Swarms with canonical active Brownian motion

Alexander Glück,[1,*] Helmuth Hüffel,[1] and Saša Ilijić[2]
[1]*Faculty of Physics, University of Vienna, Boltzmanngasse 5, A-1090 Wien, Austria*
[2]*Department of Physics, Faculty of Electrical Engineering and Computing, University of Zagreb, Unska 3, HR-10000 Zagreb, Croatia*
(Received 26 May 2010; revised manuscript received 21 September 2010; published 4 May 2011)

We present a swarm model of Brownian particles with harmonic interactions, where the individuals undergo canonical active Brownian motion, i.e., each Brownian particle can convert internal energy to mechanical energy of motion. We assume the existence of a single global internal energy of the system. Numerical simulations show amorphous swarming behavior as well as static configurations. Analytic understanding of the system is provided by studying stability properties of equilibria.

DOI: 10.1103/PhysRevE.83.051105 PACS number(s): 05.40.Jc

I. CANONICAL ACTIVE BROWNIAN MOTION

The concept of active Brownian motion was developed to describe complex motion in various systems [1,2]. Its main idea is to assume the existence of an additional *internal* degree of freedom, called the "internal energy" e, which can be taken up from the environment and converted to energy of motion. Former models only assumed an exchange between the internal energy and the *kinetic* energy of motion. In Ref. [3], we discussed a more general case, where e can be also converted to the potential energy of the particle, thus to the *full* mechanical energy $H = \frac{p^2}{2} + U(x)$. The corresponding stochastic equations for the position $\mathbf{x} \in \mathbb{R}^3$ and the momentum $\mathbf{p}$ of the particle read

$$\frac{d\mathbf{x}}{dt} = \frac{\partial H}{\partial \mathbf{p}}, \quad \frac{d\mathbf{p}}{dt} = -(1 - d_1 e)\frac{\partial H}{\partial \mathbf{x}} - (\gamma - d_2 e)\frac{\partial H}{\partial \mathbf{p}} + \boldsymbol{\eta}, \tag{1}$$

$$\frac{de}{dt} = 1 - ce - eH, \tag{2}$$

where $\boldsymbol{\eta}$ represents the stochastic noise, satisfying the correlations

$$\langle \eta^a \rangle = 0, \quad \langle \eta^a(t)\eta^b(\bar{t}) \rangle = 2\delta^{ab}\delta(t - \bar{t}), \tag{3}$$

and therefore describing a completely irregular force. The constants d_i and c remain to be fixed for concrete applications.

The above system of equations defines the dynamics of a particle subject to an external potential $U(x)$ and an irregular force $\boldsymbol{\eta}$. The friction with the environment is characterized by the friction constant γ. The dynamical equation for e has the following meaning: The first term gives the constant rate of internal energy take-up from the environment, the second term $-ce$ models the inner energy loss of the individual, and the third term $-eH$ describes the conversion of internal energy e into the mechanical energy H. Numerical simulations and analytic investigations of these dynamics were made in Ref. [3] for a single particle in harmonic external potentials. Studies of the multiparticle case were missing so far.

Most swarm models available today rely on the assumption of a small set of specific rules (see, e.g., Ref. [4]), which guide the dynamical behavior of each individual particle within the swarm. Discrete computer models can then evaluate the positions and momenta at each time step and thus can simulate the behavior of the system as a whole.

However, only a minority of current swarm models is indeed based on continuous dynamical equations. There is an ansatz in biological physics [5,6], which goes under the name of "self-driven" or "self-propelled" particles. In this case particles are driven with a constant absolute velocity field, violating, however, the net momentum conservation during their interactions. A particularly interesting many-particle model, presented in Ref. [7], uses the formalism of active Brownian motion. Swarming results in a coherent mechanical interpretation as the superposition of particle attraction to the center of mass and the active motion of each individual.

In the case of Ref. [7], however, internal energy is converted only to *kinetic* energy (corresponding to $d_1 = 0$ and H being replaced by $\frac{p^2}{2}$ in the equation for $\dot{e}$). Furthermore, an essential simplification is made by considering the internal energy e to be stationary ($\frac{de}{dt} = 0$) so that it can be eliminated from dynamical equations. As a result, only a specific form of circular swarming behavior is found. It is interesting to see how the original model [7] can be extended along the lines of Ref. [3] to describe a much bigger class of swarms from a classical mechanical point of view. We discuss in general the behavior of many particles undergoing canonical active Brownian motion, i.e., the study of particles which can convert dynamical internal energy to total mechanical energy. We will show amorphous swarming behavior as well as static configurations.

II. MANY-PARTICLE SYSTEMS

We present a simple model for a swarm of active Brownian particles by assuming the existence of a single global internal energy e of the swarm. If we think of symbiotic biological systems, this assumption becomes physically relevant. When the individual particles can exchange their internal energy, there is only one global internal energy depot, which defines the state of the system as a whole. Swarm particles give and take back energy with respect to this energy depot. Our additional variable thus defines the internal energy of the

*alexander.glueck@chello.at

system as a whole, hence the swarm is modeled as a single living organism.

Additionally, the global internal energy variable can be used to distinguish open subsystems from its surroundings, e.g., a heat bath. We are looking forward to applying our results also in statistical physics. The internal energy then functions as an external variable, which defines the state of an open subsystem, whose behavior under the influence of its surroundings can be studied by means of thermodynamics.

Consider the movement of n active Brownian particles, enumerated by the index i, with unit mass in d space dimensions coupled to the internal energy e. The position vectors are $\mathbf{x}_i \in \mathbb{R}^d$ with $i = 1,\ldots,n$ and the stochastic equations of motion read

$$\frac{d\mathbf{x}_i}{dt} = \frac{\partial H}{\partial \mathbf{p}_i}, \tag{4}$$

$$\frac{d\mathbf{p}_i}{dt} = -(1-d_1 e)\frac{\partial H}{\partial \mathbf{x}_i} - (\gamma - d_2 e)\frac{\partial H}{\partial \mathbf{p}_i} + \boldsymbol{\eta}_i, \tag{5}$$

$$w\frac{de}{dt} = 1 - ce - eH, \tag{6}$$

where the total mechanical energy is given by

$$H = \sum_{i=1}^{n}\left(\frac{\mathbf{p}_i^2}{2} + U_i^{(\mathrm{in})} + U_i^{(\mathrm{ex})}\right). \tag{7}$$

The potential energy of one particle is composed of a potential $U_i^{(\mathrm{in})}$ describing the interaction with all the other particles, and an external potential $U_i^{(\mathrm{ex})}$ modeling the environment of the swarm. We study the case where all individuals are attracted to the center of mass $\mathbf{X} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i$ and are moving in a landscape of harmonic shape. The total mechanical energy therefore reads

$$H = \sum_{i=1}^{n}\left(\frac{\mathbf{p}_i^2}{2} + \sum_{j=1}^{n}\frac{k_1}{4n}(\mathbf{x}_i - \mathbf{x}_j)^2 + \frac{k_2}{2}\mathbf{x}_i^2\right), \tag{8}$$

(with positive constants k_1 and k_2) so that the swarm dynamics is finally given by the equations

$$\dot{\mathbf{x}}_\mathbf{i} = \mathbf{p}_i, \tag{9}$$

$$\dot{\mathbf{p}}_\mathbf{i} = -(1-d_1 e)(k_1(\mathbf{x}_i - \mathbf{X}) + k_2\mathbf{x}_i) - (\gamma - d_2 e)\mathbf{p}_i + \eta_i, \tag{10}$$

$$\dot{e} = 1 - ce - e\sum_{i=1}^{n}\left(\frac{\mathbf{p}_i^2}{2} + \sum_{j=1}^{n}\frac{k_1}{4n}(\mathbf{x}_i - \mathbf{x}_j)^2 + \frac{k_2}{2}\mathbf{x}_i^2\right). \tag{11}$$

A. Variable transformation

The above system of coupled stochastic nonlinear differential equations is not easily accessible with direct analytic procedures. Nonetheless, predictions for the long-time behavior can be made by transforming to the new variables,

$$K = \sum_{i=1}^{n}\frac{\mathbf{p}_\mathbf{i}^2}{2}, \quad U = \sum_{i=1}^{n}\frac{\mathbf{x}_\mathbf{i}^2}{2}, \quad S = \sum_{i=1}^{n}\mathbf{x}_i\mathbf{p}_i, \tag{12}$$

$$L = \left(\sum_{i=1}^{n}\mathbf{p}_i\right)^2, \quad V = \left(\sum_{i=1}^{n}\mathbf{x}_i\right)^2, \quad T = \left(\sum_{i=1}^{n}\mathbf{x}_i\right)\left(\sum_{i=1}^{n}\mathbf{p}_i\right). \tag{13}$$

K represents the total kinetic energy of the swarm, and U the total external potential energy, whereas the variables V and L are quadratic forms of the center of mass $\mathbf{X}$, respectively its time derivative $\mathbf{P} \equiv \frac{d\mathbf{X}}{dt}$: $V = n^2\mathbf{X}^2$ and $L = n^2\mathbf{P}^2$. Additionally, also the variable T can be written in terms of $\mathbf{X}$ and $\mathbf{P}$: $T = n^2\mathbf{X}\mathbf{P}$. In these new variables, when omitting the noise influences, the system reads

$$\dot{K} = -(1-d_1 e)\left((k_1+k_2)S - \frac{k_1}{n}T\right) - 2(\gamma - d_2 e)K, \tag{14}$$

$$\dot{U} = S, \tag{15}$$

$$\dot{S} = 2K - (1-d_1 e)\left(2(k_1+k_2)U - \frac{k_1}{n}V\right) - (\gamma - d_2 e)S, \tag{16}$$

$$\dot{e} = 1 - ce - e\left(K + (k_1+k_2)U - k_1\frac{V}{2n}\right), \tag{17}$$

$$\dot{T} = L - (1-d_1 e)k_2 V - (\gamma - d_2 e)T, \tag{18}$$

$$\dot{V} = 2T, \tag{19}$$

$$\dot{L} = -2(1-d_1 e)k_2 T - 2(\gamma - d_2 e)L. \tag{20}$$

We have thus reduced our $(2dn+1)$-dimensional problem to an analysis of the dynamics of "only" seven variables. In these new variables, equilibrium states do not necessarily correspond to equilibrium states in the original variables. However, we will be able to characterize *swarming* states as stationary points in the variables (K,U,S,e,T,V,L), which is a large benefit of this method. Bifurcation analysis then provides us with the information under which parameter values our swarm is actually moving or eventually collapsing. Additionally, our new variables are independent of the spatial dimension of the system. The following analysis therefore holds for two as well as for three space dimensions.

B. Equilibrium solutions

To handle the above nonlinear system, we first calculate stationary points, i.e., values of $z = (K,U,S,e,T,V,L)$, where $\dot{z} = 0$ and linearize the equations around these equilibrium solutions. By calculating eigenvalues of the Jacobian, we can analyze stability properties. We find three distinct equilibrium points E_i:

$$E_1 : e_0 = \frac{1}{c}, \quad K_0 = U_0 = S_0 = e_0 = T_0 = V_0 = L_0 = 0.$$

$$E_2 : U_0 = \frac{d_1 - c + \frac{k_1 V_0}{2n}}{(k_1+k_2)}, \quad e_0 = \frac{1}{d_1}, \quad K_0 = S_0 = T_0 = L_0 = 0.$$

$$E_3 : U_0 = \frac{d_2(d_2 - c\gamma) + \frac{\gamma k_1}{2n}(2d_2 - \gamma d_1)V_0}{\gamma(k_1+k_2)(2d_2 - \gamma d_1)}, \quad e_0 = \frac{\gamma}{d_2},$$

$$L_0 = k_2\left(1-\gamma\frac{d_1}{d_2}\right)V_0, \quad S_0 = T_0 = 0,$$
$$K_0 = \frac{(\gamma d_1 - d_2)(\gamma c - d_2)}{\gamma(2d_2 - \gamma d_1)}.$$

Equilibria E_1 and E_2 are truly *static* configurations of the swarm, since $K_0 = 0$ in both cases, whereas equilibrium E_3 has a nonvanishing equilibrium value of K_0, which means that swarm particles are actually moving.

Calculating the stability conditions of all three equilibria allows us to decide under which parameter conditions the swarm either collapses to a point (E_1), freezes to a certain pattern (E_2), or moves amorphously (E_3). For equilibrium E_1, all seven eigenvalues of the Jacobian are accessible, and it is found to be stable under the conditions

$$\gamma > \frac{d_2}{c}, \quad c > d_1, \quad 4c(c-d_1)(k_1+k_2) \leqslant (\gamma c - d_2)^2. \tag{21}$$

If these inequalities are satisfied, all swarm particles will collapse into the origin and will remain motionless, apart from stochastic fluctuations.

The evolution of such a collapse, according to Eqs. (14)–(20) is shown in Fig. 1. More interesting is equilibrium E_2, which is stable under the conditions $\gamma > \frac{d_2}{d_1}$, $c < d_1$, and

$$2(d_1 - c)(k_1 + k_2) < (d_1\gamma - d_2)\left(d_1 + \gamma - \frac{d_2}{d_1}\right). \tag{22}$$

We used the Routh-Hurwitz theorem [8] to determine the stability properties. Exactly one eigenvalue is identical to zero independent of the parameter choice. E_2 is in fact not an isolated fixpoint, but a one-dimensional continuum of equilibria, since U_0 and V_0 are linearly dependent. This corresponds to exactly one zero eigenvalue of the Jacobian, whereas the other six eigenvalues are stable under the conditions given above. According to stable manifold theory, for every chosen value of V_0, there exists a stable manifold on which the trajectory

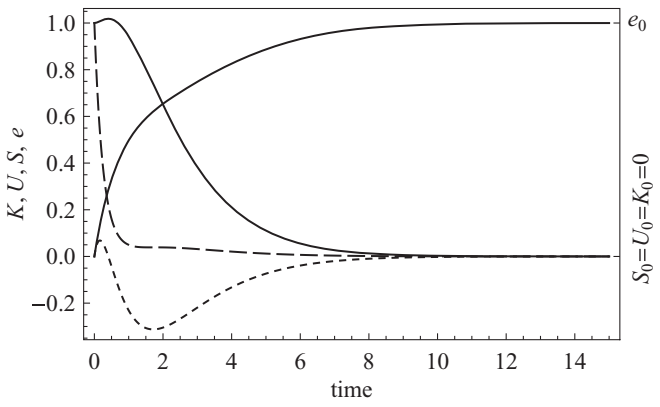

FIG. 1. Time evolution of the deterministic equations (14)–(20) for K (long dashed line), U (solid line), S (short dashed line), and e (solid line) for the collapse of the swarm to the minimum of the external potential (E_1). Simulation parameters are $c = 1$, $d_1 = 1/2$, $d_2 = 1$, $\gamma = 2$, $k_1 = 1/4$, and $k_2 = 1/4$. The initial conditions are $U(0) = K(0) = 1/c = 1$ and $S(0) = e(0) = T(0) = V(0) = L(0) = 0$. ($T$, V, and L remain equal to zero in time and are not shown for simplicity of presentation, also in the following simulations.)

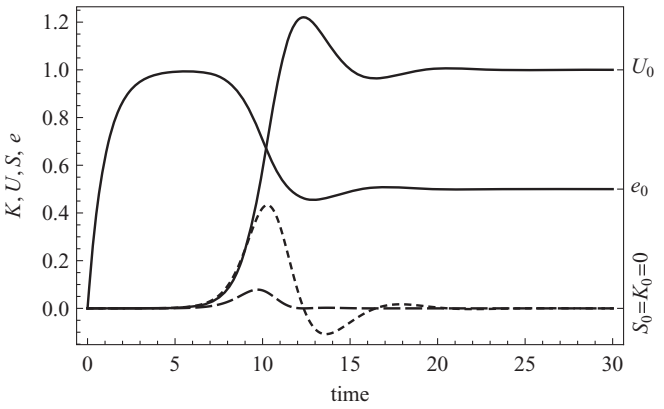

FIG. 2. Relaxation to a jellylike state (E_2), according to Eqs. (14)–(20). Simulation parameters are $c = 1$, $d_1 = 2$, $d_2 = 1$, $\gamma = 2$, $k_1 = 1/2$, and $k_2 = 1/2$. The initial conditions are $U(0) = 10^{-4}$ and the remaining six variables equal zero.

converges asymptotically to equilibrium E_2. (In the following, we have set $V_0 = 0$.) In this second equilibrium state, the whole swarm is frozen to a static configuration, which satisfies $\sum_{i=1}^{n} \mathbf{x}_i^2 = \text{const}$.

One example of such a frozen state is shown in Fig. 2. If the swarm is subject to noise influences, the individual particles will oscillate around their fixed positions.

Dynamic states of the swarm are realized when reaching equilibrium E_3, since the sum of all kinetic energies then has a nonvanishing value. E_3 is found to be stable under the conditions $d_1 < 0$, $\gamma < \frac{d_1}{c}$, and

$$\gamma d_1(k_1 + k_2) < \frac{d_2^2(d_2 - d_1\gamma)\left(2d_2^2 - 2d_1d_2\gamma + cd_1\gamma\right)}{\left(4d_2^3 - 6\gamma d_1 d_2^2 + 3\gamma^2 d_1^2 d_2 - c\gamma^3 d_1^2\right)}. \tag{23}$$

Again, one zero eigenvalue appears and the Ruth-Hurwitz theorem was applied to determine the stability of the system. Note that in E_3 the two equations for U_0 and L_0 define two planes in the space of variables (U, V, L), whose intersection gives a line, i.e., a manifold of equilibria with dimension one, which matches the overall number of zero eigenvalues. Hence the theory of stable manifolds again ensures that we have converging trajectories to E_3 under the conditions given above. If these inequalities are satisfied by a specific parameter choice, the swarm reaches equilibrium E_3 and appears to simulate the amorphous behavior of insect swarms.

In Fig. 3, the equilibrating process is explicitly shown. Numerical simulation of our original equations (9)–(11) gives the full N-particle time evolution. Video examples for the corresponding motion of swarm particles are available online [9] for all three distinct cases E_1 (collapse), E_2 (jellylike state), and E_3 (amorphous swarming). Simulations of the stochastic system (9)–(11) in two dimensions are shown, which—apart from small noise perturbations—agree nicely with our simulations of the deterministic equations (14)–(20).

III. SUMMARY AND OUTLOOK

We have formulated swarm dynamics based on canonical active Brownian motion. We assumed the existence of a single global internal energy of the swarm and postulated its coupling to the full Hamiltonian of the system. We studied

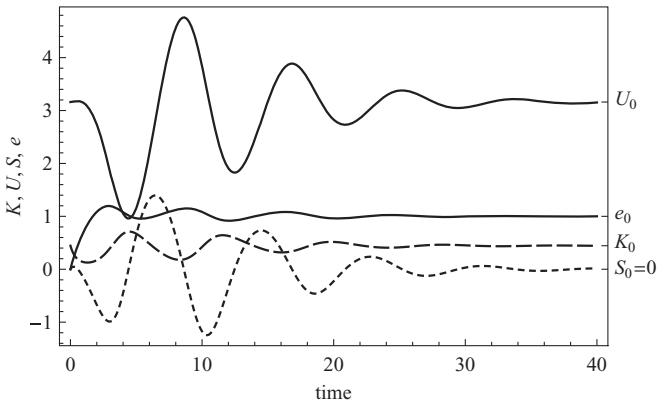

FIG. 3. State of amorphous swarming (E_3), according to Eqs. (14)–(20). Simulation parameters are $c = 1/3, d_1 = -1, d_2 = 1$, $\gamma = 1, k_1 = 1/16$, and $k_2 = 1/128$. The initial conditions for U and K equal the expected equilibrium values while the remaining five variables equal zero.

the case where all particles were harmonically attracted to their center of mass and were all moving in a landscape of harmonic shape. The system was simulated numerically and analytic understanding was obtained by studying stability properties of equilibria. We found three distinct forms of the large time behavior of the swarm, showing collapse to a point, freezing to a static configuration, and amorphous swarming behavior, respectively.

Several applications of our present investigation appear rewarding for future activities. One of them is the study of particle swarms in given (genuinely complicated) potentials and the development of a new algorithm for particle swarm optimization theory. Particle swarm optimization is a stochastic optimization technique inspired by social behavior of bird flocking or fish schooling. Systems are typically made up of a population of simple agents interacting locally with one another and with their environment. It is precisely here that the concept of active Brownian multiparticle systems will be applied. A new type of swarm will be generated, with its time evolution directed toward the minimum of the given potential.

Furthermore, we want to mention possible applications in statistical physics on which we are currently working. The global internal energy characterizes a subsystem and distinguishes it from the environment. By that we have a specific model of energy exchange between an extended system and its surroundings. Addressing the question of quantizing open systems, we are presently applying the Ford-Kac-Mazur formalism [10] to this situation. So far, to the best of our knowledge, no quantized version of active Brownian motion exists and we hope to make a first step in this direction. It eventually will lead to a unique quantum Langevin description of open systems.

In a different approach we envisage the study of crystal or lattice systems in solid-state physics. Complex motion patterns due to canonical active Brownian motion will generalize the dynamics of usual lattice vibrations. In the vicinity of equilibrium points we attempt to perform a semiclassical quantization of these lattice systems.

ACKNOWLEDGMENTS

We thank Josef Hofbauer for valuable advice on dynamical systems. In addition, we are grateful for financial support within the Agreement on Cooperation between the Universities of Vienna and Zagreb.

[1] F. Schweitzer, W. Ebeling, and B. Tilch, Phys. Rev. Lett. **80**, 5044 (1998).
[2] F. Schweitzer, *Brownian Agents and Active Particles* (Springer, Berlin, 2003).
[3] A. Glück, H. Hüffel, and S. Ilijić, Phys. Rev. E **79**, 021120 (2009).
[4] J. Kennedy, R. C. Eberhart, and Y. Shi, *Swarm Intelligence* (Academic, San Diego, 2001).
[5] T. Vicsek, A. Czirok, E. Ben-Jacob, I. Cohen, and O. Shochet, Phys. Rev. Lett. **75**, 1226 (1995).
[6] A. Czirok, A. L. Barabasi, T. Vicsek, Phys. Rev. Lett. **82**, 209 (1999).
[7] F. Schweitzer, W. Ebeling, and B. Tilch, Phys. Rev. E **64**, 021110 (2001).
[8] E. X. DeJesus and Ch. Kaufman, Phys. Rev. A **35**, 5288 (1987).
[9] [http://sail.zpf.fer.hr/active/swarm].
[10] G. W. Ford, M. Kac, and P. Mazur, J. Math. Phys. **6**, 447 (1965).