ON THE FORMAL NEIGHBORHOOD OF DEGENERATE ARCS

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Abstract. For any k-point x of a variety X and its image $\overline{x}$ in the arc space $\mathcal{L}(X)$ we prove a structure theorem for the formal neighborhood $\mathcal{L}(X)_{\overline{x}}$ in the case $\mathrm{char}(k) = 0$. In particular, our result implies that, for x singular, there does not exist a decomposition for $\mathcal{L}(X)_{\overline{x}}$ as in the theorem of Grinberg, Kazhdan and Drinfeld.

1. Introduction

Let X be a scheme of finite type over a field k and let $\mathcal{L}(X)$ be the arc space of X. Any k-point of $\mathcal{L}(X)$ corresponds to an arc $\mathrm{Spec}(k[[t]]) \to X$ on X. Call $\gamma \in \mathcal{L}(X)$ *non-degenerate* if the image of γ is not entirely contained in $\mathrm{Sing}(X)$. The following result - proven first by Grinberg and Kazhdan for characteristic 0 and later by Drinfeld in general - is an important step in understanding the singularities of $\mathcal{L}(X)$.

Theorem 1.1 ([9], [7])**.** *Let $\gamma \in \mathcal{L}(X)(k)$ be non-degenerate and write $\mathcal{L}(X)_\gamma$ for the formal neighborhood of γ. Then there exists a scheme Y of finite type over k and $y \in Y$ such that $\mathcal{L}(X)_\gamma \cong Y_y \times \mathrm{D}^\infty$, where D^∞ is the product of countably many copies of $\mathrm{D} = \mathrm{Spf}(k[[t]])$.*

The formal scheme Y_y is sometimes called a *finite-dimensional formal model* of $\mathcal{L}(X)$ at γ. The existence of finite-dimensional formal models is used in [4] to define a canonical function on the space of non-degenerate arcs, which should correspond to the Frobenius trace on the (hypothetical) intersection complex of $\mathcal{L}(X)$. In [5] and [6] stronger versions of Theorem 1.1 appeared which give a better description of the local structure at non-degenerate arcs. See also [11] for a version which extends the isomorphism of Theorem 1.1 to the strata of a certain stratification of $\mathcal{L}(X)$.

The assumption that γ is non-degenerate in Theorem 1.1 is essential, as [2] shows. There the authors proved that, assuming that the base field k does not contain a root of -1, the Theorem 1.1 fails for the constant arc in $0 \in V(x^2 + y^2)$. Furthermore, they ask whether the same holds for the constant arc in $0 \in V(y^2 - x^3)$. Our main result clarifies the situation for degenerate constant arcs on arbitrary varities in characteristic 0. We say that the formal neighborhood X_x *has a smooth factor* if there exists an isomorphism $X_x \cong Z \times \mathrm{D}$ (with the product taken in the category of formal k-schemes). We then have:

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Theorem 1.2. *Let k be a field of characteristic 0. Let X a scheme locally of finite type over k and $x \in X(k)$. Assume that $X_x \cong Y_y \times D^r$ and Y_y has no smooth factor. Let $\gamma \in \mathcal{L}(X)(k)$ be such that $\gamma(0) = x$ and its restriction to Y is the constant arc $\overline{y}$. Then $\mathcal{L}(X)_\gamma = \mathcal{L}(Y)_{\overline{y}} \times \mathrm{D}^\infty$ and $\mathcal{L}(Y)_{\overline{y}}$ admits no smooth factor.*

In particular, the theorem implies that for any $x \in \mathrm{Sing}(X)$ there exists no decomposition for $\mathcal{L}(X)_{\overline{x}}$ as in Theorem 1.1, see Corollary 3.3. The assumption in Theorem 1.2 that the arc γ is constant on Y is essential, as Example 3.5 shows. Nevertheless, in Example 3.6 we prove that there does not exist a finite-dimensional formal model for any degenerate arc on the Whitney umbrella $V(y^2 - x^2 z)$.

The proof of Theorem 1.2 uses a version of the criterion for triviality of germs of analytic varieties, which we are going to discuss in the next section.

Conventions. Our definition of a formal k-scheme is in agreement with [10]. A *test-ring* is a local k-algebra A with nilpotent maximal ideal $\mathfrak{m}$ and such that $A/\mathfrak{m} \cong k$. In particular, any test-ring is a discrete topological ring with respect to the $\mathfrak{m}$-adic topology. As in [7], for any k-scheme Y and $y \in Y(k)$ the formal neighborhood $Y_y = \mathrm{Spf}(\widehat{\mathcal{O}_{Y,y}})$ can be described by its functor of points from the category of test-rings to sets.

2. Derivations and the Triviality criterion

Throughout this paper let k be a field of characteristic 0. Let A be any index set and denote by $k[x^A]$ the polynomial ring in the variables x_α, $\alpha \in A$. By $k[[x^A]]$ we denote the completion of $k[x^A]$ with respect to the (x^A)-adic topology. Its elements are formal power series of the form $f = \sum_{n \geq 0} f_n$, where each f_n is a homogeneous polynomial in $k[x^A]$ of degree n. Observe that $k[[x^A]]$ is local with maximal ideal $\mathfrak{m}$ consisting of all power series of order ≥ 1. Furthermore, when endowed with the inverse limit topology $k[[x^A]]$ becomes an admissible topological ring (see [10, 0;(7.1.2)]) with a system of neighborhoods for 0 given by

$$\overline{\mathfrak{m}^n} = \ker(k[[x^A]] \to k[x^A]/(x^A)^n).$$

Remark 2.1. If A is infinite, then $k[[x^A]]$ is not Noetherian and the inverse limit topology is no longer equivalent to the $\mathfrak{m}$-adic one. Moreover, $k[[x^A]]$ is not flat as a module over $k[x^A]$ and we have $\mathfrak{m} \supsetneq (x^A) \cdot k[[x^A]]$. For more details, see [13, Example 1.8].

Let us remark that the following version of the inverse function theorem holds independent of the size of the index set A. The proof is the same as in the finite-dimensional case.

Lemma 2.2 (Formal inverse function theorem)**.** *Let $\varphi : k[[x^A]] \to k[[x^A]]$ be a (continuous) endomorphism. Then φ is an isomorphism if and only if the induced map $\mathfrak{m}/\overline{\mathfrak{m}^2} \to \mathfrak{m}/\overline{\mathfrak{m}^2}$ is an isomorphism of k-vector spaces.*

For any topological k-algebra R define $\mathrm{Der}_k^{\mathrm{cont}}(R)$ to be the module of continuous k-derivations $R \to R$. Observe that $\mathrm{Der}_k^{\mathrm{cont}}(k[[x^A]])$ is isomorphic to the topological module of sequences $k[[x^A]]^A$ via $d \mapsto (d(x_\alpha))_{\alpha \in A}$. We write each $d \in \mathrm{Der}_k^{\mathrm{cont}}(k[[x^A]])$ as a (possibly infinite) sum

$$d = \sum_{\alpha \in A} f_\alpha \partial_\alpha,$$

where $f_\alpha = d(x_\alpha)$ and ∂_α the derivation given by $\partial_\alpha(x_\beta) = \delta_{\alpha\beta}$. Such a derivation d is called *regular* if $d \not\equiv 0 \mod \mathfrak{m}$, or equivalently, if there exists $\alpha \in A$ with $f_\alpha \in k[[x^A]]^*$.

Let $R = k[[x^A]]/I$ be some quotient, then for any $d \in \mathrm{Der}_k^{\mathrm{cont}}(R)$ there exists a derivation $d' \in \mathrm{Der}_k^{\mathrm{cont}}(k[[x^A]])$ with $d'(I) \subset I$ making the following diagram commute:

$$\begin{array}{ccc} k[[x^A]] & \xrightarrow{d'} & k[[x^A]] \\ \downarrow & & \downarrow \\ k[[x^A]]/I & \xrightarrow{d} & k[[x^A]]/I. \end{array}$$

We say that d is regular if any lifting of it is.

If $\varphi : R \to S$ is an isomorphism of quotients of power series rings, then φ induces a map

$$\varphi_* : \mathrm{Der}_k^{\mathrm{cont}}(S) \to \mathrm{Der}_k^{\mathrm{cont}}(R),\ d \mapsto \varphi^{-1} \circ d \circ \varphi\,.$$

It is obvious that $\varphi_*(d)$ is regular if and only if d is regular.

Let Z be any formal k-scheme. Recall that Z has a smooth factor if there exists some formal k-scheme Z' and an isomorphism $Z \cong Z' \times \mathrm{D}$, where $\mathrm{D} = \mathrm{Spf}(k[[t]])$ denotes the formal disk. Now let Y be a scheme over k, $y \in Y(k)$ and Y_y be the formal neighborhood of y. The following proposition is well-known in the case of Y being locally of finite type (see for example [8, p. 91]). In the general case the proof is exactly the same and we shall reproduce it here.

Proposition 2.3 (Triviality criterion)**.** *Y_y has a smooth factor if and only if there exists a regular derivation $d \in \mathrm{Der}_k^{\mathrm{cont}}(\widehat{\mathcal{O}_{Y,y}})$.*

Proof. Write $S = \widehat{\mathcal{O}_{Y,y}}$. There exists an index set A such that $S \cong k[[x^A]]/I$ (as topological rings), for some closed ideal I.

Assume now that $Y_y \cong Z \times \mathrm{D}$, then there exists an isomorphism $\varphi : S \to S' \widehat{\otimes} k[[t]]$ with S' an admissible local k-algebra. Observe that $d' = \mathrm{id}_{S'} \widehat{\otimes} \partial_t$ gives a regular derivation of $S' \widehat{\otimes} k[[t]]$. Then $d := \varphi_*(d) \in \mathrm{Der}_k^{\mathrm{cont}}(S)$ is regular by definition and we are done.

Now assume that there exists $d = \sum f_\alpha \partial_\alpha \in \mathrm{Der}_k^{\mathrm{cont}}(k[[x^A]])$ with $d(I) \subset I$ and $\beta \in A$ with $f_\beta \in k[[x^A]]^*$. Applying Lemma 2.4 to the system

$$\frac{\partial \phi_\alpha}{\partial x_\beta} = f_\alpha(\underline{\phi}), \phi_\alpha|_{x_\beta = 0} = (1 - \delta_{\alpha\beta}) x_\alpha,$$

shows that there exists an endomorphism $\phi : k[[x^A]] \to k[[x^A]]$ which is an isomorphism by Lemma 2.2 and such that $\phi_*(\partial_\beta) = d$. In particular, for any $g \in I$ we have that $\partial_\beta(\phi(g)) = \phi(d(g)) \in I$. Taking Taylor expansion with respect to x_β we see that every $\phi(g)$ can be expressed as a power series in x_β with coefficients of the form $\phi(\tilde{g})|_{x_\beta=0}$ with $\tilde{g} \in I$. Thus

$$\phi(I) \subset \overline{(\phi(g)|_{x_\beta=0}, g \in I)}.$$

The converse inclusion follows from the formula

$$h|_{x_\beta=0} = \sum_n \frac{(-1)^n}{n!} \frac{\partial^n h}{\partial x_\beta^n} x_\beta^n.$$

So we can assume that I is the closure of an ideal generated by elements $g \in k[[x^A]]$ which do not depend on the variable x_β. Consider the image I' of I under the map $k[[x^A]] \to k[[x^{A-\{\beta\}}]]$, it follows that

$$k[[x^A]]/I \cong k[[x^{A-\{\beta\}}]]/I' \widehat{\otimes} k[[x_\beta]].$$

Since $S' = k[[x^{A-\{\beta\}}]]/I'$ is again admissible we can define $Z = \mathrm{Spf}(S')$.

Now all that is left to prove is the following lemma:

Lemma 2.4. *Let $g_\alpha \in k[[x^A]]$ for $\alpha \in A$ and write $\mathfrak{m}$ for the maximal ideal in $k[[x^A]]$. For any collection of elements $c_\alpha \in \mathfrak{m}$ there exists a solution $\phi = (\phi_\alpha)_{\alpha \in A}$ in $k[[x^A]]\widehat{\otimes}k[[t]]$ of the following initial value problem:*

$$\frac{\partial \phi_\alpha}{\partial t} = g_\alpha(\underline{\phi}),\ \phi_\alpha(0) = c_\alpha.$$

Note that the composition $g_\alpha(\underline{\phi})$ is well-defined since $\phi_\alpha(0) \in \mathfrak{m}$.

Proof. Write $\phi_\alpha = \sum_{d \geq 0} \phi_{\alpha,d} t^d$, then the above gives a triangular system of equations for the coefficients $\phi_{\alpha,d}$. Namely, assume we have constructed $\phi_\alpha^n = \sum_{d=0}^n \phi_{\alpha,d} t^d$ such that

$$\frac{\partial \phi_\alpha^n}{\partial t} = g_\alpha(\underline{\phi^n}) \bmod (t)^n$$

holds. Now write

$$g_\alpha(\underline{\phi^n}) - \frac{\partial \phi_\alpha}{\partial t} = \widetilde{g_\alpha} t^n + t^{n+1}(\ldots)$$

and observe that $\widetilde{g_\alpha} \in R$ is polynomial in $\phi_{\alpha,d}$, $d \leq n$. Set $\phi_{\alpha,n+1} := \frac{1}{n+1}\widetilde{g_\alpha}$ and $\phi_\alpha^{n+1} := \sum_{d=0}^{n+1} \phi_{\alpha,d} t^d$ and proceed by induction. □

This finishes the proof of the proposition.

□

An easy application of the triviality criterion yields the following lemma.

Lemma 2.5. *Let Y, Y' be two k-schemes and $y \in Y(k)$, $y' \in Y'(k)$. Suppose there exist index sets A, B such that*

$$Y_y \times \mathrm{D}^A \cong Y'_{y'} \times \mathrm{D}^B .$$

If Y_y and $Y'_{y'}$ admit no smooth factors, then the above isomorphism induces an isomorphism of k-vector spaces

$$\mathfrak{m}_y/\overline{\mathfrak{m}_y^2} \cong \mathfrak{m}_{y'}/\overline{\mathfrak{m}_{y'}^2},$$

where $\mathfrak{m}_y$ and $\mathfrak{m}_{y'}$ denote the maximal ideals of $\widehat{\mathcal{O}_{Y,y}}$ resp. $\widehat{\mathcal{O}_{Y',y'}}$.

Remark 2.6. Lemma 2.5 also appears as a weaker version in [1, Lemma 7.5], proven only for Y, Y' locally of finite type. We shall note here that the proof there does not immediately generalize to arbitrary schemes Y and Y'. In particular, the proof makes use of [10, 0;(7.2.4)], which states that for any ideal I of an admissible ring R which contains an ideal of definition the I-adic topology on R is separated and complete. As discussed in [13], this does not hold for $R = k[[x^A]]$ and I the maximal ideal of R.

Proof. By assumption we have an isomorphism of complete topological rings

$$\varphi : \widehat{\mathcal{O}_{Y,y}} \widehat{\otimes}_k k[[t^A]] \to \widehat{\mathcal{O}_{Y',y'}} \widehat{\otimes}_k k[[u^B]].$$

Now assume there exists $f \in \mathfrak{m}_y$ with $\varphi(f) = \ldots + cu_\beta + \ldots$, with $c \in k^*$, $\beta \in B$. Then $d = \varphi_*(\partial_{u_\beta})$ is a regular derivation since $d(f) = c \bmod \overline{\mathfrak{m}_y^2}$. But Proposition 2.3 yields a contradiction, hence the module $\mathfrak{m}_y/\overline{\mathfrak{m}_y^2}$ maps into $\mathfrak{m}_{y'}/\overline{\mathfrak{m}_{y'}^2}$ via φ. Applying the same argument in the opposite direction yields the desired statement. □

Let us also mention here that from Lemma 2.5 we can easily deduce the following *cancellation theorem*: if Y and Y' are schemes locally of finite type over k and $y \in Y(k), y' \in Y'(k)$ such that both Y_y and $Y'_{y'}$ do not have smooth factors, then the existence of an isomorphism

$$Y_y \times \mathrm{D}^\infty \cong Y'_{y'} \times \mathrm{D}^\infty$$

already implies that Y_y and $Y'_{y'}$ are isomorphic (see also [1, 7.1] and [4, Proposition 1.2], which are proven also for $\mathrm{char}(k) > 0$). The result was finally extended to arbitrary Y and Y' in [3].

3. Application to the formal neighborhoods of arcs

From now on let X be a scheme locally of finite type over k. Then, for any arc $\gamma \in \mathcal{L}(X)(k)$, the formal neighborhood $\mathcal{L}(X)_\gamma$ depends only on the formal structure at $\gamma(0) \in X$. More precisely, for every test-ring A there is a natural bijection between the set of A-deformations of γ and the set

$$\{\widetilde{\gamma} : \mathrm{Spf}(A[[t]]) \to X_{\gamma(0)}, \widetilde{\gamma} \equiv \gamma \mod \mathfrak{m}_A\}.$$

In particular, we see that for an isomorphism $X_{\gamma(0)} \cong Y_y$ of formal neighborhoods, the arc γ induces an arc $\gamma' \in \mathcal{L}(Y)(k)$ with $\gamma'(0) = y$. With these observations the following lemma is immediate:

Lemma 3.1. *Let $\gamma \in \mathcal{L}(X)(k)$ and suppose there exist an isomorphism of formal neighborhoods $X_{\gamma(0)} \cong Y_y \times Y'_{y'}$. Let (γ_1, γ_2) denote the arc on $Y \times_k Y'$ corresponding to γ. Then $\mathcal{L}(X)_\gamma \cong \mathcal{L}(Y)_{\gamma_1} \times \mathcal{L}(Y')_{\gamma_2}$.*

The main result of this section is the following proposition:

Proposition 3.2. *Let $x \in X(k)$ and write $\overline{x}$ for the corresponding constant arc. Then X_x has a smooth factor if and only if $\mathcal{L}(X)_{\overline{x}}$ does so.*

It is clear that Proposition 3.2 together with Lemma 3.1 are enough to prove Theorem 1.2. Before we come to its proof let us quickly mention one consequence which is a partial converse of the Grinberg-Kazhdan-Drinfeld theorem:

Corollary 3.3. *Let $x \in X(k)$ and $X_x \cong Y_y \times \mathrm{D}^r$, with Y having no smooth factor. Let $\gamma \in \mathcal{L}(X)(k)$ be an arc which is constant on Y. Then $\mathcal{L}(X)$ has no finite-dimensional formal model at γ, i.e. there does not exist a decomposition $\mathcal{L}(X)_\gamma \cong Y_y \times \mathrm{D}^\infty$ with Y of finite type.*

Proof. By Theorem 1.2, $\mathcal{L}(X)_\gamma \cong \mathcal{L}(Y)_{\bar{y}} \times \mathrm{D}^\infty$ and $\mathcal{L}(Y)_{\bar{y}}$ admits no smooth factor. Assume that $\mathcal{L}(X)_\gamma$ has a decomposition as above, then

$$\mathcal{L}(Y)_{\bar{y}} \times \mathrm{D}^\infty \cong Z_z \times \mathrm{D}^\infty .$$

Using Lemma 2.5 yields a contradiction to Z_z being of finite dimension. □

Proof of Proposition 3.2. Suppose that X_x has a smooth factor. By Lemma 3.1 we may assume that X_x is isomorphic to D^r, i.e. x is a smooth point of X. But then $\mathcal{L}(X)_{\overline{x}}$ is isomorphic to D^∞, so we are done.

For the converse, we first assume $X = V(I) \subset \mathbb{A}^N$ is affine and $x = 0$. Then $X_x = \mathrm{Spf}(R_0)$, where

$$R_0 = k[[x_1, \ldots, x_N]]/I_0,$$

with $I_0 = I \cdot k[[x_1, \ldots, x_N]]$. Recall that $\mathcal{L}(X)$ can be constructed as follows: let $f \in I$ and consider the power series expansion

$$f(\sum_{n\geq 0} x_{1,n}t^n, \ldots, \sum_{n\geq 0} x_{N,n}t^n) = \sum_{n\geq 0} F_n t^n,$$

with $F_n \in k[x_1^{\mathbb{N}}, \ldots, x_N^{\mathbb{N}}]$. Then $\mathcal{L}(X) = \mathrm{Spec}(k[x_1^{\mathbb{N}}, \ldots, x_N^{\mathbb{N}}]/I_\infty)$, where $I_\infty = (F_n, f \in I)$.

Remark 3.4. Let $f = \sum_{\alpha\in\mathbb{N}^N} c_\alpha x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ be a polynomial in $k[x_1, \ldots, x_N]$. Then F_n is of the form

$$F_n = \sum_{\alpha\in\mathbb{N}^N} c_\alpha \sum_{\substack{\lambda_i\in\mathbb{N}^{\alpha_i},\\ \sum_{i\in[N]}|\lambda_i|=n}} \prod_{i\in[N]} x_{i,\lambda_i^1} \cdots x_{i,\lambda_i^{\alpha_i}}.$$

Note that there occurs no cancellation of monomials in F_n, since k has characteristic 0.

Finally, we have that $\mathcal{L}(X)_{\overline{x}} = \mathrm{Spf}(R_\infty)$ with

$$R_\infty = k[[x_1^{\mathbb{N}}, \dots, x_N^{\mathbb{N}}]]/I_{\infty,0},$$

where $I_{\infty,0} = \overline{I_\infty}$ denotes the topological closure of I_∞. Note that we have maps

$$k[[x_1, \dots, x_N]] \xrightarrow{\iota} k[[x_1^{\mathbb{N}}, \dots, x_N^{\mathbb{N}}]] \xrightarrow{\sigma} k[[x_1, \dots, x_N]] \ ,$$

with ι being the inclusion $x_i \mapsto x_{i,0}$ and σ its retraction given by $x_{ij} = 0$ for $j \geq 1$. Observe that $\sigma(I_{\infty,0}) = I_0$.

Now assume $\mathcal{L}(X)_0$ has a smooth factor. By Proposition 2.3 there exists a regular derivation d of $k[[x_1^{\mathbb{N}}, \dots, x_N^{\mathbb{N}}]]$ with $d(I_{\infty,0}) \subset I_{\infty,0}$. Write

$$d = \sum_{i \in [N], j \in \mathbb{N}} g_{i,j} \partial_{x_{i,j}},\ g_{i,j} \in k[[x_1^{\mathbb{N}}, \dots, x_N^{\mathbb{N}}]].$$

The regularity of d means that there exists an index $l \in \mathbb{N}$ such that some $g_{i,l}$ is invertible. Set $g_i := \sigma(g_{i,l}) \in k[[x_1, \dots, x_N]]$. Then we define a derivation $d_0 \in \mathrm{Der}_k^{\mathrm{cont}}(k[[x_1, \dots, x_N]])$ by

$$d_0 = \sum_{i=1}^{N} g_i \partial_{x_i}.$$

It is regular by construction. If we prove that $d_0(I) \subset I_0$, then we can use Proposition 2.3 and we are done. Let $f \in I$. Then

$$d(F_l) = \sum_{i \in [N], j \in [l]} g_{i,j} \partial_{x_{i,j}}(F_l) = \sum_{i \in [N]} g_{i,l} \partial_{x_{i,l}}(F_l) + \widetilde{d}(F_l) \in I_{\infty,0}$$

From Remark 3.4 we see that the only terms in F_n involving $x_{i,n}$ are of the form

$$c_\alpha \alpha_i x_{1,0}^{\alpha_1} \cdots x_{i,0}^{\alpha_i - 1} x_{i,n} \cdots x_{N,0}^{\alpha_N},$$

with $c_\alpha x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ a term of f. Hence we have $\partial_{x_{i,l}}(F_l) = \partial_{x_{i,0}}(F_0)$ for all i. Furthermore, by a similar argument, we see that $\sigma(\widetilde{d}(F_l)) = 0$. Applying σ to $d(F_l)$ we get that

$$\sum_{i \in [N]} \sigma(g_{i,l}) \sigma(\partial_{x_{i,0}}(F_0)) = \sum_{i \in [N]} g_i \partial_{x_i}(f) = d_0(f) \in \sigma(I_{\infty,0}) = I_0.$$

□

Example 3.5. The proposition does not hold true for non-constant arcs, in the following sense: let X be as in the proof, take any arc γ on X and write it as

$$\gamma = (\sum_n \gamma_{1,n} t^n, \dots, \sum_n \gamma_{N,n} t^n).$$

After applying the automorphism $x_{i,n} \mapsto x_{i,n} + \gamma_{i,n}$ we can assume

$$\widehat{\mathcal{O}_{\mathcal{L}(X),\gamma}} = k[[x_1^{\mathbb{N}}, \dots, x_N^{\mathbb{N}}]]/I.$$

Then it is not hard to verify that the derivation

$$d = \sum_{i,n}(n+1)(x_{i,n+1} + \gamma_{i,n+1})\partial_{x_{i,n}} \in \mathrm{Der}_k^{\mathrm{cont}}(k[[x_1^{\mathbb{N}}, \ldots, x_N^{\mathbb{N}}]])$$

fulfills $d(I) \subset I$ (in fact d is just the pushforward of the universal derivation coming from the differential algebra description of $\mathcal{L}(X)$, see [12, 3.3.1]). Observe that the arc γ being nonconstant is equivalent to d being regular. Hence, by Proposition 2.3 the formal neighborhood $\mathcal{L}(X)_\gamma$ has a smooth factor, irrespective of the structure of $X_{\gamma(0)}$.

Example 3.6. Consider the Whitney umbrella $X = \mathrm{Spec}(k[x,y,z]/(y^2 - x^2 z))$. Any degenerate arc is of the form $\gamma(t) = (0,0,z(t))$, with $z(t) \in k[[t]]$. If $z(0) \neq 0$, then we can apply our previous result to obtain

$$\mathcal{L}(X)_\gamma \cong \mathcal{L}(Y)_0 \widehat{\times} \mathrm{Spf}(k[[t^{\mathbb{N}}]]),$$

where $Y = \mathrm{Spec}(k[x,y]/(xy))$. Now assume that $z(0) = 0$. By a suitable change of coordinates we may assume that $\gamma(t) = (0,0,t^n)$ for some $n \geq 0$. If γ is just the constant arc at 0 the formal neighborhood $\mathcal{L}(X)_\gamma$ has no smooth factors, whereas for $n \geq 1$ Example 3.5 shows there exists at least one smooth factor. Let us now prove that the conclusion of the theorem of GKD still fails in the case $n = 1$. The formal neighborhood of $\gamma = (0,0,t)$ is given by the equations

$$\begin{aligned} G_0 &= y_0^2 + x_0^2 z_0, \\ G_1 &= 2y_0 y_1 - 2x_0 x_1 z_0 - x_0^2(z_1 + 1), \\ G_2 &= y_1^2 + 2y_0 y_2 - x_1^2 z_0 - 2x_0 x_2 z_0 - 2x_0 x_1(z_1 + 1) - x_0^2 z_2, \\ &\ldots \end{aligned}$$

The universal derivation d is regular (see Example 3.5) with $d(z_0)$ invertible. Thus $\mathcal{L}(X)_\gamma$ is isomorphic to $\mathrm{Spf}(k[[x^{\mathbb{N}}, y^{\mathbb{N}}, z^{\mathbb{N}-\{0\}}]]/I')$ with I' being the closure of the ideal generated by $G_i|_{z_0=0}$. Assume that d' is a regular derivation with $d'(I') \subset I'$. Now it is easy to show that $d'(y_n)$ cannot be invertible, which implies that d' remains regular after factoring out $\overline{(y_n, n \in \mathbb{N})}$. Hence d' yields a smooth factor of X_0', with $X' = \mathrm{Spec}(k[x,z]/(x^2(z+1)))$ and 0 denoting the constant arc. Together with Proposition 3.2 we obtain a contradiction, thus the singularity of X_γ is infinite-dimensional.

A similar argument works for all $n > 1$. Hence, in this case, we see that those k-arcs whose formal neighborhood has only a finite-dimensional singular part are precisely those which are non-degenerate.

In Example 3.6 we see that the converse of Theorem 1.1 holds for $X = V(x^2 - y^2 z)$, i.e. that the formal neighborhood of any degenerate arc on X has infinite-dimensional singularities. However, it is not clear how to extend the arguments of the example to arbitrary varieties. To the authors' knowledge, the question of

whether the converse to Theorem 1.1 holds is still open in general, and in particular in the case of positive characteristic.

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