has* no smooth factors*, that is, there does not exist an isomorphism* $\widehat{Y}_y \simeq \widehat{Z}_z \times \widehat{(\mathbb{A}^1)}_0$.

The assertion of Theorem B fails in positive characteristics, as can be seen in the example of the Whitney umbrella $f = x^p + y^p z$ in characteristic $p > 0$, c.f. Example 2.6. Let us mention that the proof of Theorem B crucially relies on the characteristic 0 assumption in two separate places. First, it makes use of the Nagata–Zariski-Lipman criterion (see Theorem 2.4), which expresses the existence of smooth factors in terms of the existence of *regular* derivations. This part of the argument could potentially be extended to positive characteristics by means of *Hasse–Schmidt derivations*. The second important assumption which holds only in characteristic 0 is that the orbit map associated to the (truncated) contact group $\mathcal{K}_\beta$ is separable for all β large enough. In fact, the fact that this fails in positive characteristic was first exhibited by Greuel and Pham in [6, 5] for certain isolated singularities. As we will see in Example 2.7, the results of Section 2 give new examples of this phenomenon, including the singularity $f = x^p + y^p z$ mentioned above. Moreover, it prompts the following question:

Question C. *Let* k *be an algebraically closed field and* $0 \in X \subset \mathbb{A}^N$ *a hypersurface singularity given by* $f \in k[x_1, \ldots, x_N]$*. Assume that there exists* $\beta_0 > 0$ *such that the orbit map* $\mathcal{K}_\beta \to o(f_\beta)$ *is separable for* $\beta \geq \beta_0$ *(see Section 2). Does the conclusion of Theorem B hold in this case?*

Note that we restrict the question to the hypersurface case as even then we do not know of any counterexample.

Conventions. Throughout this paper we always assume X to be an algebraic variety, that is, a separated scheme of finite type over an algebraically closed field k. For any point $x \in X(k)$ we denote by $\widehat{X}_x$ the formal neighborhood of X at x. That is, $\widehat{X}_x$ is given by the formal completion of X along the closed subscheme $\{x\}$ and is thus isomorphic to $\mathrm{Spf}(\widehat{\mathcal{O}_{X,x}})$.

1. Contact equivalence and proof of Theorem A

Let k be an algebraically closed field. We start by repeating the main definition: for any point $x \in X(k)$ the *isosingular locus* of X at x is defined to be the set

$$\mathrm{Iso}(X,x) := \{x' \in X \mid \widehat{X}_x \simeq \widehat{X}_{x'}\}.$$

If X is smooth at x, then $\mathrm{Iso}(X,x)$ is clearly an open subset of $X(k)$.

Remark 1.1. Let us briefly discuss the situation when k is not algebraically closed. Assume $\operatorname{char} k = 0$ and consider the Whitney umbrella X defined by x^2+y^2z in $\mathbb{A}^3_k$. Clearly $\operatorname{Sing} X$ is just given by the z-axis. Taking any point of the form $x = (0, 0, t)$ for $t \neq 0$, we see that the formal completion $\widehat{X}_x$ is isomorphic to the union of two hyperplanes if and only if $\sqrt{t} \in k$. Thus the isosingular loci of X might not even be constructible as subsets of $X(k)$.

Since both Theorems A and B are local statements on X, we may assume for the rest of this paper that $X \subset \mathbb{A}^N$ is affine, given by polynomials $f_1, \dots, f_n \in k[x_1, \dots, x_N]$. To simplify notation, let us write $\underline{x} = (x_1, \dots, x_N)$ and $f = (f_1, \dots, f_n)$.

Let us start by recalling a general version of the formal inverse function theorem:

Lemma 1.2. *Let R be any ring and $S = R[[\underline{x}]]$ or $S = R[\underline{x}]/(\underline{x})^\beta$ for some $\beta > 0$. Let $\varphi : S \to S$ be a map of R-algebras given by $\varphi_i := \varphi(x_i)$ and satisfying $\varphi_i(0) = 0$. Then φ is an isomorphism if and only if $\det(\frac{\partial \varphi_i}{\partial x_j}(0))_{i,j \leq N} \in R^*$.*

Proof. Follows from considering the maps on the associated graded rings. □

To describe the action of the contact group we fix some notation. All schemes are considered relative to the base field k; in particular, R will always denote a k-algebra. By $\mathcal{O}_N$ we denote the (infinite-dimensional) scheme whose R-points are formal series in $R[[\underline{x}]]$. Similarly, for $\beta \geq 0$ we define the (finite-dimensional) scheme $\mathcal{O}_{N,\beta}$ via $\mathcal{O}_{N,\beta}(R) := R[\underline{x}]/(\underline{x})^{\beta+1}$. Clearly we have $\mathcal{O}_N = \varprojlim_\beta \mathcal{O}_{N,\beta}$ via the natural maps.

For any ring R we consider the group $\operatorname{Aut}_R(R[[\underline{x}]])$ of local R-automorphisms. By Lemma 1.2 we have

$$\operatorname{Aut}_R(R[[\underline{x}]]) = \{\varphi \in R[[\underline{x}]]^N \mid \varphi(0) = 0,\ \det(\frac{\partial \varphi_i}{\partial x_j}(0))_{i,j \leq N} \in R^*\}. \tag{1a}$$

Thus the assignment $R \mapsto \operatorname{Aut}_R(R[[\underline{x}]])$ defines a group scheme $\operatorname{Aut} \mathcal{O}_N$. Similarly, for each $\beta \geq 0$ we obtain an algebraic group $\operatorname{Aut} \mathcal{O}_{N,\beta}$ whose R-points are elements of $\operatorname{Aut}_R(R[\underline{x}]/(\underline{x})^{\beta+1})$.

For any $n > 0$ the group scheme $\operatorname{Aut} \mathcal{O}_N$ acts on $\mathcal{O}_N^n$ as follows: given $f \in R[[\underline{x}]]^n$ and $\varphi \in \operatorname{Aut}_R(R[[\underline{x}]])$, we define $\varphi \cdot f := f \circ \varphi = f(\varphi_1, \dots, \varphi_N)$. For $\beta \geq 0$ an action of $\operatorname{Aut} \mathcal{O}_{N,\beta}$ on $\mathcal{O}_{N,\beta}^n$ is defined in the same way. These actions are compatible via the diagram

$$\begin{array}{ccc} \operatorname{Aut} \mathcal{O}_N \times \mathcal{O}_N^n & \longrightarrow & \mathcal{O}_N^n \\ \downarrow & & \downarrow \\ \operatorname{Aut} \mathcal{O}_{N,\beta} \times \mathcal{O}_{N,\beta}^n & \longrightarrow & \mathcal{O}_{N,\beta}^n. \end{array} \tag{1b}$$

Definition 1.3. For $N, n > 0$ the *contact group* $\mathcal{K} = \mathcal{K}_{N,n}$ is defined as the group scheme $\mathrm{GL}_n(\mathcal{O}_N) \rtimes \operatorname{Aut} \mathcal{O}_N$. That is, for each R we have

$$\mathcal{K}(R) = \mathrm{GL}_n(R[[\underline{x}]]) \rtimes \operatorname{Aut}_R(R[[\underline{x}]]),$$

where the semidirect product is taken with respect to the group homomorphism $\operatorname{Aut}_R(R[[\underline{x}]]) \to \operatorname{Aut} \mathrm{GL}_n(R[[\underline{x}]])$ given by

$$\varphi \mapsto (M = (m_{i,j})_{i,j} \mapsto M \circ \varphi = (m_{i,j}(\varphi_1, \dots, \varphi_N))_{i,j}).$$

Similarly, for $\beta \geq 0$ the *truncated contact group* $\mathcal{K}_\beta$ is defined as the algebraic group $\mathrm{GL}_n(\mathcal{O}_{N,\beta}) \rtimes \operatorname{Aut} \mathcal{O}_{N,\beta}$ and we have $\mathcal{K} = \varprojlim_\beta \mathcal{K}_\beta$.

The action of $\operatorname{Aut}\mathcal{O}_N$ on $\mathcal{O}_N^n$ together with the natural action of $\mathrm{GL}_n(\mathcal{O}_N)$ defines an action $\rho : \mathcal{K} \times \mathcal{O}_N^n \to \mathcal{O}_N^n$ which is given by

$$\rho : (M, \varphi; f) \mapsto M \cdot (f \circ \varphi).$$

Similarly we obtain an action ρ_β of $\mathcal{K}_\beta$ on $\mathcal{O}_{N,\beta}^n$ and Eq. (1b) implies that ρ and ρ_β are compatible via the diagram

$$\begin{array}{ccc} \mathcal{K} \times \mathcal{O}_N^n & \xrightarrow{\rho} & \mathcal{O}_N^n \\ \downarrow & & \downarrow \\ \mathcal{K}_\beta \times \mathcal{O}_{N,\beta}^n & \xrightarrow{\rho_\beta} & \mathcal{O}_{N,\beta}^n. \end{array} \tag{1c}$$

As a first remark we note that $\mathcal{K}_\beta$ is smooth independent of the characteristic of the base field k.

Lemma 1.4. *For each $\beta \geq 0$ the truncated contact group $\mathcal{K}_\beta$ is smooth over k. Moreover, for any $f \in \mathcal{O}_{N,\beta}(k)$ the map $(\rho_\beta)_f : \mathcal{K}_\beta \to \mathcal{O}_{N,\beta}^n$ given by $(M, \varphi) \mapsto M \cdot (f \circ \varphi)$ has locally closed image $o(f)$, which is smooth when considered as a reduced subscheme of $\mathcal{O}_{N,\beta}^n$ (we call $o(f)$ the* orbit *of f).*

Proof. For the first assertion note that the underlying space of $\mathcal{K}_\beta$ is $\mathrm{GL}_n(\mathcal{O}_{N,\beta}) \times \operatorname{Aut}\mathcal{O}_{N,\beta}$ and that we have open immersions $\mathrm{GL}_n(\mathcal{O}_{N,\beta}) \hookrightarrow \mathbb{A}^{n^2 M}$ and $\operatorname{Aut}\mathcal{O}_{N,\beta} \hookrightarrow \mathbb{A}^{NM}$, where $M := \binom{N+\beta}{N}$. The second assertion then follows from [12, Proposition 7.4]. $\square$

The main idea behind the use of the contact group in the proof of Theorem A is that two points of X have isomorphic formal neighborhoods if and only if the respective Taylor expansions of f at those points lie in the same orbit of $\mathcal{K}$. We want to review this classical argument in our setting. First, let $\gamma : X \to \mathcal{O}_N^n$ be the morphism defined as follows: for each k-algebra R the map $\gamma(R)$ is given by

$$a \in X(R) \subset R^N \mapsto f(\underline{x} + a) \in R[[\underline{x}]].$$

In an analogous way we obtain morphisms $\gamma_\beta : X \to \mathcal{O}_{N,\beta}^n$ for $\beta \geq 0$.

Lemma 1.5. *Let $x_1, x_2 \in X(k)$. Then $\widehat{X}_{x_1} \simeq \widehat{X}_{x_2}$ if and only if $\gamma(x_2) \in o(\gamma(x_1))$.*

Proof. We may assume that $x_1 = 0$ and $x_2 = a \in k^N$. Then

$$\widehat{\mathcal{O}_{X,x_2}} \simeq k[[\underline{x}]]/(f(\underline{x} + a)).$$

The existence of an isomorphism $\widehat{\mathcal{O}_{X,x_1}} \simeq \widehat{\mathcal{O}_{X,x_2}}$ is equivalent to the existence of an isomorphism $\varphi \in \operatorname{Aut}_k(k[[\underline{x}]])$ such that the following equality of ideals holds:

$$(\varphi(f(\underline{x}))) = (f(\underline{x} + a)).$$

To proceed we make use of a trick of Mather:

Lemma 1.6. *Let $A, B \in k^{n \times n}$. Then there exists $C \in k^{n \times n}$ such that $C(1 - AB) + B$ is invertible.*

Proof. Let $r = \operatorname{rk}(B)$ and choose a basis $\{e_i\}_i$ for k^n such that $Be_1, \ldots, Be_r$ are linearly independent and $Be_i = 0$ for $i > r$. Let $e'_{r+1}, \ldots, e'_n$ be such that Be_i, e'_j form a basis. Then C is the matrix representing the linear map given by $e_i \mapsto 0$ for $i \leq r$ and $e_i \mapsto e'_i$ for $i > r$. $\square$

Write $R = k[[\underline{x}]]$ and $\mathfrak{m} = (\underline{x})$. The above equality of ideals implies the existence of $A, B \in R^{n\times n}$ with $A \cdot f(\varphi(\underline{x})) = f(\underline{x} + a)$ and $B \cdot f(\underline{x} + a) = f(\varphi(\underline{x}))$. Then, by Lemma 1.6, there exists $C \in k^{n\times n}$ with $D = C(1 - AB) + B \in R^{n\times n}$ invertible modulo $\mathfrak{m}$, which implies that D is invertible in $R^{n\times n}$. Clearly, $D \cdot f(\underline{x} + a) = f(\varphi(\underline{x}))$. □

Note the contact group $\mathcal{K}$ is not of finite type and neither is the scheme $\mathcal{O}_N^n$. In order to be able to apply Lemma 1.4 for the action of the truncated contact group $\mathcal{K}_\beta$ we make use of a variant of Artin's celebrated approximation results, which is commonly referred to as *Universal Strong Artin Approximation.*

Theorem 1.7 ([1, Theorem 6.1])**.** *Let k be a field. For each tuple (n, m, d, α) of non-negative integers there exists a $\beta \geq 0$ satisfying the following property: let $\underline{x} = (x_1, \dots, x_n)$ and $\underline{y} = (y_1, \dots, y_m)$ be two sets of variables and $f = (f_1, \dots, f_r) \in k[\underline{x}, \underline{y}]^r$ with $\deg(f_i) \leq d$. Assume there exist polynomials $\bar{y} = (\bar{y}_1, \dots, \bar{y}_m) \in k[\underline{x}]^m$ with*

$$f(\underline{x}, \bar{y}(\underline{x})) \equiv 0 \mod (\underline{x})^\beta.$$

Then there exist algebraic power series $y = (y_1, \dots, y_m)$ with $f(\underline{x}, y(\underline{x})) = 0$ and

$$\bar{y} \equiv y(\underline{x}) \mod (\underline{x})^\alpha.$$

Corollary 1.8. *Let $x_1, x_2 \in X(k)$. Then there exists a $\beta > 0$ such that we have $\gamma(x_2) \in o(\gamma(x_1))$ if and only if $\gamma_\beta(x_2) \in o(\gamma_\beta(x_1))$.*

Proof. Assume $x_1 = 0$ and let x_2 be given by $a = (a_1, \dots, a_n) \in k^n$. Consider the system of equations in variables $M = (M_{ij})$ and $\underline{y} = (y_1, \dots, y_N)$ over $k[\underline{x}]$ given by

$$M \cdot f(\underline{y}) - f(\underline{x} + a) = 0.$$

The condition $\gamma_\beta(x_2) \in o(\gamma_\beta(x_1))$ is equivalent to the existence of $(\bar{M}(\underline{x}), \bar{y}(\underline{x}))$ such that $\det(\bar{M}(0))$, $\det(\frac{\partial \bar{y}_i}{\partial x_j}(0)) \in k^*$. Take $\alpha \gg 0$. By Theorem 1.7 there a $\beta > 0$ such that, for each solution $(\bar{M}(\underline{x}), \bar{y}(\underline{x}))$ of this system module $(\underline{x})^\beta$, there exists a solution $(M(\underline{x}), y(\underline{x}))$ with $\bar{M}(\underline{x}) - M(\underline{x}), \bar{y}(\underline{x}) - y(\underline{x}) \in (\underline{x})^\alpha$. Thus, in particular $\det(M(0))$, $\det(\frac{\partial y_i}{\partial x_j}(0)) \in k^*$, which in turn implies that $\gamma(x_2) \in o(\gamma(x_1))$. The other direction follows from the diagram Eq. (1c). □

Thus we obtain the first assertation of Theorem A:

Proposition 1.9. *Let X be a variety over an algebraically closed field k and let $x \in X(k)$. Then $\operatorname{Iso}(X, x)$ is locally closed as a subset of $X(k)$.*

Proof. Follows from Lemmas 1.4 and 1.5 and Corollary 1.8. □

Remark 1.10. Let us mention that the isosingularity loci $\operatorname{Iso}(X, x)$ do not give a stratification of X in general, since X might have infinitely many points with distinct singularities. Consider the classical example by Whitney [14, Example 13.2], which is $X \subset \mathbb{A}^3$ defined by $f = xy(x + y)(x + zy)$. For each point $x = (0, 0, a) \in \operatorname{Sing} X$ the associated tangent cone is a union of four planes, which have a well-defined cross-ratio depending on a. As any formal isomorphism induces a linear isomorphism of tangent cones we see that $\operatorname{Iso}(X, x) = \{x\}$.

Proposition 1.9 leads us the make the following definition:

Definition 1.11. Let X be a scheme of finite type over an algebraically closed field k. For each $x \in X(k)$ we define $X^{(x)}$ to be the unique reduced subscheme of X whose k-points equal $\operatorname{Iso}(X,x)$ and call it the *isosingularity scheme* associated to x.

To finish the proof of Theorem A we aim to show that $X^{(x)}$ is smooth. As we will see in the next section, we cannot conclude directly using that the orbit $o(\gamma(x))$ is smooth. Our strategy is therefore to use generic smoothness to establish the existence of $x' \in \operatorname{Iso}(X,x)$ such that $X^{(x)}$ is smooth at x' and then extend the isomorphism $\widehat{X}_x \simeq \widehat{X}_{x'}$ étale-locally. To that end, recall that an *étale neighborhood* (U,y) of $x \in X(k)$ is an étale morphism $u : U \to X$ and $y \in U(k)$ with $u(y) = x$. Artin's approximation results then imply the following corollary:

Lemma 1.12 ([1, Corollary 2.5])**.** *Let $x \in X(k)$ and $x' \in \operatorname{Iso}(X,x)$, that is, $\widehat{X}_x \simeq \widehat{X}_{x'}$. Then there exists a common étale neighborhood (U,y) of x and x', that is, a diagram of étale morphisms*

$$\begin{array}{ccc} & U & \\ {}^{u}\swarrow & & \searrow^{u'} \\ X & & X \end{array}$$

and $y \in U(k)$ with $u(y) = x$ and $u'(y) = x'$.

Lemma 1.13. *Let $f : U \to X$ be étale and $y \in U(k)$ with $x = f(y) \in X(k)$. Then $f^{-1}(\operatorname{Iso}(X,x)) = \operatorname{Iso}(U,y)$ and the restriction $U^{(y)} \to X^{(x)}$ is étale.*

Proof. The first assertion follows from the fact that, for $y' \in U(k)$ and $x' = f(y')$ the morphism f induces an isomorphism on completions $\widehat{U}_{y'} \simeq \widehat{X}_{x'}$. To see the second claim, consider the fiber diagram

$$\begin{array}{ccc} U \times_X X^{(x)} & \longrightarrow & X^{(x)} \\ \downarrow & & \downarrow \\ U & \xrightarrow[f]{} & X. \end{array}$$

As a base change of f the morphism $U \times_X X^{(x)} \to X^{(x)}$ is étale again and in particular, since $X^{(x)}$ is reduced, so is $U \times_X X^{(x)}$. Thus $U^{(y)} \simeq U \times_X X^{(x)}$ and we are done. ☐

Proposition 1.14. *Let X be a variety over an algebraically closed field k and let $x \in X(k)$. Then $X^{(x)}$ is smooth over k.*

Proof. By definition $X^{(x)}$ is geometrically reduced and thus the subset of k-smooth points of $X^{(x)}$ is dense open (see [13, Tag 056V]). Thus there exists $x' \in X^{(x)}(k) = \operatorname{Iso}(X,x)$ smooth over k. By Lemma 1.12 there exists a common étale neighborhood (U,y) of x and x'. Thus, by Lemma 1.13, we have $\widehat{(X^{(x)})}_x \simeq \widehat{(U^{(y)})}_y \simeq \widehat{(X^{(x')})}_{x'}$. Clearly $X^{(x')} \simeq X^{(x)}$, and thus $\widehat{(X^{(x)})}_x$ is formally smooth over k, which in turn implies that $X^{(x)}$ is smooth at x. ☐

2. The proof of Theorem B and separability of the orbit map

This section is devoted to the proof of Theorem B, which will involve studying the orbit map $\mathcal{K} \to o(f)$. As both sides are non-Noetherian schemes of infinite dimension, we first need to introduce the right notion of smoothness in this setting.

Definition 2.1. Let k be a ring and $(R, \mathfrak{m})$ be a local k-algebra. We say that R is *formally smooth* over k if for every k-algebra C with nilpotent ideal $J \subset C$ and every diagram

$$\begin{array}{ccc} R & \xrightarrow{\bar{\psi}} & C/J \\ \uparrow & \overset{\psi}{\searrow} & \uparrow \\ k & \longrightarrow & C \end{array}$$

such that $\bar{\psi}(\mathfrak{m}^n) = 0$ in C/J there exists a diagonal arrow $\psi : R \to C$ making the diagram commute.

Note that this is equivalent to saying that $(R, \mathfrak{m})$ considered as a topological ring with respect to its $\mathfrak{m}$-adic topology is formally smooth in the sense of [7, (19.3.1)].

Lemma 2.2. *Let $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ be local k-algebras with $R/\mathfrak{m} = S/\mathfrak{n} = k$.*

(1) *Assume $R = \varinjlim_n R_n$ with $\{(R_n, \mathfrak{m}_n)\}_{n \in \mathbb{N}}$ a direct system of local k-algebras smooth over k. Then R is formally smooth over k.*

(2) *Assume $\varphi : R \to S$ is a local map and R, S are formally smooth over k. If the induced cotangent map $T^*\varphi : \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2$ is injective, then there exists a retraction $\widehat{S} \to \widehat{R}$ of the completion map $\widehat{\varphi} : \widehat{R} \to \widehat{S}$.*

Compare the second assertion with the fact that any submersion between smooth manifolds allows for a local section.

Proof. Let us start by proving (1). Since R_n is smooth over k, it is in particular formally smooth over k. By [7, (19.5.4)] this is equivalent to the natural map

$$\mathrm{Sym}_k(\mathfrak{m}_n/\mathfrak{m}_n^2) \to \mathrm{gr}(R_n)$$

being a bijection. The colimit of these maps is given by

$$\mathrm{Sym}_k(\mathfrak{m}/\mathfrak{m}^2) \to \mathrm{gr}(R)$$

and thus this map is a bijection, which, again by [7, (19.5.4)], implies that R is formally smooth over k.

To prove (2), let $x_j \in S$, $j \in J$, be elements whose images form a basis for $\mathfrak{n}/\mathfrak{n}^2$ and such that for $I \subset J$ the images of x_i, $i \in I$, form a basis for the subspace $\mathfrak{m}/\mathfrak{m}^2$. Then the bijection $\mathrm{Sym}_k(\mathfrak{m}/\mathfrak{m}^2) \simeq \mathrm{gr}(R)$ from before induces an isomorphism

$$k[[x_i \mid i \in I]] := \varprojlim_n k[x_i \mid i \in I]/(x_i \mid i \in I)^n \to \widehat{R}$$

and similarly for $\widehat{S}$. In particular, the map $\widehat{\varphi}$ is given by the inclusion

$$k[[x_i \mid i \in I]] \hookrightarrow k[[x_j \mid j \in J]],$$

which has a obvious retraction defined by $x_j \mapsto 0$ for $j \in J \setminus I$. □

Let us now go back to the situation of Theorem B, that is, X is a variety over an algebraically closed field k and for $x \in X(k)$ we let $X^{(x)}$ be the associated isosingularity scheme. We keep the notation of the last section. Our main result will establish the existence of "enough" regular derivations on $\widehat{\mathcal{O}_{X,x}}$. Recall that a derivation $d \in \mathrm{Der}_k(k[[\underline{x}]])$ is called *regular* if there exists $g \in k[[\underline{x}]]$ with $d(g) \in k[[\underline{x}]]^*$.

Lemma 2.3. *Assume that there exists $\beta_0 > 0$ such that the orbit map $\mathcal{K}_\beta \to o(f_\beta)$ is separable for all $\beta \geq \beta_0$. Then, for every tangent vector $a \in T_x X^{(x)}$ there exists a regular derivation $d_a \in \mathrm{Der}_k(k[[\underline{x}]])$ satisfying $d_a(f) \subset (f)$ and $d_a(x) = a$.*

Proof. Write $Z_\beta := o(f_\beta)$ and let $\beta \geq \beta_0$. By Section 1, $\mathcal{K}_\beta$ and Z_β are nonsingular varieties over an algebraically closed field and thus the orbit map $\mathcal{K}_\beta \to Z_\beta$ is generically smooth. Since it is $\mathcal{K}_\beta$-equivariant (and the action of $\mathcal{K}_\beta$ on both sides is obviously transitive) this implies that it is smooth everywhere. In particular, the tangent map $T_1\mathcal{K}_\beta \to T_{f_\beta} Z_\beta$ is surjective for all $\beta \geq \beta_0$. Setting $Z := o(f) \subset \mathcal{O}^n$, we get that $T_1\mathcal{K} \to T_f Z$ is surjective. As both $\mathcal{K}$ and Z are colimits of schemes smooth over k, by Lemma 2.2 the morphism of formal schemes

$$\Phi : \widehat{\mathcal{K}}_1 \to \widehat{Z}_f$$

admits a section $\widetilde{\Psi} : \widehat{Z}_f \to \widehat{\mathcal{K}}_1$. Write Ψ for the composition of the map $\gamma : \widehat{(X^{(x)})}_x \to \widehat{Z}_f$ with $\widetilde{\Psi}$; this gives a factorization of γ as

$$\widehat{(X^{(x)})}_x \xrightarrow{\Psi} \widehat{\mathcal{K}}_1 \xrightarrow{\Phi} \widehat{Z}_f.$$

Let us analyze what this factorization means for the tangent map of γ. We may assume for convenience's sake that $x = 0$. To proceed we consider the functorial description of formal neighborhoods via *test rings* $(A, \mathfrak{m})$, that is, A is a local k-algebra with $A/\mathfrak{m} = k$ and $\mathfrak{m}^n = 0$ for some n. Let $a = (a_1, \ldots, a_N)$ be an A-point of $\widehat{(X^{(x)})}_x$, that is, $a_i \in \mathfrak{m}$ and $f(a_1, \ldots, a_N) = 0$. The map $\Psi(A)$ is given by

$$a \mapsto (M(a), \varphi(a)),\ M(a) \in \mathrm{GL}_n(A[[\underline{x}]]), \varphi(a) \in \mathrm{Aut}_k(A[[\underline{x}]]),$$

where $M(a) \equiv 1 \mod \mathfrak{m}$ and $\varphi(a) \equiv \underline{x} \mod \mathfrak{m}$. Composing with $\Phi(A)$ gives $\gamma(A)$ and thus the identity

$$f(\underline{x} + a) = M(a) \cdot f(\varphi(a)). \tag{2a}$$

To compute $T_0\gamma$ we take $A = k[\varepsilon]/(\varepsilon^2)$ and let $a = \widetilde{a}\varepsilon$ with $\widetilde{a} \in k^N$. Taking Taylor expansions on both sides of (2a) gives

$$f(\underline{x}) + \frac{\partial f}{\partial x}(\underline{x}) \cdot \widetilde{a}\varepsilon = (1 + \widetilde{M}(\widetilde{a})\varepsilon)(f(\underline{x}) + \frac{\partial f}{\partial \underline{x}}(\underline{x}) \cdot \widetilde{\varphi}(\widetilde{a})\varepsilon),$$

with $\widetilde{M}(\widetilde{a}) \in k[[\underline{x}]]^{n\times n}$ and $\widetilde{\varphi}(\widetilde{a}) \in k[[\underline{x}]]^n$. Simplifying yields

$$\frac{\partial f}{\partial x}(\underline{x}) \cdot (\widetilde{a} - \widetilde{\varphi}(\widetilde{a})) = \widetilde{M}(\widetilde{a}) \cdot f(\underline{x}). \tag{2b}$$

Define $d := \frac{\partial}{\partial \underline{x}} \cdot (\widetilde{a} - \widetilde{\varphi}(\widetilde{a})) \in \mathrm{Der}_k(k[[\underline{x}]])$. As $\underline{x} + \widetilde{\varphi}(\widetilde{a}) \in \mathrm{Aut}_k(k[[\underline{x}]])$, it follows that $d(0) = \widetilde{a}$ and by (2b) we have $d(f) \subset (f)$. □

Now the proof of Theorem B follows from Lemma 2.3 together with the following variant of the classical Nagata–Zariski–Lipman criterion:

Theorem 2.4. *Assume that k is of characteristic 0 and X is a variety over k. Let $X' \subset X$ be a subvariety which is smooth of dimension m at a point $x \in X'(k)$. Assume that for a choice of local coordinates $x'_1, \ldots, x'_m$ for X' at x the associated derivations $dx'_1, \ldots, dx'_m$ lift to derivations on X. Then*

$$\widehat{X}_x \simeq \widehat{(X')}_x \times \widehat{Y}_y,$$

for some variety Y and $y \in Y(k)$.

Proof. See for example [11, Theorem 30.1]. □

Proof of Theorem B. Assume $\operatorname{char} k = 0$, then the orbit map $\mathcal{K}_\beta \to o(f_\beta)$ is separable since it is dominant. Thus Lemma 2.3 and Theorem 2.4 together imply that

$$\widehat{X}_x \simeq \widehat{(X^{(x)})}_x \times \widehat{Y}_y,$$

for some variety Y and $y \in Y(k)$. Using Lemma 2.5 below it follows that $\widehat{Y}_q$ itself has no smooth factors. □

Lemma 2.5. *Let X be a scheme of finite type over an algebraically closed field k and let $x \in X(k)$. Assume that $\widehat{X}_x \simeq \widehat{Y}_y \times \Delta^m$. Then $\dim_x X^{(x)} \geq m$.*

Proof. By Lemma 1.12 we can find a common étale neighborhood U for $x \in X$ and $y' = (y, 0) \in Y \times \mathbb{A}^m$. Clearly $\operatorname{Iso}(Y \times \mathbb{A}^m, y') \supset \{y\} \times \mathbb{A}^m$ and thus we are done using Lemma 1.13. □

We now want to discuss the existence of a decomposition as in Theorem B for k of positive characteristic. As the following example shows, this fails in general.

Example 2.6. Let k be of characteristic $p > 0$ and X be the Whitney umbrella given by $x^p + y^p z$ in $\mathbb{A}^3_k$. We claim that $\widehat{X}_0$ is isomorphic to $\widehat{X}_x$, where $x = (0, 0, t)$ for some $t \neq 0$. As k is algebraically closed, there exists $s \in k$ with $s^p = t$. Consider now the change of coordinates φ given by

$$x \mapsto x + ys,\ y \mapsto y,\ z \mapsto z.$$

Then $\varphi(x^p + y^p z) = x^p + y^p(z + t)$ and thus $\widehat{X}_0 \simeq \widehat{X}_x$. In particular, we have that $\operatorname{Iso}(X, 0)$ is just the z-axis.

We claim that $\widehat{X}_0$ has no smooth factors and sketch the argument here. By an extension of Theorem 2.4 (see for example [11]) it is sufficient to show that there does not exist a regular continuous *Hasse–Schmidt derivation* $D \in \operatorname{Der}^\infty_k(\widehat{\mathcal{O}_{X,0}})$. That is, there does not exist a map $D : \widehat{\mathcal{O}_{X,0}} \to \widehat{\mathcal{O}_{X,0}}[[t]]$ of $\widehat{\mathcal{O}_{X,0}}$-algebras which is continuous for the respective adic topologies and such that there exists an element $g \in \widehat{\mathcal{O}_{X,0}}$ with $g(0) = 0$ and $D(g)$ invertible. To that avail, suppose such a map D is given by

$$D(x) = \sum_{i \geq 0} \widetilde{x}_i t^i,\ D(y) = \sum_{i \geq 0} \widetilde{y}_i t^i,\ D(z) = \sum_{i \geq 0} \widetilde{z}_i t^i,$$

satisfying $\widetilde{x}_0 = x$ and so on. Applying D to the equation $x^p + y^p z = 0$ yields a system of equations for $\widetilde{x}_i, \widetilde{y}_i, \widetilde{z}_i$. For simplicity we will give them explicitly only in the case $p = 2$:

$$y^2 \widetilde{z}_1 = 0$$
$$\widetilde{x}_1^2 + y^2 \widetilde{z}_2 + \widetilde{y}_1^2 z = 0$$
$$\ldots$$

From the first equation it follows that $\widetilde{z}_1 = 0$ and from the second that $\widetilde{x}_1(0) = 0$. Now suppose that $\widetilde{y}_1$ is invertible. Then the second equation gives

$$z = -\frac{y^2\widetilde{z}_2 + \widetilde{x}_1^2}{\widetilde{y}_1^2}.$$

Note that the right hand side has order ≥ 2, which gives a contradiction. Thus it follows that for any $g \in \widehat{\mathcal{O}_{X,0}}$ with $g(0) = 0$ we have that $D(g)$ is not invertible.

One of the main assumptions in the proof of Theorem B was the separability of the orbit map $\mathcal{K}_\beta \to o(f_\beta)$ for $\beta \gg 0$. As observed already in [6, Example 2.9], this fails in positive characteristics for general isolated singularities. The example provided there was the cusp singularity $f = y^2 + x^3$ for $\operatorname{char}(k) = 2$. While obviously not applicable to the case of isolated singularities, Lemma 2.3 can be used to construct related examples where the separability of the orbit map fails.

Example 2.7. Let k be algebraically closed with $\operatorname{char} k = 2$ and consider the deformation $\widetilde{f} = x^2 + y^3 + zy^2 \in k[x, y, z]$ of the cusp singularity $f = x^2 + y^3$. Set $X = V(\widetilde{f}) \subset \mathbb{A}^3$; we claim that $X^{(0)} = V(x, y) \simeq \mathbb{A}^1$. If $t \neq 0$ and $x = (0, 0, t)$, then an isomorphism between $\widehat{X}_0$ and $\widehat{X}_x$ is given by the map

$$x \mapsto x + ys,\ y \mapsto y,\ z \mapsto z,$$

where $s \in k$ with $s^2 = t$. However, there does not exist $d \in \operatorname{Der}_k(k[[x, y, z]])$ satisfying $d(\widetilde{f}) \subset (\widetilde{f})$ and $d(0) = (0, 0, 1)$, as can be verified with an argument similar to the one in Example 2.6. Therefore, by Lemma 2.3 the orbit map $\mathcal{K}_\beta \to o(\widetilde{f}_\beta)$ is inseparable for infinitely many $\beta > 0$.

Note that the same argument also works for $f = x^p + y^p z$ and $\operatorname{char} k = p$, as in Example 2.6.

As mentioned in the introduction, these examples prompt the question whether inseparability of the orbit map is the main obstruction to extending Theorem B to positive characteristics. We expect a further investigation into this problem to shed more light on the formal structure of singularities for $\operatorname{char} k = p$.

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(C. Chiu) Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, A-1090 Wien (Österreich)

Email address: `christopher.heng.chiu@univie.ac.at`

(H. Hauser) Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, A-1090 Wien (Österreich)

Email address: `herwig.hauser@univie.ac.at`