Blowups and Resolution

Herwig Hauser

To the memory of Sheeram Abhyankar,
with great respect.

The present manuscript originates from a series of lectures the author[1] gave at the Clay Summer School on Resolution of Singularities at Obergurgl in the Tyrolean Alps in June 2012. A hundred young students gathered for four weeks to hear and learn about resolution of singularities. Their interest and dedication became essential for the success of the school.

The idealistic reader for this article is an algebraist or geometer having a rudimentary acquaintance with the main techniques and results in resolution of singularities – wishing to get quick and concrete reference about specific topics in the field. As such, the article is modelled like a dictionary and not particularly suited to be read from the beginning to the end (except for those who like to read dictionaries). Also, in order to enable to read and understand selected portions of the text without having to browe the whole earlier material, a certain redundance of definitions and assertions had to be accepted.

Background informations on the historic development and the motivations behind the various constructions can be found in the cited literature, especially in [Obe00], [Hau03], [Hau10a], [FH10], [Cut04], [Kol07], [Lip75]. Complete proofs of several more technical results can be found in [EH02].

All statements are formulated for varieties and morphism between them. In certain cases, the respective statements for schemes are indicated separately.

Each chapter concludes with a broad selection of examples, ranging from computational exercises to suggestions for additional material which could not be covered in the text. Some more challenging problems are marked accordingly. The examples should be especially useful for people planning to give a graduate course on the resolution of singularities. Occasionally the examples repeat or specialize statements which have appeared in the text and which are worth to be done personally before looking at the given proof.

Several results appear without proof, due to lack of time and energy of the author. Precise references are given whenever possible. The various survey articles contain complementary bibliography.

We are indebted to the Clay Mathematics Institute for choosing resolution of singularities as the topic of the 2012 summer school. It has been a particular pleasure to cooperate in this endeavour with its research director David Ellwood, whose

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enthusiasm and interpretation of the school largely coincided with the approach of the organizers, thus creating a wonderful working atmosphere. His sensitiveness of how to plan and realize the event has been exceptional.

The CMI director Jim Carlson and the CMI secretary Julie Feskoe supported very efficiently the preparation and realization of the school.

Xudong Zheng provided a preliminary write-up of the lectures, Stefan Perlega and Eleonore Faber completed several missing parts in earlier drafts of the manuscript. Perlega worked out many technical details, and Faber produced the two visualizations. Barbara Beeton from the AMS took care of the TeX-layout. All this was very helpful, thanks!

1. Lecture I: An Incomprehensible Example

Let X be the zeroset in affine three space $\mathbb{A}^3$ of the polynomial

$$f = 27x^2y^3z^2 + (x^2 + y^3 - z^2)^3$$

over a field $\mathbb{K}$ of characteristic different from 2 and 3. This is an algebraic surface with possibly singular points and curves, and certain symmetries. For instance, the origin 0 is singular on X, and X is symmetric with respect to the automorphisms of $\mathbb{A}^3$ given by replacing x by $-x$ or z by $-z$, and also by replacing y by $-y$ while interchanging x with z. Sending x, y and z to t^3x , t^2y and t^3z for $t \in \mathbb{K}$ also preserves X. See figure 1 for a plot of the real points of X. The intersections of X with the three coordinate hyperplanes of $\mathbb{A}^3$ are given as the zerosets of the equations

$$\begin{aligned} x &= (y^3 - z^2)^3 = 0, \\ y &= (x^2 - z^2)^3 = 0, \\ z &= (x^2 + y^3)^3 = 0. \end{aligned}$$

These intersections are plane curves: two perpendicular cusps lying in the xy- and yz-plane, respectively the union of the two diagonals in the xz-plane. The singular locus $\mathrm{Sing}(X)$ of X is given as the zeroset of the partial derivatives of f. This yields for $\mathrm{Sing}(X)$ the equations

$$\begin{aligned} x \cdot [9y^3z^2 + (x^2 + y^3 - z^2)^2] &= 0, \\ y^2 \cdot [9x^2z^2 + (x^2 + y^3 - z^2)^2] &= 0, \\ z \cdot [9x^2y^3 - (x^2 + y^3 - z^2)^2] &= 0. \end{aligned}$$

Therefore the singular locus of X has six irreducible components, defined by

$$\begin{aligned} x = y^3 - z^2 &= 0, \\ z = x^2 + y^3 &= 0, \\ y = x + z &= 0, \\ y = x - z &= 0, \\ x^2 - y^3 = x + \sqrt{-1}z &= 0, \\ x^2 - y^3 = x - \sqrt{-1}z &= 0. \end{aligned}$$

The first four components of $\mathrm{Sing}(X)$ coincide with the four curves given by the three coordinate hyperplane sections of X. The last two components are plane cusps in the hyperplanes given by $x \pm \sqrt{-1}z = 0$. At points $a \neq 0$ on the first two singular components of $\mathrm{Sing}(X)$, the intersections of X with a plane through a and transversal to the component are again cuspidal curves.

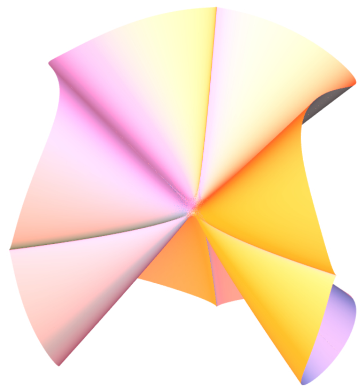

FIGURE 1. The surface *Camelia*: $27x^2y^3z^2 + (x^2 + y^3 - z^2)^3 = 0$.

Consider now the surface Y in $\mathbb{A}^4$ which is given as the cartesian product $C \times C$ of the plane cusp $C : x^2 - y^3 = 0$ in $\mathbb{A}^2$ with itself. It is defined by the equations $x^2 - y^3 = z^2 - w^3 = 0$. The singular locus $\mathrm{Sing}(Y)$ is the union of the two cusps $C_1 = C \times 0$ and $C_2 = 0 \times C$ defined by $x^2 - y^3 = z = w = 0$, respectively $x = y = z^2 - w^3 = 0$. The surface Y admits the parametrization

$$\gamma : \mathbb{A}^2 \to \mathbb{A}^4, (s, t) \mapsto (s^3, s^2, t^3, t^2).$$

The image of γ is Y. The composition of γ with the linear projection

$$\pi : \mathbb{A}^4 \to \mathbb{A}^3, (x, y, z, w) \mapsto (x, -y + w, z)$$

yields the map

$$\delta = \pi \circ \gamma : \mathbb{A}^2 \to \mathbb{A}^3, (s, t) \mapsto (s^3, -s^2 + t^2, t^3).$$

Replacing in f the variables x, y, z by s^3, $-s^2 + t^2$, t^3 gives 0. This shows that the image of δ lies in X. As X is irreducible of dimension 2 and δ has rank 2 outside 0 the image of δ is dense in X. Therefore the image of Y under π is dense in X: This interprets X as a contraction of Y by means of the projection π from $\mathbb{A}^4$ to $\mathbb{A}^3$. The two surfaces are not isomorphic; for instance, their singular loci have a different number of components. The simple geometry of Y as a cartesian product of two plane curves is scrambled up under the projection.

The point blowup of Y in the origin produces a surface Y_1 whose singular locus has two components. They map to the two components C_1 and C_2 of $\mathrm{Sing}(Y)$ and are regular and transversal to each other. The blowup X_1 of X at 0 will still be the image of Y_1 under a suitable projection. The four singular components of $\mathrm{Sing}(X)$ will become regular in X_1 and will either meet pairwise transversally or not at all. The two regular components of $\mathrm{Sing}(X)$ will remain regular in X_1 but will no longer meet each other, cf. figure 2.

The point blowup of Y_1 in the intersection point of the two curves of $\mathrm{Sing}(Y_1)$ separates the two curves and yields a surface Y_2 whose singular locus consists of two disjoint regular curves. Blowing up these separately yields a regular surface Y_3 and thus resolves the singularities of Y. The resolution of the singularities of X is more complicated, see the examples below.

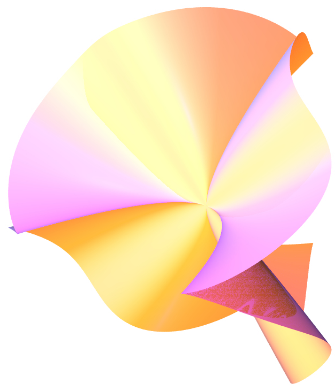

FIGURE 2. The surface obtained from *Camelia* by a point blowup .

EXAMPLE 1.1. Show that the surface X defined in $\mathbb{A}^3$ by $27x^2y^3z^2+(x^2+y^3-z^2)^3=0$ is the image of the cartesian product Y of the cusp $C: x^3-y^2=0$ with itself under the projection from $\mathbb{A}^4$ to $\mathbb{A}^3$ given by $(x,y,z,w)\mapsto(x,-y+w,z)$.

EXAMPLE 1.2. Find additional symmetries of X aside from those mentioned in the text.

EXAMPLE 1.3. Produce a real visualization of the surface obtained from *Camelia* by replacing in the equation z by $\sqrt{-1}z$. Determine the components of the singular locus.

EXAMPLE 1.4. Consider at the point $a=(0,1,1)$ of X the plane $P: 2y+3z=5$ through a. It is transversal to the component of the singular locus of X passing through a (i.e., this component is regular at a and P intersects its tangent line in a point). Determine the singularity of $X\cap P$ at a. The normal vector $(0,2,3)$ to P is the tangent vector at $t=1$ of the parametrization $(0,t^2,t^3)$ of the component of $\mathrm{Sing}(X)$ defined by $x=y^3-z^2$.

EXAMPLE 1.5. The point a on X with coordinates $(1,1,\sqrt{-1})$ is a singular point of X, and a non-singular point of the curve of $\mathrm{Sing}(X)$ passing through it. A plane transversal to $\mathrm{Sing}(X)$ at a is given e.g. by

$$P: 3x+2y+3\sqrt{-1}\cdot z=9.$$

Determine the singularity of $X\cap P$ at a. The normal vector $(3,2,3\sqrt{-1})$ to P is the tangent vector at $s=1$ of the parametrization $(s^3,s^2,\sqrt{-1}s^3)$ of the curve of $\mathrm{Sing}(X)$ through a defined by $x^2-y^3=x-\sqrt{-1}z=0$.

EXAMPLE 1.6. Blow up X and Y in the origin, describe exactly the geometry of the transforms X_1 and Y_1 and produce visualizations of X and X_1 over $\mathbb{R}$ in all coordinate charts. For Y_1 the equations are in the x-chart $1-xy^3=z^3x-w^2=0$ and in the y-chart $x^2-y=z^3y-w^2=0$.

EXAMPLE 1.7. Blow up X_1 and Y_1 along their singular loci. Compute a resolution of X_1 and compare it with the resolution of Y_1.

2. Lecture II: Varieties and Schemes

The following summary of basic concepts shall clarify the terminology used in later sections. Detailed definitions are available in [Mum99, Sha94, Har77, EH00, Liu02, Kem11, GW10, Gro61, ZS75, Nag75, Mat89, AM69].

Definition 2.1. Write $\mathbb{A}^n = \mathbb{A}^n_{\mathbb{K}}$ for the *affine n-space* over some field $\mathbb{K}$, equipped with the Zariski topology whose closed sets are defined as the algebraic subsets of $\mathbb{A}^n$, i.e., as the zerosets $V(I)$ of ideals I of $\mathbb{K}[x_1, \ldots, x_n]$. The *coordinate ring* $\mathbb{K}[\mathbb{A}^n]$ of $\mathbb{A}^n$ is the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ in n variables over $\mathbb{K}$. The set of prime ideals of $\mathbb{K}[x_1, \ldots, x_n]$ is denoted by $\mathrm{Spec}(\mathbb{K}[x_1, \ldots, x_n])$ and equipped with the Zariski topology whose closed sets $V(I)$ are formed by the prime ideals containing a given ideal I of $K[x_1, \ldots, x_n]$. This gives rise to the scheme-theoretic version of affine space $\mathbb{A}^n$.

Definition 2.2. A *polynomial map* $g : \mathbb{A}^n \to \mathbb{A}^m$ is given by a vector $g = (g_1, \ldots, g_m)$ of polynomials $g_i(x) \in \mathbb{K}[x_1, \ldots, x_n]$. It induces a $\mathbb{K}$-algebra homomorphism $f^* : \mathbb{K}[y_1, \ldots, y_m] \to \mathbb{K}[x_1, \ldots, x_n]$ sending h to $h \circ f$. A *rational map* $g : \mathbb{A}^n \to \mathbb{A}^m$ is given by a vector $g = (g_1, \ldots, g_m)$ of rational functions $g_i(x) \in \mathbb{K}(x_1, \ldots, x_n) = \mathrm{Quot}(\mathbb{K}[x_1, \ldots, x_n])$. A rational map need not be defined on whole $\mathbb{A}^n$; it is only defined on the complement of the union of the zerosets of the denominators of the functions g_i.

Definition 2.3. An *affine algebraic variety* X is a subset of affine space $\mathbb{A}^n$ defined as the zeroset $V(I)$ of a radical ideal I of $\mathbb{K}[x_1, \ldots, x_n]$ and equipped with the topology induced by the Zariski topology of $\mathbb{A}^n$. The ideal I need not be prime, hence X is not required to be irreducible. The *affine coordinate ring of* X is the factor ring $\mathbb{K}[X] = \mathbb{K}[x_1, \ldots, x_n]/I$. As I is radical, $\mathbb{K}[X]$ is reduced, i.e., has no nilotent elements. If X is irreducible, I is a prime ideal and $\mathbb{K}[X]$ is an integral domain. The *function field* of an irreducible variety X is the quotient field $\mathbb{K}(X) = \mathrm{Quot}(\mathbb{K}[X])$. The local ring of a variety X at a point a is the localization $\mathcal{O}_{X,a} = \mathbb{K}[X]_{m_{X,a}}$ of $\mathbb{K}[X]$ at the maximal ideal $m_{X,a}$ defining a in X.

Definition 2.4. A *quasi-affine variety* is an open subset of an affine variety. A *principal open subset* of X is the complement in X of the zeroset $V(g)$ of a single polynomial g.

Definition 2.5. Write $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{K}}$ for the *projective n-space* over some field $\mathbb{K}$, equipped with the Zariski topology whose closed sets are defined as the algebraic subsets of $\mathbb{P}^n$, i.e., as the zerosets $V(I)$ of homogeneous ideals I of $\mathbb{K}[x_0, \ldots, x_n]$. The projective space is covered by the affine open subsets $U_i \simeq \mathbb{A}^n$ formed by the points whose i-th projective coordinate does not vanish. The *homogeneous coordinate ring* of $\mathbb{P}^n$ is the graded polynomial ring $\mathbb{K}[x_0, \ldots, x_n]$ in $n+1$ variables over $\mathbb{K}$, the grading given by the degree. The set of homogeneous prime ideals of $\mathbb{K}[x_0, \ldots, x_n]$ not containing the ideal $(x_0, \ldots, x_n)$ is denoted by $\mathrm{Proj}(\mathbb{K}[x_0, \ldots, x_n])$ and equipped with the Zariski topology whose closed sets $V(I)$ are formed by the homogeneous prime ideals containing a given ideal I of $K[x_0, \ldots, x_n]$. This gives rise to the scheme-theoretic version of projective space $\mathbb{P}^n$.

Definition 2.6. A *projective algebraic variety* X is a subset of projective space $\mathbb{P}^n$ defined as the zeroset $V(I)$ of a homogeneous radical ideal $I = I_X$ of $\mathbb{K}[x_1, \ldots, x_n]$ and equipped with the Zariski topology. The *projective coordinate* *ring of* X is the factor ring $\mathbb{K}[X] = \mathbb{K}[x_1, \ldots, x_n]/I$ equipped with the grading given by degree.

Definition 2.7. A *quasi-projective variety* is an open subset of a projective variety. A *principal open subset* is the complement of the zeroset $V(g)$ of a single homogeneous polynomial g.

Remark 2.8. Abstract algebraic varieties are obtained by gluing affine algebraic varieties along principal open subsets, cf. [Mum99] I, §3, §4, [Sha94] V, §3. This allows to develop the category of algebraic varieties with the usual constructions therein.

Definition 2.9. Let X and Y be two affine algebraic varieties $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$, and let U be an open subset of X. A *regular map* from U to Y is a map $f : U \to \mathbb{A}^n$ sending U into Y with components rational functions whose denominators do not vanish on U. A *morphism* $f : X \to Y$ is a regular map defined on whole X. It induces a $\mathbb{K}$-algebra homomorphism $f^* : \mathbb{K}[Y] \to \mathbb{K}[X]$ between the coordinate rings, which, in turn, determines f. Over algebraically closed fields, a morphism is the restriction to X of a polynomial map $f : \mathbb{A}^m \to \mathbb{A}^n$ sending X into Y, i.e., such that $f^*(I_Y) \subset I_X$ [Har77].

Definition 2.10. A *rational map* $f : X \to Y$ between affine varieties is a morphism $f : U \to Y$ defined on some dense open subset U of X. Equivalently, a rational map is given by a $\mathbb{K}$-algebra homomorphism $\alpha_f : \mathbb{K}(Y) \to \mathbb{K}(X)$. A *birational map* $f : X \to Y$ is a regular map $f : U \to Y$ on some dense open subset U of X such that $V = f(U)$ is open in Y and such that $f_U : U \to V$ is a regular isomorphism, i.e., admits an inverse morphism. In this case U and V are called *biregularly isomorphic*, and X and Y are *birationally isomorphic*. Equivalently, a birational map is given by an isomorphism $\alpha_f : \mathbb{K}(Y) \to \mathbb{K}(X)$ of the function fields. A *birational morphism* $f : X \to Y$ is a birational map which is defined on whole X, i.e., a regular morphism $f : X \to Y$ with rational inverse defined on a dense open subset of Y.

Definition 2.11. A morphism $f : X \to Y$ between algebraic varieties is called *separated* if the diagonal $\Delta \subset X \times_Y X$ is closed in the fibre product. A morphism $f : X \to Y$ between varieties is *proper* if it is separated and universally closed, i.e., for any variety Z and morphism $Z \to Y$ the induced morphism $g : X \times_Y Z \to Y$ is closed.

Definition 2.12. The *germ of a variety* X *at a point* a, denoted by (X, a), is the equivalence class of on open neighbourhoods U of a in X where two neighbourhoods of a are said to be equivalent if they coincide on a (possibly smaller) neighbourhood of a. Equivalently, the germ of a variety is given by the localization $\mathcal{O}_{X,a} = \mathbb{K}[X]_{m_{X,a}}$ of the coordinate ring $\mathbb{K}[X]$ of X at the maximal ideal $m_{X,a}$ of $\mathbb{K}[X]$ defining a in X.

Definition 2.13. Let X and Y be two varieties, and let a and b be points of X and Y. The *germ of a morphism* $f : (X, a) \to (Y, b)$ at a is the equivalence class of a morphism $\widetilde{f} : U \to Y$ defined on an open neighbourhood U of a in X and sending a to b; here, two morphisms defined on neighbourhoods of a in X are said to be equivalent if they coincide on a (possibly smaller) neighbourhood of a. The morphism $\widetilde{f} : U \to Y$ is called a *representative of* f *on* U. Equivalently, the germ of a morphism is given by a local $\mathbb{K}$-algebra homomorphism $\alpha_f = f^* : \mathcal{O}_{Y,b} \to \mathcal{O}_{X,a}$.

DEFINITION 2.14. The *tangent space* T_aX to a variety X at a point a is defined as the $\mathbb{K}$-vector space

$$\mathrm{T}_aX = (m_{X,a}/m_{X,a}^2)^* = \mathrm{Hom}(m_{\mathrm{X,a}}/m_{\mathrm{X,a}}^2, \mathbb{K})$$

with $m_{X,a} \subset \mathcal{O}_{X,a}$ the maximal ideal of a. The *tangent map* $\mathrm{T}_af : \mathrm{T}_aX \to \mathrm{T}_bY$ of a morphism $f : X \to Y$ sending a point $a \in X$ to $b \in Y$ is defined as the map induced naturally by $f^* : \mathcal{O}_{Y,b} \to \mathcal{O}_{X,a}$.

DEFINITION 2.15. The *embedding dimension* $\mathrm{embdim_aX}$ of a variety X at a point a is defined as the $\mathbb{K}$-dimension of T_aX.

DEFINITION 2.16. An *affine scheme* X is a commutative ring R with 1, called the *coordinate ring* of X or *ring of global sections*. The set $\mathrm{Spec}(R)$, also denoted by X and called the *spectrum* of R, is defined as the set of prime ideals of R. Here, R does not count as a prime ideal, but 0 does if it is prime, i.e., if R is an integral domain. The spectrum is equipped with a topology, the *Zariski topology*, whose closed sets $V(I)$ are formed by the prime ideals containing a given ideal I of R, and with a sheaf of rings $\mathcal{O}_X$, the *structure sheaf of* X, whose stalks are the localizations of R at prime ideals [Mum99, Har77, Sha94]. An *affine scheme of finite type over some field* $\mathbb{K}$ is an affine scheme whose coordinate ring R is a finitely generated $\mathbb{K}$-algebra, i.e., a factor ring $\mathbb{K}[x_1, \ldots, x_n]/I$ of a polynomial ring by some arbitrary ideal I.

DEFINITION 2.17. Let X be an affine scheme of coordinate ring R. A *closed subscheme* Y of X is a factor ring $S = R/I$ of R by some ideal I of R, together with the canonical homomorphism $R \to R/I$. The set $Y = \mathrm{Spec}(R/I)$ of prime ideals of R/I can be identified with the closed subset $V(I)$ of $X = \mathrm{Spec}(R)$ of prime ideals of R containing I. It carries the topology induced by the Zariski topology of X. A *point* a of X is a prime ideal I of R, considered as an element of $\mathrm{Spec}(R)$. To a point one associates a closed subscheme of X via the factor ring R/I. The point a is *closed* if I is a maximal ideal of R. A *principal open subset of* X is the affine scheme U defined by the ring of fractions R_g of R with respect to the multiplicatively closed set $\{1, g, g^2, \ldots\}$ for some non-zero divisor $g \in R$. The natural ring homomorphism $R \to R_g$ sending f to $f/1$ interprets $U = \mathrm{Spec}(R_g)$ as an open subset of $X = \mathrm{Spec}(R)$.

REMARK 2.18. Abstract schemes are obtained by gluing affine schemes along principal open subsets [Mum99] II, §1, §2, [Har77] II.2, [Sha94] V, §3. For local considerations, one can often restrict to affine schemes.

DEFINITION 2.19. A *morphism* $f : X \to Y$ between affine schemes $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$ is a ring homomorphism $\alpha = \alpha_f : S \to R$. It induces a continuous map $\mathrm{Spec}(R) \to \mathrm{Spec}(S)$ by sending a prime ideal I of R to the prime ideal $\alpha^{-1}(I)$ of S. Morphisms between arbitary schemes are defined by choosing coverings by affine schemes and defining the morphism locally. A *rational map* $f : X \to Y$ between schemes is a morphism $f : U \to Y$ defined on a dense open subscheme U of X. It need not be defined on whole X. A rational map is *birational* if the morphism $f : U \to Y$ admits on a dense open subscheme V of Y an inverse morphism $V \to U$, i.e., if $f_{|U} : U \to V$ is an isomorphism. A *birational morphism* $f : X \to Y$ is a morphism $f : X \to Y$ which is also a birational map, i.e., induces an isomorphism $U \to V$ of dense open subschemes. In contrast to birational maps, a birational morphism is defined on whole X, while its inverse map is only defined on a dense open subset of Y.

Definition 2.20. Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a graded ring. The set $X = \mathrm{Proj}(R)$ of homogeneous prime ideals of R not containing the *irrelevant ideal* $M = \oplus_{i \geq 1}^{\infty} R_i$ is equipped with a topology, the *Zariski topology*, and with a sheaf of rings $\mathcal{O}_X$, the *structure sheaf* of X [Mum99, Har77, Sha94]. It thus becomes a scheme. Typically, R is generated as an R_0-algebra by the homogeneous elements $g \in R_1$ of degree 1. An open covering of X is then given by the affine schemes $X_g = \mathrm{Spec}(R_g^\circ)$, where R_g° denotes, for any non-zero $g \in R_1$, the subring of elements of degree 0 in the ring of fractions R_g.

Remark 2.21. A graded ring homomorphism $S \to R$ induces a morphism of schemes $\mathrm{Proj}(R) \to \mathrm{Proj}(S)$.

Definition 2.22. Equip the polynomial ring $\mathbb{K}[x_0, \ldots, x_n]$ with the natural grading given by the degree. The scheme $\mathrm{Proj}(\mathbb{K}[x_0, \ldots, x_n])$ is called n-dimensional *projective space* over $\mathbb{K}$, denoted by $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{K}}$. As a variety, it coincides with the classical definition of projective space as the quotient $\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\})/\mathbb{K}^*$, as introduced in def. 2.5.

Definition 2.23. A morphism $f : X \to Y$ is called *projective* if it factors, for some k, into a closed embedding $X \hookrightarrow Y \times \mathbb{P}^k$ followed by the projection $Y \times \mathbb{P}^k \to Y$ on the first factor [Har77] II.4, p. 103.

Definition 2.24. Let R be a ring and I an ideal of R. The powers I^k of I induce natural homomorphisms $R/I^{k+1} \to R/I^k$. The *I-adic completion* of R is the inverse limit $\widehat{R} = \varprojlim R/I^k$, together with the canonical homomorphism $R \to \widehat{R}$. If R is a local ring with maximal ideal m, the m-adic completion $\widehat{R}$ is called the *completion* of R. For an R-module M, one defines the *I-adic completion* $\widehat{M}$ of M as the inverse limit $\widehat{M} = \varprojlim M/I^k \cdot M$.

Lemma 2.25. Let R be a noetherian ring with prime ideal I and I-adic completion $\widehat{R}$.

(1) If J is another ideal of R with $I \cap J$-adic completion $\widehat{J}$, then $\widehat{J} = J \cdot \widehat{R}$ and $\widehat{R/J} \simeq \widehat{R}/\widehat{J}$.
(2) If J_1, J_2 are two ideals of R and $J = J_1 \cdot J_2$, then $\widehat{J} = \widehat{J_1} \cdot \widehat{J_2}$.
(3) If R is a local ring with maximal ideal m, the completion $\widehat{R}$ is a noetherian local ring with maximal ideal $\widehat{m} = m \cdot R$, and $\widehat{m} \cap R = m$.
(4) If J is an arbitrary ideal of a local ring R, then $\widehat{J} \cap R = \bigcap_{i \geq 0}(J + m^i)$.
(5) Passing to the I-adic completion defines an exact functor on finitely generated R-modules.
(6) $\widehat{R}$ is a flat R-algebra.
(7) If M a finitely generated R-module, then $\widehat{M} = M \otimes_R \widehat{R}$.

Proof. (1) By [ZS75] VIII, Thm. 6, Cor. 2, p. 258, it suffices to verify that J is closed in the $I \cap J$-adic topology. The closure $\overline{J}$ of J equals $\overline{J} = \bigcap_{i \geq 0}(J + (I \cap J)^i) = J$ by [ZS75] VIII, Lem. 1, p. 253.

(2) By definition, $\widehat{J} = J_1 \cdot J_2 \cdot \widehat{R} = J_1 \cdot \widehat{R} \cdot J_2 \cdot \widehat{R} = \widehat{J}_1 \cdot \widehat{J}_2$.

(3), (4) The ring $\widehat{R}$ is noetherian and local by [AM69] Prop. 10.26, p. 113, and Prop. 10.16, p. 109. The ideal $\widehat{I} \cap R$ equals the closure $\overline{I}$ of I in the m-adic topology. Thus, $\widehat{I} \cap R = \bigcap_{i \geq 0}(I + m^i)$ and $\widehat{m} \cap R = m$.

(5) – (7) [AM69] Prop. 10.12, p. 108, Prop. 10.14, p. 109, Prop. 10.14, p. 109. □

Definition 2.26. Let X be a variety or a scheme and let a be a point of X. The local ring $\mathcal{O}_{X,a}$ of X at a is equipped with the $m_{X,a}$-adic topology whose basis of neighbourhoods of 0 is given by the powers $m_{X,a}^k$ of $m_{X,a}$. The induced completion $\widehat{\mathcal{O}}_{X,a}$ of $\mathcal{O}_{X,a}$ is called the *complete local ring of X at a* [Nag75] II, [ZS75] VIII, §1, §2. The scheme defined by $\widehat{\mathcal{O}}_{X,a}$ is called the *formal neighbourhood* or *formal germ* of X at a, denoted by $(\widehat{X}, a)$. A *formal subvariety* $(\widehat{Y}, a)$ of $(\widehat{X}, a)$ is a factor ring $\widehat{\mathcal{O}}_{X,a}/I$ by an ideal I of $\widehat{\mathcal{O}}_{X,a}$. A map $f : (\widehat{X}, a) \to (\widehat{Y}, b)$ between two formal germs is defined as a local algebra homomorphism $\alpha_f : \widehat{\mathcal{O}}_{Y,b} \to \widehat{\mathcal{O}}_{X,a}$. It is also called a *formal map* between X and Y at a. A *formal coordinate change* of X at a is an automorphism of $\widehat{\mathcal{O}}_{X,a}$.

Remark 2.27. The inverse function theorem does not hold for algebraic varieties and regular maps between them, but it holds in the category of formal germs and formal maps: A formal map $f : (\widehat{X}, a) \to (\widehat{Y}, b)$ is an isomorphism if and only if its tangent map $\mathrm{T}_a f : \mathrm{T}_a X \to \mathrm{T}_b Y$ is a linear isomorphism.

Definition 2.28. A morphism $f : X \to Y$ between varieties is called *étale* if for all $a \in X$ the induced maps of formal germs $f : (\widehat{X}, a) \to (\widehat{Y}, f(a))$ are isomorphisms, or, equivalently, if all tangent maps $\mathrm{T}_a f$ of f are isomorphisms.

Definition 2.29. A morphism $f : X \to Y$ between varieties is called *smooth* if for all $a \in X$ the induced maps of formal germs $f : (\widehat{X}, a) \to (\widehat{Y}, f(a))$ are submersions, i.e., the tangent maps $\mathrm{T}_a f$ of f are surjective.

Remark 2.30. The category of formal germs and formal maps admits the usual concepts and constructions as e.g. the decomposition of a variety in irreducible components, intersections of germs, inverse images, or fibre products. Similarly, when working of $\mathbb{R}$, $\mathbb{C}$ or any complete valued field $\mathbb{K}$, one can develop, based on rings of convergent power series, the category of analytic varieties and analytic spaces, respectively of their germs, and analytic maps between them [dJP00].

Remark 2.31. The natural ring homomorphism $\mathcal{O}_{X,a} \to \widehat{\mathcal{O}}_{X,a}$ defines a map $(\widehat{X}, a) \to (X, a)$ of the formal neighbourhood into the germ of X at a when considered as schemes.

Example 2.32. Compare the algebraic varieties X satisfying $(X, a) \cong (\mathbb{A}^d, 0)$ for all $a \in X$ (where $\cong$ stands for biregularly isomorphic Zariski-germs) with those where the isomorphism is just formal. Varieties with the first property are called *plain*, cf. def. 3.11.

Example 2.33. It can be shown that any complex regular and rational surface (i.e., regular and birationally isomorphic to affine space over $\mathbb{C}$) is plain [BHSV08] Prop. 3.2. Show directly that X defined in $\mathbb{A}^3$ by $x - (x^2 + z^2)y = 0$ is plain by exhibiting a local isomorphism at 0 with $\mathbb{A}^2$. Is there an algorithm to construct such a local isomorphism for any complex regular and rational surface?

Example 2.34. Let X be an algebraic variety and $a \in X$ be a point. What is the difference between the concept of regular system of parameters in $\mathcal{O}_{X,a}$ and $\widehat{\mathcal{O}}_{X,a}$?

Example 2.35. The formal neighbourhood of affine space $\mathbb{A}^n$ at a point $a = (a_1, \ldots, a_n)$ is given by the formal power series ring $\widehat{\mathcal{O}}_{\mathbb{A}^n,a} \simeq \mathbb{K}[[x_1-a_1, \ldots, x_n-a_n]]$. If $X \subset \mathbb{A}^n$ is an affine algebraic variety defined by the ideal I of $\mathbb{K}[x_1, \ldots, x_n]$, the formal neighbourhood $(\widehat{X}, a)$ is given by the factor ring $\widehat{\mathcal{O}}_{X,a} = \widehat{\mathcal{O}}_{\mathbb{A}^n,a}/\widehat{I} \simeq \mathbb{K}[[x_1 - a_1, \ldots, x_n - a_n]]/\widehat{I}$, where $\widehat{I} = I \cdot \widehat{\mathcal{O}}_{\mathbb{A}^n,a}$ denotes the extension of I to $\widehat{\mathcal{O}}_{\mathbb{A}^n,a}$.

Example 2.36. The map $f : \mathbb{A}^1 \to \mathbb{A}^2$ given by $t \mapsto (t^2, t^3)$ is a regular morphism which induces a birational isomorphism onto the curve X in $\mathbb{A}^2$ defined by $x^3 = y^2$. The inverse $(x, y) \mapsto y/x$ is rational on X and regular on $X \setminus \{0\}$.

Example 2.37. The map $f : \mathbb{A}^1 \to \mathbb{A}^2$ given by $t \mapsto (t^2 - 1, t(t^2 - 1))$ is a regular morphism which induces a birational isomorphism onto the curve X in $\mathbb{A}^2$ defined by $x^2 + x^3 = y^2$. The inverse $(x, y) \mapsto y/x$ is rational on X and regular on $X \setminus \{0\}$. The germ $(X, 0)$ of X at 0 is not isomorphic to the germ $(Y, 0)$ of the union Y of the two diagonals in $\mathbb{A}^2$ defined by $x^2 = y^2$. The formal germs $(\widehat{X}, 0)$ and $(\widehat{Y}, 0)$ are isomorphic via the map $(x, y) \mapsto (x\sqrt{1 + x}, y)$.

Example 2.38. The map $f : \mathbb{A}^2 \to \mathbb{A}^2$ given by $(x, y) \mapsto (xy, y)$ is a birational morphism with inverse the rational map $(x, y) \mapsto (\frac{x}{y}, y)$. The inverse defines a regular moprhism outside the x-axis.

Example 2.39. The maps $\varphi_{ij} : \mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$ given by

$$(x_0, \ldots, x_n) \mapsto (\frac{x_0}{x_j}, \ldots, \frac{x_{i-1}}{x_j}, 1, \frac{x_{i+1}}{x_j}, \ldots, \frac{x_n}{x_j}).$$

are birational maps for each $i, j = 0, \ldots, n$. They are the transition maps between the affine charts $U_j = \mathbb{P}^n \setminus V(x_j) \simeq \mathbb{A}^n$ of projective space $\mathbb{P}^n$.

Example 2.40. The maps $\varphi_{ij} : \mathbb{A}^n \to \mathbb{A}^n$ given by

$$(x_1, \ldots, x_n) \mapsto \Big(\frac{x_1}{x_j}, \ldots, \frac{x_{i-1}}{x_j}, \frac{1}{x_j}, \frac{x_{i+1}}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, x_i x_j, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_n}{x_j}\Big)$$

are birational maps for $i, j = 1, \ldots, n$. They are the transition maps between the affine charts of the blowup $\widetilde{\mathbb{A}}^n \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$ of $\mathbb{A}^n$ at the origin.

Example 2.41. The elliptic curve X defined in $\mathbb{A}^2$ by $y^2 = x^3 - x$ is formally isomorphic at each point a of X to $(\widehat{\mathbb{A}}^1, 0)$, whereas the germs (X, a) are not biregularly isomorphic to $(\mathbb{A}^1, 0)$.

Example 2.42. The projection $(x, y) \mapsto x$ from the hyperbola X defined in $\mathbb{A}^2$ by $xy = 1$ to the x-axis $\mathbb{A}^1$ is not proper, and the image of X is not closed.

Example 2.43. The curves X and Y defined in $\mathbb{A}^2$ by $x^3 = y^2$, respectively $x^5 = y^2$ are not formally isomorphic to each other at 0, whereas the curve Z defined in $\mathbb{A}^2$ by $x^3 + x^5 = y^2$ is formally isomorphic to X at 0.

3. Lecture III: Singularities

Let X be an affine algebraic variety defined over a field $\mathbb{K}$. Analogous concepts as the ones given below can be defined for abstract varieties and schemes.

DEFINITION 3.1. A noetherian local ring R with maximal ideal m is called a *regular ring* if m can be generated by n elements, where n is the vector space dimension of m/m^2 over the residue field R/m, or, equivalently, the Krull dimension of R.

DEFINITION 3.2. Let R be a noetherian regular local ring with maximal ideal m. A minimal set of generators of $x_1, \dots, x_n$ of m is called a *regular system of parameters* or *local coordinates* of R.

REMARK 3.3. A regular system of parameters of a local ring R is also a regular system of parameters of its completion $\widehat{R}$, but not conversely, cf. ex. 3.27.

DEFINITION 3.4. A point a of X is a *regular* or *non-singular point* of X if the local ring $\mathcal{O}_{X,a}$ of X at a is a regular ring. Equivalently, the $m_{X,a}$-adic completion $\widehat{\mathcal{O}}_{X,a}$ of $\mathcal{O}_{X,a}$ is isomorphic to the completion $\widehat{\mathcal{O}}_{\mathbb{A}^d,0} \simeq \mathbb{K}[[x_1, \dots, x_d]]$ of the local ring of some affine space $\mathbb{A}^d$ at 0. Otherwise a is called a *singular point* or a *singularity* of X. The set of singular points of X is denoted by $\mathrm{Sing}(X)$, its complement by $\mathrm{Reg}(X)$. The variety X is *regular* or *non-singular* if all its points are regular. Over perfect fields regular varieties are also called *smooth*.

PROPOSITION 3.5. A subvariety X of a regular variety W is regular at a if and only if there are local coordinates $x_1, \dots, x_n$ of W at a so that X can be defined locally at a by $x_1 = \dots = x_k = 0$ where k is the codimension of X in W at a.

PROOF. [dJP00] Cor. 4.3.20, p. 155. □

REMARK 3.6. Even though regular points are defined through the completion of the local rings, the regular parameter system as in the proposition can be taken in the local ring. The germ of a variety at a regular point need not be biregularly isomorphic to the germ of an affine space at 0. This only holds for the formal neighbourhoods, cf. ex. 3.28.

PROPOSITION 3.7. Assume that the field $\mathbb{K}$ is perfect. Let X be a hypersurface defined in a regular variety W by the square-free equation $f = 0$. The point $a \in X$ is singular if and only if all partial derivatives $\partial_{x_1} f, \dots, \partial_{x_n} f$ of vanish at a.

PROOF. [Zar47] Thm. 7, [Har77] I.5, [dJP00] 4.3. □

REMARK 3.8. A similar statement holds for non-hypersurfaces, using instead of the partial derivatives the $k \times k$-minors of the Jacobian matrix of a system of equations of X in W, with k the codimension of X in W at a [dJP00]. The choice of the system of defining polynomials of the variety is significant: The ideal generated by them has to be radical, otherwise the concept of singularity has to be developed scheme-theoretically, allowing non-reduced schemes. Over non-perfect fields, the criterion of the proposition does not hold [Zar47].

DEFINITION 3.9. The characterization of singularities by the vanishing of the partial derivatives is known as the *Jacobian criterion for smoothness*.

COROLLARY 3.10. The singular locus $\mathrm{Sing}(X)$ of X is closed in X.

Definition 3.11. A point a of X is a *plain point* of X if the local ring $\mathcal{O}_{X,a}$ is isomorphic to the local ring $\mathcal{O}_{\mathbb{A}^d,0} \simeq \mathbb{K}[x_1,\ldots,x_d]_{(x_1,\ldots,x_d)}$ of some affine space $\mathbb{A}^d$ at 0 (with d the dimension of X at a). Equivalently, there exists an open neighbourhood U of a in X which is biregularly isomorphic to an open subset V of some affine space $\mathbb{A}^d$. The variety X is *plain* if all its points are plain.

Theorem 3.12. (Bodnár-Hauser-Schicho-Villamayor) The blowup of a plain variety defined over an infinite field along a regular center is again plain.

Proof. [BHSV08] Thm. 4.3. □

Definition 3.13. A variety X is *rational* if it has a dense open subset U which is biregularly isomorphic to a dense open subset V of some affine space $\mathbb{A}^d$. Equivalently, the function field $\mathbb{K}(X) = \mathrm{Quot}(\mathbb{K}[X])$ is isomorphic to the field of rational functions $K(x_1,\ldots,x_d)$ of $\mathbb{A}^d$.

Remark 3.14. A plain complex variety is smooth and rational. The converse is true for curves and surfaces, and unknown in arbitrary dimension [BHSV08].

Definition 3.15. A point a is a *normal crossings point* of X if the formal neighbourhood $(\widehat{X}, a)$ is isomorphic to the formal neighbourhood $(\widehat{Y}, 0)$ of a union Y of coordinate subspaces of $\mathbb{A}^n$ at 0. The point a is a *simple normal crossings point* of X if it is a normal crossings point and all components of X passing through a are regular at a. The variety X has *normal crossings*, respectively *simple normal crossings*, if the property holds at all of its points.

Remark 3.16. In the case of schemes, a normal crossings scheme may be non-reduced in which case the components of the scheme Y are equipped with multiplicities. Equivalently, Y is defined locally at 0 in $\mathbb{A}^n$ by a monomial ideal of $\mathbb{K}[x_1,\ldots,x_n]$.

Proposition 3.17. A point a is a normal crossings point of a subvariety X of a regular ambient variety W if and only if there exists a regular system of parameters $x_1,\ldots,x_n$ of $\mathcal{O}_{W,a}$ so that the germ (X,a) is defined in (W,a) by a radical monomial ideal in $x_1,\ldots,x_n$, i.e., each component of (X,a) is defined by a subset of the coordinates.

Remark 3.18. In the case of schemes, the monomial ideal need not be radical.

Definition 3.19. Two subvarieties X and Y of a regular ambient variety W meet *transversally* at a point a of W if they are regular at a and if the union $X \cup Y$ has normal crossings at a.

Remark 3.20. The definition differs from the corresponding notion in differential geometry, where it is required that the tangent spaces of the two varieties at intersection points sum up to the tangent space of the ambient variety at the respective point. In the present context, an inclusion $Y \subset X$ of two regular varieties is considered as being transversal, and also any two coordinate subspaces in $\mathbb{A}^n$ meet transversally. In the case of schemes, the union $X \cup Y$ has to be defined by the product of the defining ideals, not their intersection.

Definition 3.21. A variety X is a *cylinder over a subvariety $Z \subset X$ at a point a of Z* if the formal neighborhood $(\widehat{X}, a)$ is isomorphic to a cartesian product $(\widehat{Y}, a) \times (\widehat{Z}, a)$ for some (positive-dimensional) subvariety Y of X which is regular at a. One also says that X is *trivial* or a *cylinder along Y at a with transversal section Z*.

Definition 3.22. Let X and F be varieties, and let 0 be a distinguished point on F. The set S of points a of X where the formal neighborhood $(\widehat{X}, a)$ is isomorphic to $(\widehat{F}, 0)$ is called the *triviality locus of X of singularity type* $(\widehat{F}, 0)$.

Theorem 3.23. (Ephraim, Hauser-Müller) For any variety F and point 0 on F, the triviality locus S of X of singularity type $(\widehat{F}, 0)$ is locally closed and regular in X, and X is a cylinder along S. Any subvariety Z of X such that $(\widehat{X}, a) \simeq (\widehat{S}, a) \times (\widehat{Z}, a)$ is unique up to formal isomorphism at a.

Proof. [Eph78] Thm. 2.1, [HM89] Thm. 1. □

Corollary 3.24. The singular locus $\mathrm{Sing}(X)$ and the non-normal crossings locus of a variety X are closed subvarieties.

Proof. [Mum99] III.4, Prop. 3, p. 170, [Bod04]. □

Definition 3.25. A variety X is a *cartesian product*, if there exist positive dimensional varieties Y and Z such that X is biregularly isomorphic to $Y \times Z$. Analogous definitions hold for germs (X, a) and formal germs $(\widehat{X}, a)$.

Theorem 3.26. (Hauser-Müller) Let X, Y and Z be varieties with points a, b and c on them, respectively. The formal germs of $X \times Z$ at (a, c) and $Y \times Z$ at (b, c) are isomorphic if and only if $(\widehat{X}, a)$ and $(\widehat{Y}, b)$ are isomorphic.

Proof. [HM90], Thm. 1. □

Example 3.27. The element $x\sqrt{1+x}$ is a regular parameter system of $\mathbb{K}[[x]]$ which does not stem from a regular parameter system of the local ring $K[x]_{(x)}$.

Example 3.28. The elements $y^2 - x^3 - x$ and y form a regular parameter system of $\mathcal{O}_{\mathbb{A}^2,0}$ but the zeroset X of $y^2 = x^3 - x$ is not locally at 0 isomorphic to $\mathbb{A}^1$. However, $(\widehat{X}, 0)$ is formally isomorphic to $(\widehat{\mathbb{A}}^1, 0)$.

Example 3.29. The set of plain points of a variety X is Zariski open.

Example 3.30. Let X be the plane cubic curve defined by $x^3 + x^2 - y^2 = 0$ in $\mathbb{A}^2$. The origin $0 \in X$ is a normal crossings point, but X is not locally at 0 biregularly isomorphic to the union of the two diagonals of $\mathbb{A}^2$.

Example 3.31. Let X be the surface in $\mathbb{A}^3$ defined by $x^2 - y^2 z = 0$, the *Whitney umbrella* or *pinch point singularity*. The singular locus is the z-axis. The origin is not a normal crossings point of X. For $a \neq 0$ a point on the z-axis, a is a normal crossings point but not a simple normal crossings point of X. The formal neighbourhoods of X at points $a \neq 0$ on the z-axis are isomorphic to each other, since they are isomorphic to the formal neighbourhood at 0 of the union of two transversal planes in $\mathbb{A}^3$.

Example 3.32. Prove that the non-normal crossings locus of a variety is closed. Try to find equations for it [Bod03].

Example 3.33. Find a local invariant that measures reasonably the distance of a point a of X from being a normal crossings point.

EXAMPLE 3.34. Do the varieties defined by the following equations have normal crossings, respectively simple normal crossings, at the origin? Vary the ground field.

(a) $x^2 + y^2 = 0$,
(b) $x^2 - y^2 = 0$,
(c) $x^2 + y^2 + z^2 = 0$,
(d) $x^2 + y^2 + z^2 + w^2 = 0$,
(e) $xy(x - y) = 0$,
(f) $xy(x^2 - y) = 0$,
(g) $(x - y)z(z - x) = 0$.

EXAMPLE 3.35. Visualize the zeroset of $(x - y^2)(x - z)z = 0$ in $\mathbb{A}^3_{\mathbb{R}}$.

EXAMPLE 3.36. $*$ Formulate and then prove the theorem of local analytic triviality in positive characteristic, cf. Thm. 1 of [HM89].

EXAMPLE 3.37. Find a coordinate free description of normal crossings singularities. Find an algorithm that tests for normal crossings [Fab11, Fab12].

EXAMPLE 3.38. The singular locus of a cartesian product $X \times Y$ is the union of $\mathrm{Sing}(X) \times Y$ and $X \times \mathrm{Sing}(Y)$.

EXAMPLE 3.39. Call a finite union $X = \bigcup X_i$ of regular varieties *mikado* if all intersections are (scheme-theoretically) regular. The intersections are defined by the sums of the ideals, not taking their radical. Find the simplest example of a variety which is not mikado but for which all pairwise intersections are non-singular.

EXAMPLE 3.40. Interpret the family given by taking the germs of a variety X at varying points $a \in X$ as the germs of the fibers of a morphism of varieties, equipped with a section.

4. Lecture IV: Blowups

Blowups, also known as monoidal transformations, can be introduced in several ways. The respective equivalences will be proven in the second half of this section. All varietes are reduced but not necessarily irreducible, and subvarieties are closed if not mentioned differently. Schemes will be noetherian but not necessarily of finite type over a field. To easen the exposition they are often assumed to be affine, i.e., of the form $X = \mathrm{Spec}(R)$ for some ring R. Points of varieties are closed, points of schemes can also be non-closed. The coordinate ring of an affine variety X is denoted by $\mathbb{K}[X]$ and the structure sheaf of a scheme X by $\mathcal{O}_X$, with local rings $\mathcal{O}_{X,a}$ at points $a \in X$.

References providing additional material on blowups are, among many others, [Hir64], chap. III, and [EH00], chap. IV.2.

DEFINITION 4.1. A subvariety Z of a variety X is called a *hypersurface in X* if the codimension of Z in X at any point a of Z is 1,

$$\dim_a Z = \dim_a X - 1.$$

REMARK 4.2. In the case where X is non-singular and irreducible, a hypersurface is locally defined at any point a by a single non-trivial equation, i.e., an equation given by a non-zero and non-invertible element h of $\mathcal{O}_{X,a}$. This need not be the case for singular varieties, see ex. 4.21. Hypersurfaces are a particular case of *effective Weil divisors* [Har77] II, Rmk. 6.17.1, p. 145.

DEFINITION 4.3. A subvariety Z of an irreducible variety X is called a *Cartier divisor* in X at a point $a \in Z$ if Z can be defined locally at a by a single equation $h = 0$ for some non-zero element $h \in \mathcal{O}_{X,a}$. If X is not assumed to be irreducible, h is required to be a non-zero divisor of $\mathcal{O}_{X,a}$. This excludes the possibility that Z is a component or a union of components of X. The subvariety Z is called a *Cartier divisor* in X if it is a Cartier divisor at each of its points. The empty subvariety is considered as a Cartier divisor. A (non-empty) Cartier divisor is a hypersurface in X, but not conversely, and its complement $X \setminus Z$ is dense in X. Cartier divisors are, in a certain sense, the largest closed and properly contained subvarieties of X. If Z is Cartier in X, the ideal I defining Z in X is called *invertible* or *locally free of rank* 1 [Har77] II, Prop. 6.13, p. 144, [EH00] III.2.5, p. 117.

DEFINITION 4.4. (Blowup via universal property) A variety $\widetilde{X}$ together with a morphism $\pi : \widetilde{X} \to X$ is called a *blowup of X with center a (closed) subvariety Z of X*, or a *blowup of X along Z*, if the inverse image $E = \pi^{-1}(Z)$ of Z is a Cartier divisor in $\widetilde{X}$ and π is universal with respect to this property: For any morphism $\tau : X' \to X$ such that $\tau^{-1}(Z)$ is a Cartier divisor in X', there exists a unique morphism $\sigma : X' \to \widetilde{X}$ so that τ factors through σ, say $\tau = \pi \circ \sigma$,

$$\begin{tikzcd} X' \arrow[r, dashed, "\exists_1 \sigma"] \arrow[dr, "\tau"'] & \widetilde{X} \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

The morphism π is also called the *blowup map*. The subvariety E of $\widetilde{X}$ is a Cartier divisor, in particular a hypersurface, and called the *exceptional divisor* or *exceptional locus* of the blowup. One says that π *contracts E to Z*.

REMARK 4.5. By the universal property, a blowup of X along Z is unique up to unique isomorphism. It is therefore called *the* blowup of X along Z. If Z is already a Cartier divisor in X, then $\widetilde{X} = X$ and π is the identity by the universal property. In particular, this is the case when X is non-singular and Z is a hypersurface in X.

DEFINITION 4.6. The *Rees algebra* of an ideal I of a commutative ring R is the graded R-algebra

$$\mathrm{Rees}(I) = \bigoplus_{i=0}^{\infty} I^i = \bigoplus_{i=0}^{\infty} I^i \cdot t^i \subset R[t],$$

where I^i denotes the i-fold power of I, with I^0 set equal to R. The variable t is given degree 1. Write $\widetilde{R}$ for $\mathrm{Rees}(I)$ when I is clear from the context. The Rees algebra of I is generated by elements of degree 1, and R embeds naturally into $\widetilde{R}$ by sending an element g of R to the degree 0 element $g \cdot t^0$. If I is finitely generated by elements $g_1, \ldots, g_k \in R$, then $\widetilde{R} = R[g_1 t, \ldots, g_k t]$. The Rees algebras of the zero-ideal $I = 0$ and of the whole ring $I = R$ equal R, respectively $R[t]$. If I is a principal ideal generated by a non-zero divisor of R, the Rees algebra is isomorphic to R. The Rees algebras of an ideal I and its k-th power I^k are isomorphic as graded R-algebras, for any $k \geq 1$.

DEFINITION 4.7. (Blowup via Rees algebra) Let $X = \mathrm{Spec}(R)$ be an affine scheme and let $Z = \mathrm{Spec}(R/I)$ be a closed subscheme of X defined by an ideal I of R. Denote by $\widetilde{R} = \mathrm{Rees}(I)$ the Rees algebra of I over R, equipped with the induced grading. The *blowup of X along Z* is the scheme $\widetilde{X} = \mathrm{Proj}(\widetilde{R})$ together with the morphism $\pi : \widetilde{X} \to X$ given by the natural graded ring homomorphism $R \to \widetilde{R}$. The subscheme $E = \pi^{-1}(Z)$ of $\widetilde{X}$ is called the *exceptional divisor* of the blowup.

Definition 4.8. (Blowup via secants) Let $W = \mathbb{A}^n$ be affine space over $\mathbb{K}$ and let p be a fixed point of $\mathbb{A}^n$. Equip $\mathbb{A}^n$ with a vector space structure by identifying it with its tangent space $\mathrm{T}_p\mathbb{A}^n$. For a point $a \in \mathbb{A}^n$ different from p, denote by $g(a)$ the secant line in $\mathbb{A}^n$ through p and a, considered as an element of projective space $\mathbb{P}^{n-1} = \mathbb{P}(\mathrm{T}_p\mathbb{A}^n)$. The morphism

$$\gamma : \mathbb{A}^n \setminus \{p\} \to \mathbb{P}^{n-1}, a \mapsto g(a),$$

is well defined. The Zariski closure $\widetilde{X}$ of the graph Γ of γ inside $\mathbb{A}^n \times \mathbb{P}^{n-1}$ together with the restriction $\pi : \widetilde{X} \to X$ of the projection map $\mathbb{A}^n \times \mathbb{P}^{n-1} \to \mathbb{A}^n$ is the *point blowup of $\mathbb{A}^n$ with center p*.

Definition 4.9. (Blowup via closure of graph) Let X be an affine variety with coordinate ring $\mathbb{K}[X]$ and let $Z = \mathrm{V}(I)$ be a subvariety of X defined by an ideal I of $\mathbb{K}[X]$ generated by elements $g_1, \ldots, g_k$. The morphism

$$\gamma : X \setminus Z \to \mathbb{P}^{k-1}, a \mapsto (g_1(a) : \cdots : g_k(a)),$$

is well defined. The Zariski closure $\widetilde{X}$ of the graph Γ of γ inside $X \times \mathbb{P}^{k-1}$ together with the restriction $\pi : \widetilde{X} \to X$ of the projection map $X \times \mathbb{P}^{k-1} \to X$ is the *blowup of X along Z*. It does not depend, up to isomorphism over X, on the choice of the generators g_i of I.

Definition 4.10. (Blowup via equations) Let X be an affine variety with coordinate ring $\mathbb{K}[X]$ and let $Z = \mathrm{V}(I)$ be a subvariety of X defined by an ideal I of $\mathbb{K}[X]$ generated by elements $g_1, \ldots, g_k$. Assume that $g_1, \ldots, g_k$ form a regular sequence in $\mathbb{K}[X]$. Let $(u_1 : \ldots : u_k)$ be projective coordinates on $\mathbb{P}^{k-1}$. The subvariety $\widetilde{X}$ of $X \times \mathbb{P}^{k-1}$ defined by the equations

$$u_i \cdot g_j - u_j \cdot g_i = 0, \qquad i, j = 1, \ldots k,$$

together with the restriction $\pi : \widetilde{X} \to X$ of the projection map $X \times \mathbb{P}^{k-1} \to X$ is the *blowup of X along Z*. It does not depend, up to isomorphism over X, on the choice of the generators g_i of I.

Remark 4.11. This is a special case of the preceding definition of blowup as the closure of a graph. If $g_1, \ldots, g_k$ do not form a regular sequence, the subvariety $\widetilde{X}$ of $X \times \mathbb{P}^{k-1}$ may require more equations, see ex. 4.43.

Definition 4.12. (Blowup via affine charts) Let $W = \mathbb{A}^n$ be affine space over $\mathbb{K}$ with coordinates $x_1, \ldots, x_n$. Let $Z \subset W$ be a coordinate subspace defined by equations $x_j = 0$ for j in some subset $J \subset \{1, \ldots n\}$. Set $U_j = \mathbb{A}^n$ for $j \in J$, and glue two affine charts U_j and U_ℓ via the transition maps

$$\begin{aligned}
x_i &\mapsto x_i/x_j, && \text{if } i \in J \setminus \{j, \ell\},\\
x_j &\mapsto 1/x_\ell, &&\\
x_\ell &\mapsto x_j x_\ell, &&\\
x_i &\mapsto x_i, && \text{if } i \notin J.
\end{aligned}$$

This yields a variety $\widetilde{W}$. Define a morphism $\pi : \widetilde{W} \to W$ by the chart expressions $\pi_j : \mathbb{A}^n \to \mathbb{A}^n$ for $j \in J$ as follows,

$$\begin{aligned} x_i &\mapsto x_i, && \text{if } i \notin J \setminus \{j\}, \\ x_i &\mapsto x_i x_j, && \text{if } i \in J \setminus \{j\}. \end{aligned}$$

The variety $\widetilde{W}$ together with the morphism $\pi : \widetilde{W} \to W$ is the *blowup of* $W = \mathbb{A}^n$ *along* Z. The map $\pi_j : \mathbb{A}^n \to \mathbb{A}^n$ is called the *j-th affine chart* of the blowup map, or the *x_j-chart*. It depends on the choice of coordinates.

Definition 4.13. (Blowup via ring extensions) Let I be a non-zero ideal in a noetherian integral domain R, generated by non-zero elements $g_1, \ldots, g_k$ of R. The *blowup of R in I* is given by the ring extensions

$$R \hookrightarrow R_j = R\left[\frac{g_1}{g_j}, \ldots, \frac{g_k}{g_j}\right], \qquad j = 1, \ldots, k,$$

inside the rings $R_{g_j} = R[\frac{1}{g_j}]$, where the rings R_j are glued pairwise via the inclusions $R_{g_j}, R_{g_\ell} \subset R_{g_j g_\ell}$, for $j, \ell = 1, \ldots, k$. The blowup does not depend, up to isomorphism over X, on the choice of the generators g_i of I.

Remark 4.14. The seven notions of blowup given in the preceding definitions are all essentially equivalent. It will be convenient to prove the equivalence in the language of schemes, though the substance of the proofs is inherent to varieties.

Definition 4.15. (Local blowup via germs) Let $\pi : X' \to X$ be the blowup of X along Z, and let a' be a point of X' mapping to the point $a \in X$. The *local blowup of X along Z at a'* is the morphism of germs $\pi : (X', a') \to (X, a)$. It is given by the dual homomorphism of local rings $\pi^* : \mathcal{O}_{X,a} \to \mathcal{O}_{X',a'}$. This terminology is also used for the completions of the local rings giving rise to a morphism of formal neighborhoods $\pi : (\widehat{X}', a') \to (\widehat{X}, a)$ with dual homomorphism $\pi^* : \widehat{\mathcal{O}}_{X,a} \to \widehat{\mathcal{O}}_{X',a'}$.

Definition 4.16. (Local blowup via localizations of rings) Let $R = (R, m)$ be a local ring and let I be an ideal of R with Rees algebra $\widetilde{R} = \bigoplus_{i=0}^\infty I^i$. Any non-zero element g of I defines a homogeneous element of degree 1 in $\widetilde{R}$, also denoted by g. The quotient ring $\widetilde{R}_g$ inherits from $\widetilde{R}$ the structure of a graded ring. The set of degree 0 elements of $\widetilde{R}_g$ forms a ring, denoted by $R[Ig^{-1}]$, and consists of fractions f/g^ℓ with $f \in I^\ell$ and $\ell \in \mathbb{N}$. The localization $R[Ig^{-1}]_p$ of $R[Ig^{-1}]$ at any prime ideal p of $R[Ig^{-1}]$ containing the maximal ideal m of R together with the natural ring homomorphism $\alpha : R \to R[Ig^{-1}]_p$ is called the *local blowup* of R with center I associated to g and p.

Remark 4.17. The same definition can be made for non-local rings R, giving rise to a local blowup $R_q \to R_q[Ig^{-1}]_p$ with respect to the localization R_q of R at the prime ideal $q = p \cap R$ of R.

Theorem 4.18. Let $X = \mathrm{Spec}(R)$ be an affine scheme and Z a closed subscheme defined by an ideal I of R. The blowup $\pi : \widetilde{X} \to X$ of X along Z when defined as $\mathrm{Proj}(\widetilde{R})$ of the Rees algebra $\widetilde{R}$ of I satisfies the universal property of blowups.

Proof. (a) It suffices to prove the universal property on the affine charts of an open covering of $\mathrm{Proj}(\widetilde{R})$, since on overlaps the local patches will agree by their uniqueness. This in turn reduces the proof to the local situation.

(b) Let $\beta : R \to S$ be a homomorphism of local rings such that $\beta(I) \cdot S$ is a principal ideal of S generated by a non-zero divisor, for some ideal I of R. It suffices to show that there is a unique homomorphism of local rings $\gamma : R' \to S$ such that R' equals a localization $R' = R[Ig^{-1}]_p$ for some non-zero element g of I and a prime ideal p of $R[Ig^{-1}]$ containing the maximal ideal of R, and such that the diagram

$$\begin{tikzcd} R \arrow[r, "\alpha"] \arrow[d, "\beta"] & R' \arrow[dl, "\gamma"] \\ S & \end{tikzcd}$$

commutes, where $\alpha : R \to R'$ denotes the local blowup of R with center I specified by the choice of g and p. The proof of the local statement goes in two steps.

(c) There exists an element $f \in I$ such that $\beta(f)$ generates $\beta(I) \cdot S$: Let $h \in S$ be a non-zero divisor generating $\beta(I) \cdot S$. Write $h = \sum_{i=1}^n \beta(f_i)s_i$ with elements $f_1, \ldots, f_n \in I$ and $s_1, \ldots, s_n \in S$. Write $\beta(f_i) = t_i h$ with elements $t_i \in S$. Thus, $h = \big(\sum_{i=1}^n s_i t_i\big)h$. Since h is a non-zero divisor, the sum $\sum_{i=1}^n s_i t_i$ equals 1. In particular, there is an index i for which t_i does not belong to the maximal ideal of S. This implies that this t_i is invertible in S, so that $h = t_i^{-1}\beta(f_i)$. In particular, $(\beta(f_i))$ generates $\beta(I) \cdot S$.

(d) Let $f \in I$ be as in (c). By assumption, the image $\beta(f)$ is a non-zero divisor in S. For every $\ell \geq 0$ and every $h_\ell \in I^\ell$, there is an element $a_\ell \in S$ such that $\beta(h_\ell) = a_\ell \beta(f)^\ell$. Since $\beta(f)$ is a non-zero divisor, a_ℓ is unique with this property. For an arbitrary element $\sum_{\ell=0}^n h_\ell/g^\ell$ of $R[Ig^{-1}]$ with $h_\ell \in I^\ell$ set $\delta(\sum_{\ell=0}^n h_\ell/g^\ell) = \sum_{\ell=0}^n a_\ell$. This defines a ring homomorphism $\delta : R[Ig^{-1}] \to S$ that restricts to β on R. By definition, δ is unique with this property. Let $p \subset R[Ig^{-1}]$ be the inverse image under δ of the maximal ideal of S. This is a prime ideal of $R[Ig^{-1}]$ which contains the maximal ideal of R. By the universal property of localization, δ induces a homomorphism of local rings $\gamma : R[Ig^{-1}]_p \to S$ that restricts to β on R, i.e., satisfies $\gamma \circ \alpha = \beta$. By construction, γ is unique. □

Theorem 4.19. Let $X = \mathrm{Spec}(R)$ be an affine scheme and let Z a closed subscheme defined by an ideal I of R. The blowup of X along Z defined by $\mathrm{Proj}(\widetilde{R})$ can be covered by affine charts as described in the respective definition.

Proof. For $g \in I$ denote by $R[Ig^{-1}] \subset R_g$ the subring of the quotient ring R_g generated by homogeneous elements of degree 0 of the form h/g^ℓ with $h \in I^\ell$ and $\ell \in \mathbb{N}$. This gives an injective ring homomorphism $R[Ig^{-1}] \to \widetilde{R}_g$. Let now $g_1, \ldots, g_k$ be generators of I. Then $X = \mathrm{Proj}(\widetilde{R})$ is covered by the principal open sets $\mathrm{Spec}(R[Ig_i^{-1}]) = \mathrm{Spec}(R[g_1/g_i, \ldots, g_k/g_i])$. The chart expression $\mathrm{Spec}(R[g_1/g_i, \ldots, g_k/g_i]) \to \mathrm{Spec}(R)$ of the blowup map $\pi : \widetilde{X} \to X$ follows now by computation. □

Theorem 4.20. Let $X = \mathrm{Spec}(R)$ be an affine scheme and Z a closed subscheme defined by the ideal I of R with generators $g_1, \ldots, g_k$. The blowup of X along Z defined by $\mathrm{Proj}(\widetilde{R})$ of the Rees algebra equals the closure of the graph Γ of $\gamma : X \setminus Z \to \mathbb{P}^{k-1}, a \mapsto (g_1(a) : \cdots : g_k(a))$. If $g_1, \ldots, g_k$ form a regular sequence, the closure is defined as a subscheme of $X \times \mathbb{P}^{k-1}$ by equations as indicated in the respective definition.

Proof. (a) Let $U_j \subset \mathbb{P}^{k-1}$ be the affine chart given by $u_j \neq 0$, with isomorphism $U_j \simeq \mathbb{A}^{k-1}, (u_1 : \ldots : u_k) \mapsto (u_1/u_j, \ldots, u_k/u_j)$. The j-th chart expression of γ equals $a \mapsto (g_1(a)/g_j(a), \ldots, g_k(a)/g_j(a))$ and is defined on the principal open set $g_j \neq 0$ of X. The closure of the graph of γ in U_j is therefore given by $\mathrm{Spec}(R[g_1/g_i, \ldots, g_k/g_i])$. The preceding theorem then establishes the required equality.

(b) If $g_1, \ldots, g_k$ form a regular sequence, their only linear relations over R are the trivial ones, so that

$$R[Ig_i^{-1}] = R[g_1/g_i, \ldots g_k/g_i] \simeq R[t_1, \ldots, t_k]/(g_i t_j - g_j, j = 1, \ldots, k).$$

The i-th chart expression of $\pi : \widetilde{X} \to X$ is then given by the ring inclusion $R \to R[t_1, \ldots, t_k]/(g_i t_j - g_j, j = 1, \ldots, k)$. □

Example 4.21. For the cone X in $\mathbb{A}^3$ of equation $x^2 + y^2 = z^2$, the line Y in $\mathbb{A}^3$ defined by $x = y - z = 0$ is a hypersurface of X at each point. It is a Cartier divisor at any point $a \in Y \setminus \{0\}$ but it is not a Cartier divisor at 0. The double line Y' in $\mathbb{A}^3$ defined by $x^2 = y - z = 0$ is a Cartier divisor of X since it can be defined in X by $y - z = 0$. The subvariety Y'' in $\mathbb{A}^3$ consisting of the two lines defined by $x = y^2 - z^2 = 0$ is a Cartier divisor of X since it can be defined in X by $x = 0$.

Example 4.22. For the surface $X : x^2y - z^2 = 0$ in $\mathbb{A}^3$, the subvariety Y defined by $y^2 - xz = x^3 - yz = 0$ is the singular curve parametrized by $t \mapsto (t^3, t^4, t^5)$. It is everywhere a Cartier divisor except at 0: there, the local ring of Y in X is $\mathbb{K}[x, y, z]_{(x,y,z)}/(x^2y - z^2, y^2 - xz, x^3 - yz)$, which defines a singular cubic. It is of codimension 1 in X but a non-complete intersection. Thus it is not a Cartier divisor. At any other point $a = (t^3, t^4, t^5)$, $t \neq 0$ of Y, one has $\mathcal{O}_{Y,a} = S/(x^2y - z^2, y^2/x - z, x^3 - yz) = S/(x^2y - z^2, y^2/x - z)$ with $S = \mathbb{K}[x, y, z]_{(x-t^3, y-t^4, z-t^5)}$ the localization of $\mathbb{K}[x, y, z]$ at a. Hence Y is Cartier divisor there.

Example 4.23. Let Z be one of the axes of the cross $X : xy = 0$ in $\mathbb{A}^2$, e.g., the x-axis. At any point a on the y-axis except the origin, Z is Cartier: one has $\mathcal{O}_{X,a} = \mathbb{K}[x, y]_{(x,y-a)}/(xy) = \mathbb{K}[x, y]_{(x,y-a)}/(x)$ and the element $h = y$ defining Z is a unit in this local ring. At the origin 0 of $\mathbb{A}^2$ the local ring of X is $\mathbb{K}[x, y]_{(x,y)}/(xy)$ and Z is locally defined by $h = y$ which is a zero-divisor in $\mathcal{O}_{X,0}$. Thus Z is not Cartier in X at 0.

Example 4.24. Let Z be the origin of $X = \mathbb{A}^2$. Then Z is not a hypersurface and hence not Cartier in X.

Example 4.25. The origin Z of $X = V(xy, x^2)$ in $\mathbb{A}^2$ is not a Cartier divisor in X.

Example 4.26. Are $Z = V(x^2)$ in $\mathbb{A}^1$ and $Z = V(x^2y)$ in $X = \mathbb{A}^2$ Cartier divisors?

Example 4.27. Let Z be the reduced origin of $X = V(xy, x^2)$ in $\mathbb{A}^2$. Show that Z is not a Cartier divisor in X.

Example 4.28. Let $X = \mathbb{A}^1$ be the affine line with coordinate ring $R = \mathbb{K}[x]$ and let $Z = 0$ be the origin. The Rees algebra $\widetilde{R} = \mathbb{K}[x, xt] \subset \mathbb{K}[x, t]$ with respect to Z is isomorphic, as a graded ring, to a polynomial ring $\mathbb{K}[u, v]$ in two variables with $\deg u = 0$ and $\deg v = 1$. Therefore, the point blowup $\widetilde{X}$ of X is isomorphic to $\mathbb{A}^1 \times \mathbb{P}^0 = \mathbb{A}^1$, and $\pi : \widetilde{X} \to X$ is the identity.

More generally, let $X = \mathbb{A}^n$ be n-dimensional affine space with coordinate ring $R = \mathbb{K}[x_1, \ldots, x_n]$and let $Z \subset X$ be a hypersurface defined by some non-zero $g \in \mathbb{K}[x_1, \ldots, x_n]$. The Rees algebra $\widetilde{R} = \mathbb{K}[x_1, \ldots, x_n, gt] \subset \mathbb{K}[x_1, \ldots, x_n, t]$ with respect to Z is isomorphic, as a graded ring, to a polynomial ring $\mathbb{K}[u_1, \ldots, u_n, v]$ in $n+1$ variables with $\deg u_i = 0$ and $\deg v = 1$. Therefore the blowup $\widetilde{X}$ of X along Z is isomorphic to $\mathbb{A}^n \times \mathbb{P}^0 = \mathbb{A}^n$, and $\pi : \widetilde{X} \to X$ is the identity.

Example 4.29. Let $X = \mathrm{Spec}(R)$ be an affine scheme and $Z \subset X$ be defined by some non-zero-divisor $g \subset R$. The Rees algebra $\widetilde{R} = R[gt] \subset R[t]$ with respect to Z is isomorphic, as a graded ring, to $R[v]$ with $\deg r = 0$ for $r \in R$ and $\deg v = 1$. Therefore the blowup $\widetilde{X}$ of X along Z is isomorphic to $X \times \mathrm{Proj}^0 = X$, and $\pi : \widetilde{X} \to X$ is the identity.

Example 4.30. Take $X = V(xy) \subset \mathbb{A}^2$ with coordinate ring $R = K[x, y]/(xy)$. Let Z be defined in X by $y = 0$. The Rees algebra of R with respect to Z equals $\widetilde{R} = R[yt] \cong K[x, y, u]/(xy, xu)$ with $\deg x = \deg y = 0$ and $\deg u = 1$.

Example 4.31. Let $X = \mathrm{Spec}(R)$ be an affine scheme and let $Z \subset X$ be defined by some zero-divisor $g \neq 0$ in R, say $h \cdot g = 0$ for some non-zero $h \in R$. The Rees algebra $\widetilde{R} = R[gt] \subset R[t]$ with respect to Z is isomorphic, as a graded ring, to $R[v]/(h \cdot v)$ with $\deg r = 0$ for $r \in R$ and $\deg v = 1$. Therefore, the blowup $\widetilde{X}$ of X along Z equals the closed subvariety of $X \times \mathbb{P}^0 = X$ defined by $h \cdot v = 0$, and $\pi : \widetilde{X} \to X$ is the inclusion map.

Example 4.32. Take in the situation of the preceding example $X = V(xy) \subset \mathbb{A}^2$ and $g = y$, $h = x$. Then $R = \mathbb{K}[x, y]/(xy)$ and $\widetilde{R} = R[yt] \simeq \mathbb{K}[x, y, u]/(xy, xu)$ with deg $x =$ deg $y = 0$ and deg $u = 1$.

Example 4.33. Let $X = \mathrm{Spec}(R)$ be an affine scheme and let $Z \subset X$ be defined by some nilpotent element $g \neq 0$ in R, say $g^k = 0$ for some $k \geq 1$. The Rees algebra $\widetilde{R} = R[gt] \subset R[t]$ with respect to Z is isomorphic, as a graded ring, to $R[v]/(v^k)$ with $\deg r = 0$ for $r \in R$ and $\deg v = 1$. The blowup $\widetilde{X}$ of X is the closed subvariety of $X \times \mathbb{P}^0$ defined by $v^k = 0$.

Example 4.34. Let $X = \mathbb{A}^2$ and let Z be defined in X by $I = (x, y)$. The Rees algebra $\widetilde{R} = \mathbb{K}[x, y, xt, yt] \subset \mathbb{K}[x, y, t]$ of $R = \mathbb{K}[x, y]$ with respect to Z is isomorphic, as a graded ring, to the factor ring $\mathbb{K}[x, y, u, v]/(xv - yu)$, with $\deg x = \deg y = 0$ and $\deg v = \deg u = 1$. It follows that the blowup $\widetilde{X}$ of X along Z embeds naturally as the closed and regular subvariety of $\mathbb{A}^2 \times \mathbb{P}^1$ defined by $xv - yu = 0$, the morphism $\pi : \widetilde{X} \to X$ being given by the restriction to $\widetilde{X}$ of the first projection $\mathbb{A}^2 \times \mathbb{P}^1 \to \mathbb{A}^2$.

Example 4.35. Let $X = \mathbb{A}^2$ and let Z be defined in X by $I = (x, y^2)$. The Rees algebra $\widetilde{R} = \mathbb{K}[x, y, xt, y^2t] \subset \mathbb{K}[x, y, t]$ of $R = \mathbb{K}[x, y]$ with respect to Z is isomorphic, as a graded ring, to the factor ring $\mathbb{K}[x, y, u, v]/(xv - y^2u)$, with $\deg x = \deg y = 0$ and $\deg u = \deg v = 1$. It follows that the blowup $\widetilde{X}$ of X along Z embeds naturally as the closed and singular subvariety of $\mathbb{A}^2 \times \mathbb{P}^1$ defined by $xv - y^2u = 0$, the morphism $\pi : \widetilde{X} \to X$ being given by the restriction to $\widetilde{X}$ of the first projection $\mathbb{A}^2 \times \mathbb{P}^1 \to \mathbb{A}^2$.

Example 4.36. Let $X = \mathbb{A}^3$ and let Z be defined in X by $I = (xy, z)$. The Rees algebra $\widetilde{R} = \mathbb{K}[x, y, z, xyt, zt] \subset \mathbb{K}[x, y, z, t]$ of $R = \mathbb{K}[x, y, z]$ with respect to Z is isomorphic, as a graded ring, to $\mathbb{K}[u, v, w, r, s]/(uvs - wr)$, with $\deg u = \deg v = \deg w = 0$ and $\deg r = \deg s = 1$. It follows that the blowup $\widetilde{X}$ of X along Z embeds naturally as the closed and singular subvariety of $\mathbb{A}^3 \times \mathbb{P}^1$ defined by $uvs - wr = 0$, the morphism $\pi : \widetilde{X} \to X$ being given by the restriction to $\widetilde{X}$ of the first projection $\mathbb{A}^3 \times \mathbb{P}^1 \to \mathbb{A}^3$.

Example 4.37. Let $X = \mathbb{A}^1_{\mathbb{Z}}$ be the affine line over the integers and let Z be defined in X by $I = (x, p)$ for a prime $p \in \mathbb{Z}$. The Rees algebra $\widetilde{R} = \mathbb{Z}[x, xt, pt] \subset \mathbb{Z}[x, t]$ of $R = \mathbb{Z}[x]$ with respect to Z is isomorphic, as a graded ring, to $\mathbb{Z}[u, v, w]/(xw - pv)$, with $\deg u = 0$ and $\deg v = \deg w = 1$. It follows that the blowup $\widetilde{X}$ of X along Z embeds naturally as the closed and regular subvariety of $\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}}$ defined by $xw - pv$, the morphism $\pi : \widetilde{X} \to X$ being given by the restriction to $\widetilde{X}$ of the first projection $\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}} \to \mathbb{A}^1_{\mathbb{Z}}$.

Example 4.38. Let $X = \mathbb{A}^2_{\mathbb{Z}_{20}}$ with $\mathbb{Z}_{20} = \mathbb{Z}/20\mathbb{Z}$, and let Z be defined in X by $I = (x, 2y)$. The Rees algebra $\widetilde{R} = \mathbb{Z}_{20}[x, y, xt, 2yt] \subset \mathbb{Z}_{20}[x, y, t]$ of $R = \mathbb{Z}_{20}[x]$ with respect to Z is isomorphic, as a graded ring, to $\mathbb{Z}_{20}[u, v, w, z]/(xz - 2vw)$, with $\deg u = \deg v = 0$ and $\deg w = \deg z = 1$.

Example 4.39. Let $R = \mathbb{Z}_6$ with $\mathbb{Z}_6 = \mathbb{Z}/6\mathbb{Z}$, and let Z be defined in X by $I = (2)$. The Rees algebra $\widetilde{R} = \mathbb{Z}_6[2t] \subset \mathbb{Z}_6[t]$ of $R =$ with respect to Z is isomorphic, as a graded ring, to $\mathbb{Z}_3[u]$ with $\deg u = 1$.

Example 4.40. Let $R = \mathbb{K}[[x, y]]$ be a formal power series ring in two variables x and y, and let Z be defined in X by I be the ideal generated by $e^x - 1$ and $\ln(y)$. The Rees algebra $\widetilde{R}$ of R with respect to Z equals $R[(e^x - 1)t, \ln(y)t] = \mathbb{K}[[x, y]][(e^x - 1)t, \ln(y)t] \cong \mathbb{K}[[x, y]][u, v]/(\ln(y)u - (e^x - 1)v)$ with $\deg u = \deg v = 1$.

Example 4.41. Let $X = \mathbb{A}^n$. The point blowup $\widetilde{X} \subset X \times \mathbb{P}^{n-1}$ of X at the origin is defined by the ideal $(u_i x_j - u_j x_i, i, j = 1, \dots n)$ in $\mathbb{K}[x_1, \dots, x_n, u_1, \dots, u_n]$. This is a graded ring, where $\deg x_i = 0$ and $\deg u_i = 1$ for all $i = 1, \dots, n$. But the ring

$$\mathbb{K}[x_1, \dots, x_n, u_1, \dots, u_n]/(u_i x_j - u_j x_i, i, j = 1, \dots n)$$

is isomorphic, as a graded ring, to $\widetilde{R} = \mathbb{K}[x_1, \dots, x_n, x_1 t, \dots, x_n t] \subset \mathbb{K}[x_1, \dots, x_n, t]$ with $\deg x_i = 0$, $\deg t = 1$. The ring $\widetilde{R}$ is the Rees-algebra of the ideal $(x_1, \dots, x_n)$ of $\mathbb{K}[x_1, \dots, x_n]$. Thus $\widetilde{X}$ is isomorphic to $\mathrm{Proj}\big(\bigoplus_{d \geq 0} (x_1, \dots, x_n)^d\big)$.

Example 4.42. The i-th affine chart of the blowup $\widetilde{\mathbb{A}}^n$ of $\mathbb{A}^n$ in the ideal $I = (x_1, \dots, x_k)$ is isomorphic to $\mathbb{A}^n$, for $i = 1, \dots, k$, via the ring isomorphism

$$\mathbb{K}[x_1, \dots, x_n] \simeq \mathbb{K}[x_1, \dots, x_n, t_1, \dots, \widehat{t_i}, \dots t_k]/(x_i t_j - x_j, j = 1, \dots, k, j \neq i),$$

where $x_j \mapsto t_j$ for $j = 1, \dots, k$, $j \neq i$, respectively $x_j \mapsto x_j$ for $j = k+1, \dots, n$ and $j = i$. The inverse is given by $x_j \mapsto x_i x_j$ for $j = 1, \dots, k$, $j \neq i$, respectively $x_j \mapsto x_j$ for $j = k+1, \dots, n$ and $j = i$, respectively $t_j \mapsto x_j$ for $j = 1, \dots, k$, $j \neq i$.

Example 4.43. Let $X = \mathbb{A}^2$ be the affine plane and let $g_1 = x^2, g_2 = xy, g_3 = y^3$ be generators of the ideal $I \subset \mathbb{K}[x, y]$. The g_i do not form a regular sequence. The subvariety of $\mathbb{A}^2 \times \mathbb{P}^2$ defined by all $g_i u_j - g_j u_i = 0$ is singular, but the blowup of $\mathbb{A}^2$ in I is regular.

EXAMPLE 4.44. Compute the following blowups:
(a) $\mathbb{A}^2$ in the center $(x,y)(x,y^2)$,
(b) $\mathbb{A}^3$ in the centers (x,yz) and $(x,yz)(x,y)(x,z)$,
(c) $\mathbb{A}^3$ in the center (x^2+y^2-1,z).
(d) The plane curve $x^2=y$ in the origin.
Use affine charts and ring extensions to determine at which points the resulting varieties are regular or singular.

EXAMPLE 4.45. Blow up $\mathbb{A}^3$ in 0 and compute the inverse image of $x^2+y^2=z^2$.

EXAMPLE 4.46. Blow up $\mathbb{A}^2$ in the point $(0,1)$. What is the inverse image of the lines $x+y=0$ and $x+y=1$?

EXAMPLE 4.47. Compute the chart transition map for the blowup of $\mathbb{A}^3$ in the z-axis.

EXAMPLE 4.48. Blow up the cone $x^2+y^2=z^2$ in one of its lines.

EXAMPLE 4.49. Blow up $\mathbb{A}^3$ in the circle $x^2+(y+2)^2-1=z=0$ and in the elliptic curve $y^2-x^3-x=z=0$.

EXAMPLE 4.50. Interpret the blowup of $\mathbb{A}^3$ in $(x,yz)(x,y)(x,z)$ as a composition of blowups in regular centers [Hau00, FW11, Lev01].

EXAMPLE 4.51. Show that the blowup of $\mathbb{A}^n$ along a coordinate subspace Z equals the cartesian product of the point-blowup in a transversal subspace V of $\mathbb{A}^n$ of complementary dimension (with respect to Z) with the identity map on Z.

EXAMPLE 4.52. Show that the ideals $(x_1,\dots,x_n)$ and $(x_1,\dots,x_n)^m$ define the same blowup of $\mathbb{A}^n$ when taken as center.

EXAMPLE 4.53. Let E be a normal crossings subvariety of $\mathbb{A}^n$ and let Z be a subvariety of $\mathbb{A}^n$ such that $E\cup Z$ also has normal crossings (the union being defined by the product of ideals). Show that the inverse image of E under the blowup of $\mathbb{A}^n$ along Z has again normal crossings. Show by an example that the assumption on Z cannot be dropped in general.

EXAMPLE 4.54. Draw a real picture of the blowup of $\mathbb{A}^2$ at the origin.

EXAMPLE 4.55. Show that the blowup $\widetilde{\mathbb{A}}^n$ of $\mathbb{A}^n$ with center a point is nonsingular.

EXAMPLE 4.56. Describe the geometric construction of the blowup $\widetilde{\mathbb{A}}^3$ of $\mathbb{A}^3$ with center 0. What is $\pi^{-1}(0)$? For a chosen affine chart W' of $\widetilde{\mathbb{A}}^3$, consider all cylinders Y over a circle in W' centered at the origin and parallel to a coordinate axis. What is the image of Y under π in $\mathbb{A}^3$?

EXAMPLE 4.57. Compute the blowup of the Whitney umbrella $X=V(x^2-y^2z)\subset\mathbb{A}^3$ with center 0 and the three coordinate axes respectively.

EXAMPLE 4.58. Determine the locus of points of the Whitney umbrella $X=V(x^2-y^2z)$ where the singularities are normal crossings, respectively simple normal crossings. Blow up the complements of these loci and compare with the preceding example [Kol07] ex. 3.6.1, p. 123, [BDMP12] Thm. 3.4.

EXAMPLE 4.59. Let Z be a regular center in $\mathbb{A}^n$, with induced blowup $\pi : \widetilde{\mathbb{A}}^n \to \mathbb{A}^n$, and let a' be a point of $\widetilde{\mathbb{A}}^n$ mapping to a point $a \in Z$. Show that it is possible to choose local formal coordinates at a, i.e., a regular parameter system of $\widehat{\mathcal{O}}_{\mathbb{A}^n,a}$, so that the center is a coordinate subspace, and so that a' the origin of one of the affine charts of $\widetilde{\mathbb{A}}^n$. Is this also possible with a regular parameter system of $\mathcal{O}_{\mathbb{A}^n,a}$?

EXAMPLE 4.60. Let $X = V(f)$ be a hypersurface in $\mathbb{A}^n$ and Z a regular closed subvariety which is contained in the locus S of points of X where f attains its maximal order. Let $\pi : \widetilde{\mathbb{A}}^n \to \mathbb{A}^n$ be the blowup along Z and let $X^s = V(f^s)$ be the strict transform of X, defined as the Zariski closure of $\pi^{-1}(X \setminus Z)$ in $\widetilde{\mathbb{A}}^n$. Show that for points $a \in Z$ and $a' \in E = \pi^{-1}(Z)$ with $\pi(a') = a$ the inequality $\mathrm{ord}_{a'} f' \leq \mathrm{ord}_a f$ holds. Do the same for subvarieties defined by arbitrary ideals.

EXAMPLE 4.61. Find an example of a variety X for which the dimension of the singular locus increases under the blowup of a closed regular center Z that is contained in the top locus of X [Hau98] ex. 9.

EXAMPLE 4.62. Let $\pi : (\widetilde{\mathbb{A}}^n, a') \to (\mathbb{A}^n, a)$ be the local blowup of $\mathbb{A}^n$ with center the point a, considered at a point $a' \in E$. Let be given a local coordinates $x_1, \ldots, x_n$ at a. Determine the coordinate changes in $(\mathbb{A}^n, a)$ which make the chart expression of π monomial.

EXAMPLE 4.63. Let $x_1, \ldots, x_n$ be local coordinates at a so that the local blowup $\pi : (\widetilde{\mathbb{A}}^n, a') \to (\mathbb{A}^n, a)$ is monomial with respect to them. Determine the formal automorphisms of $(\widetilde{\mathbb{A}}^n, a')$ which commute with the local blowup.

EXAMPLE 4.64. Compute the blowup of $X = \mathrm{Spec}(\mathbb{Z}[x])$ in the ideals (x, p) and (x, pq) where p and q are primes.

EXAMPLE 4.65. The zeroset in $\mathbb{A}^3$ of the non-reduced ideal $(x, yz)(x, y)(x, z) = (x^3, x^2y, x^2z, xyz, y^2z)$ is the union of the y- and the z-axis. Taken as center, the resulting blowup of $\mathbb{A}^3$ equals the composition of two blowups: The first blowup has center the ideal (x, yz) in $\mathbb{A}^3$, giving a three-fold W_1 in a regular four-dimensional ambient variety with one singular point of local equation $xy = zw$. The second blowup is the point blowup of W_1 with center this singular point [Hau00] Prop. 3.5.

EXAMPLE 4.66. Let $R := \mathbb{K}[x, y, z]$ be the polynomial ring in three variables and let $I = (x, yz) \subset R$. The blowup of R along I corresponds to the ring extensions:

$$R \hookrightarrow R\left[\frac{yz}{x}\right] \cong \mathbb{K}[s, t, u, v]/(sv - tu),$$

$$R \hookrightarrow R\left[\frac{x}{yz}\right] \cong \mathbb{K}[s, t, u, v]/(s - tuv).$$

The first ring extension defines a singular variety while the second one defines a non-singular one. Let $W = \mathbb{A}^3$ and $Z = V(I) \subset W$ be the affine space and the subvariety defined by I. The blowup of R along I coincides with the blowup W' of W along Z.

(a) Compute the chart expressions of the blowup maps.
(b) Determine the exceptional divisor.
(c) Apply one more blowup to W' to get a non-singular variety W''.

(d) Express the composition of the two blowups as a single blowup in a properly chosen ideal.

(e) Blow up $\mathbb{A}^3$ along the three coordinate axes. Show that the resulting variety is non-singular.

(f) Show the same for the blowup of $\mathbb{A}^n$ along the n coordinate axes.

EXAMPLE 4.67. * Consider the blowup $\widetilde{\mathbb{A}}^n$ of $\mathbb{A}^n$ in a monomial ideal I of $\mathbb{K}[x_, \dots, x_n]$. Show that $\widetilde{\mathbb{A}}^n$ may be singular. What types of singularities will occur? Find a natural saturation procedure $I \rightsquigarrow \overline{I}$ so that the blowup of $\mathbb{A}^n$ in $\overline{I}$ is regular and equal to a (natural) resolution of the singularities of $\widetilde{\mathbb{A}}^n$ [FW11].

EXAMPLE 4.68. The *Nash modification* of a subvariety X of $\mathbb{A}^n$ is the closure of the graph of the map which associates to each non-singular point its tangent space, taken as an element of the Grassmanian of d-dimensional linear subspaces of $\mathbb{K}^n$, where $d = \dim(X)$. For a hypersurface X defined in $\mathbb{A}^n$ by $f = 0$, the Nash modification coincides with the blowup of X in the Jacobian ideal of f generated by the partial derivatives of f.

5. Lecture V. Properties of Blowup

PROPOSITION 5.1. Let $\pi : \widetilde{X} \to X$ be the blowup of X along a subvariety Z, and let $\varphi : Y \to X$ be a morphism, the base change. Denote by $p : \widetilde{X} \times_X Y \to Y$ the projection from the fibre product to the second factor. Let $S = \varphi^{-1}(Z) \subset Y$ be the inverse image of Z under φ, and let $\widetilde{Y}$ be the Zariski closure of $p^{-1}(Y \setminus S)$ in $\widetilde{X} \times_X Y$. The restriction $\tau : \widetilde{Y} \to Z$ of p to $\widetilde{Y}$ equals the blowup of Y along S.

$$\begin{tikzcd}
F \arrow[r, hook] \arrow[dr] & \widetilde{Y} \arrow[r, hook] \arrow[dr, "\tau"] & \widetilde{X} \times_X Y \arrow[r, "q"] \arrow[d, "p"] & \widetilde{X} \arrow[d, "\pi"] \\
& S \arrow[r, hook] & Y \arrow[r, "\varphi"] & X
\end{tikzcd}$$

PROOF. To show that $F = \tau^{-1}(S)$ is Cartier in $\widetilde{Y}$, consider the projection $q : \widetilde{X} \times_X Y \to \widetilde{X}$ onto the first factor. By the commutativity of the diagram,

$$F = p^{-1}(S) = p^{-1} \circ \varphi^{-1}(Z) = q^{-1} \circ \pi^{-1}(Z) = q^{-1}(E).$$

As E Cartier in $\widetilde{X}$ and q is a projection, F is locally defined by a principal ideal. The associated primes of $\widetilde{Y}$ are the associated primes of Y not containing the ideal of S. Thus, the local defining equation of E in $\widetilde{X}$ cannot pull back to a zero divisor on $\widetilde{Y}$. This proves that F is Cartier.

To show that $\tau : \widetilde{Y} \to Y$ fulfills the universal property of blowups, let $\psi : Y' \to Y$ be a morphism such that $\psi^{-1}(S)$ is a Cartier divisor in Y'.

$$\begin{tikzcd}
& & \widetilde{X} \arrow[dr, "\pi"] & \\
Y' \arrow[r, "\sigma"] \arrow[urr, bend left, "\rho"] \arrow[drr, bend right, "\psi"'] & \widetilde{X} \times_X Y \arrow[ur, "q"] \arrow[dr, "p"] & & X \\
& & Y \arrow[ur, "\varphi"] &
\end{tikzcd}$$

Since $\psi^{-1}(S) = \psi^{-1}(\varphi^{-1}(Z)) = (\varphi \circ \psi)^{-1}(Z)$, there exists by the universal property of the blowup $\pi : \widetilde{X} \to X$ a unique map $\rho : Y' \to \widetilde{X}$ such that $\varphi \circ \psi =$

$\pi \circ \rho$. By the universal property of fibre products, there exists a unique map $\sigma : Y' \to \widetilde{X} \times_X Y$ such that $q \circ \sigma = \rho$ and $p \circ \sigma = \psi$.

It remains to show that $\sigma(Y')$ lies in $\widetilde{Y} \subset \widetilde{X} \times_X Y$. Since $\psi^{-1}(S)$ is Cartier in Y', its complement $Y' \setminus \psi^{-1}(S)$ is dense in Y'. From

$$Y' \setminus \psi^{-1}(S) = \psi^{-1}(Y \setminus S) = (p \circ \sigma)^{-1}(Y \setminus S) = \sigma^{-1}(p^{-1}(Y \setminus S))$$

follows that $\sigma(Y' \setminus \psi^{-1}(S)) \subset p^{-1}(Y \setminus S)$. But $\widetilde{Y}$ is the closure of $p^{-1}(Y \setminus S)$ in $\widetilde{X} \times_X Y$, so that $\sigma(Y') \subset \widetilde{Y}$ as required. $\square$

Corollary 5.2. (a) Let $\pi : X' \to X$ be the blowup of X along a subvariety Z, and let Y be a closed subvariety of X. Denote by Y' the Zariski closure of $\pi^{-1}(Y \setminus Z)$ in X'. The restriction $\tau : Y' \to Y$ of π to Y' is the blowup of Y along $Y \cap Z$. In particular, if $Z \subset Y$, then τ is the blowup of Y along Z.

(b) Let $U \subset X$ be an open subvariety, and let $Z \subset X$ be a closed subvariety, so that $U \cap Z$ is closed in U. Let $\pi : X' \to X$ be the blowup of X along Z. The blowup of U along $U \cap Z$ equals the restriction of π to $U' = \pi^{-1}(U)$.

(c) Let $a \in X$ be a point. Write (X, a) for the germ of X at a, and $(\widehat{X}, a)$ for the formal neighbourhood. There are natural maps

$$(X, a) \to X \quad \text{and} \quad (\widehat{X}, a) \to X$$

corresponding to the localization and completion homomorphisms $\mathcal{O}_X \to \mathcal{O}_{X,a} \to \widehat{\mathcal{O}}_{X,a}$. Take a point a' above a in the blowup X' of X along a subvariety Z containing a. This gives local blowups of germs and formal neighborhoods

$$\pi_{a'} : (X', a') \to (X, a), \qquad \widehat{\pi}_{a'} : (\widehat{X}', a') \to (\widehat{X}, a).$$

The blowup of a local ring is not local in general; to get a local blowup one needs to localize also on X'.

(d) If $X_1 \to X$ is an isomorphism between varieties sending a subvariety Z_1 to Z, the blowup X_1' of X_1 along Z_1 is canonically isomorphic to the blowup X' of X along Z. This also holds for local isomorphisms.

(e) If $X = Z \times Y$ is a cartesian product of two varieties, and a is a given point of Y, the blowup $\pi : X' \to X$ of X along $Z \times \{a\}$ is isomorphic to the cartesian product $\mathrm{Id}_Z \times \tau : Z \times Y' \to Z \times Y$ of the identity on Z with the blowup $\tau : Y' \to Y$ of Y in a.

Proposition 5.3. Let $\pi : X' \to X$ be the blowup of X along a regular subvariety Z with exceptional divisor E. Let Y be a subvariety of X, and $Y^* = \pi^{-1}(Y)$ its preimage under π. Let Y' be the Zariski closure of $\pi^{-1}(Y \setminus Z)$ in X'. If Y is transversal to Z, i.e., $Y \cup Z$ has normal crossings at all points of the intersection $Y \cap Z$, also Y^* has normal crossings at all points of $Y^* \cap E$. In particular, if Y is regular and transversal to Z, also Y' is regular and transversal to E.

Proof. Having normal crossings is defined locally at each point through the completions of local rings. The assertion is proven by a computation in local coordinates for which the blowup is monomial, cf. Prop. 5.4 below. $\square$

Proposition 5.4. Let W be a regular variety of dimension n with a regular subvariety Z of codimension k. Let $\pi : W' \to W$ denote the blowup of W along Z, with exceptional divisor E. Let V be a regular hypersurface in W containing Z, let D be a (not necessarily reduced) normal crossings divisor in W having normal crossings with V. Let a be a point of $V \cap Z$ and let $a' \in E$ be a point lying above a. There exist local coordinates $x_1, \ldots, x_n$ of W at a such that

(1) a has components $a = (0, \ldots, 0)$.

(2) V is defined in W by $x_{n-k+1} = 0$.

(3) Z is defined in W by $x_{n-k+1} = \ldots = x_n = 0$.

(4) $D \cap V$ is defined in V locally at a by a monomial $x_1^{q_1} \cdots x_n^{q_n}$, for some $q = (q_1, \ldots, q_n) \in \mathbb{N}^n$ with $q_{n-k+1} = 0$.

(5) The point a' lies in the x_n-chart of W'. The chart expression of π in the x_n-chart is of the form

$$\begin{aligned} x_i &\mapsto x_i && \text{for } i \leq n-k \text{ and } i = n, \\ x_i &\mapsto x_i x_n && \text{for } n-k+1 \leq i \leq n-1, \end{aligned}$$

(6) In the induced coordinates of the x_n-chart, the point a' has components $a' = (0, \ldots, 0, a'_{n-k+2}, \ldots, a'_{n-k+d}, 0, \ldots, 0)$ with non-zero entries $a'_j \in \mathbb{K}$ for $n-k+2 \leq j \leq n-k+d$, where d is the number of components of D whose strict transforms do not pass through a'.

(7) The strict transform (def. 6.2) V^s of V in W' is given in the induced coordinates locally at a' by $x_{n-k+1} = 0$.

(8) The local coordinate change φ in W at a given by $\varphi(x_i) = x_i + a'_i \cdot x_n$ makes the local blowup $\pi : (W', a') \to (W, a)$ monomial. It preserves the defining ideals of Z and V in W.

(9) If condition (4) is not imposed, the coordinates $x_1, \ldots, x_n$ at a can be chosen with (1) to (3) and so that a' is the origin of the x_n-chart.

Proof. [Hau10b]. □

Theorem 5.5. Any projective birational morphism $\pi : X' \to X$ is a blowup of X in an ideal I.

Proof. [Har77] Thm. 7.14. □

Example 5.6. Let X be a normal crossings subvariety of $\mathbb{A}^n$ and Z a regular closed subscheme which is transversal to X. Show that the blowup X' of X along Z is again a normal crossings scheme.

Example 5.7. * Prove that plain varieties remain plain under blowup in regular centers [BHSV08] Thm. 4.3.

Example 5.8. * Is any rational and regular variety plain?

Example 5.9. Consider the blowup $\pi : W' \to W$ of a regular variety W along a closed subvariety Z. Show that the exceptional divisor $E = \pi^{-1}(Z)$ is a hypersurface in W'.

Example 5.10. The composition of two blowups $W'' \to W'$ and $W' \to W$ is a blowup of W in a suitable center [Bod03].

Example 5.11. A *fractional ideal* I over an integral domain R is an R-submodule of $\mathrm{Quot}(R)$ such that $rI \subset R$ for some non-zero element $r \in R$. Blowups can be defined via Proj also for centers which are fractional ideals [Gro61]. Let I and J be two (ordinary) non-zero ideals of R. The blowup of R along I is isomorphic to the blowup of R along J if and only if there exist positive integers k, ℓ and fractional ideals K, L over R such that $JK = Ik$ and $IL = Jl$ [Moo01] Cor. 2.

EXAMPLE 5.12. For the surface $x^2y - z^2 = 0$ in $\mathbb{A}^3$, the subvariety Z defined by $y^2 - xz = x^3 - yz = 0$ is the singular curve parametrized by $t \to (t^3, t^4, t^5)$. It is everywhere Cartier except at 0. Prove that Z is not Cartier at 0.

EXAMPLE 5.13. Determine the equations of the blowup X' in $X \times \mathbb{P}^2$ of $X = \mathbb{A}^3$ along the image Z of the monomial curve (t^3, t^4, t^5) of equations $g_1 = y^2 - xz$, $g_2 = yz - x^3$, $g_3 = z^2 - x^2y$.

EXAMPLE 5.14. Show that the blowup of the cone $X = V(x^2 - yz)$ in $\mathbb{A}^3$ along the line $Z = V(x, y)$ is an isomorphism locally at all points outside 0, but not globally on X.

EXAMPLE 5.15. Blow up the non-reduced point $X = V(x^2)$ in $\mathbb{A}^2$ in the (reduced) origin $Z = 0$.

EXAMPLE 5.16. Blow up the subscheme $X = V(x^2, xy)$ of $\mathbb{A}^2$ in the reduced origin.

EXAMPLE 5.17. Blow up the subvariety $X = V(xz, yz)$ of $\mathbb{A}^3$ first in the origin, then in the x-axis, and determine the points where the resulting morphisms are local isomorphisms.

EXAMPLE 5.18. The blowups of $\mathbb{A}^3$ along the union of the x- with the y-axis, respectively along the cusp, with ideals (xy, z) and $(x^3 - y^2, z)$, are singular.

6. Lecture VI: Transforms of Ideals and Varieties under Blowup

Throughout this section, $\pi : W' \to W$ denotes the blowup of a variety W along a closed subvariety Z, with exceptional divisor $E = \pi^{-1}(Z)$ defined by the principal ideal I_E of $\mathcal{O}_{W'}$. Let $\pi^* : \mathcal{O}_W \to \mathcal{O}_{W'}$ be the dual homomorphism of π. Let $X \subset W$ be a closed subvariety, and let I be an ideal on W.

DEFINITION 6.1. The inverse image $X^* = \pi^{-1}(X)$ of X and the extension $I^* = \pi^*(I) = I \cdot \mathcal{O}_{W'}$ of I are called the *total transform* of X and I under π. For $f \in \mathcal{O}_W$, denote by f^* its image $\pi^*(f)$ in $\mathcal{O}_{W'}$. The ideal I^* is generated by all transforms f^* for f varying in I. If I is the ideal defining X in W, the ideal I^* defines X^* in W'. In particular, the total transform of the center Z equals the exceptional divisor E, and $W^* = W'$. The total transform can be non-reduced when considered as a subscheme of W'.

DEFINITION 6.2. The Zariski closure of $\pi^{-1}(X \setminus Z)$ in W' is called the *strict transform* of X under π and denoted by X^s, also known as the *proper* or *birational transform*. The strict transform is a closed subvariety of the total transform X^*. The difference $X^* \setminus X'$ is contained in the exceptional locus E. If the center Z equals whole X, the strict transform is empty.

Let I_U denote the restriction of an ideal I to the open set $U = W \setminus Z$. Set $U' = \pi^{-1}(U) \subset W'$ and let $\tau : U' \to U$ be the restriction of π to U'. The *strict transform* I^s of the ideal I is defined as $\tau^*(I_U) \cap \mathcal{O}_{W'}$. If the ideal I defines X in W, the ideal I^s defines X^s in W'. It equals the union of colon ideals

$$I^s = \bigcup_{i \geq 0} I^* : I_E^i.$$

Let h be an element of $\mathcal{O}_{W',a'}$ defining E locally in W' at a point a'. Then, locally at a',

$$I^s = (f^s,\, f \in I),$$

where the strict transform f^s of f is defined at a' and up to multiplication by invertible elements in $\mathcal{O}_{W',a'}$ through $f^* = h^k \cdot f^s$ with maximal exponent k. The value of k is the order of f^* along E, i.e., the maximal number k such that $f^* \in I_E^k$. By abuse of notation this is written as $f^s = h^{-k} \cdot f^* = h^{-\mathrm{ord}_Z f} \cdot f^*$.

Lemma 6.3. Let $f : R \to S$ be a ring homomorphism, I an ideal of R and s an element of R, with induced ring homomorphism $f_s : R_s \to S_{f(s)}$. Let $I^e = f(I) \cdot S$ and $(I \cdot R_s)^e = f_s(I \cdot R_s) \cdot S_{f(s)}$ denote the respective extensions of the ideals. Then $\bigcup_{i \geq 0} I^e : f(s)^i = (I \cdot R_s)^e \cap S$.

Proof. Let $u \in S$. Then $u \in \bigcup_{i \geq 0} I^e : f(s)^i$ if and only if $uf(s)^i \in I^e$ for some $i \geq 0$, say $uf(s)^i = \sum_j a_j f(x_j)$ for elements $x_j \in I$ and $a_j \in S$. Rewrite this as $u = \sum_j a_j f(\frac{x_j}{s^i})$. This just means that $u \in (I \cdot R_s)^e$. $\square$

Remark 6.4. If I is generated locally by elements $f_1, \ldots, f_k$ of $\mathcal{O}_W$, then I^s contains the ideal generated by the strict transforms $f_1^s, \ldots, f_k^s$ of $f_1, \ldots, f_k$, but the inclusion can be strict, see the examples below.

Definition 6.5. Let $\mathbb{K}[x] = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring over $\mathbb{K}$, considered with the natural grading given by the degree. Denote by $\mathrm{in}(g)$ the homogeneous form of lowest degree of a non-zero polynomial g of $\mathbb{K}[x]$, called the *initial form* of g. Set $\mathrm{in}(0) = 0$. For a non-zero ideal I, denote by $\mathrm{in}(I)$ the ideal generated by all initial forms $\mathrm{in}(g)$ of elements g of I, called the *initial ideal* of I. Elements $g_1, \ldots, g_k$ of an ideal I of $\mathbb{K}[x]$ are a *Macaulay basis* of I if their initial forms $\mathrm{in}(g_1), \ldots, \mathrm{in}(g_k)$ generate $\mathrm{in}(I)$. In [Hir64] III.1, def. 3, p. 208, such a basis was called a *standard basis*, which is now used for a slightly more specific concept. By noetherianity of $\mathbb{K}[x]$, any ideal possesses a Macaulay basis.

Proposition 6.6. (Hironaka) The strict transform of an ideal under blowup in a regular center is generated by the strict transforms of the elements of a Macaulay basis of the ideal.

Proof. ([Hir64], III.2, Lem. 6, p. 216, and III.6, Thm. 5, p. 238) If $I \subset J$ are two ideals of $\mathbb{K}[[x]]$ such that $\mathrm{in}(I) = \mathrm{in}(J)$ then they are equal, $I = J$. This holds at least for degree compatible monomial orders, due to the Grauert-Hironaka-Galligo division theorem. Therefore it has to be shown that $\mathrm{in}(g_1^s), \ldots, \mathrm{in}(g_k)$ generate $\mathrm{in}(I^s)$. But $\mathrm{in}(I^s) = (\mathrm{in}(I))^s$, and the assertion follows. $\square$

Remark 6.7. The strict transform of a Macaulay basis at a point a' of W' need no longer be a Macaulay basis. This is however the case if the Macaulay basis is *reduced* and the sequence of its orders has remained constant at a', cf. [Hir64] III.8, Lem. 20, p. 254. More generally, taking on $\mathbb{K}[x]$ a grading so that all homogeneous pieces are one-dimensional and generated by monomials (i.e., a grading induced by a monomial order on $\mathbb{N}^n$), the initial form of a polynomial and the initial ideal are both monomial. In this case Macaulay bases are called *standard bases*. A standard basis $g_1, \ldots, g_k$ is *reduced* if no monomial of the tails $g_i - \mathrm{in}(g_i)$ belongs to $\mathrm{in}(I)$. If the monomial order is degree compatible, i.e., the induced grading a refinement of the natural grading of $\mathbb{K}[x]$ by degree, the strict transforms of a standard basis of I generate the strict transform of the ideal.

Definition 6.8. Let I be an ideal on W and let $c \geq 0$ be a natural number less than or equal to the order d of I along the center Z. Then I^* has order $\geq c$

along E. There exists a unique ideal $I^!$ of $\mathcal{O}_{W'}$ such that $I^* = I_E^d \cdot I^!$, called the *controlled transform* of I with respect to the *control* c. It is not defined for values of $c > d$. In case that $c = d$ attains the maximal value, $I^!$ is denoted by $I^\curlyvee$ and called the *weak transform* of I. It is written as $I^\curlyvee = I_E^{-d} \cdot I^* = I_E^{-\mathrm{ord}_Z I} \cdot I^*$.

Remark 6.9. The inclusions $I^* \subset I^\curlyvee \subset I^! \subset I^s$ are obvious. The components of $V(I^\curlyvee)$ which are not contained in $V(I^s)$ lie entirely in the exceptional divisor E, but can be strictly contained. For principal ideals, $I^\curlyvee$ and I^s coincide. When the transforms are defined scheme-theoretically, the reduction X^*_{red} of the total transform X^* of X consists of the union of E with the strict transform X^s.

Definition 6.10. A *local flag* $\mathcal{F}$ on W at a is a chain $F_0 = \{a\} \subset F_1 \subset \ldots \subset F_n = W$ of regular closed subvarieties, respectively subschemes, F_i of dimension i of an open neighbourhood U of a in W. Local coordinates $x_1, \ldots, x_n$ on W at a are called *subordinate to the flag* $\mathcal{F}$ if $F_i = V(x_{i+1}, \ldots, x_n)$ locally at a. The flag $\mathcal{F}$ at a is *transversal* to a regular subvariety Z of W if each F_i is transversal to Z at a [Hau04, Pan06].

Proposition 6.11. Let $\pi : W' \to W$ be the blowup of W along a center Z transversal to a flag $\mathcal{F}$ at $a \in Z$. Let $x_1, \ldots, x_n$ be local coordinates on W at a subordinate to $\mathcal{F}$. At each point a' of E above a there exists a unique local flag $\mathcal{F}'$ such that the coordinates $x'_1, \ldots, x'_n$ on W' at a' induced by $x_1, \ldots, x_n$ are subordinate to $\mathcal{F}'$ [Hau04], Thm. 1.

Definition 6.12. The flag $\mathcal{F}'$ is called the *transform of* $\mathcal{F}$ *under* π.

Proof. It suffices to define F'_i at a' by $x'_{i+1}, \ldots, x'_n$. For point blowups in W, the transform of $\mathcal{F}$ is defined as follows. The point $a' \in E$ is determined by a line L in the tangent space $\mathrm{T}_a W$ of W at a. Let $k \le n$ be the minimal index for which $\mathrm{T}_a F_k$ contains L. For $i < k$, choose a regular $(i+1)$-dimensional subvariety H_i of W with tangent space $L + \mathrm{T}_a F_i$ at a. In particular, $\mathrm{T}_a H_{k-1} = \mathrm{T}_a F_k$. Let H_i^s be the strict transform of H_i in W'. Then set

$$F'_i = E \cap H_i^s \qquad \text{for } i < k,$$
$$F'_i = F_i^s \qquad \text{for } i \ge k,$$

to get the required flag $\mathcal{F}'$ at a' . □

Example 6.13. Blow up $\mathbb{A}^2$ in 0 and compute the inverse image of the zerosets of $x^2 + y^2 = 0$, $xy = 0$ and $x(x - y^2) = 0$, as well as their strict transforms.

Example 6.14. Compute the strict transform of $X = V(x^2 - y^3, xy - z^3) \subset \mathbb{A}^3$ under the blowup of the origin.

Example 6.15. Determine the total, weak and strict transform of $X = V(x^2 - y^3, z^3) \subset \mathbb{A}^3$ under the blowup of the origin. Clarify the algebraic and geometric differences between them.

Example 6.16. Blow up $\mathbb{A}^3$ along the curve $y^2 - x^3 + x = z = 0$ and compute the strict transform of the lines $x = z = 0$ and $y = z = 0$.

Example 6.17. Let I_1 and I_2 be ideals of $\mathbb{K}[x]$ of order c_1 and c_2 at 0. Take the blowup of $\mathbb{A}^n$ at zero. Show that the weak transform of $I_1^{c_2} + I_2^{c_1}$ is the sum of the weak transforms of $I_1^{c_2}$ and $I_2^{c_1}$.

Example 6.18. For $\dim(W) = 3$, a flag at a point a consists of a regular curve F_1 through a and contained in a regular surface F_2. Blow up the point a in W, so that $E \simeq \mathbb{P}^2$ is the projective plane. At the intersection point p_1 of the strict transform $C_1 = F_1^s$ of F_1 with E, the transformed flag $\mathcal{F}'$ is given by $F_1^s \subset F_2^s$. Along the intersection C_2 of F_2^s with E, the flag $\mathcal{F}'$ is given at each point p_2 different from p_1 by $C_2 \subset F_2^s$. At any point p_3 not on C_2 the flag $\mathcal{F}'$ is given by $C_3 \subset E$, where C_3 is the projective line in E through p_1 and p_3.

Example 6.19. Blowing up a regular curve Z in W transversal to $\mathcal{F}$ there occur six possible configurations of Z with respect to $\mathcal{F}$. Denoting by L the plane in the tangent space T_aW of W at a corresponding to the point a' in E above a, these are: (1) $Z = F_1$ and $L = \mathrm{T}_aF_2$, (2) $Z = F_1$ and $L \neq \mathrm{T}_aF_2$, (3) $Z \neq F_1$, $Z \subset F_2$ and $L = \mathrm{T}_aF_2$, (4) $Z \neq F_1$, $Z \subset F_2$ and $L \neq \mathrm{T}_aF_2$, (5) $Z \not\subset F_2$ and $\mathrm{T}_aF_1 \subset L$, (6) $Z \not\subset F_2$ and $\mathrm{T}_aF_1 \not\subset L$. Determine in each case the flag $\mathcal{F}'$.

Example 6.20. Let $\widetilde{\mathbb{A}}^n \to \mathbb{A}^n$ be the blowup of $\mathbb{A}^n$ in 0 and let a' be the origin of the x_n-chart of $\widetilde{\mathbb{A}}^n$. Compute the total and strict transforms g^* and g^s for $g = x_1^d + \ldots + x_{n-1}^d + x_n^e$ for $e = d, 2d-1, 2d, 2d+1$ and $g = \prod_{i \neq j}(x_i - x_j)$.

Example 6.21. Determine the total, weak and strict transform of $X = V(x^2 - y^3, z^3) \subset \mathbb{A}^3$ under the blowup of $\mathbb{A}^3$ at the origin. Point out the geometric differences between the three types of transforms.

Example 6.22. Compute the strict transform of the ideal $(x^2 - y^3, xy - z^3)$ under the blowup of $\mathbb{A}^3$ at the origin.

Example 6.23. The inclusion $I^\curlyvee \subset I^s$ can be strict. Blow up $\mathbb{A}^2$ at 0, and consider in the y-chart of $\widetilde{\mathbb{A}}^2$ the transforms of $I = (x^2, y^3)$. Show that $I^\curlyvee = (x^2, y)$, $I^s = (x^2, 1) = \mathbb{K}[x, y]$.

Example 6.24. Let $X = V(f)$ be a hypersurface in $\mathbb{A}^n$ and Z a regular closed subvariety which is contained in the locus of points of X where f attains its maximal order. Let $\pi : \widetilde{\mathbb{A}}^n \to \mathbb{A}^n$ be the blowup along Z and let $X' = V(f')$ be the strict transform of X. Show that for points $a \in Z$ and $a' \in E$ with $\pi(a') = a$ the inequality $\mathrm{ord}_{a'} f' \leq \mathrm{ord}_a f$ holds.

Example 6.25. Find an example of a variety X for which the dimension of the singular locus increases under blowup of a closed regular center Z that is contained in the singular locus of X [Hau98].

7. Lecture VII: Resolution Statements

Definition 7.1. A *non-embedded resolution* of a variety X is a non-singular variety $\widetilde{X}$ together with a proper birational morphism $\pi : \widetilde{X} \to X$ which is a regular isomorphism $\pi : \widetilde{X} \setminus E \to X \setminus \mathrm{Sing}(X)$ outside $E = \pi^{-1}(\mathrm{Sing}(X))$.

Remark 7.2. Requiring properness implies the surjectivity of π and excludes trivial cases as e.g. taking for $\widetilde{X}$ the locus of regular points of X and for π the inclusion map. One may ask for additional properties: (a) Any automorphism $\varphi : X \to X$ of X shall lift to an automorphism $\widetilde{\varphi} : \widetilde{X} \to \widetilde{X}$ of $\widetilde{X}$ which commutes with π, say $\pi \circ \widetilde{\varphi} = \varphi \circ \pi$. (b) If X is defined over $\mathbb{K}$ and $\mathbb{K} \subset \mathbb{L}$ is a field extension, any resolution of $X_{\mathbb{L}}$ should induce a resolution of $X = X_{\mathbb{K}}$.

DEFINITION 7.3. A *local non-embedded resolution* of a variety X at a point a is the germ $(\widetilde{X}, a')$ of a non-singular variety $\widetilde{X}$ together with a local morphism $\pi : (\widetilde{X}, a') \to (X, a)$ inducing an isomorphism of the function fields of $\widetilde{X}$ and X.

DEFINITION 7.4. Let X be an affine irreducible variety with coordinate ring $R = \mathbb{K}[X]$. A (local, ring-theoretic) *non-embedded resolution* of X is a ring extension $R \hookrightarrow \widetilde{R}$ of R into a regular ring $\widetilde{R}$ having the same quotient field as R.

DEFINITION 7.5. Let X be a subvariety of a regular ambient variety W. An *embedded resolution* of X consists of a proper birational morphism $\pi : \widetilde{W} \to W$ from a regular variety $\widetilde{W}$ onto W which is an isomorphism over $W \setminus \mathrm{Sing}(X)$ such that the total transform $X^* = \pi^{-1}(X)$ of X is a subvariety of $\widetilde{W}$ with normal crossings.

DEFINITION 7.6. A *strong resolution* of a variety X is, for each closed embedding of X into a regular variety W, a birational proper morphism $\pi : \widetilde{W} \to W$ satisfying the following five properties [EH02]:

Embeddedness. The variety $\widetilde{W}$ is regular and the total transform $X^* = \pi^{-1}(X)$ of X in $\widetilde{W}$ has simple normal crossings.

Equivariance. Let $W' \to W$ be a smooth morphism and let X' be the inverse image of X in W'. The morphism $\pi' : \widetilde{W}' \to W'$ induced by $\pi : \widetilde{X} \to W$ by taking fiber product of $\widetilde{X}$ and W' over W is an embedded resolution of X'. This implies the economy of the resolution and also that π commutes with open immersions, localization, completion, automorphisms of W stabilizing X and taking cartesian products with regular varieties.

Excision. The restriction $\tau : \widetilde{X} \to X$ of π to X does not depend on the choice of the embedding of X in W.

Explicitness. The morphism π is a composition of blowups along regular centers transversal to the exceptional loci of the previous sequence blowups.

Effectiveness. There exists, for all varieties X, a local upper semicontinuous invariant $\mathrm{inv}_a(X)$ in a well-ordered set Γ which attains its minimal value if and only if X is regular (or a normal crossings variety) and such that the top locus S of $\mathrm{inv}_a(X)$ is closed and regular in X and blowing up X along S makes $\mathrm{inv}_a(X)$ drop all points a' above $a \in S$.

REMARK 7.7. One may require in addition that the centers of a resolution are transversal to the inverse images of a given normal crossings divisor E in W. This is known as the *boundary condition.*

DEFINITION 7.8. Let I be an ideal on a regular ambient variety W. A *log-resolution* of I is a proper birational morphism $\pi : \widetilde{W} \to W$ from a regular variety $\widetilde{W}$ onto W which is an isomorphism over $W \setminus \mathrm{Sing}(I)$ such that $I^* = \pi^{-1}(I)$ is a locally monomial ideal on $\widetilde{W}$.

EXAMPLE 7.9. The blowup of the cusp $X : x^2 = y^3$ in $\mathbb{A}^2$ at 0 produces a non-embedded resolution. Further blouwps are necessary to make it to an embedded resolution.

Example 7.10. Determine the geometry at 0 of the hypersurfaces in $\mathbb{A}^4$ defined by the following equations:

(a) $x + x^7 - 3yw + y^7z^2 + 17yzw^5 = 0$,

(b) $x + x^5y^2 - 3yw + y^7z^2 + 17yzw^5 = 0$,

(c) $x^3y^2 + x^5y^2 - 3yw + y^7z^2 + 17yzw^5 = 0$.

For the first two equation, the implicit function theorem shows that the zerosets are regular at 0 and gives a formal parametrization. The zeroset of the third equation is singular at 0, and one cannot use the theorem to describe it at 0.

Example 7.11. Let Z be the variety in $\mathbb{A}^3_{\mathbb{C}}$ given by the equation $(x^2 - y^3)^2 = (z^2 - y^2)^3$. Show that the map $\alpha : \mathbb{A}^3 \to \mathbb{A}^3 : (x, y, z) \to (u^2z^3, uyz^2, uz^2)$, with $u(x, y) = x(y^2 - 1) + y$, resolves the singularities of Z. What is the inverse image $\alpha^{-1}(Z)$? Produce instructive pictures over $\mathbb{R}$.

Example 7.12. Consider the inverse images of the cusp $X = V(y^2 - x^3)$ in $\mathbb{A}^2$ under the maps $\pi_x, \pi_y : \mathbb{A}^2 \to \mathbb{A}^2$, $\pi_x(x, y) = (x, xy)$, $\pi_y(x, y) = (xy, y)$. Factor the maximal power of x, respectively y, from the equation of the inverse image of X and show that the resulting equation defines in both cases a regular variety. Apply the same process to the variety $E_8 = V(x^2 + y^3 + z^5) \subset \mathbb{A}^3$ repeatedly until all resulting equations define non-singular varieties.

Example 7.13. Let R be the coordinate ring of an irreducible plane algebraic curve X. The integral closure $\widetilde{R}$ of R in the field of fractions of R is a regular ring and thus resolves R. The resulting curve $\widetilde{X}$ is the normalization of X [Mum99] III.8, [dJP00] 4.4.

Example 7.14. Consider a cartesian product $X = Y \times Z$ with Z a regular variety. Show that a resolution of X can be obtained from a resolution Y' of Y by taking the cartesian product $Y' \times Z$ of Y' with Z.

Example 7.15. Let X and Y be two varieties (schemes, analytic spaces) with singular loci $\mathrm{Sing}(X)$ and $\mathrm{Sing}(Y)$ respectively. Suppose that X' and Y' are regular varieties (schemes, analytic spaces, within the same category as X and Y) together with proper birational morphisms $\pi : X' \to X$ and $\tau : Y' \to Y$ which define resolutions of X and Y respectively, and which are isomorphisms outside $\mathrm{Sing}(X)$ and $\mathrm{Sing}(Y)$. There is a uniquely determined proper birational morphism $f : X' \times Y' \to X \times Y$. So $X' \times Y'$ gives rise to a resolution of $X \times Y$.

8. Lecture VIII: Invariants of Singularities

Definition 8.1. A *stratification* of an algebraic variety X is a decomposition of X into finitely many disjoint locally closed subvarieties X_i, called the *strata*,

$$X = \dot{\bigcup_i} X_i,$$

such that the *boundaries* $\overline{X}_i \setminus X_i$ of strata are unions of strata. This last property is called the *frontier condition.* Two strata are called *adjacent* if one lies in the closure of the other.

Definition 8.2. A *local invariant* on an algebraic variety X is a function $\mathrm{inv}(X) : X \to \Gamma$ from X to a well-ordered set $(\Gamma, \leq)$ which associates to each point $a \in X$ an element $\mathrm{inv}_a(X)$ depending only on the formal isomorphism class of X at

a: If (X, a) and (X, b) are formally isomorphic, viz $\widehat{\mathcal{O}}_{X,a} \simeq \widehat{\mathcal{O}}_{X,b}$, then $\mathrm{inv}_a(X) = \mathrm{inv}_b(X)$. Usually, the ordering on Γ will also be total: for any $c, d \in \Gamma$ either $c \leq d$ or $d \leq c$ holds. The invariant is *upper semicontinuous along a subvariety* S of X if for all $c \in \Gamma$, the sets

$$\mathrm{top}_S(\mathrm{inv}, c) = \{a \in S, \, \mathrm{inv}_a(X) \geq c\}$$

are closed in S. If $S = X$, the map $\mathrm{inv}(X)$ is called *upper semicontinuous.*

Remark 8.3. The upper semicontinuity signifies that the value of $\mathrm{inv}(X)$ can only go up or remain the same when passing to a limit point. In the case of schemes, the value of $\mathrm{inv}(X)$ has also to be taken into account at non-closed points of X.

Definition 8.4. Let X be a subvariety of a not necessarily regular ambient variety W defined by an ideal I, and let Z be an irreducible subvariety of W defined by the prime ideal J. The *order* of X or I in W along Z or J is the maximal integer $k = \mathrm{ord}_Z(X) = \mathrm{ord}_Z(I)$ such that $I_Z \in J_Z^k$, where $I_Z = I \cdot \mathcal{O}_{W,Z}$ and $J_Z = J \cdot \mathcal{O}_{W,Z}$ denote the ideals generated by I and J in the localization $\mathcal{O}_{W,Z}$ of W along Z. If $Z = \{a\}$ is a point of W, the order of X and I at a is denoted by $\mathrm{ord}_a(X) = \mathrm{ord}_a(I)$ or $\mathrm{ord}_{m_a}(X) = \mathrm{ord}_{m_a}(I)$.

Definition 8.5. Let R be a local ring with maximal ideal m. Let $k \in \mathbb{N}$ be an integer. The k-th symbolic power $J^{(k)}$ of a prime ideal J is defined as the ideal generated by all elements $x \in R$ for which there is an element $y \in R \setminus J$ such that $y \cdot x^k \in J^k$. Equivalently, $J^{(k)} = J^k \cdot R_J \cap R$.

Remark 8.6. The symbolic power is the smallest J-primary ideal containing J^k. If J is a complete intersection, the ordinary power J^k and the symbolic power $J^{(k)}$ coincide [ZS75] IV, §12, [Pel88].

Proposition 8.7. Let X be a subvariety of a not necessarily regular ambient variety W defined by an ideal I, and let Z be an irreducible subvariety of W defined by the prime ideal J. The order of X along Z is the maximal integer k such that $I \subset J^{(k)}$.

Proof. This follows from the equality $J^k \cdot R_J = (J \cdot R_J)^k$. □

Proposition 8.8. Let R be a noetherian local ring with maximal ideal m, and let $\widehat{R}$ denote its completion with maximal ideal $\widehat{m} = m \cdot \widehat{R}$. Let I be an ideal of R, with completion $\widehat{I} = I \cdot \widehat{R}$. Then $\mathrm{ord}_m(I) = \mathrm{ord}_{\widehat{m}}(\widehat{I})$.

Proof. If $I \subset m^k$, then also $\widehat{I} \subset \widehat{m}^k$, and hence $\mathrm{ord}_m(I) \leq \mathrm{ord}_{\widehat{m}}(\widehat{I})$. Conversely, $m = \widehat{m} \cap R$ and $\bigcap_{i \geq 0}(I + m^i) = \widehat{I} \cap R$ by Lem. 2.25. Hence, if $\widehat{I} \subset \widehat{m}^k$, then $I \subset \widehat{I} \cap R \subset \widehat{m}^k \cap R = m^k$, so that $\mathrm{ord}_m(I) \geq \mathrm{ord}_{\widehat{m}}(\widehat{I})$. □

Definition 8.9. Let X be a subvariety of a regular variety W, and let a be a point of W. The *local top locus* $\mathrm{top}_a(X)$ of X at a with respect to the order is the stratum S of points of an open neighborhood U of a in W where the order of X equals the order of X at a. The *top locus* $\mathrm{top}(X)$ of X with respect to the order is the (global) stratum S of points of W where the order of X attains its maximal value. For $c \in \mathbb{N}$, define $\mathrm{top}_a(X, c)$ and $\mathrm{top}(X, c)$ as the local and global stratum of points of W where the order of X is at least c.

Remark 8.10. The analogous definition holds for ideals on W and can be made for other local invariants. By the upper semicontinuity of the order, the local top locus of X at a is locally closed in W, and the top locus of X is closed in W.

Proposition 8.11. The order of a variety X or an ideal I of a regular variety W at points of W defines an upper semicontinuous local invariant on W.

Proof. In characteristic zero, the assertion follows from the next proposition. For the case of positive characteristic, see [Hir64] III.3, Cor. 1, p. 220. $\square$

Proposition 8.12. Over fields of zero characteristic, the local top locus $\mathrm{top}_a(I)$ of an ideal I at a is defined by the vanishing of all partial derivatives of elements of I up to order $o-1$, where o is the order of I at a.

Proof. In zero characteristic, a polynomial has order o at a point a if and only if all its partial derivatives up to order $o-1$ vanish at a. $\square$

Proposition 8.13. Let X be a subvariety of a regular ambient variety W. Let Z be a non-singular subvariety of W and a a point on Z such that locally at a the order of X is constant along Z, say equal to $d = \mathrm{ord}_a X = \mathrm{ord}_Z X$. Consider the blowup $\pi : W' \to W$ of W along Z with exceptional divisor $E = \pi^{-1}(Z)$. Let a' be a point on E mapping under π to a. Denote by X^*, $X^\curlyvee$ and X^s the total, weak and strict transform of X, respectively. Then, locally at a', the order of X^* along E is d, and

$$\mathrm{ord}_{a'} X^s \leq \mathrm{ord}_{a'} X^\curlyvee \leq d.$$

Proof. By Prop. 5.4 there exist local coordinates $x_1, \ldots, x_n$ of W at a such that Z is defined locally at a by $x_1 = \ldots = x_k = 0$ for some $k \leq n$, and such that a' is the origin of the x_1-chart of the blowup. Let $I \subset \mathcal{O}_{W,a}$ be the local ideal of X in W, and let f be an element of I. It has an expansion $f = \sum_{\alpha\in\mathbb{N}^n} c_\alpha x^\alpha$ with respect to the coordinates $x_1, \ldots, x_n$, with coefficients $c_\alpha \in \mathbb{K}$.

Set $\alpha_+ = (\alpha_1, \ldots, \alpha_k)$. Since the order of X is constant along Z, the inequality $|\alpha_+| = \alpha_1 + \ldots + \alpha_k \geq d$ holds whenever $c_\alpha \neq 0$. There is an element f with an exponent α such that $c_\alpha \neq 0$ and such that $|\alpha| = |\alpha_+| = d$. The total transform X^* and the weak transform $X^\curlyvee$ are given locally at a' by the ideal generated by all

$$f^* = \sum_{\alpha\in\mathbb{N}^n} c_\alpha x_1^{|\alpha_+|} x_2^{\alpha_2} \ldots x_n^{\alpha_n},$$

respectively

$$f^\curlyvee = \sum_{\alpha\in\mathbb{N}^n} c_\alpha x_1^{|\alpha_+|-d} x_2^{\alpha_2} \ldots x_n^{\alpha_n},$$

for f varying on I. The exceptional divisor E is given locally at a' by the equation $x_1 = 0$. This implies that, locally at a', $\mathrm{ord}_E f^* \geq d$ for all $f \in I$ and $\mathrm{ord}_E f^* = d$ for the special f chosen above with $|\alpha| = |\alpha_+| = d$. Therefore $\mathrm{ord}_E X^* = d$ locally at a'.

Since $\mathrm{ord}_{a'} f^\curlyvee \leq |\alpha_-| \leq d$ for the chosen f, it follows that $\mathrm{ord}_{a'} X^\curlyvee \leq d$. The ideal I^s of the strict transform X^s contains the ideal $I^\curlyvee$ of the weak transform $X^\curlyvee$, thus also $\mathrm{ord}_{a'} X^s \leq \mathrm{ord}_{a'} X^\curlyvee$. $\square$

Corollary 8.14. If the order of X is globally constant along Z, the order of X^* along E is globally equal to d.

Definition 8.15. Let X be a subvariety of a regular variety W, and let $W' \to W$ be a blowup with center Z. Denote by X' the strict or weak transform in W'. A point $a' \in W'$ above $a \in Z$ is called *infinitesimally near to a* or *equiconstant* if $\mathrm{ord}_{a'} X' = \mathrm{ord}_a X$.

Definition 8.16. Let X be a variety defined over a field $\mathbb{K}$, and let a be a point of W. The *Hilbert-Samuel function* $\mathrm{HS}_a(X) : \mathbb{N} \to \mathbb{N}$ of X at a is defined by

$$\mathrm{HS}_a(X)(k) = \dim_{\mathbb{K}}(m_{X,a}^k / m_{X,a}^{k+1}),$$

where $m_{X,a}$ denotes the maximal ideal of the local ring $\mathcal{O}_{X,a}$ of X at a. If X is a subvariety of a regular variety W defined by an ideal I, with local ring $\mathcal{O}_{X,a} = \mathcal{O}_{W,a}/I$, one also writes $\mathrm{HS}_a(I)$ for $\mathrm{HS}_a(X)$.

Remark 8.17. The Hilbert-Samuel function does not depend on the embedding of X in W. There exists a univariate polynomial $P(t) \in \mathbb{Q}[t]$, called the *Hilbert-Samuel polynomial* of X at a, such that $HS_a(X)(k) = P(k)$ for all sufficiently large k [Ser00].

Theorem 8.18. The Hilbert-Samuel function of a subvariety X of a regular variety W defines an upper semicontinuous local invariant on X with respect to the lexicographic ordering of integer sequences.

Proof. [Ben70] Thm. 4, p. 82. □

Theorem 8.19. Let $\pi : W' \longrightarrow W$ be the blowup of W along a non-singular center Z. Let I be an ideal of $\mathcal{O}_W$. Assume that the Hilbert-Samuel function of I is constant along Z, and denote by I^s the strict transform of I in W'. Let $a' \in E$ be a point in the exceptional divisor mapping under π to a. Then

$$\mathrm{HS}_{a'}(I^s) \leq \mathrm{HS}_a(I)$$

holds with respect to the lexicographic ordering of integer sequences.

Proof. [Ben70] Thm. 0. □

Theorem 8.20. (Zariski-Nagata, [Hir64] III.3, Thm. 1, p. 218) Let $S \subset T$ be closed irreducible subvarieties of a closed subvariety X of a regular ambient variety W. The order of X along T is less than or equal to the order of X along S.

Remark 8.21. In the case of schemes, the assertion says that if X is embedded in a regular ambient scheme W, and a, b are points of X such that a lies in in the closure of b, then $\mathrm{ord}_b X \leq \mathrm{ord}_a X$.

Proof. The proof goes in several steps and relies on the non-embedded resolution of curves. Let R be a regular local ring, m its maximal ideal, p a prime ideal in R and $I \neq 0$ a non-zero ideal in R. Denote by $\nu_p(I)$ the maximal integer $\nu \geq 0$ such that $I \subset p^\nu$. Recall that $\mathrm{ord}_p(I)$ is the maximal integer n such that $IR_p \subset p^n R_p$ or, equivalently, $I \subset p^{(\nu)}$, where $p^{(\nu)} = p^\nu R_p \cap R$ denotes the ν-th symbolic power of p. Thus $\nu_p(I) \leq \mathrm{ord}_p(I)$ and the inequality can be strict if p is not a complete intersection, in which case the usual and the symbolic powers of p may differ. Write $\nu(I) = \nu_m(I)$ for the maximal ideal m so that $\nu_p(I) \leq \nu(I)$ and $\nu_p(I) \leq \nu(IR_p) = \mathrm{ord}_p(I)$.

The assertion of the theorem is equivalent to showing that $\nu(IR_p) \leq \nu(I)$ taking for R the local ring $\mathcal{O}_{W,S}$ of W along S, for I the ideal of R defining X in W along S and for p the ideal defining T in W along S.

If R/p is regular, then $\nu_p(I) = \nu(IR_p)$ because $p^n R_p \cap R = p^n$: The inclusion $p^n \subseteq p^n R_p \cap R$ is clear, so suppose that $x \in p^n R_p \cap R$. Choose $y \in p^n$ and $s \notin p$ such that $xs = y$. Then $n \leq \nu_p(y) = \nu_p(xs) = \nu_p(x) + \nu_p(s) = \nu_p(x)$, hence $x \in p^n$. It follows that $\nu(IR_p) \leq \nu(I)$.

It remains to prove the inequality in the case that R/p is not regular. Since R is regular, there is a chain of prime ideals $p = p_0 \subset \ldots \subset p_k = m$ in R with $\dim(R_{p_i}/p_{i-1}R_{p_i}) = 1$ for $1 \leq i \leq n$. By induction on the dimension of R/p, it therefore suffices to prove the inequality in the case $\dim(R/p) = 1$. Since the order remains constant under completion of a local ring by Prop. 8.8, it can be assumed that R is complete.

By the embedded resolution of curve singularities there exists a sequence of complete regular local rings $R = R_0 \to R_1 \to \ldots \to R_k$ with prime ideals $p_0 = p$ and $p_i \subset R_i$ with the following properties:

- R_{i+1} is a monoidal transform of R_i with center the maximal ideal m_i of R_i.
- $(R_{i+1})_{p_{i+1}} = (R_i)_{p_i}$.
- R_k/p_k is regular.

Set $I_0 = I$ and let I_{i+1} be the weak transform of I_i under $R_i \to R_{i+1}$. Set $R' = R_k$, $I' = I_k$ and $p' = p_k$. The second condition on the blowups R_i implies $\nu(IR_p) = \nu(I'R'_{p'})$. By Prop. 8.13 one knows that $\nu(I') \leq \nu(I)$. Further, since R'/p' is regular, $\nu(I'R'_{p'}) \leq \nu(I')$. Combining these inequalities yields $\nu(IR_p) \leq \nu(I)$. □

Proposition 8.22. An upper semicontinuous local invariant $\mathrm{inv}(X) : X \to \Gamma$ with values in a well and totally ordered set Γ induces, up to refinement, a stratification of X with strata $X_c = \{a \in X, \mathrm{inv}_a(X) = c\}$, for $c \in \Gamma$.

Proof. For a given value $c \in \Gamma$, let $S = \{a \in X, \mathrm{inv}_a(X) \geq c\}$ and $T = \{a \in X, \mathrm{inv}_a(X) = c\}$. The set S is a closed subset of X. If c is a maximal value of the invariant on X, the set T equals S and is thus a closed stratum. If c is not maximal, let $c' > c$ be an element of Γ which is minimal with $c' > c$. Such elements exist since Γ is well-ordered. As Γ is totally ordered, c' is unique. Therefore

$$S' = \{a \in X, \mathrm{inv}_a(X) \geq c'\} = \{a \in X, \mathrm{inv}_a(X) > c\} = S \setminus T$$

is closed in X. Therefore T is open in S and hence locally closed in X. As S is closed in X and $T \subset S$, the closure $\overline{T}$ is contained in S. It follows that the boundary $\overline{T} \setminus T$ is contained in S' and closed in X. Refine the strata by replacing S' by $\overline{T} \setminus T$ and $S' \setminus \overline{T}$ to get a stratification. □

Example 8.23. The singular locus $S = \mathrm{Sing}(X)$ of a variety is closed and properly contained in X. Let $X_1 = \mathrm{Reg}(X)$ be the set of regular points of X. It is open and dense in X. Repeat the procedure with $X \setminus X_1 = \mathrm{Sing}(X)$. By noetherianity, the process eventually stops, yielding a stratification of X in regular strata. The strata of locally minimal dimension are closed and non-singular. The frontier condition holds, since the regular points of a variety are dense in the variety, and hence $\overline{X}_1 = X = X_1 \dot\cup \mathrm{Sing}(X)$.

Example 8.24. There exists a threefold X whose singular locus $\mathrm{Sing}(X)$ consists of two components, a non-singular surface and a singular curve meeting the surface at a singular point of the curve. The stratification by iterated singular loci satisfies the frontier condition.

EXAMPLE 8.25. Give an example of a variety whose stratification by the iterated singular loci has four different types of strata.

EXAMPLE 8.26. Find interesting stratifications for three three-folds.

EXAMPLE 8.27. The stratification of an upper semicontinuous invariant need not be finite. Take for Γ the set X underlying a variety X with the trivial (partial) ordering $a \leq b$ if and only if $a = b$.

EXAMPLE 8.28. Let X be the non-reduced scheme defined by $xy^2 = 0$ in $\mathbb{A}^2$. The order of X at points of the y-axis outside 0 is 1, at points of the x-axis outside 0 it is 2, and at the origin it is 3. The local embedding dimension equals 1 at points of the y-axis outside 0, and 2 at all points of the x-axis.

EXAMPLE 8.29. Determine the stratification given by the order for the following varieties. If the smallest stratum is regular, blow it up and determine the stratification of the strict transform. Produce pictures and describe the geometric changes.
(a) Cross: $xyz = 0$,
(b) Whitney umbrella: $x^2 - yz^2 = 0$,
(c) Kolibri: $x^3 + x^2z^2 - y^2 = 0$,
(d) Xano: $x^4 + z^3 - yz^2 = 0$,
(e) Cusp & Plane: $(y^2 - x^3)z = 0$.

EXAMPLE 8.30. Consider the order $\mathrm{ord}_Z I$ of an ideal I in $\mathbb{K}[x_1, \ldots, x_n]$ along a closed subvariety Z of $\mathbb{A}^n$, defined as the order of I in the localization of $\mathbb{K}[x_1, \ldots, x_n]_J$ of $K[x_1, \ldots, x_n]$ at the ideal J defining Z in $\mathbb{A}^n$. Express this in terms of the symbolic powers $J^{(k)} = J^k \cdot \mathbb{K}[x_1, \ldots, x_n]_J \cap \mathbb{K}[x_1, \ldots, x_n]$ of J. Give an example to show that $\mathrm{ord}_Z I$ need not coincide with the maximal power k such that $I \subset J^k$. If J defines a complete intersection, the ordinary powers J^k and the symbolic powers $J^{(k)}$ coincide. This holds in particular when Z is a coordinate subspace of $\mathbb{A}^n$ [ZS75] IV, §12, [Pel88].

EXAMPLE 8.31. [2] The ideal $I = (y^2 - xz, yz - x^3, z^2 - x^2y)$ of $\mathbb{K}[x, y, z]$ has symbolic square $I^{(2)}$ strictly containing I^2.

EXAMPLE 8.32. The variety defined by $I = (y^2 - xz, yz - x^3, z^2 - x^2y)$ in $\mathbb{A}^3$ has parametrization $t \mapsto (t^3, t^4, t^5)$. Let $f = x^5 + xy^3 + z^3 - 3x^2yz$. For all maximal ideals m which contain I, $f \in m^2$ and thus, $\mathrm{ord}_m f \geq 2$. But $f \notin I^2$. Since $xf \in I^2$ and x does not belong to I, it follows that $f \in I^2 R_I$ and thus $\mathrm{ord}_I f \geq 2$, in fact, $\mathrm{ord}_I f = 2$.

EXAMPLE 8.33. The order of an ideal depends on the embedding of X in W. If X is not minimally embedded locally at a, the order of X at a is 1 and not significant.

EXAMPLE 8.34. Associate to a stratification of a variety the so called *Hasse diagram*, i.e., the directed graph whose nodes and edges correspond to strata, respectively to the adjacency of strata. Determine the Hasse diagram for the surface X given in $\mathbb{A}^4$ as the cartesian product of the cusp $C : x^2 = y^3$ with the node $D : x^2 = y^2 + y^3$. Then project X to $\mathbb{A}^3$ by means of $\mathbb{A}^4 \to \mathbb{A}^3, (x, y, z, w) \mapsto (x, y+z, w)$ and compute the Hasse diagram of the image Y of X under this projection.

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[2] This example, due to Macaulay, was kindly communicated by M. Hochster.

Example 8.35. Show that the order of a hypersurface, the dimension and the Hilbert-Samuel function of a variety, the embedding-dimension of a variety and the local number of irreducible components are invariant under local formal isomorphisms, and determine whether they are upper or lower semicontinuous. How does each of these invariants behave under localization and completion?

Example 8.36. Take $\mathrm{inv}_a(X) = \dim_a(X)$, the dimension of X at a. It is constant on irreducible varieties, and upper semicontinuous on arbitrary ones, because at an intersection point of several components, $\dim_a(X)$ is the maximum of the dimensions of the components.

Example 8.37. Take $\mathrm{inv}_a(X) =$ the number of irreducible components of X passing through a. If the components carry multiplicities as e.g. a divisor, one may alternatively take the sum of the multiplicities of the components passing through a. Both options produce an upper semicontinuous local invariant.

Example 8.38. Take $\mathrm{inv}_a(X) = \mathrm{embdim}_a(X) = \dim \mathrm{T}_a X$, the local embedding dimension of X at a. It is upper semicontinuous. At regular points, it equals the dimension of X at a, at singular points it exceeds this dimension.

Example 8.39. Consider for a given coordinate system $x, y_1, \ldots, y_n$ on $\mathbb{A}^n$ a polynomial of order c at 0 of the form

$$g(x, y_1, \ldots, y_n) = x^c + \sum_{i=0}^{c-1} g_i(y) \cdot x^i.$$

Express the order and the top locus of g nearby 0 in terms of the orders of the coefficients g_i.

Example 8.40. Take $\mathrm{inv}_a(X) = \mathrm{HS}_a(X)$, the Hilbert-Samuel function of X at a. The lexicographic order on integer sequences defines a well-ordering on $\Gamma = \{\gamma : \mathbb{N} \to \mathbb{N}\}$. Find two varieties X and Y with points a and b where $\mathrm{HS}_a(X)$ and $\mathrm{HS}_b(Y)$ only differ from the fourth entry on.

Example 8.41. Take $\mathrm{inv}_a(X) = \nu_a^*(X)$ the increasingly ordered sequence of the orders of a minimal Macaulay basis of the ideal I defining X in W at a [Hir64] III.1, def. 1 and Lem. 1, p. 205. It is upper semicontinuous but does not behave well under specialization [Hir64] III.3, Thm. 2, p. 220 and the remark after Cor. 2, p. 220, see also [Hau98], ex. 12.

Example 8.42. Let be given a monomial order $<_\varepsilon$ on $\mathbb{N}^n$, i.e., a total ordering with minimal element 0 which is compatible with addition in $\mathbb{N}^n$. The *initial ideal* $\mathrm{in}(I)$ of an ideal I of $\mathbb{K}[[x_1, \ldots, x_n]]$ with respect to $<_\varepsilon$ is the ideal generated by all *initial monomials* of elements f of I, i.e., the monomials with minimal exponent with respect to $<_\varepsilon$ in the series expansion of f. The initial monomial of 0 is 0.

The initial ideal is a monomial ideal in $\mathbb{K}[[x_1, \ldots, x_n]]$ and depends on the choice of coordinates. If the monomial order $<_\varepsilon$ is compatible with degree, $\mathrm{in}(I)$ determines the Hilbert-Samuel function $\mathrm{HS}_a(I)$ of I [Hau04].

Order monomial ideals totally by comparing their increasingly ordered unique minimal monomial generator system lexicographically, where any two monomial generators are compared with respect to $<_\varepsilon$. If two monomial ideals have generator systems of different length, complete the sequences of their exponents by a symbol

∞ so as to be able to compare them. This defines a well-order on the set of monomial ideals.

Take for $\mathrm{inv}_a(X)$ the minimum $\min(I)$ or the maximum $\max(I)$ of the initial ideal of the ideal I of X at a, the minimum and maximum being taken over all choices of local coordinates, say regular parameter systems of $\mathbb{K}[[x_1, \ldots, x_n]]$. Both exist and define local invariants which are upper semicontinuous with respect to localization [Hau04] Thms. 3 and 8.

(a) The minimal initial ideal $\min(I) = \min_x\{\mathrm{in}(I)\}$ over all choices of regular parameter systems of $\mathbb{K}[[x_1, \ldots, x_n]]$ is achieved for almost all regular parameter systems.

(b) * Let I be an ideal in $\mathbb{K}[x_1, \ldots, x_n]$ and denote by I_a the induced ideal in $\mathbb{K}[[x_1 - a_n, \ldots, x_n - a_n]]$. The minimal initial ideal $\min(I_a)$ is upper semicontinuous when the point a varies.

(c) Compare the induced stratification of $\mathbb{A}^n$ with the stratification by the Hilbert-Samuel function of I.

(d) Let Z be a regular center inside a stratum of the $\mathrm{in}(I)$ stratification, and consider the induced blowup $\widetilde{\mathbb{A}}^n \to \mathbb{A}^n$ along Z. Let $a \in Z$ and $a' \in W'$ be a point above a. Assume that the monomial order $<_\varepsilon$ is compatible with degree. Show that $\min_a(I)$ and $\max_a(I)$ do not increase when passing to the strict transform of I_a at a' [Hau04] Thm. 6.

Example 8.43. Let $(W', a') \to (W, a)$ be the composition of two monomial point blowups of $W = \mathbb{A}^2$ with respect to coordinates y, z, defined as follows. The first is the blowup of $\mathbb{A}^2$ with center 0 considered at the origin of the y-chart, the second has as center the origin of the y-chart and is considered at the origin of the z-chart. Show that the order of the weak transform $g'(y, z)$ at a' of any non zero polynomial $g(y, z)$ in W is at most the half of the order of $g(y, z)$ at a.

Example 8.44. * Let $(W', a') \to (W, a)$ be a composition of local blowups in regular centers such that a' lies in the intersection of n exceptional components where n is the dimension of W at a. Let $f \in \mathcal{O}_{W,a}$ and assume that the characteristic is zero. Show that the order of f has dropped between a and a'.

Example 8.45. * Show the same in positive characteristic.

Example 8.46. (B. Schober) Let $\mathbb{K}$ be a non perfect field of characteristic 3, let $t \in \mathbb{K} \setminus \mathbb{K}^2$ be an element which is not a square. Stratify the hypersurface $X : x^2 + y(z^2 + tw^2) = 0$ in $\mathbb{A}^4$ according to its singularities. Show that this stratification does not provide suitable centers for a resolution.

Example 8.47. Let $I = (x^2 + y^{17})$ be the ideal defining an affine plane curve singularity X with singular locus the origin of $W = \mathbb{A}^2$. The order of X at 0 is 2. The blowup $\pi : W' \to W$ of W at the origin with exceptional divisor E is covered by two affine charts, the x- and the y-chart. The total and strict transform of I in the x-chart are as follows:

$$\begin{aligned} I^* &= (x^2 + x^{17}y^{17}), \\ I^s &= (1 + x^{15}y^{17}). \end{aligned}$$

At the origin of this chart, the order of I^s has dropped to zero, so the strict transform X^s of X does not contain this point. Therefore it suffices to consider the complement of this point in E, which lies entirely in the y-chart. There, one obtains the following transforms:

$$\begin{aligned} I^* &= (x^2y^2 + y^{17}), \\ I^s &= (x^2 + y^{15}). \end{aligned}$$

The origin a' of the y-chart is the only singular point of X^s. The order of X^s at a' has remained constant equal to 2. Find a local invariant of X which has improved at a'. Make sure that it does not depend on any choices.

Example 8.48. The ideal $I = (x^2+y^{16})$ has in the y-chart the strict transform $I^s = (x^2 + y^{14})$. If the ground field has characteristic 2, the y-exponents 16 and 14 are irrelevant because the coordinate change $x \mapsto x + y^8$ transforms I into (x^2).

Example 8.49. The ideal $I = (x^2 + 2xy^7 + y^{14} + y^{17})$ has in the y-chart the strict transform $I^s = (x^2 + 2xy^6 + y^{12} + y^{15})$. Here the drop of the y-exponent of the last monomial from 17 to 15 is significant, whereas the terms $2xy^7 + y^{14}$ can be eliminated in any characteristic by the coordinate change $x \mapsto x + y^7$.

Example 8.50. The ideal $I = (x^2+xy^9) = (x)(x+y^9)$ defines the union of two non-singular curves in $\mathbb{A}^2$ which have a common tangent line at their intersection point 0. The strict transform is $I^s = (x^2 + xy^8)$. The degree of tangency, viz the intersection multiplicity, has decreased.

Example 8.51. Let X and Y be two non-singular curves in $\mathbb{A}^2$, meeting at one point a. Show that there exists a sequence of point blowups which separates the two curves, i.e., so that the strict transforms of X and Y do not intersect.

Example 8.52. Take $I = (x^2 + g(y))$ where g is a polynomial in y with order at least 3 at 0. The strict transform in the y-chart is $I^s = x^2 + y^{-2}g(y)$, with order at 0 equal to 2 again. This suggests to take the order of g as a secondary invariant. In characteristic 2 it may depend on the choice of coordinates.

Example 8.53. Take $I = (x^2 + xg(y) + h(y))$ where g and h are polynomials in y of order at least 1, respectively 2, at 0. The order of I at 0 is 2. The strict transform equals $I^s = (x^2 + xy^{-1}g(y) + y^{-2}h(y))$ of order 2 at 0. Here it is less clear how to detect a secondary invariant which represents an improvement.

Example 8.54. Take $I = (x^2 + y^3z^3)$ in $\mathbb{A}^3$, and apply the blowup of $\mathbb{A}^3$ in the origin. The strict transform of I in the y-chart equals $I^s = (x^2 + y^4z^3)$ and the singularity seems to have got worse.

Example 8.55. Let X be a surface in three-space, and S its top locus. Assume S is singular at a, and let X' be the blowup of X in a. Determine the top locus of X' [Zar44, Hau00].

9. Lecture IX. Hypersurfaces of Maximal Contact

Proposition 9.1. (Zariski) Let X be a subvariety of a regular variety W, defined over a field of arbitrary characteristic. Let $W' \to W$ be the blowup of W along a regular center Z contained in the top locus of X, and let a be a point of Z. There exists, in a neighbourhood U of a, a regular closed hypersurface V of U whose strict transform V^s in W' contains all points a' of W' lying above a where the order of the strict transform X^s of X has remained constant.

Proof. Choose local coordinates $x_1, \dots, x_n$ of W at a. The associated graded ring of $\mathcal{O}_{W,a}$ can be identified with $\mathbb{K}[x_1, \dots, x_n]$. Let $\mathrm{in}(I) \subset \mathbb{K}[x_1, \dots, x_n]$ denote the ideal of inital forms of elements of I at a. Apply a linear coordinate change so that generators of $\mathrm{in}(I)$ are expressed with the minimal number of variables, say $x_1, \dots, x_k$, for some $k \leq n$. Choose any $1 \leq i \leq k$ and define V in W at a by $x_i = 0$. It follows that the local top locus of X at a is contained in V. Hence $Z \subset V$, locally at a. Let a' be a point of W' above a where the order of X has remained constant. By Prop. 5.4 the local blowup $(W', a') \to (W, a)$ can be made monomial by a suitable coordinate change. The assertion then follows by computation, cf. ex. 9.8 and [Zar44]. $\square$

Definition 9.2. Let X be a subvariety of a regular variety W, and let a be a point of W. A *hypersurface of maximal contact for X at a* is a regular closed hypersurface V of an open neighborhood U of a in W such that

(1) V contains the local top locus S of X at a, i.e., the points of U where the order of X equals the order of X at a.

(2) The strict transform V^s of V under any blowup of U along a regular center Z contained in S contains all points a' above a where the order of X^s has remained constant equal to the order of X at a, i.e., V is a directrix of X at a.

(3) Property (2) is preserved in any sequence of blowups with regular centers contained in the successive top loci of the strict transforms of X along which the order of X has remained constant.

$$\begin{array}{ccccccc} a' \in & E & \cap & V^s & \subset & W' \\ \downarrow & \downarrow & & & & \downarrow \pi \\ a \in & Z & \subset & V & \subset & W \end{array}$$

Definition 9.3. Assume that the characteristic of the ground field is zero. Let X be a subvariety of a regular variety W defined locally at a point a of W by the ideal I. Let o be the order of X at a. An *osculating hypersurface for X at a* is a regular closed hypersurface V of a neighbourhood U of a in W defined by a derivative of order $o-1$ of an element f of I [EH02].

Remark 9.4. The element f has necessarily order o at a, and its $(o-1)$-st derivative has order 1 at a, so that it defines a regular hypersurface at a. The concept is due to Abhyankar and Zariski [AZ55]. Abhyankar called the local isomorphism constructing an osculating hypersurface V from a given regular hypersurface H of W *Tschirnhaus transformation.* If H is given by $x_n = 0$, this transformation eliminates from f all monomials whose x_n-exponent is $o-1$. The concept was exploited by Hironaka in his proof of characteristic zero resolution [Hir64].

For each point a in X, osculating hypersurfaces contain locally at a the local top locus $S = \mathrm{top}_a(X)$ and the equiconstant points above.

Proposition 9.5. (Abhyankar, Hironaka) Let X be a subvariety of a regular variety W, and let a be a point of W. For ground fields of characteristic zero there exist, locally at a in W, hypersurfaces of maximal contact for X. Any osculating hypersurface V at a has maximal contact with X at a.

Proof. [EH02]. $\square$

Remark 9.6. The assertion of the proposition does not hold over fields of positive characteristic: Narasimhan gave an example of a hypersurface X in $\mathbb{A}^4$ over a field of characteristic 2 whose top locus is not contained at 0 in any regular local hypersurface, and Kawanoue describes a whole family of such varieties [Nar83, Kaw13], [Hau98], ex. 8. See also ex. 12.1 below. In Narasimhan's example, there is a sequence of point blowups for which there is no regular local hypersurface V of $\mathbb{A}^4$ at 0 whose strict transforms contain all points where the transforms of X have order 2 as at the beginning [Hau03] II.14, ex. 2, p. 388.

Remark 9.7. The existence of hypersurfaces of maximal contact in zero characteristic suggests to associate to X locally at a point a a variety Y or an ideal J in V whose transforms under the blowup of V and W along a center Z locally contained in V equals the variety Y' associated to the strict transform X^s of X in V^s at points a' above a where the order of X^s has remained constant. The variety Y or the ideal J and their transforms may then help to measure the improvement of X^s at a' by looking at their respective orders. This is precisely the way how the proof of resolution in zero characteristic proceeds. The reasoning is also known as *descent in dimension*. The main problem in this approach is to define properly the variety Y, respectively the ideal I, and to show that the local construction is independent of the choice of V and patches to give a global resolution algorithm.

$$\begin{array}{ccccccc}
a' \in & E & \subset W' & \rightsquigarrow & V' & \supset Y' \\
\downarrow & \downarrow & \downarrow \pi & & \downarrow & \downarrow \pi_{|Y'} \\
a \in & Z & \subset W & \rightsquigarrow & V & \supset Y
\end{array}$$

Example 9.8. Let $\pi : (W', a') \to (W, a)$ be a local blowup and let $x_1, \ldots, x_n$ be local coordinates on W at a such that π is monomial. Assume that x_1 appears in the initial form of an element $f \in \mathcal{O}_{W,a}$, and let $V \subset W$ be the local hypersurface at a defined by $x_1 = 0$. If the order of the strict transform f^s of f at a' has remained constant equal to the order of f at a, the point a' belongs to the strict transform V^s of V.

Example 9.9. Let the characteristic of the ground field be different from 3. Apply the second order differential operator $\partial = \frac{\partial^2}{\partial x^2}$ to $f = x^3 + x^2yz + z^5$ so that $\partial f = 6x + 2yz$. This defines a hypersurface of maximal contact for f at 0. Replacing in f the variable x by $x - \frac{1}{3}yz$ gives

$$g = x^3 - \frac{1}{3}xy^2z^2 + \frac{2}{27}y^3z^3 + z^5.$$

The term of degree 2 in x has been eliminated, and $x = 0$ defines an osculating hypersurface for g at 0.

Example 9.10. Let $X \subset \mathbb{A}^n$ be a hypersurface defined locally at the origin by a polynomial $f = x_n^d + \sum_{i=0}^{d-1} a_i(y)x_n^i$ where $y = (x_1, \ldots, x_{n-1})$ and $\mathrm{ord}_0 a_i(y) \geq d-i$. Make the change of coordinates $x_n \mapsto x_n - \frac{1}{d}a_{d-1}(y)$. Show that after this change, the hypersurface defined by $x_n = 0$ has maximal contact with X at the origin. What prevents this technique from working in positive characteristic?

Example 9.11. Consider the hypersurface $X \subset \mathbb{A}^3$ given by the equation $x^2y + xy^2 - x^2z + y^2z - xz^2 - yz^2$. Show that the hypersurface V given by $x = 0$

does not have maximal contact with X at 0. In particular, consider the blowup of $\mathbb{A}^3$ in the origin. Find a point a' on the exceptional divisor that lies in the x-chart of the blowup such that the order of the strict transform of X has order 3 at a'. Then show that the strict transform of V does not contain this point.

Example 9.12. Hypersurfaces of maximal contact are only defined locally. They need not patch to give a globally defined hypersurface of maximal contact on W. Find an example for this.

Example 9.13. Consider $f = x^4 + y^4 + z^6$, $g = x^4 + y^4 + z^{10}$ and $h = xy + z^{10}$ under point blowup. Determine, according to the characteristic of the ground field, the points where the order of f has remained constant.

Example 9.14. Let $f = x^c + g(y_1, \ldots, y_m) \in K[[x, y_1, \ldots, y_m]]$ be a formal power series with g a series of order $\geq c$ at 0. Show that there exists a formal coordinate change maximizing the order of g.

Example 9.15. Let $f = x^c + g(y_1, \ldots, y_m) \in K[x, y_1, \ldots, y_m]$ be a polynomial with g a polynomial of order $\geq c$ at 0. Does there exist a local coordinate change maximizing the order of g?

Example 9.16. Let f be a polynomial or power series in n variables $x_1, \ldots, x_n$ of order c at 0. Assume that the ground field is infinite. There exists a linear coordinate change after which $f(0, \ldots, 0, x_n)$ has order c at 0. Such polynomials and series, called *x_n-regular of order c* at 0, appear in the Weierstrass preparation theorem, which was frequently used by Abhyankar.

10. Lecture X. Coefficient Ideals

Definition 10.1. Let I be an ideal on W, let a be a point of W with open neighbourhood U, and let V be a regular closed hypersurface of U containing a. Let $x_1, \ldots, x_n$ be coordinates on U such that V is defined in U by $x_n = 0$. The restrictions of $x_1, \ldots, x_{n-1}$ to V form coordinates on V and will be abbreviated by x'. For $f \in \mathcal{O}_U$, denote by $\sum a_{f,i}(x') \cdot x_n^i$ the expansion of f with respect to x_n, with coefficients $a_{f,i} = a_{f,i}(x') \in \mathcal{O}_V$. The *coefficient ideal of I in V at a* is the ideal $J_V(I)$ on V defined by

$$J_V(I) = \sum_{i=0}^{o} (a_{f,i}, f \in I)^{\frac{o!}{o-i}}.$$

Remark 10.2. The coefficient ideal is defined on whole V. It depends on the choice of the coordinates $x_1, \ldots, x_n$ on U, even so the notation only refers to V. Actually, $J_V(I)$ depends on the choice of a section $\mathcal{O}_V \to \mathcal{O}_U$ of the restriction map $\mathcal{O}_U \to \mathcal{O}_V$. The same definition applies to stalks of ideals in W at points a, giving rise to an ideal, also denoted by $J_V(I)$, in the local ring $\mathcal{O}_{V,a}$. The weights $\frac{o!}{o-i}$ in the exponents are chosen so as to obtain a systematic behaviour of the coefficient ideal under blowup, cf. Prop. 884 below. The chosen algebraic definition of the coefficient ideal is modelled so as to commute with blowups [EH02], but is less conceptual than definitions through differential operators proposed and used by Encinas-Villamayor, Bierstone-Milman, Włodarczyk, Kawanoue-Matsuki and Hironaka [EV00, EV98, BM97, Wło05, KM10, Hir03].

Proposition 10.3. The passage to the coefficient ideal $J_V(I)$ of I in V commutes with taking germs along the local top locus $S = \mathrm{top}_a(I)$ of points of V where the order of I in W is equal to the order of I in W at a: Let $x_1, \dots x_n$ be coordinates of W at a, defined on an open neighborhood U of a, and let V be closed and regular in U, defined by $x_n = 0$. The stalks of $J_V(I)$ at points b of S inside U coincide with the coefficient ideals of the stalks of I at b.

Proof. Clear from the definition of coefficient ideals. □

Corollary 10.4. In the above situation, for any fixed closed hypersurface V in $U \subset W$ open, the order of $J_V(I)$ at points of S is upper semicontinuous along S, locally at a.

Remark 10.5. In general, V need not contain, locally at a, the top locus of I in W. This can, however, be achieved in zero characteristic by choosing for V an osculating hypersurface, cf. Prop. 10.9 below. In this case, the order of $J_V(I)$ at points of S does not depend on the choice of the hypersurface, cf. Prop. 10.13. In arbitrary characteristic, a local hypersurface V will be chosen separately at each point $b \in S$ in order to maximize the order of $J_V(I)$ at b, cf. Prop. 10.11. In this case, the order of $J_V(I)$ at b does not depend on the choice of V, and its upper semicontinuity as b moves along S holds again, but is more difficult to prove [Hau04].

Proposition 10.6. The passage to the coefficient ideal $J_V(I)$ of I at a commutes with blowup: Let $\pi : W' \to W$ be the blowup of W along a regular center Z contained locally at a in $S = \mathrm{top}_a(I)$. Let V be a local regular hypersurface of W at a containing Z and such that V^s contains all points a' above a where the order of the weak transform $I^\curlyvee$ has remained constant equal to the order of I at a. For any such point a', the coefficient ideal $J_{V^s}(I^\curlyvee)$ of $I^\curlyvee$ equals the controlled transform $J_V(I)^! = I_E^{-c} \cdot J_V(I)^*$ of $J_V(I)$ with respect to the control $c = o!$ with $o = \mathrm{ord}_a(I)$.

Proof. Write V' for V^s, and let $h = 0$ be a local equation of $E \cap V'$ in V'. For $f \in I$ of order o at a write the strict transform of f (up to multiplication by units in the local ring) as $f^s = h^{-o} \cdot f^*$, where * denotes the total transform. The coefficients $a_{f,i}$ of the monomials x^α of the expansion of an element f of I of order o at a in the coordinates $x_1, \dots, x_n$ satisfy $a_{f^\curlyvee,i} = h^{i-o} \cdot (a_{f,i})^*$. Then

$$J_{V'}(I^\curlyvee) = J_{V'}(\sum_{i<o'} a_{f^s,i} \cdot x_n^i,\ f^\curlyvee \in I^\curlyvee) =$$

$$= J_{V'}(\sum_{i<o'} a_{h^{-o}\cdot f^*,i} \cdot x_n^i,\ f^\curlyvee \in I^\curlyvee) =$$

$$= J_{V'}(\sum_{i<o} h^{-o} \cdot (a_{f,i} \cdot x_n^i)^*,\ f \in I) =$$

$$= \sum_{i<o} h^{-o!} \cdot (a_{f,i}^*,\ f \in I)^{o!/(o-i)} =$$

$$= h^{-o!} \cdot (\sum_{i<o} (a_{f,i},\ f \in I)^{o!/(o-i)})^* =$$

$$= h^{-o!} \cdot (J_V I)^* = (J_V I)^!.$$

This proves the claim. □

REMARK 10.7. The definitions of the coefficient ideal used in [EV00, EV98, BM97, Wło05, KM10, Hir03] produce a weaker commutativity property with respect to blowups, typically only for the radicals of the coefficient ideals.

REMARK 10.8. The order of the coefficient ideal $J_V(I)$ of I is not directly suited as a secondary invariant after the order of I since, by the proposition, the coefficient ideal passes under blowup to its controlled transform, and thus its order may increase. It is appropriate to decompose $J_V(I)$ and $(J_V(I))^!$ into products of ideals with an exceptional monomial factor and a second, possibly singular factor, called the *residual factor*, which passes under blowup to its weak transform. The order of this second factor does not increase under blowup by Prop. 8.13 and can thus serve as a secondary invariant.

PROPOSITION 10.9. Assume that the characteristic of the ground field is zero. Let I have maximal order o at a point $a \in W$, and let V be a regular hypersurface for I at a, with coefficient ideal $J = J_V(I)$. The locus $\mathrm{top}_a(J, o!)$ of points of V where J has order $\geq o!$ coincides with $\mathrm{top}_a(I)$,

$$\mathrm{top}_a(J, o!) = \mathrm{top}_a(I).$$

PROOF. Choose local coordinates $x_1, \dots, x_n$ in W at a so that V is defined by $x_n = 0$. Expand the elements f of I with respect to x_n with coefficients $a_{f,i} \in \mathcal{O}_{V,a}$, and choose representatives of them on a suitable neighbourhood of a. Let b be a point in a sufficiently small neighborhood of a. Then, by the upper semicontinuity of the order, b belongs to $\mathrm{top}_a(I)$ if and only if $\mathrm{ord}_b I \geq o$, which is equivalent to $\sum_{i<o} a_{f,i} \cdot x_n^i$ having order $\geq o$ at b for all $f \in I$. This, in turn, holds if and only if $a_{f,i}$ has order $\geq o - i$ at b, say $a_{f,i}^{\frac{o!}{o-i}}$ has order $\geq o!$ at b. Hence $b \in \mathrm{top}_a(I)$ if and only if $b \in \mathrm{top}_a(J_V(I), o!)$. □

COROLLARY 10.10. Assume that the characteristic of the ground field is zero. Let a be a point in W and set $S = \mathrm{top}_a(I)$. Let U be a neighbourhood of a on which there exists a closed regular hypersurface V which is osculating for I at all points of $S \cap U$. Let $J_V(I)$ be the coefficient ideal of I in V.

(a) The top locus $\mathrm{top}_a(J_V(I))$ of $J_V(I)$ on V is contained in $\mathrm{top}_a(I)$.

(b) The blowup of U along a regular locally closed subvariety Z of $\mathrm{top}_a(J_V(I))$ Z commutes with the passage to the coefficient ideals of I and $I^\curlyvee$ in V and V^s.

PROOF. Assertion (a) is immediate from the proposition, and (b) follows from Prop. 10.6. □

PROPOSITION 10.11. (Encinas-Hauser) Assume that the characteristic of the ground field is zero. The order of the coefficient ideal $J_V(I)$ of I at a with respect to an osculating hypersurface V at a attains the maximal value of the orders of the coefficient ideals over all local regular hypersurfaces. In particular, it is independent of the choice of V.

PROOF. Choose local coordinates $x_1, \dots, x_n$ in W at a such that the appropriate derivative of the chosen element $f \in I$ is given by x_n. Let o be the order of f at a. The choice of coordinates implies that the expansion of f with respect to x_n has a monomial x_n^o with coefficient 1 and no monomial in x_n of degree $o - 1$. Any other local regular hypersurface U is obtained from V by a local isomorphism φ of W at a. Assume that the order of $J_V(\varphi^*(I))$ is larger than the order of

$J_V(I)$. Let $g = \varphi^*(f)$. This signifies that the order of all coefficients $a_{g,i}^{\frac{o!}{o-i}}$ is larger than the order of $J_V(I)$. Therefore φ^* must eliminate from f the terms of $a_{f,i}$ for which $a_{f,i}^{\frac{o!}{o-i}}$ has order equal to the order of $J_V(I)$. But then φ^* produces from x_n^o a non-zero coefficient $a_{g,o-1}$ such that $a_{g,o-1}^{o!}$ has order equal to the order of $J_V(I)$, contradiction. $\square$

Remark 10.12. This result suggests to look in positive characteristic for local regular hypersurfaces which maximize the order of the associated coefficient ideal, called *hypersurfaces of weak maximal contact* [EH02]. They are used in recent approaches to resolution of singularities in positive characteristic [Hir12, Hau10a], based on the work of Moh [Moh87].

Proposition 10.13. Let the characteristic of the ground field be arbitrary. The supremum in $\mathbb{N}\cup\{\infty\}$ of the orders of the coefficient ideals $J_V(I)$ of I in local regular hypersurfaces V in W at a is realized by a formal hypersurface V in W at a. If the supremum is finite, it can be realized by a local regular hypersurface V in W at a.

Proof. If the supremum is finite, the existence of some V realizing this value is obvious. If the supremum is infinite, one uses the m-adic completeness of $\widehat{\mathcal{O}}_{W,a}$ to construct V, see [EH02, Hau04]. $\square$

Definition 10.14. A formal local regular hypersurface V realizing the supremum of the order of the coefficient ideal $J_V(I)$ of I at a is called a *hypersurface of weak maximal contact of I at a*. If the supremum is finite, it will always be assumed to be a local hypersurface.

Proposition 10.15. (Zariski) Let V be a local formal regular hypersurface in W at a of weak maximal contact with I at a. Let $\pi : W' \to W$ be the blowup of W along a closed regular center Z contained locally at a in $S = \mathrm{top}_a(I)$. The points $a' \in W'$ above a for which the order of the weak transform $I^\curlyvee$ of I at a' has not decreased are contained in the strict transform V^s of V.

Proof. By definition of weak maximal contact, the variable x_n defining V in W at a appears in the initial form of some element f of I of order $o = \mathrm{top}_a(I)$ at a, cf. ex. 10.23 below. The argument then goes analogous to the proof of Prop. 9.1. $\square$

Remark 10.16. In characteristic zero and if V has been chosen osculating at a, the hypersurface V' is again osculating at points a' above a where $\mathrm{ord}_{a'}(I') = \mathrm{ord}_a(I)$, hence it has again weak maximal contact with I' at such points a'. In positive characteristic this is no longer true, cf. ex. 10.24.

Example 10.17. Determine in all characteristics the points of $\widetilde{\mathbb{A}}^2$ where the strict transform of $g = x^4 + kx^2y^2 + y^4 + 3y^7 + 5y^8 + 7y^9$ under the blowup of $\mathbb{A}^2$ at 0 has order 4, for any $k \in \mathbb{N}$.

Example 10.18. Determine for all characteristics the maximal order of the coefficient ideal of $I = (x^3 + 5y^3 + 3(x^2y^2 + xy^4) + y^6 + 7y^7 + y^9 + y^{10})$ in regular local hypersurfaces V at 0.

Example 10.19. Same as before for $I = (xy + y^4 + 3y^7 + 5y^8 + 7y^9)$.

Example 10.20. Compute the coefficient ideal of $I = (x^5 + x^2y^4 + y^k)$ in the hypersurfaces $x = 0$, respectively $y = 0$. According to the value of k and the characteristic, which hypersurface is osculating or has weak maximal contact?

Example 10.21. Consider $f = x^2 + y^3z^3 + y^7 + z^7$. Show that $V \subset \mathbb{A}^3$ defined by $x = 0$ is a local hypersurface of weak maximal contact for f. Blow up $\mathbb{A}^3$ at the origin, respectively along a coordinate axis. How does the order of the coefficient ideal of f in V behave under these blowups at the points where the order of f has remained constant? Factorize the controlled transform of the coefficient ideal with respect to the exceptional factor and observe the behaviour of the order of the residual factor.

Example 10.22. Assume that, for a given coordinate system $x_1, \dots, x_n$ in W at a, the coefficient ideal of a polynomial f in the hypersurface V defined by $x_n = 0$ is a principal monomial ideal. Show that there is a sequence of blowups in coordinate subspaces of the induced affine charts which eventually makes the order of f drop.

Example 10.23. The variable x_n defining a hypersurface V in W at a of weak maximal contact with an ideal I appears in the initial form of some element f of I of order $o = \mathrm{ord}_a(I)$ at a.

Example 10.24. In positive characteristic, a hypersurface of weak maximal contact for an ideal I need not have again weak maximal contact after blowup with the weak transform $I^\curlyvee$ at points a' where the order of $I^\curlyvee$ has remained constant.

Example 10.25. Compute in the following situations the coefficient ideal of I in W at a with respect to the given local coordinates x, y, z in $\mathbb{A}^3$ and the hypersurface V. Determine in each case whether the order of the coefficient ideal is maximal. If not, find a coordinate change which maximizes it.

(a) $a = 0 \in \mathbb{A}^1$, x, $V : x = 0$, $I = (x)$ and $I = (x + x^2)$.
(b) $a = 0 \in \mathbb{A}^2$, x, y, $V : x = 0$, $I = (x)$, $I = (x + y^2)$, $I = (y + x^2)$, $I = (xy)$.
(c) $a = (1, 0, 0) \in \mathbb{A}^3$, x, y, z, $V : y + z = 0$, $I = (x^2)$, $I = (xy)$, $I = (x^3 + z^3)$.
(d) $a = 0 \in \mathbb{A}^3$, x, y, z, $V : x = 0$, $I = (xyz)$, $I = (x^2 + y^3 + z^5)$.
(e) $a = 0 \in \mathbb{A}^2$, x, y, $V : x = 0$, $I = (x^2 + y^4, y^4 + x^2)$.

Example 10.26. Blow up in each of the preceding examples the origin and determine the points of the exceptional divisor E where the order of the weak transform $I^\curlyvee$ of I has remained constant. Check at these points whether commutativity holds for the descent to the coefficient ideal and its controlled transform.

Example 10.27. * Show that the maximum of the order of the coefficient ideal $J_V(I)$ of a principal idela I over all regular parameter systems of $\widehat{\mathcal{O}}_{\mathbb{A}^n,0}$ is attained (it might be ∞).

Example 10.28. * Show that this maximum, when taken at any point of the top locus S of $\mathbb{A}^n$ where I has maximal order o, defines an upper semicontinuous function on S.

Example 10.29. Let V be the hypersurface $x_n = 0$ of $\mathbb{A}^n$ and let $V' \to V$ be the blowup with center Z in V. Let f be a polynomial of order o at a point a of Z, with strict transform f'. Assume that $\mathrm{ord}_{a'} f' = \mathrm{ord}_a f$ at the origin a' of the x_n-chart. Compare the total transform of the coefficent ideal $J_V(f)$ of f with its controlled transform with respect to the control $c = o!$.

Example 10.30. Show that, in characteristic zero, the top locus of an ideal I of W, when taken locally at a point a, is contained in a local regular hypersurface V through a, and that this hypersurface maximizes the order of the coefficient ideal.

Example 10.31. Show that the supremum of the order of the coefficient ideal of an ideal I in W at a point a can be realized, if the supremum is finite, by a regular system of parameters of the local ring $\mathcal{O}_{W,a}$ without passing to the completion.

Example 10.32. Consider $f = x^3 + y^2z$ in $\mathbb{A}^3$ and the point blowup of $\mathbb{A}^3$ at the origin. Find, according to the characteristic, at all points of the exceptional divisor a hypersurface of weak maximal contact for the strict transform of f.

Example 10.33. Consider surfaces of the form $f = x^o + y^a z^b \cdot g(y, z)$ where $y^a z^b$ is considered as an exceptional monomial factor of the coefficient ideal in the hypersurface V defined $x = 0$ (up to raising the coefficient ideal to the power $c = o!$). Assume that $a + b + \mathrm{ord}_0 g \geq p$. Give three examples where the order of g at 0 is not maximal over all choices of local hypersurfaces at 0, and indicate the coordinate change which makes it maximal.

Example 10.34. Consider surfaces of the form $f = x^o + y^a z^b \cdot g(y, z)$ where $y^a z^b$ is considered again as an exceptional monomial factor of the coefficient ideal in the hypersurface V defined $x = 0$. Assume that $a + b + \mathrm{ord}_0 g \geq o$. Compute the strict transform $f' = x^o + y^{a'} z^{b'} \cdot g'(y, z)$ of f under point blowup at points where the order of f has remained equal to o. Find three examples where the order of g' is not maximal over all local coordinate choices.

Example 10.35. For a flag of local regular hypersurfaces $V_{n-1} \supset \ldots \supset V_1$ at a one gets from I a chain of coefficient ideals $J_{n-1}, \ldots, J_1$ defined recursively by $J_i = J_{V_i}(I_{i+1})$, where J_{i+1} is decomposed into $J_{i+1} = M_{i+1} \cdot I_{i+1}$ with exceptional monomial factor M_{i+1} and residual factor I_{i+1}. Show that the lexicographic maximum of the vector of orders of the ideals J_i at a over all choices of flags at a can be realized stepwise, choosing first a local hypersurface V_{n-1} in W at a maximizing the order of J_{n-1} at a, then a local hypersurface V_{n-2} in V_{n-1} at a maximizing the order of J_{n-2}, and iterating this process.

11. Lecture XI: Resolution in Zero Characteristic

The inductive proof of resolution of singularities in characteristic zero requires a more detailed statement about the nature of the resolution:

Theorem 11.1. Let W be a regular ambient variety and let $E \subset W$ be a divisor with normal crossings. Assume that the characteristic of the ground field is 0. Let J be an ideal on W, together with a decomposition $J = M \cdot I$ into a principal monomial ideal M, the monomial factor of J, supported on a normal crossings divisor D transversal to E, and an ideal I, the residual factor of J. Let $c_+ \geq 1$ be a given number, the control of J.

There exists a sequence of blowups of W along regular centers Z transversal to E and D and their total transforms, contained in the locus $\mathrm{top}(J, c_+)$ of points where J and its controlled transforms with respect to c_+ have order $\geq c_+$, and satisfying the requirements *equivariance* and *excision* of a strong resolution so that the order of the controlled transform of J with respect to c_+ drops eventually at all points below c_+.

Definition 11.2. In the situation of the theorem, with prescribed divisor D and control c_+, the ideal J is called *resolved* with respect to D and c_+ if the order of J at all points of W is $< c_+$.

Remark 11.3. Once the order of the controlled transform of J has dropped below c_+, induction on the order can be applied to find an additional sequence of blowups which makes the order of the controlled transform of J drop everywhere to 0. At that stage, the controlled transform of J has become the whole coordinate ring of W, and the total transform of J, which differs from the controlled transform by a monomial exceptional factor, has become a monomial ideal supported on a normal crossings divisor D transversal to E. This establishes the existence of a strong embedded resolution of J, respectively of the singular variety X defined by J in W.

Proof. The technical details can be found in [EH02], and motivations are given in [Hau03]. The main argument is the following.

The resolution process has two different stages: In the first, a sequence of blowups is chosen through a local analysis of the singularities of J and by induction on the ambient dimension. The order of the weak transforms of I will be forced to drop eventually below the maximum of the order of I at the points a of W. By induction on the order one can then apply additional blowups until the order of the weak transform of I has become equal to 0. At that moment, the weak transform of I equals the coordinate ring/structure sheaf of the ambient variety, and the controlled transform of J has become a principal monomial ideal supported on a suitable transform of D, which will again be a normal crossings divisor meeting the respective transform of E transversally.

For simplicity, denote this controlled transform again by J. It is a principal monomial ideal supported by exceptional components. The second stage of the resolution process makes the order of J drop below c_+. The sequence of blowups is now chosen globally according to the multiplicities of the exceptional factors appearing in J. This is a completely combinatorial and quite simple procedure which can be found in many places in the literature [Hir64, **?**, EH02].

The first part of the resolution process is much more involved and will be described now. The centers of blowup are defined locally at the points where the order of I is maximal. Then it is shown that the definition does not depend on the local choices and thus defines a global, regular and closed center in W. Blowing up W along this center will improve the singularities of I, and finitely many further blowups will make the order of the weak transform of I drop.

The local definition of the center goes as follows. Let $S = \mathrm{top}(I)$ be the stratum of points in W where I has maximal order. Let a be a point of S and denote by o the order of I at a. Choose an osculating hypersurface V for I at a in an open neighbourhood U of a in W. This is a closed regular hypersurface of U given by a suitable partial derivative of an element of I of order $o - 1$. The hypersurface V maximizes the order of the coefficient ideal $J_{n-1} = J_V(I)$ over all choices of local regular hypersurfaces, cf. Prop. 10.11. Thus $\mathrm{ord}_a(J_{n-1})$ does not depend on the choice of V. There exists a closed stratum T in S at a of points $b \in S$ where the order of J_{n-1} equals the order of J_{n-1} at a.

The ideal J_{n-1} on V together with the new control $c = o!$ can now be resolved by induction on the dimension of the ambient space, applying the respective statement of the theorem. The new normal crossings divisor E_{n-1} in V is defined as $V \cap E$.

For this it is necessary that the hypersurface V can be chosen transversal to E. This is indeed possible, but will not be shown here. Also, it is used that J_{n-1} admits again a decomposition $J_{n-1} = M_{n-1} \cdot I_{n-1}$ with M_{n-1} a principal monomial ideal supported on some normal crossings divisor D_{n-1} on V, and I_{n-1} an ideal, the residual factor of J_{n-1}.

There thus exists a sequence of blowups of V along regular centers transversal to E_{n-1} and D_{n-1} and their total transforms, contained in the locus $\mathrm{top}(J_{n-1}, c)$ of points where J_{n-1} and its controlled transforms with respect to c have order $\geq c$, and satisfying the requirements of a strong resolution so that the order of the weak transform of I_{n-1} drops eventually at all points below the maximum of the order of I_{n-1} at points a of V.

All this is well defined on the open neighbourhood U of a in W, but depends a priori on the choice of the hypersurface V, since the centers are chosen locally in each V. It is not clear that the local choices patch to give a globally defined center. One can show that this is indeed the case, even though the local ideals J_{n-1} do depend on V. The argument relies on the fact that the centers of blowup are constructed as the maximum locus of an invariant associated to J_{n-1}. By induction on the ambient dimension, such an invariant exists for each J_{n-1}: In dimension 1, it is just the order. In higher dimension, it is a vector of orders of suitably defined coefficient ideals. One then shows that the invariant of J_{n-1} is independent of the choice of V and defines an upper semicontinuous function on the stratum S. Its maximum locus is therefore well defined and closed. Again by induction on the ambient dimension, it can be assumed that it is also regular and transversal to the divisors E_{n-1} and D_{n-1}.

By the assertion of the theorem in dimension $n-1$, the order of the weak transform of J_{n-1} can be made everywhere smaller than $c = o!$ by a suitable sequence of blowups. The sequence of blowups also transforms the original ideal $J = M \cdot I$, producing controlled transforms of J and weak transforms of I. The order of I cannot increase in this sequence, if the centers are always chosen inside S. To achieve this inclusion a technical adjustment of the definition of coefficient ideals is required which will be omitted here. If the order of I drops, induction applies. So one is left to consider points where possible the order of the weak transform has remained constant. There, one will use the commutativity of blowups with the passage to coefficient ideals, Prop. 10.6: The coefficient ideal of the final transform of I will equal the controlled transform of the coefficient ideal J_{n-1} of I. But the order of this controlled transform has dropped below c, hence, by Prop. 10.9, also the order of the weak transform of I must have dropped. This proves the existence of a resolution. $\square$

Example 11.4. In the situation of the theorem, take $W = \mathbb{A}^2$, $E = \emptyset$, and $J = (x^2y^3) = 1 \cdot I$ with control $c_+ = 1$. Even though being a monomial, the ideal J is not resolved yet, since it is not supported on the exceptional divisor. The ideal J has order 5 at 0, order 3 along the x-axis, and order 2 along the y-axis. Blow up $\mathbb{A}^2$ in 0. The controlled transform in the x-chart is $J^! = (x^4y^3)$ with exceptional factor $I_E = (x^4)$ and residual factor $I_1 = (y^3)$, the strict transform of J. The controlled transform in the y-chart is $J^! = (x^2y^4)$ with exceptional factor $I_E = (y^4)$ and residual factor $I_1 = (x^2)$, the strict transform of J. One additional blowup resolves J.

EXAMPLE 11.5. In the situation of the preceding example, replace $E = \emptyset$ by $E = V(x + y)$, respectively $E = V(x + y^2)$, and resolve the ideal J.

EXAMPLE 11.6. Take $W = \mathbb{A}^3$, $E = V(x + z^2)$, and $J = (x^2y^3)$ with control $c_+ = 1$. The variety X defined by J is the union of the xz- and the yz-plane. The top locus of J is the z-axis, which is tangent to E. Therefore it is not allowed to take it as the center of the first blowup. The only possible center is the origin. Applying this blowup, the top locus of the controlled transform of J and the total transform of E have normal crossings, so that the top locus can now be chosen as the center of the next blowup. Resolve the singularities of X.

REMARK 11.7. The non-transversality of the candidate center of blowup with already existing exceptional components is known as the *transversality problem*. The preceding example gives a first instance of the problem, see [EH02, Hau03] for more details.

EXAMPLE 11.8. Take $W = \mathbb{A}^3$, $E = D = \emptyset$ and $J = (x^2 + yz)$. In characteristic different from 2, the origin of $\mathbb{A}^3$ is the unique isolated singular point of the cone X defined by J. The ideal J has order 2 at 0. After blowing up the origin, the singularity is resolved and the strict transform of J defines a regular surface. It is transversal to the exceptional divisor.

EXAMPLE 11.9. Take $W = \mathbb{A}^3$, $E = D = \emptyset$, $J = (x^2 + y^az^b)$ with $a, b \in \mathbb{N}$. According to the values of a and b, the top locus of J is either the origin, the y- or z-axis, or the union of the y- and the z-axis. If $a + b \geq 2$, the yz-plane is a hypersurface of maximal contact for J at 0. The coefficient ideal is $J_1 = (y^az^b)$ (up to raising it to the square). This is a monomial ideal, but not supported yet on exceptional components. Resolve J.

EXAMPLE 11.10. Take $W = \mathbb{A}^4$, $E = D = V(yz)$, $J = (x^2 + y^2z^2(y + w))$. The order of J at 0 is 2. The yzw-hyperplane has maximal contact with J at 0, with coefficient ideal $J_1 = y^2z^2(y + w)$, in which the factor $M_1 = (y^2z^2)$ is exceptional and $I_1 = (y + w)$ is residual. Resolve J. The top locus of I_1 is the plane $V(y + w) \subset \mathbb{A}^3$, hence not contained in the top locus of J. This technical complication is handled by introducing an intermediate ideal, the *companion ideal* [Vil07, EH02].

EXAMPLE 11.11. Compute the first few steps of the resolution process for the three surfaces $x^2 + y^2z$, $x^2 + y^3 + z^5$ and $x^3 + y^4z^5 + z^{11}$.

EXAMPLE 11.12. Prove with all details the embedded resolution of plane curves in characteristic zero according to the above description.

EXAMPLE 11.13. Resolve the following items, taking into account different characteristics.

(a) $W = A^2$, $E = \emptyset$, $J = I = (x^2y^3)$, $c_+ = 1$.
(b) $W = A^2$, $E = V(x)$, $J = (x^2y^3)$, $c_+ = 1$.
(c) $W = A^2$, $E = V(xy)$, $J = (x^2y^3)$, $c_+ = 1$.
(d) $W = A^3$, $E = V(x + z^2)$, $J = (x^2y^3)$, $c_+ = 1$.
(e) $W = A^3$, $E = V(y + z)$, $J = (x^3 + (y + z)z^2)$, $c_+ = 3$.
(f) $W = A^3$, $E = \emptyset$, $J = I = (x^2 + yz)$, $c_+ = 2$.
(g) $W = A^3$, $E = \emptyset$, $J = I = (x^3 + y^2z^2)$, $c_+ = 3$.
(h) $W = A^3$, $E = \emptyset$, $J = I = (x^2 + xy^2 + y^5)$, $c_+ = 2$.

(i) $W = A^3$, $E = \emptyset$, $J = I = (x^2 + xy^3 + y^5)$, $c_+ = 2$.
(j) $W = A^3$, $E = V(x + y^2)$, $J = (x^3 + (x + y^2)y^4z^5)$, $c_+ = 3$.
(k) $W = A^3$, $E = \emptyset$, $J = I = (x^2 + y^2z^2(y + z))$, $c_+ = 12$.
(l) $W = A^3$, $E = V(yz)$, $J = (x^2 + y^2z^2(y + z))$, $c_+ = 2$.
(m) $W = A^4$, $E = V(yz)$, $J = (x^2 + y^2z^2(y + w))$, $c_+ = 2$.

Example 11.14. Consider $J = I = (x^2 + y^7 + yz^4)$ in $\mathbb{A}^3$, with $E = \emptyset$ and $c_+ = 2$. Consider the sequence of three point blowups with the following centers. First the origin of $\mathbb{A}^3$, then the origin of the y-chart, then the origin of the z-chart. On the last blowup, consider the midpoint between the origin of the y- and the z-chart. Show that J is resolved at this point if the characteristic is zero or > 2.

Example 11.15. What happens in the preceding example if the characteristic is equal to 2? Describe in detail the behaviour of the first two components of the resolution invariant under the three local blowups.

Example 11.16. A combinatorial version of resolution is known as Hironaka's polyhedral game: Let N be an integral convex polyhedron in $\mathbb{R}^n_+$, i.e., the positive convex hull $N = \mathrm{conv}(S + \mathbb{R}^n_+)$ of a finite set S of points in $\mathbb{N}^n$. Player A chooses a subset J of $\{1, \dots, n\}$, then player B chooses an element $j \in J$. After these moves, S is replaced by the set S' of points α defined by $\alpha'_i = \alpha_i$ if $i \neq j$ and $\alpha'_j = \sum_{k \in J} \alpha_k$, giving rise to a new polyhedron N'. Player A has won if after finitely many rounds the polyhedron has become an orthant $\alpha + \mathbb{R}^n_+$. Player B can never win, but only prevent player A from winning. Show that player A has a winning strategy, first with and then without using induction on n [Spi83, Zei06].

12. Lecture XII: Positive Characteristic

The existence of the resolution of varieties of arbitrary dimension over a field of positive characteristic is still an open problem. For curves, there exist various proofs [Kol07] chap. 1. For surfaces, the first proof of non-embedded resolution was given by Abhyankar in his thesis [Abh56]. Later, he proved embedded resolution, but the proof is scattered over several papers which sum up to over 500 pages [Abh59, Abh64, Abh66b, Abh66a, Abh67, Abh98]. Cutkosky was able to shorten and simplify the argumens substantially [Cut11]. An invariant similar to the one used by Abhyankar was developed independently by Zeillinger and Wagner [Zei05, Wag09, HW09]. Lipman gave an elegent proof of non-embedded resolution for arbitrary two-dimensional schemes [Lip78, Art86]. Hironaka proposed an invariant based on the Newton polyhedron which allows to prove embedded resolution of surfaces which are hypersurfaces [Hir84, Cos81, Hau00]. Hironaka's invariant seems to be restricted to work only for surfaces. The case of higher codimensional surfaces was settled by Cossart-Jannsen-Saito, extending Hironaka's invariant [CJS09]. Different proofs were recently proposed by Benito-Villamayor and Kawanoue-Matsuki [BO12, KM12].

For three-folds, Abhyankar and Cossart gave partial results. Quite recently, Cossart-Piltant proved non-embedded resolution by a long case-by-case study [CP08, CP09]. See also [Cut09].

Programs and techniques for resolution in arbitrary dimension and characteristic have been developed quite recently, among others, by Hironaka, Teissier, Kuhlmann, Kawanoue-Matsuki, Benito-Bravo-Villamayor, Hauser-Schicho, Cossart [Hir12, Tei03, Kuh00, Kaw13, KM10, BO12, BV10, BV11, Hau10b, HS12, Cos11].

REMARK 12.1. Let $\mathbb{K}$ be an algebraically closed field of prime characteristic $p > 0$, and let X be an affine variety defined in $\mathbb{A}^n$. The main problems already appear in the hypersurface case. Let $f = 0$ be an equation for X in $\mathbb{A}^n$. Various properties of singularities used in the characteristic zero proof fail in positive characteristic:

(a) The top locus of f of points of maximal order need not be contained locally in a regular hypersurface. Take the variety in $\mathbb{A}^4$ defined by $f = x^2+yz^3+zw^3+y^7w$ over a field of characteristic 2 [Nar83, Mul83, Kaw13, Hau98].

(b) There exist sequences of blowups for which the sequence of points above a given point$a$ where the order of the strict transforms of f has remained constant are not contained eventually in the strict transforms of any regular local hypersurface passing through a.

(c) Derivatives cannot be used to construct hypersurfaces of maximal contact.

(d) The characteristic zero invariant is no longer upper semicontinuous when translated to positive characteristic.

EXAMPLE 12.2. A typical situation where the characteristic zero resolution invariant does not work in positive characteristic is as follows. Take the polynomial $f = x^2+y^7+yz^4$ over a field of characteristic 2. There exists a sequence of blowups along which the order of the strict transforms of f remains constant equal to 2 but where eventually the order of the residual factor of the coefficient ideal of f with respect to a hypersurface of weak maximal contact increases.

The hypersurface V of $W = \mathbb{A}^3$ defined by $x = 0$ produces – up to raising the ideal to the square – as coefficient ideal the ideal on $\mathbb{A}^2$ generated by $y^7 + yz^4$. Its order at the origin is 5.

Blow up $\mathbb{A}^3$ at the origin. In the y-chart W' of the blowup, the total transform of f is given by

$$f^* = y^2(x^2 + y^5 + y^3z^4),$$

with $f' = x^2 + y^3(y^2 + z^4)$ the strict transform of f. It has order 2 at the origin of W'. The generator of the coefficient ideal of f' in $V' : x = 0$ decomposes into a monomial factor y^3 and a residual factor $y^2 + z^4$. The order of the residual factor at the origin of W' is 2. Blow up W' along the z-axis, and consider the y-chart W'', with strict transform

$$f'' = x^2 + y(y^2 + z^4),$$

of f. The residual factor of the coefficient ideal in $V'' : x = 0$ equals again $y^2 + z^4$. Blow up W'' at the origin and consider the z-chart W''', with strict transform

$$f''' = x^2 + yz(y^2 + z^2).$$

The origin of W''' is the intersection point of the two exceptional components $y = 0$ and $z = 0$. The residual factor of the coefficient ideal in $V''' : x = 0$ equals $y^2 + z^2$, of order 2 at the origin of W'''.

Blow up the origin of W''' and consider the affine chart $W^{(iv)}$ given by the coordinate transformation

$$x \mapsto xz, y \mapsto yz + z, \text{ and } z \mapsto z.$$

The origin of this chart is the midpoint of the new exceptional component. The strict transform of f''' equals

$$f^{(iv)} = x^2 + (y + 1)z^2y^2,$$

which, after the coordinate change $x \mapsto x + yz$, becomes

$$f^{(iv)} = x^2 + y^3z^2 + y^2z^2.$$

The order of the strict transforms of f has remained constant equal to 2 along the sequence of local blowups. The order of the residual factor of the associated coefficient ideal has decreased from 5 to 2 in the first blowup, then remained constant until the last blowup, where it increased from 2 to 3.

Remark 12.3. The preceding example shows that the order of the residual factor of the coefficient ideal of the defining ideal of a singularity with respect to a local hypersurface of weak maximal contact is not suited for an induction argument as in the case of zero characteristic.

Definition 12.4. A hypersurface singularity X at a point a of affine space $W = \mathbb{A}^n_{\mathbb{K}}$ over a field $\mathbb{K}$ of characteristic p is called *purely inseparable of order p^e at a* if there exist local coordinates $x_1, \ldots, x_n$ on W at a such that $a = 0$ and such that the local equation f of X at a is of the form

$$f = x_n^{p^e} + F(x_1, \ldots, x_{n-1})$$

for some $e \geq 1$ and a polynomial $F \in \mathbb{K}[x_1, \ldots, x_{n-1}]$ of order $\geq p^e$ at a.

Proposition 12.5. For a purely inseparable hypersurface singularity X at a, the polynomial F is unique up to multiplication by units in the local ring $\mathcal{O}_{W,a}$ and the addition of p^e-th powers in $\mathbb{K}[x_1, \ldots, x_{n-1}]$.

Proof. Multiplication by units does not change the local geometry of X at a. A coordinate change in x_n of the form $x_n \mapsto x_n + a(x_1, \ldots, x_{n-1})$ with $a \in \mathbb{K}[x_1, \ldots, x_{n-1}]$ transforms f into $f = x_n^{p^e} + a(x_1, \ldots, x_{n-1})^{p^e} + F(x_1, \ldots, x_{n-1})$. This implies the assertion. $\square$

Definition 12.6. Let affine space $W = \mathbb{A}^n$ be equipped with an exceptional normal crossings divisor E produced by earlier blowups with multiplicities $r_1, \ldots, r_n$. Let $x_1, \ldots, x_n$ be local coordinates at a point a of W such that E is defined at a by $x_1^{r_1} \cdots x_n^{r_n} = 0$. Let $f = x_n^{p^e} + F(x_1, \ldots, x_{n-1})$ define a purely inseparable singularity X of order p^e at the origin $a = 0$ of $\mathbb{A}^n$ such that F factorizes into

$$F(x_1, \ldots, x_{n-1}) = x_1^{r_1} \cdots x_{n-1}^{r_{n-1}} \cdot G(x_1, \ldots, x_{n-1}).$$

The *residual order of X at a with respect to E* is the maximum of the orders of the polynomials G at a over all choices of local coordinates such that f has the above form [Hir12, Hau10b].

Remark 12.7. The residual order can be defined for arbitrary singularities [Hau10b]. In characteristic zero, the definition coincides with the second component of the local resolution invariant, defined by the choice of an osculating hypersurface or, more generally, of a hypersurface of weak maximal contact and the factorization of the associated coefficient ideal.

Remark 12.8. In view of the preceding example, one is led to investigate the behaviour of the residual order under blowup at points where the order of the singularity remains constant. Moh showed that it can increase at most by p^{e-1} [Moh87]. Abhyankar seems to have observed already this bound in the case of surfaces. He defines a correction term ε taking values equal to 0 or p^{e-1} which is added to the residual order according to the situation in order to make up for the occasional increases of the residual order [Abh67, Cut11]. A similar construction has been proposed by Zeillinger and Hauser-Wagner [Zei05, Wag09, HW09]. This allows, at least for surfaces, to define a secondary invariant after the order of the singularity, the *modified residual order of the coefficient ideal*, which does not increase under blowup. The problem then is to handle the case where the order of the singularity and the modified residual order remain constant. It is not clear how to define a third invariant which manifests the improvement of the singularity.

REMARK 12.9. Following ideas of Giraud, Cossart has studied the behaviour of the order of the Jacobian ideal of f, defined by certain partial derivatives of f. Again, it seems that hypersurfaces of maximal contact do not exist for this invariant [Gir75, Cos11]. There appeared promising recent approaches by Hironaka, using the machinery of differential operators in positive characteristic, by Villamayor and collaborators using instead of the restriction to hypersurfaces of maximal contact projections to regular hypersurfaces via elimination algebras, and by Kawanoue-Matsuki using their theory of idealistic filtrations and differential closures. None of these proposals has been able to produce an invariant or a resolution strategy which works in positive characteristic for all dimensions.

REMARK 12.10. Another approach consists in analyzing the singularities and blowups for which the residual order increases under blowup. This leads to the notion of kangaroo singularities:

DEFINITION 12.11. A hypersurface singularity X defined at a point a of affine space $W = \mathbb{A}^n_{\mathbb{K}}$ over a field $\mathbb{K}$ of characteristic p by a polynomial equation $f = 0$ is called *a kangaroo singularity* if there exists a local blowup $\pi : (\widetilde{W}, a') \to (W, a)$ of W along a regular center Z contained in the top locus of X and transversal to an already existing exceptional normal crossings divisor E such that the order of the strict transform of X remains constant at a' but the residual order of the strict transform of f increases at a'. The point a' is then called *kangaroo point of X above a*.

REMARK 12.12. Kangaroo singularities can be defined for arbitrary singularities. They have been characterized in all dimensions by Hauser [Hau10a, Hau10b]. However, the knowledge of the algebraic structure of these singularities did not yet give any hint how to overcome the obstruction caused by the increase of the residual order.

PROPOSITION 12.13. If a polynomial $f = x_n^{p^e} + F(x_1, \dots, x_{n-1}) = x_n^{p^e} + x_1^{r_1} \cdots x_{n-1}^{r_{n-1}} \cdot G(x_1, \dots, x_{n-1})$ defines a kangaroo singularity of order p^e at 0 the sum of the r_i and of the order of G at 0 is divisible by p^e, the sum of the residues of the exceptional multiplicities r_i modulo p^e is bounded by $m \cdot p^e$ with m the number of exceptional multiplicities not divisible by p^e, and the initial form of F equals a specific homogeneous polynomial prescribed by the situation. Any kangaroo point a' of X above a lies outside the strict transform of the components of the exceptional divisor at a whose multiplicities are not a multiple of p^e.

REMARK 12.14. A more detailed description of kangaroo singularities and a further discussion of typical characteristic p phenomena can be found in [Hau10a, Hau10b].

EXAMPLE 12.15. Prove the resolution of plane curves in arbitary characteristic by using the order and the residual order as the resolution invariants.

Example 12.16. Let $\mathbb{K}$ be an algebraically closed field of characteristic $p > 0$. Develop a significant notion of resolution for elements of the quotient of rings $\mathbb{K}[x, y]/\mathbb{K}[x^p, y^p]$. Then prove that such a resolution always exists.

Example 12.17. Consider the polynomial $f = x^2 + yz^3 + zw^3 + y^7w$ on $\mathbb{A}^4$ over a ground field of characteristic 2. Its maximal order is 2, and the respective top locus is the image of the monomial curve $(t^{32}, t^7, t^{19}, t^{15}), t \in \mathbb{K}$. The image curve has embedding dimension 4 at 0 and cannot be embedded locally at 0 into a regular hypersurface of $\mathbb{A}^4$. Hence there is no hypersurface of maximal contact with f locally at the origin.

Example 12.18. Find a surface X in positive characteristic and a sequence of point blowups starting at $a \in X$ so that some of the points above a where the order of the weak transforms of X remains constant eventually leave the transforms of any local regular hypersurface passing through a. *Hint*: You may use 72. or cook up your own example.

Example 12.19. Show that $f = x^2 + yz^3 + zw^3 + y^7w$ has in characteristic 2 top locus $\mathrm{top}(f)$ equal to the parametrized curve $(t^{32}, t^7, t^{19}, t^{15})$ in $\mathbb{A}^4$ [Nar83, Mul83, Kaw13].

Example 12.20. Show that f is not contained in the square of the ideal defining the parametrized curve $(t^{32}, t^7, t^{19}, t^{15})$.

Example 12.21. Find the defining ideal for the image of the monomial curve $(t^{32}, t^7, t^{19}, t^{15})$ in $\mathbb{A}^4$. What is the local embedding dimension at 0?

Example 12.22. Show that $f = x^2 + yz^3 + zw^3 + y^7w$ admits in characteristic 2 at the point 0 no local regular hypersurface of permanent maximal contact (i.e., whose successive strict transforms contain all points where the order of f has remained constant in any sequence of blowups with regular centers inside the top locus).

Example 12.23. Consider $f = x^2 + y^7 + yz^4$ in characteristic 2. Show that there exists a sequence of point blowups for which f admits at the point 0 no local regular hypersurface whose transforms have weak maximal contact with the transforms of f as long as the order of f remains equal to 2.

Example 12.24. Define the p-th order derivative of polynomials in $\mathbb{K}[x_1, \ldots, x_n]$ for K a field of characteristic p.

Example 12.25. Construct a surface of order p^5 in $\mathbb{A}^3$ for which the residual order increases under blowup.

Example 12.26. Show that in $f = x^p + y^pz$ with $E = \emptyset$ the residual order along the (closed) points of the z-axis is not equal to its value at the generic point.

Example 12.27. Let $y_1, \ldots, y_m$ be fixed coordinates, and consider a homogeneous polynomial $G(y) = y^r \cdot g(y)$ with $r \in \mathbb{N}^m$ and $g(y)$ homogeneous of degree k. Let $G^+(y)$ be the polynomial obtained from G by the linear coordinate change $y_i \to y_i + y_m$ for $i = 1, \ldots, m-1$. Show that the order of G^+ along the y_m-axis is at most k.

Example 12.28. Express the assertion of the preceding example through the invertibility of a matrix of multinomial coefficients.

EXAMPLE 12.29. Consider $G(y,z) = y^r z^s \sum_{i=0}^k \binom{k+r}{i+r} y^i (tz-y)^{k-i}$. Compute for $t \in K^*$ the polynomial $G^+(y,z) = G(y+tz,z)$ and its order with respect to y modulo p-th power polynomials.

EXAMPLE 12.30. * Determine all homogeneous polynomials $G(y,z) = y^r z^s g(y,z)$ so that $G^+(y,z)$ has order $k+1$ with respect to y modulo p-th power polynomials.

EXAMPLE 12.31. * Find a new sytematic proof for the embedded resolution of surfaces in three-space.

EXAMPLE 12.32. Let $G(x)$ be a polynomial in one variable over a field K of characteristic p, of degree d and order k at 0. Let $t \in K$, and consider the equivalence class $\overline{G}$ of $K(x+t)$ in $K[x]/K[x^p]$ (i.e., consider $K(x+t)$ modulo p-th power polynomials). What is the maximal order of $\overline{G}$ at 0? Describe all examples where this maximum is achieved.

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Fakultät für Mathematik,
University of Vienna, and
Institut für Mathematik,
University of Innsbruck,
Austria
herwig.hauser@univie.ac.at