STOCHASTIC QUANTIZATION

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Abstract:

Stochastic quantization provides a novel and interesting connection between quantum field theory and statistical mechanics, with new applications also in numerical simulations of field theories.

This review article tries to present as broad as possible the most relevant features of the Parisi–Wu approach of stochastic quantization. It covers scalar, gauge, tensor and string field theories, it discusses fermions and explores the intrinsic connection of stochastic quantization to supersymmetry. Further topics are the large N-limit, stochastic quantization in Minkowski space and stochastic regularization. Finally we describe some rather recent developments concerning the numerical possibilities associated with stochastic quantization.

1. Introduction

Stochastic quantization is a comparatively new method for the quantization of field theories. It was first proposed in a highly original paper by Parisi and Wu [1.1], and has since grown into a useful tool in several areas of quantum field theory. *The main idea of stochastic quantization is to view Euclidean quantum field theory as the equilibrium limit of a statistical system coupled to a thermal reservoir. This system evolves in a new fictitious time direction t until it reaches the equilibrium limit as $t \to \infty$. The coupling to a heat reservoir is simulated by means of a stochastic noise field which forces the original Euclidean field to wander randomly on its manifold. In the equilibrium limit stochastic averages become identical to ordinary Euclidean vacuum expectation values.*

Since the original paper by Parisi and Wu in 1981, a large number of new results, generalized schemes, new applications, etc., have been found. It is, therefore, appropriate at this stage to try to summarize these more recent developments, while at the same time explaining the original ideas in, hopefully, a somewhat new light.

To begin, we first give in section 2 a brief introduction to stochastic processes as they have been developed in statistical mechanics. We try to avoid the use of unnecessary jargon from this field, and concentrate just on those aspects of stochastic methods which are really needed for stochastic quantization.

In section 3 following this short review of stochastic processes, we go through the stochastic quantization of a scalar field theory in detail. Stochastic perturbation theory is set up, and we present an induction proof of the equivalence to ordinary perturbation theory. Also discussed is a formal *non-perturbative* proof of the equivalence to ordinary path integral quantization. This proof is based on the Fokker–Planck equation.

In section 4 we turn to the question of (Abelian and non-Abelian) gauge fields, one of the main motivations of the original study by Parisi and Wu. An example from scalar QED is treated in detail, thus exemplifying most of the main features of stochastic quantization in connection with gauge fields. If no gauge-fixing term is introduced (and one of the surprising advantages of stochastic quantization is that this is not necessary) then gauge *non*-invariant Green functions are found not to have an equilibrium limit – at least order by order in perturbation theory. However, *gauge invariant quantities*, when computed in stochastic quantization, *do* have equilibrium limits. These equilibrium limits have always been found to agree with vacuum expectation values as they are computed in ordinary (gauge fixed) perturbation theory. In the non-Abelian case we show how ghost fields are avoided, and how, nevertheless, their 'contribution' is included automatically. We then turn to a discussion of stochastic gauge fixing, and axial gauge Langevin equations.

In the following section 5 we discuss in detail higher rank tensor and string fields, and in section 6 fermions within stochastic quantization schemes. We then proceed in section 7 to connect stochastic quantization with supersymmetry (through the Nicolai map), and discuss superfield formulations of stochastic quantization.

The large-N limit of field theories, be they gauged or not, turns out to simplify considerably in the language of stochastic quantization; this is treated in some detail in section 8. Attempts to formulate stochastic quantization directly in Minkowski space are discussed in section 9.

Next we review in section 10 some of the main features of stochastic *regularization*, a regularization scheme based directly on stochastic quantization. Its principal advantage is that it can be made to respect manifestly all symmetries during the regularization.

Finally, in section 11, we describe some rather recent developments concerning the *numerical* possibilities associated with stochastic quantization. Section 12 contains our conclusions.

Our selection of topics included is of course slightly biased by our own interests, and by the necessarily subjective judgement of what has been the most important developments. As a matter of fact the amount of results within the Parisi–Wu approach of stochastic quantization has grown to impressive proportions and a detailed exposition consequently acquires considerable volume. We therefore decided not to cover, for example, many interesting alternative approaches. The microcanonical quantization prescription [1.2] is very similar to stochastic quantization, and various other 'fifth-time' approaches, which share many features with stochastic quantization, have been suggested [1.3, 1.4]. We regret that due to lack of space we have not been able to include a description of these somewhat similar techniques. This applies as well to the stochastic quantization method of 'dimensional reduction' [1.5], which has been developed shortly after the original paper of Parisi and Wu, and to the operator formulation of stochastic quantization [1.6]. Nelson's stochastic description of quantum mechanics [1.7] and its generalizations to field theory [1.8] will not be covered either. Several good reviews on this subject already exist; see, for example, ref. [1.9]. Let us remark also that the concept of stochasticity used in this review does not refer to quantum stochastic processes [1.10] nor to the constructive approach for quantum fields by Euclidean Markov fields; see, e.g. [1.11] and references therein.

Instead, we shall here focus exclusively on stochastic quantization as it arises from the original formulation of Parisi and Wu. Let us mention finally that a few other reviews exist on this subject; see ref. [1.12].

2. Basic concepts of stochastic processes

The concept of stochasticity plays a key role in the field of statistical mechanics. It is crucial in the theory of Brownian motion, it enters into microscopic descriptions of fluid mechanics, and it can be used to model the approach towards equilibrium of a large set of physical systems. In general, stochastic variables are used to represent the (random) thermal fluctuations arising from a heat reservoir background.

There exists already an enormous amount of literature and a long list of excellent reviews on this subject. For this, we can refer the reader to the partial selection of literature we have listed in refs. [2.1–2.3]. References to the original literature on this subject can be found there too.

The aim of this section is to introduce the reader to only those basic concepts of stochastic processes which we find are needed in order to understand the method of stochastic quantization. Most of these concepts (the Langevin equation, the Fokker–Planck equation, Markovian and non-Markovian processes, etc.) were developed at the beginning of this century. Through stochastic quantization they have now been 'rediscovered', and found a new domain of applicability in quantum field theory. Before getting to that point, the 'classical' theory of stochastic processes should be understood first. This section, then, can serve as a brief introduction to that theory.

In order to define the concept of a stochastic process we need to acquire a little vocabulary of termini technici. Let us start with the notion of a *stochastic variable*. This is an object X defined by a set of possible values x (its 'range' or 'sample space') and a probability distribution $P(x)$ over this set. We will mainly be interested in stochastic variables with a continuous range I and define the probability density as non-negative

$$P(x) \geq 0 \tag{2.1}$$

and normalized

$$\int_{\mathrm{I}} P(x)\, \mathrm{d}x = 1\,. \tag{2.2}$$

We allow generally for P being defined in the sense of tempered distributions. The probability that X has a value between x and $x + \mathrm{d}x$ is given by $P(x)\,\mathrm{d}x$.

If X is a well-defined stochastic variable, any quantity Y related to X by $Y = f(X)$ is also a well-defined stochastic variable. These quantities Y may be any kinds of mathematical objects (for a rigorous definition, see e.g. [2.3]), in particular also functions of an additional variable t, usually associated with time

$$Y(t) = f(t, X)\,. \tag{2.3}$$

We call such an object a *stochastic process*. Thus a stochastic process is simply a function of the time and a stochastic variable.

Next we define the expectation value of Y by

$$\langle Y(t) \rangle = \int Y_x(t)\, P(x)\, \mathrm{d}x\,. \tag{2.4}$$

Here $Y_x(t)$ is obtained from Y by inserting for X one of its possible values x

$$Y_x(t) = f(t, x)\,. \tag{2.5}$$

We call $Y_x(t)$ a sample function or realization of the stochastic process.

We need now to define the joint probability distribution

$$P_n(y_1, t_1; y_2, t_2; \ldots y_n, t_n) = \int \delta(y_1 - Y_x(t_1))\, \delta(y_2 - Y_x(t_2)) \cdots \delta(y_n - Y_x(t_n))\, P(x)\, \mathrm{d}x \tag{2.6}$$

which is the probability density for Y to take the value y_1 at t_1, y_2 at $t_2, \ldots,$ and y_n at t_n. Similarly one defines the conditional probability

$$P_{l|k}(y_{k+1}, t_{k+1}; \ldots y_{k+l}, t_{k+l} | y_1, t_1; \ldots y_k, t_k)$$
$$= \frac{P_{k+l}(y_1, t_1; \ldots y_k, t_k; y_{k+1}, t_{k+1}; \ldots y_{k+l}, t_{k+l})}{P_k(y_1, t_1; \ldots y_k, t_k)} \tag{2.7}$$

which is the probability density for Y to take the values Y_{k+1} at $t_{k+1}, \ldots, Y_{k+l}$ at t_{k+l}, provided that its value is y_1 at $t_1, \ldots,$ and y_k at t_k.

To go through all these definitions seems quite unavoidable if we now want to describe the important class of stochastic processes that have the Markov property. We define a *Markov process* as the stochastic process where the conditional probability that Y takes the value y_n at t_n is uniquely determined by just the value y_{n-1} at t_{n-1}; it is not affected by any values at earlier times,

$$P_{1|n-1}(y_n, t_n | y_1, t_1; y_2, t_2; \ldots y_{n-1}, t_{n-1}) = P_{1|1}(y_n, t_n | y_{n-1}, t_{n-1}) . \tag{2.8}$$

The oldest and best known example of a Markov process in physics is Brownian motion, as described by Langevin's [2.4] equation

$$m \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{v}(t) = -\alpha\, \boldsymbol{v}(t) + \boldsymbol{\eta}(t) . \tag{2.9}$$

Here m denotes the mass of a particle immersed in a fluid with a friction coefficient α, and η is the stochastic force vector. $\eta(t)$ provides a phenomenological representation of the many collisions between the Brownian particle and the molecules of the fluid. Each of these collisions implies a change of the particle's velocity. The number of collisions will essentially depend just on the present velocity of the particle and not on its earlier values. Thus the velocity of the Brownian particle is a Markov process and, conversely, so is the stochastic force vector η. It may further be assumed that η has a Gaussian distribution which implies together with the Markov property that

$$P(\eta) = \frac{\exp\{-(1/4\lambda) \int \mathrm{d}\tau\, \eta^2(\tau)\}}{\int \mathrm{D}\eta \exp\{-(1/4\lambda) \int \mathrm{d}\tau\, \eta^2(\tau)\}} \tag{2.10}$$

where λ is a constant, to be specified later. It then follows from the Gaussian distribution (2.10) that

$$\begin{aligned}
&\langle \eta_i(t) \rangle = 0 \\
&\langle \eta_i(t)\, \eta_k(t') \rangle = 2\lambda\, \delta_{ik}\, \delta(t-t') \\
&\langle \eta_1(t_1) \cdots \eta_{2n+1}(t_{2n+1}) \rangle = 0 \\
&\langle \eta_1(t_1) \cdots \eta_n(t_n) \rangle = \sum_{\substack{\text{pair}\\ \text{combin.}}} \prod_{\text{pairs}} \langle \eta_j(t_j)\, \eta_k(t_k) \rangle .
\end{aligned} \tag{2.11}$$

The presence of the Dirac delta function in the noise correlations (2.11) reflects the Markov property (2.8) of the stochastic process in a complementary point of view (for an explicit presentation of this connection, see, for example, ref. [2.5]). It should be added that from a strict mathematical point of view the formulation of a stochastic process with delta-correlated noise (which actually is defined on a sample space of distributions) is problematic. The appropriate way to deal with this difficulty is to introduce the mathematically well-defined *Wiener process*

$$W(t) = \int_0^t \eta(\tau)\, \mathrm{d}\tau \tag{2.12}$$

with correlations obtained from (2.11) as

$$\begin{aligned} &\langle W(t) \rangle = 0 \\ &\langle W(t)\, W(t') \rangle = \min\,(t, t')\,. \end{aligned} \tag{2.13}$$

We interpret the Langevin equation (2.9) as an integral equation

$$v(t) - v(t_0) = -\int_{t_0}^{t} \alpha\, v(\tau)\, \mathrm{d}\tau + \int_{t_0}^{t} \mathrm{d}W \tag{2.14}$$

where we have symbolically inserted

$$\mathrm{d}W = \eta\, \mathrm{d}\tau\,.$$

Whereas the first integral on the right-hand side of eq. (2.14) can be understood as a familiar Riemann–Stieltjes integral, more care is required for a proper definition of the second one. In fact, it can be shown [2.3] that in the general case of an integral of the type $\int_{t_0}^{t} G(W)\, \mathrm{d}W$ the approximation of the integral by appropriate sums of the form

$$S_n = \sum_{i=1}^{n} G(\tau_i)\, (W(t_i) - W(t_{i-1})) \tag{2.15}$$

with

$$t_0 \le t_1 \le \ldots \le t_n = t, \qquad t_{i-1} \le \tau_i \le t_i \tag{2.16}$$

is *not* unique and depends on the choice of the intermediate points. Apart from the pure mathematical argument we may understand this feature intuitively as a consequence of the unbounded variation of the sample functions of W.

Therefore, in order to obtain a unique definition of the integral it is necessary to *define* specific intermediate points τ_i. For example, if we put

$$\tau_i = (1 - a)\, t_{i-1} + a t_i, \qquad 0 \le a \le 1 \tag{2.17}$$

two standard choices have been discussed, namely

$$a = 0 \qquad \text{or} \qquad \tau_i = t_{i-1} \tag{2.18}$$

which defines the *Ito stochastic calculus* [2.6] and

$$a = \tfrac{1}{2} \quad \text{or} \quad \tau_i = (t_{i-1} + t_i)/2 \tag{2.19}$$

corresponding to the *Stratanovich stochastic calculus* [2.7]. It turns out that from the rigorously mathematical point of view the Ito calculus is preferred; for more details we refer the reader to ref. [2.3]. A peculiar feature of this calculus is that, symbolically,

$$\mathrm{d}W(t)^2 = \mathrm{d}t, \qquad \mathrm{d}W(t)^{2+n} = 0 \tag{2.20}$$

[which is properly defined as an integral over an arbitrary function $G(W)$]. This then implies among other things that the usual rules of differentiating and performing variable transformations are changed, because differentials have to be expanded up to 'order $(\mathrm{d}W)^2$'.

On the contrary, the Stratanovich calculus allows [2.3] one to formally perform the usual manipulations of differential calculus. Moreover, from a physical point of view, the midpoint prescription appears naturally when approaching the singular white noise limit from a smeared, symmetric distribution.

Our strategy for this report is as follows: if not otherwise explicitly stated (and this happens in fact only in section 5.2) all our manipulations are understood in the sense of Stratanovich. We will furthermore even revert formally to a formulation in terms of the noise field η.

If a rigorous mathematical interpretation of all the forthcoming manipulations is desired, one could introduce the Wiener process $W(t)$ as in eq. (2.14) and convert (by standard techniques, see [2.3]) to the Ito formulation.

With all these interpretational reservations in mind, we may now nevertheless continue innocently and formally solve eq. (2.9) as

$$v(t) = \exp\left\{-\frac{\alpha}{m}\, t\right\} v(0) + \frac{1}{m} \int_0^t \exp\left\{-\frac{\alpha}{m}\,(t-\tau)\right\} \eta(\tau)\, \mathrm{d}\tau\,. \tag{2.21}$$

We assume for simplicity that $v(0) = 0$, and we may then calculate the average kinetic energy of the Brownian particle as

$$\begin{aligned} \tfrac{1}{2} m \langle v(t)\, v(t)\rangle &= \tfrac{1}{2} m\, \frac{1}{m^2} \int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\tau \exp\left\{-\frac{\alpha}{m}\,(2t - \tau - \tau')\right\} \langle \eta(\tau)\, \eta(\tau')\rangle \\ &= \frac{3}{2}\, \frac{\lambda}{\alpha} \left(1 - \exp\left\{-\frac{2\alpha}{m}\, t\right\}\right). \end{aligned} \tag{2.22}$$

It is crucial to observe that for large times the particle comes to thermal equilibrium with the fluid and should have the average kinetic energy of $\frac{3}{2}\, kT$. This in turn then implies from eq. (2.22)

$$\lambda = kT\alpha \tag{2.23}$$

which is known as the fluctuation-dissipation theorem. This theorem generally relates the noise correlation to the drift term.

We close this short introduction by observing that the conditional probability distribution

$P(vt|v_0t_0) \equiv P(v, t)$ satisfies the so-called Fokker–Planck equation [2.8] (we put $m = \lambda = \alpha = 1$)

$$\frac{\partial}{\partial t} P(v, t) = \frac{\partial}{\partial v}\left(v + \frac{\partial}{\partial v}\right) P(v, t) . \tag{2.24}$$

An easy derivation of the Fokker–Planck equation from the Langevin equation (2.9) can be given as follows:

Let us consider the average of an arbitrary function $f(v(t))$, with $v(t)$ as a solution of (2.9). We can write the average either as an integral over the white noise measure or over the probability distribution P

$$\langle f(v(t))\rangle = \int \mathrm{D}\eta \exp\left\{-\tfrac{1}{4}\int \eta^2(\tau)\, \mathrm{d}\tau\right\} f(v(t)) = \int \mathrm{d}v\, f(v)\, P(v, t) . \tag{2.25}$$

Taking the time derivative and substituting the Langevin equation we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle f(v(t))\rangle = \left\langle \frac{\delta f}{\delta v}\, \dot{v}\right\rangle = \left\langle \frac{\delta f}{\delta v}\,(-v + \eta)\right\rangle . \tag{2.26}$$

We now observe that generally

$$\langle g(v(t))\, \eta(t)\rangle = 2\left\langle \frac{\delta g}{\delta v(t)}\, \frac{\delta v(t)}{\delta \eta(t)}\right\rangle \tag{2.27}$$

which follows from

$$\langle g\eta\rangle = \int \mathrm{D}\eta \exp\left\{-\tfrac{1}{4}\int \eta^2(\tau)\, \mathrm{d}\tau\right\} g\eta = -2\int \mathrm{D}\eta \left(\frac{\delta}{\delta \eta} \exp\left\{-\tfrac{1}{4}\int \eta^2\, \mathrm{d}\tau\right\}\right) g \tag{2.28}$$

and a partial integration. Furthermore,

$$v(t) = \int_0^t \mathrm{e}^{-(t-\tau)}\, \eta(\tau)\, \mathrm{d}\tau = \int_0^\infty \theta(t-\tau)\, \mathrm{e}^{-(t-\tau)} \eta(\tau)\, \mathrm{d}\tau \tag{2.29}$$

so that

$$\delta v(t)/\delta \eta(t) = \theta(0) = \tfrac{1}{2} \tag{2.30}$$

using the midpoint prescription. So, it follows that

$$\langle g(v(t))\, \eta(t)\rangle = \langle \delta g/\delta v(t)\rangle . \tag{2.31}$$

We introduce the probability distribution P and arrive at

$$\int \mathrm{d}v \left[-\frac{\partial f}{\partial v}\, v + \frac{\partial^2 f}{\partial v^2}\right] P = \int \mathrm{d}v\, f(v)\, \dot{P} \tag{2.32}$$

which can be recast upon a partial integration into

$$\int \mathrm{d}v\, f(v)\, \frac{\partial}{\partial v}\left(v + \frac{\partial}{\partial v}\right) P = \int \mathrm{d}v\, f(v)\, \dot{P} \tag{2.33}$$

and implies the Fokker–Planck equation (2.24).

As a final remark we read off the stationary solution of (2.24)

$$P^{\mathrm{eq}} = \exp(-v^2/2) \tag{2.34}$$

which is (up to a normalization) the equilibrium Boltzmann distribution for the Brownian particle.

3. Stochastic quantization of scalar field theory

3.1. The approach of Parisi and Wu

In this section we would like to explain the main features of the stochastic quantization method of Parisi and Wu [3.1].

The starting point of our discussion is the important analogy between Euclidean quantum field theory and classical statistical mechanics. There exist various excellent and rigorous presentations of this analogy (see, for example [3.2]) and we would like just to recall the most essential and basic features of it: put into simple words the Euclidean path integral measure is closely related to the Boltzmann distribution of a statistical system in equilibrium. This implies that Euclidean Green functions may be interpreted as correlation functions of a statistical system in equilibrium. Specifically, the Euclidean Green functions are obtained as

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \frac{\int \mathrm{D}\phi \exp\{-(1/\hbar) S_{\mathrm{E}}\}\, \phi(x_1) \cdots \phi(x_n)}{\int \mathrm{D}\phi \exp\{-(1/\hbar) S_{\mathrm{E}}\}} \tag{3.1}$$

where S_{E} denotes the Euclidean action. If we identify $1/\hbar = 1/kT$ we may, however, interpret eq. (3.1) as a statistical expectation value with respect to a system in equilibrium at a temperature T (in the following we will choose units such that $\hbar = kT = 1$).

The basic idea of stochastic quantization is to consider the Euclidean path integral measure $\exp\{-(1/\hbar)\, S_{\mathrm{E}}\} / \int \mathrm{D}\phi \exp\{-(1/\hbar)\, S_{\mathrm{E}}\}$ *as the stationary distribution of a stochastic process.*

In order to get a better intuitive understanding of this idea, let us observe that in a computer simulation of Euclidean field theory (as, for example, in Monte Carlo calculations) equilibrium configurations have to be generated, which are used in an averaging procedure to obtain the correlation functions. It now happens that these equilibrium configurations can only be generated after some lapse of computer (!) time. One may then take a different attitude, and calculate correlation functions while the system is still in non-equilibrium. Their desired equilibrium values can be extracted by a careful study of their time evolution up to large computer times (for details and the actual realization of such a program, see section 11).

Abstracting these ideas, Parisi and Wu formulated the following concept of stochastic quantization [3.1].

i) One supplements the fields $\phi(x)$ with an additional coordinate, the 'fictitious' time t

$$\phi(x) \to \phi(x, t) . \tag{3.2}$$

Here $x = (x_0, x_1, \ldots, x_{n-1})$ denotes a vector in the n-dimensional Euclidean space. The fictitious time t should not be confused with the usual Euclidean time x_0. One imagines the fields being coupled to a (fictitious) heat reservoir at temperature T. The system should then reach an equilibrium distribution for large fictitious time t.

ii) One demands that the fictitious time evolution of ϕ is described by a stochastic differential equation that allows for relaxation to equilibrium, as, for example, the Langevin equation (see section 2)

$$\partial\phi(x, t)/\partial t = -\delta S_E/\delta\phi(x, t) + \eta(x, t) . \tag{3.3}$$

Here S_E denotes the Euclidean action of the theory under study, which actually is the appropriate generalization of the usual one, including also an integration over the fictitious time coordinate

$$S_E = \int dt\, d^n x\, \mathscr{L}\Big(\phi(x, t), \frac{\partial}{\partial x}\, \phi(x, t)\Big) . \tag{3.4}$$

The drift term in eq. (3.3) reads explicitly

$$\begin{aligned} \frac{\delta S_E}{\delta\phi(x, t)} = \int dt'\, d^n x' \Big[& \frac{\delta}{\delta\phi(x, t)}\, \mathscr{L}\Big(\phi(x', t'), \frac{\partial}{\partial x'}\, \phi(x', t')\Big) \\ & - \partial_\mu \frac{\delta}{\delta(\partial_\mu\phi(x, t))}\, \mathscr{L}\Big(\phi(x', t'), \frac{\partial}{\partial x'}\, \phi(x', t')\Big)\Big] . \end{aligned} \tag{3.5}$$

It should be remarked that

$$\delta S_E/\delta\phi(x, t) = 0 \tag{3.6}$$

is just the classical field equation (apart from the fact that the fields are also t-dependent)

In eq. (3.3) a Gaussian 'white' noise has been introduced with correlations given by

$$\begin{aligned} &\langle\eta(x, t)\rangle_\eta = 0 \\ &\langle\eta(x_1, t_1)\, \eta(x_2, t_2)\rangle_\eta = 2\, \delta^n(x_1 - x_2)\, \delta(t_1 - t_2) \end{aligned} \tag{3.7}$$

and in general

$$\begin{aligned} &\langle\eta(x_1, t_1) \cdots \eta(x_{2k+1}, t_{2k+1})\rangle_\eta = 0 \\ &\langle\eta(x_1, t_1) \cdots \eta(x_{2k}), t_{2k})\rangle_\eta = \sum_{\substack{\text{possible pair}\\ \text{combinations}}} \prod_{\text{pairs}} \langle\eta(x_i, t_i)\, \eta(x_j, t_j)\rangle_\eta . \end{aligned} \tag{3.8}$$

The correlations are defined by performing averages over the noise η with Gaussian distribution

$$\langle \dots \rangle_\eta = \frac{\int D\eta (\dots) \exp\left\{ -\frac{1}{4} \int d^n x \, dt \, \eta^2(x,t) \right\}}{\int D\eta \exp\left\{ -\frac{1}{4} \int d^n x \, dt \, \eta^2(x,t) \right\}} . \tag{3.9}$$

iii) Given some initial condition at $t = t_0$, one has to solve the Langevin equation (3.3). We denote the solution by $\phi_\eta(x, t)$ indicating the η dependence explicitly.

Correlation functions of ϕ_η are defined as before by performing Gaussian averages over η

$$\langle \phi_\eta(x_1, t_1) \cdots \phi_\eta(x_k, t_k) \rangle_\eta = \frac{\int D\eta \exp\left\{ -\frac{1}{4} \int d^n x \, dt \, \eta^2(x,t) \right\} \phi_\eta(x_1, t_1) \cdots \phi_\eta(x_k, t_k)}{\int D\eta \exp\left\{ -\frac{1}{4} \int d^n x \, dt \, \eta^2(x,t) \right\}} . \tag{3.10}$$

The central assertion in stochastic quantization is that in the limit $t \to \infty$ equilibrium is reached, and that the (equal time) correlation functions of ϕ_η tend to the corresponding quantum Green functions, i.e.

$$\lim_{t\to\infty} \langle \phi(x_1, t) \cdots \phi(x_k, t) \rangle_\eta = \langle \phi(x_1) \cdots \phi(x_k) \rangle . \tag{3.11}$$

This equivalence can be shown for various models by different methods, as will be discussed in detail in this report. The basic features of these equivalence proofs are stated in section 3.3 (see also section 7.3).

As explained already in section 2 one may, as an alternative to the Langevin approach, adhere to a Fokker–Planck formulation. In this way the stochastic averages are obtained with respect to a functional integration over a probability distribution $P(\phi, t)$

$$\langle \phi_\eta(x_1, t) \cdots \phi_\eta(x_k, t) \rangle_\eta = \int D\phi \, P(\phi, t) \, \phi(x_1) \cdots \phi(x_k) \tag{3.12}$$

where P satisfies the Fokker–Planck equation (which can be proven, e.g. by appropriate generalization of the procedure in section 2)

$$\frac{\partial P}{\partial t} = \int d^n x \, \frac{\delta}{\delta \phi(x,t)} \left(\frac{\delta S_E}{\delta \phi(x,t)} + \frac{\delta}{\delta \phi(x,t)} \right) P . \tag{3.13}$$

Additionally, we have to give some initial condition as, for example,

$$P(\phi, 0) = \prod_x \delta(\phi(x)) . \tag{3.14}$$

The formal connection between eqs. (3.10) and (3.12) may be obtained by defining P as

$$P(\phi, t) = \int D\eta \exp\left\{ -\frac{1}{4} \int d^n x' \, dt' \, \eta^2(x', t') \right\} \prod_y \delta[\phi(y) - \phi(y, t)] \tag{3.15}$$

and the approach to equilibrium [eq. (3.11)] (which will be discussed in section 3.3) is reflected as

$$\lim_{t\to\infty} P[\phi, t] \equiv P^{\mathrm{eq}}(\phi) = \frac{\exp(-S_{\mathrm{E}})}{\int \mathrm{D}\phi \, \exp(-S_{\mathrm{E}})} \,. \tag{3.16}$$

Let us remark that upon introduction of

$$L^* = -\int \mathrm{d}^n x \, \frac{\delta}{\delta\phi(x)} \left(\frac{\delta S_{\mathrm{E}}}{\delta\phi(x)} + \frac{\delta}{\delta\phi(x)} \right), \tag{3.17}$$

we may recast the Fokker–Planck equation in the simple form

$$\dot{P} = -L^* P \tag{3.18}$$

which has the formal solution

$$P(\phi, t) = \exp\{-L^*(t - t_0)\} \, P(\phi, t_0) \tag{3.19}$$

given some initial condition $P(\phi, t_0)$ at $t = t_0$. We note that we may as well write for an arbitrary functional $F[\phi]$ of ϕ

$$\begin{aligned} \langle F[\phi_\eta(t)] \rangle_\eta &= \int \mathrm{D}\phi \, F[\phi] \, P[\phi, t] \\ &= \int \mathrm{D}\phi \, F[\phi] \exp\{-L^*(t - t_0)\} \, P[\phi, t_0] \\ &= \int \mathrm{D}\phi \, (\exp\{-L(t - t_0)\} \, F[\phi]) \, P[\phi, t_0] \end{aligned} \tag{3.20}$$

where

$$L = -\int \mathrm{d}^n x \left(\frac{\delta}{\delta\phi(x)} - \frac{\delta S_{\mathrm{E}}}{\delta\phi(x)} \right) \frac{\delta}{\delta\phi(x)} \,. \tag{3.21}$$

It is possible to consider a more general Langevin equation by introducing a kernel K (see also sections 6 and 10),

$$\dot{\phi}(x, t) = -\int K(x, y) \, \frac{\delta S_{\mathrm{E}}}{\delta\phi(y)} \, \mathrm{d}^n y + \eta(x, t) \,. \tag{3.22}$$

Provided that the noise correlations are changed as well to

$$\langle \eta(x, t) \, \eta(x', t') \rangle_\eta = 2 \, K(x, x') \, \delta(t - t') \,, \qquad \text{etc.} \tag{3.23}$$

all the above conclusions can be shown to remain unchanged.

We would like to observe that only positive kernels are admissible, since otherwise the noise correlation (3.23) cannot be obtained by a Gaussian integral of correct sign. If we repeat the derivation of the Fokker–Planck equation and perform the appropriate changes we easily find

$$\frac{\partial P}{\partial t} = \int \mathrm{d}^4x\, \mathrm{d}^n y\, \frac{\delta}{\delta\phi(x,t)}\, K(x,y)\left(\frac{\delta S_{\mathrm{E}}}{\delta\phi(y,t)} + \frac{\delta}{\delta\phi(y,t)}\right)P \tag{3.24}$$

which indeed implies the same equilibrium limit as in (3.16).

Let us further remark that we could instead of introducing one noise field η which obeys eq. (3.23) consider *two* noise fields [3.3] η_1 and η_2 such that

$$\dot{\phi} = -\int \mathrm{d}^n y\, K(x,y)\, \frac{\delta S_{\mathrm{E}}}{\delta\phi(y)} + \eta_1 + \int \mathrm{d}^n y\, K(x,y)\, \eta_2(y,t) \tag{3.25}$$

where the only non-vanishing correlation is defined as given by

$$\langle \eta_1(x,t)\, \eta_2(x',t')\rangle_\eta = 2\, \delta^n(x-x')\, \delta(t-t')\,. \tag{3.26}$$

Interesting applications for the introduction of kernels will be discussed in section 6 on fermions and in section 10 on stochastic regularization. Whereas in the first case an indefinite kernel leads to convergence of an initially non-converging Grassmannian stochastic process, in the latter case the introduction of a kernel in the noise correlation is not accompanied by the appropriate change in the Langevin equation. This may (under specific conditions, see section 10) lead to regularized Feynman diagrams. We remark that also finite temperature Green functions can be discussed in stochastic quantization [3.21, 3.6, 9.6].

3.2. Perturbation theory and stochastic quantization

In this section we would like to discuss perturbative treatments of both the Langevin and Fokker–Planck approaches.

3.2.1. The Langevin formulation and stochastic diagrams

We will first show how to solve perturbatively the Langevin equation (3.3). We start with the case of a self-interacting scalar field, described by a Euclidean action

$$S_{\mathrm{E}} = \int \mathrm{d}^n x\, \mathrm{d}t\left[(\partial_\mu\phi)(\partial_\mu\phi) + m^2\phi^2 + \frac{\lambda}{3!}\,\phi^3\right]. \tag{3.27}$$

(The restriction to a cubic interaction is motivated just by reasons of simplicity; any polynomial interaction is allowed, see later.) The corresponding Langevin equation is easily found [remember that the drift term can directly be obtained from the classical field equation, cf. eq. (3.6)] and reads

$$\frac{\partial}{\partial t}\,\phi(x,t) = (\partial^2 - m^2)\,\phi - \frac{\lambda}{2}\,\phi^2 + \eta\,. \tag{3.28}$$

It will turn out to be convenient to work with the Fourier transform of the field. For the moment we

do not consider Fourier transforms with respect to t, so that we simply define

$$\phi(k, t) = \int \mathrm{d}^n x\, \mathrm{e}^{ikx}\, \phi(x, t)\,. \tag{3.29}$$

The Langevin equation now becomes

$$\frac{\partial}{\partial t}\, \phi(k, t) = -(k^2 + m^2)\, \phi(k, t) - \frac{\lambda}{2!} \int \frac{\mathrm{d}^n p\, \mathrm{d}^n q}{(2\pi)^n}\, \phi(p, t)\, \phi(q, t)\, \delta(k - p - q) + \eta(k, t)\,. \tag{3.30}$$

A perturbative solution of this equation can be obtained by transforming this stochastic differential equation into a stochastic integral equation and solving the latter by iteration up to some given order in the coupling constant.

As a first step, we determine the retarded Green function $G(k, t)$ of the stochastic differential equation, i.e. we solve

$$\dot{G} = -(k^2 + m^2)G + \delta(t) \tag{3.31}$$

with

$$G(k, t) = 0\,, \qquad t < 0\,. \tag{3.32}$$

One finds easily

$$G(k, t) = \exp\{-(k^2 + m^2)t\}\, \theta(t) \tag{3.33}$$

and the general solution of the free Langevin equation is given by

$$\phi(k, t) = \int_{-\infty}^{\infty} \mathrm{d}\tau\, G(k, t - \tau)\, \eta(k, \tau) + c \exp\{-(k^2 + m^2)t\} \tag{3.34}$$

where c is a constant to be fixed by an initial condition.

Writing this as

$$\phi(k, 0) = \phi_0(k) \tag{3.35}$$

we obtain

$$\int_{-\infty}^{0} \exp\{\tau(k^2 + m^2)\}\, \eta(\tau)\, \mathrm{d}\tau + c = \phi_0 \tag{3.36}$$

so that

$$\begin{aligned} \phi(k, t) &= \int_{-\infty}^{t} \exp\{-(t - \tau)\,(k^2 + m^2)\}\, \eta(\tau)\, \mathrm{d}\tau + \left(-\int_{-\infty}^{0} \exp\{\tau(k^2 + m^2)\} + \phi_0\right) \exp\{-t(k^2 + m^2)\} \\ &= \int_{0}^{t} \exp\{-(t - \tau)\,(k^2 + m^2)\}\, \eta(\tau) + \phi_0 \exp\{-t(k^2 + m^2)\}\,. \end{aligned} \tag{3.37}$$

We see that the dependence on the initial condition is damped out exponentially in t.

In the interacting case we finally obtain the exact integral equation

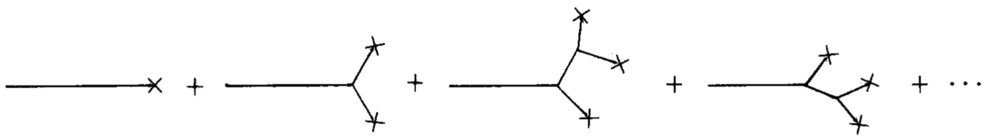
$$\phi(k,t)=\int_0^t \exp\{-(k^2+m^2)(t-\tau)\}\left[\eta(k,\tau)-\frac{\lambda}{2!}\int\frac{\mathrm{d}^n p\,\mathrm{d}^n q}{(2\pi)^n}\,\phi(p,\tau)\,\phi(q,\tau)\,\delta(k-p-q)\right]. \tag{3.38}$$

Solving this equation by iteration one arrives at a power series expansion of ϕ in the coupling λ, expressing ϕ as a certain function of the white noise η.

Symbolically we can represent ϕ as

$$\phi=\int G\eta+\frac{\lambda}{2!}\int\int\int G\,(G\eta)\,(G\eta)+\cdots \tag{3.39}$$

or graphically as

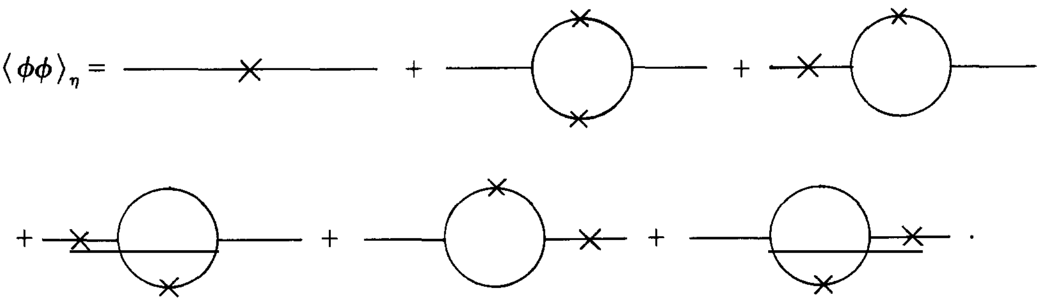
$\phi =$  (3.40)

Here we denote G by a line and η by a cross; integration over the momenta at the vertices and over the fictitious times at the vertices as well as at the crosses is included.

Next we observe that eqs. (3.7) are given in momentum space as

$$\langle\eta(k,t)\,\eta(k',t')\rangle_\eta=2(2\pi)^n\,\delta^n(k+k')\,\delta(t-t'). \tag{3.41}$$

Let us now consider an L-point function $\langle\phi_\eta(x_1,t)\cdots\phi_\eta(x_L,t)\rangle_\eta$ and substitute for ϕ its diagrammatical expansion (3.40). When the random averages over the η's are taken, all crosses are joined in all possible ways due to the Wick-decomposition property (3.8) of the white noise. In this way (we graphically denote the average over two noises by just one cross) diagrams are obtained, which we call 'stochastic diagrams',

$\langle\phi\phi\rangle_\eta =$  (3.42)

Each of these stochastic diagrams has the form of an ordinary Feynman diagram of the theory described by the action S, apart from crosses on the lines where two η's have been joined together.

Conversely, to every Feynman diagram there exist a number of stochastic diagrams with the same topology. Actually we will show in section 3.3 that the sum of all stochastic diagrams with the topology of a given Feynman diagram yields exactly this Feynman diagram.

We now calculate the lowest order contribution to the propagator

$$\begin{aligned} \mathrm{D}(k, t, t') &= \langle \phi_\eta(k, t)\, \phi_\eta(k', t')\rangle_\eta \\ &= \int_0^t \mathrm{d}\tau \int_0^{t'} \mathrm{d}\tau' \exp\{-(k^2+m^2)(t-\tau)\} \exp\{-(k'^2+m^2)(t'-\tau')\} \langle \eta(k,\tau)\,\eta(k',\tau')\rangle_\eta \\ &= (2\pi)^n\, 2 \int_0^{\min(t,t')} \mathrm{d}\tau \exp\{-(k^2+m^2)(t+t'-2\tau)\}\, \delta^n(k+k') \\ &= (2\pi)^n \frac{\delta^n(k+k')}{k^2+m^2} [\exp\{-|t-t'|\,(k^2+m^2)\} - \exp\{-(t+t')\,(k^2+m^2)\}] . \end{aligned} \tag{3.43}$$

Note that for equal fictitious times D is given by

$$D(k, t, t) = (2\pi)^n \frac{\delta^n(k+k')}{k^2+m^2} [1 - \exp\{-2t(k^2+m^2)\}] \tag{3.44}$$

which, as advertised, 'relaxes' for $t \to \infty$ to its Euclidean field theory limit

$$\lim_{t\to\infty} \langle \phi_\eta(k, t)\, \phi_\eta(k', t)\rangle_\eta = (2\pi)^n \frac{\delta^n(k+k')}{k^2+m^2} . \tag{3.45}$$

In our diagrammatic description a line in a stochastic diagram corresponds to a G-function (3.33), whereas a crossed line is just given by a D-function (3.43). The momentum δ-function in eq. (3.43) then guarantees momentum conservation at a cross, so that the momentum integration in stochastic diagrams is reduced just to the usual one of Feynman diagrams. We remark that cutting a stochastic diagram of an L-point function at all crosses gives L connected trees. Furthermore from every vertex there is a unique 'way out' by uncrossed lines.

Having shown the exponential fictitious time dependence of a crossed line – given by $D(k, t, t')$ in eqs. (3.43) (where t and t' denote the fictitious times of the neighbouring vertices) – and knowing the time dependence of an uncrossed line – given by $G(k, t-t')$ in eq. (3.33) – all that is left is to do the fictitious time integrations of all the vertices of the stochastic diagram.

In the following we will derive easy rules for how to perform these integrations, and how to take the limit $t \to \infty$ immediately [3.4].

For convenience of notation we substitute $p^2 + m^2 \to p^2$ for all momenta. Thus masses are trivially contained in our discussion. Let us consider a stochastic diagram belonging to some Feynman diagram, each vertex carrying a time τ_i. Owing to the absolute values of time differences in the D propagators it is convenient to split the fictitious time integrations into contributions of fixed-time orderings of the vertices as, for example,

$$\tau_{i_1} < \tau_{i_2} < \cdots < t . \tag{3.46}$$

Note that owing to the θ-function in the G's not all possible time orderings really contribute (see below, example 3).

Suppose now that we consider a stochastic diagram with N vertices and take, for example, the time ordering

$$\tau_1 < \tau_2 < \cdots < \tau_N < t\,. \tag{3.47}$$

We denote the set of momenta of the ith vertex by V_i. In performing the time integrations we will only keep the contributions of the upper integration limits, thus, for example, leaving out the second term in D [eqs. (3.43)]. As we will show, the neglected terms will be suppressed exponentially in t.

We start integrating over τ_1. Because of the time ordering (3.47) τ_1 is the smallest time and all exponents in G's and D's involving τ_1 have positive signs (except for non-leading terms dropped from the D's; these will not concern us for $t \to \infty$). We obtain

$$\int_0^{\tau_2} \mathrm{d}\tau_1 \exp\Big\{\tau_1 \sum_{\mathrm{V}_1} p^2\Big\} = \Big[\exp\Big\{\tau_2 \sum_{\mathrm{V}_1} p^2\Big\} - 1\Big] \Big/ \sum_{\mathrm{V}_1} p^2\,. \tag{3.48}$$

We drop the contribution from the lower boundary of integration, and continue with the τ_2 integration. Here we face two possibilities:

a) Vertex 2 is not a neighbour vertex of the first one, which means that all neighbour vertices of vertex 2 have times larger than τ_2. In this case again all τ_2 exponents have positive signs and

$$\int_0^{\tau_3} \mathrm{d}\tau_2 \exp\Big\{\tau_2 \Big(\sum_{\mathrm{V}_1} p^2 + \sum_{\mathrm{V}_2} p^2\Big)\Big\} = \Big[\exp\Big\{\tau_3 \sum_{\mathrm{V}_1 \cup \mathrm{V}_2} p^2\Big\} - 1\Big] \Big/ \sum_{\mathrm{V}_1 \cup \mathrm{V}_2} p^2\,. \tag{3.49}$$

The term coming from the lowest limit of integration will be dropped as before.

b) Vertex 2 is a neighbour of vertex 1. In this case special care has to be taken for the lines which connect vertices 1 and 2, as the corresponding G's and D's depend on both τ_1 and τ_2. They are of the form $\exp[-(\tau_2 - \tau_1)p^2]$ before τ_1 integration. From this one sees immediately that all τ_2 exponents coming from momenta, which connect vertices 1 and 2, are cancelled by the τ_1 integration. On the other hand, all other neighbour vertices of 2 will give positive τ_2 exponents. Explicitly we have

$$\int_0^{\tau_3} \mathrm{d}\tau_2 \exp\Big\{\tau_2 \Big(-\sum_{\mathrm{V}_1 \cap \mathrm{V}_2} p^2 + \sum_{\mathrm{V}_1} p^2 + \sum_{\mathrm{V}_2 - \mathrm{V}_1 \cap \mathrm{V}_2} p^2\Big)\Big\} = \Big[\exp\Big\{\tau_3 \sum_{\mathrm{W}_2} p^2\Big\} - 1\Big] \Big/ \sum_{\mathrm{W}_2} p^2 \tag{3.50}$$

where

$$\mathrm{W}_2 = \mathrm{V}_1 \cup \mathrm{V}_2 - \mathrm{V}_1 \cap \mathrm{V}_2\,. \tag{3.51}$$

Equations (3.50) and (3.51) also include eq. (3.49), so they are the general result of the second integration. Using the same arguments as in eq. (3.50) one obtains for the kth integration

$$\int_0^{\tau_{k+1}} \mathrm{d}\tau_k \exp\Big\{\tau_k \Big(-\sum_{\bigcup_{1\le i<k} (\mathrm{V}_i \cap \mathrm{V}_k)} p^2 + \sum_{\mathrm{W}_{k-1}} p^2 + \sum_{\mathrm{V}_k - \bigcup_{1\le i<k} (\mathrm{V}_i \cap \mathrm{V}_k)} p^2\Big)\Big\} = \Big[\exp\Big\{\tau_{k+1} \sum_{\mathrm{W}_k} p^2\Big\} - 1\Big] \Big/ \sum_{\mathrm{W}_k} p^2 \tag{3.52}$$

and it can easily be shown by induction that W_k, defined by

$$W_k = \Big[W_{k-1} - \bigcup_{1\le i<k} (V_i \cap V_k)\Big] \cup \Big[V_k - \bigcup_{1\le i<k} (V_i \cap V_k)\Big] \tag{3.53}$$

is actually given by

$$W_k = \bigcup_{1\le i\le k} V_i - \bigcup_{1\le i<j\le k} (V_i \cap V_j)\,. \tag{3.54}$$

Note the special values

$$W_1 \equiv V_1\,, \qquad W_N = \{\text{external lines}\}\,. \tag{3.55}$$

We would like to mention that the procedure is independent of the number of momenta leading to a vertex so we may consider, for example, any polynomially interacting theory.

The contribution from the Nth time integration is given by

$$\Big[\exp\Big\{+t \sum_{\substack{\text{external}\\ \text{lines}}} p^2\Big\} - 1\Big] \Big/ \sum_{\substack{\text{external}\\ \text{lines}}} p^2 \tag{3.56}$$

so that the negative t-exponents of the G's and D's from the external lines are exactly cancelled.

We see at this point that the neglect of non-leading contributions is indeed justified due to their exponential falloff in t.

Given a time-ordered stochastic diagram we finally get for $t \to \infty$

$$\prod_{k=1}^{N} \frac{1}{\sum_{W_k} p^2} \prod_{\text{crosses}} \frac{1}{p^2}\, \delta^n\Big(\sum_{\substack{\text{external}\\ \text{lines}}} p\Big) \tag{3.57}$$

where the second product of momenta arises simply from the denominators of the D's which were neglected so far.

To obtain the full contribution of a stochastic diagram one has to sum (3.57) over all time orderings.

This discussion was quite general. Let us now give some examples:

1) A contribution to the lowest-order 3-point vertex (see fig. 3.1): here there is just one vertex-time so that we get trivially

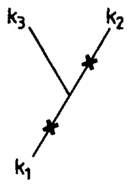


Fig. 3.1. An example of a stochastic diagram contributing to the 3-point vertex to lowest order.

$$\frac{1}{k_1^2 + k_2^2 + k_3^2} \frac{1}{k_1^2} \frac{1}{k_2^2} \delta^n(k_1 + k_2 + k_3) . \tag{3.58}$$

2) A self-energy contribution (see fig. 3.2).

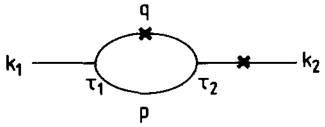


Fig. 3.2. An example of a stochastic diagram contributing to the one-loop correction of the propagator.

Two time orderings have to be considered

$$\tau_1 < \tau_2 , \quad \frac{1}{k_1^2 + p^2 + q^2} \frac{1}{k_1^2 + k_2^2} \frac{1}{q^2} \frac{1}{k_2^2} \delta^n(k_1 + k_2) \tag{3.59}$$

$$\tau_2 < \tau_1 , \quad \frac{1}{k_2^2 + p^2 + q^2} \frac{1}{k_1^2 + k_2^2} \frac{1}{q^2} \frac{1}{k_2^2} \delta^n(k_1 + k_2) . \tag{3.60}$$

3) A 1-loop contribution to the 3-point vertex (see fig. 3.3):

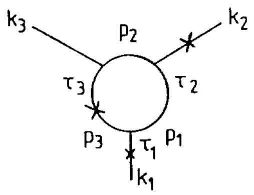


Fig. 3.3. An example of a stochastic diagram contributing to the one-loop correction of the 3-point vertex.

$$\tau_1 < \tau_2 < \tau_3 , \quad \frac{1}{k_1^2 + p_1^2 + p_3^2} \frac{1}{k_1^2 + k_2^2 + p_2^2 + p_3^2} \frac{1}{k_1^2 + k_2^2 + k_3^2} \frac{1}{k_1^2} \frac{1}{k_2^2} \frac{1}{p_3^2} \delta^n(k_1 + k_2 + k_3) . \tag{3.61}$$

$\tau_1 < \tau_3 < \tau_2$ gives zero because of the θ-function in $G\,(p_2, \tau_3 - \tau_2)$. In a similar way all the other contributions can be evaluated; we will refrain from a more detailed presentation.

To repeat in simple words [3.5] the result of performing the time integrations in the $t \to \infty$ limit (given some fixed time ordering $\tau_{i_1} < \tau_{i_2} < \cdots < t$) we have to

a) write in the denominator and multiply
- sum of $(\text{momenta})^2$ around τ_{i_1},
- sum of $(\text{momenta})^2$ around τ_{i_1} and τ_{i_2} but *not* those between τ_{i_1} and τ_{i_2},
- sum of $(\text{momenta})^2$ around $\tau_{i_1}, \tau_{i_2}, \tau_{i_3}$ but *not* those between τ_{i_1} and τ_{i_2}, τ_{i_1} and τ_{i_3} or τ_{i_2} and τ_{i_3},
- etc.,
- sum of external $(\text{momenta})^2$;

b) for every crossed line divide by the corresponding $(\text{momentum})^2$;
c) sum over all time orderings;
d) add overall momentum conservation.

This completes our discussion on how to calculate straightforwardly stochastic diagrams.

We can get a slightly different insight into stochastic dynamics when performing a Fourier transform with respect to the fictitious time as well. The Fourier transforms of the fields are then defined by

$$\phi(k, w) = \int \mathrm{d}^n x \, \mathrm{d}t \, \mathrm{e}^{\mathrm{i}kx + \mathrm{i}wt} \phi(x, t)$$

and the basic Langevin equation (3.30) becomes

$$(-\mathrm{i}w + k^2 + m^2)\, \phi(k, w) = -\frac{\lambda}{2!} \int \frac{\mathrm{d}^n p_1 \, \mathrm{d}^n p_2}{(2\pi)^{n+1}} \, \mathrm{d}w_1 \, \mathrm{d}w_2 \, \delta^n(k - p_1 - p_2) \delta(w - w_1 - w_2)$$
$$\times \phi(p_1, w_1)\, \phi(p_2, w_2) + \eta(k, w) . \tag{3.62}$$

Similarly the Fourier transforms of eqs. (3.7) are

$$\langle \eta(k, w)\, \eta(k', w') \rangle_\eta = 2\,(2\pi)^{n+1}\, \delta^n(k + k')\, \delta(w + w') . \tag{3.63}$$

It is important to notice that by taking the Fourier transform with respect to t, we have suppressed the part of $\phi(x, t)$ which depends on whatever initial condition we impose on the solution of the Langevin equation. As we saw in eq. (3.37) this part of the solution is transient, i.e. dies away exponentially like $\exp\{-(k^2 + m^2)t\}$. Because it explodes exponentially for large negative t, its Fourier transform would not exist and our procedure does not take account of it. Suppressing this transient part of $\phi(x, t)$ amounts, at least formally, to imposing the initial condition

$$\phi(x, t) \to 0 \qquad \text{as} \qquad t \to -\infty \tag{3.64}$$

so that the stochastic process relaxes to equilibrium already at finite fictitious time t, which represents a considerable simplification (remember that in the last section we had always to neglect terms coming from lower integration boundaries of the fictitious time). We shall indeed find that our procedure yields stochastic averages that are independent of t at finite t. In a similar fashion as in the last section we may introduce a stochastic Green function

$$G(k, w) = (-\mathrm{i}w + k^2 + m^2)^{-1} \tag{3.65}$$

and solve eq. (3.62) iteratively

$$\phi(k, w) = G(k, w) \Big[\eta(k, w) - \frac{\lambda}{2!} \int \frac{\mathrm{d}^n p \, \mathrm{d}^n q}{(2\pi)^{n+1}} \, \mathrm{d}w_1 \, \mathrm{d}w_2$$
$$\cdot \delta^n(k - p - q)\, \delta(w - w_1 - w_2)\, \phi(p, w_1)\, \phi(q, w_2) \Big] . \tag{3.66}$$

For an explicit demonstration of the above statements concerning the suppression of the transient part of $\phi(x, t)$ we may directly Fourier transform back the free part of eq. (3.66) and compare it with eq. (3.37). We get easily by contour integration

$$\phi(k,t)=\int\frac{\mathrm{d}w}{2\pi}\,\mathrm{e}^{-\mathrm{i}wt}\,\frac{1}{-\mathrm{i}w+k^2+m^2}\,\eta(k,w)$$
$$=\int_{-\infty}^{t}\mathrm{d}\tau\,\exp\{-(k^2+m^2)(t-\tau)\}\,\eta(k,\tau) \tag{3.67}$$

which clearly exhibits no transient component.

We may now proceed to a perturbative evaluation of stochastic averages [3.6]

$$\langle\phi(k_1,w_1)\cdots\phi(k_L,w_L)\rangle_\eta\,.$$

Owing to the noise correlation (3.63) all momentum-flows and inflows at crosses and vertices are conserved and each average contains the δ-function $\delta(w_1+\cdots w_L)$. As a consequence, the t-space stochastic averages which are obtained by applying a multiple Fourier transform to $\langle\phi(k_1,w_1)\cdots\phi(k_L,w_L)\rangle_\eta$

$$\langle\phi(k_1,t)\cdots\phi(k_L,t)\rangle_\eta=\frac{1}{(2\pi)^L}\int\mathrm{d}w_1\cdots\mathrm{d}w_L\exp\{-\mathrm{i}(w_1+\cdots w_L)t\}$$
$$\times\langle\phi(k_1,w_1)\cdots\phi(k_L,w_L)\rangle_\eta \tag{3.68}$$

are in fact independent of t and represent already the equilibrium Green functions.

We see that we must integrate over all the w-variables, including those on the external lines of the stochastic diagram; the integrand is a product of poles so that the w-integrations just give a sum of residues.

As the simplest example, consider the free propagator

$$\langle\phi(k_1,t)\,\phi(k_2,t)\rangle_\eta=\int\frac{\mathrm{d}w_1\,\mathrm{d}w_2}{(2\pi)^2}\,\frac{1}{-\mathrm{i}w_1+k_1^2}\,\frac{1}{-\mathrm{i}w_2+k_2^2}\,\langle\eta(k_1,w_1)\,\eta(k_2,w_2)\rangle_\eta$$
$$=\frac{2\delta^n(k+k')}{2\pi}\int\mathrm{d}w_1\,\frac{(2\pi)^n}{(w_1+\mathrm{i}k^2)\,(w_1-\mathrm{i}k^2)}$$
$$=\frac{2\,\delta^n(k+k')}{2\pi}\,\frac{2\pi\mathrm{i}}{2\mathrm{i}k^2}\,(2\pi)^n=\frac{\delta^n(k+k')}{k^2}\,(2\pi)^n \tag{3.69}$$

where we have closed the contour of the w_1 integration with an infinite semicircle in, for example, the lower half complex plane.

In general, Green functions soon become quite lengthy in their evaluation using the w-space technique. However, we would like to remark that this technique provides a compact method for showing the equivalence of stochastic quantization perturbation theory to the ordinary one [3.6].

3.2.2. The functional-integral approach

In the last section we relied in our formulation of stochastic quantization mainly on the Langevin equation (3.28) and developed the corresponding perturbation expansion.

In this section we would like to reformulate stochastic quantization in terms of the Fokker–Planck

approach. Specifically, we will discuss the associated functional integral formulation [3.7, 3.8] which will be shown to involve a local action $S_{FP}[\phi]$, and which allows us to calculate the stochastic averages (3.9) by conventional path integral methods. The perturbation expansion within the functional approach can then be obtained by a standard perturbation procedure with respect to S_{FP}.

To begin, let us consider the partition function for the Langevin dynamics (including a *constant* kernel k)

$$Z = \int D\eta \exp\left\{ -\frac{1}{4k} \int \eta^2(x,t)\, d^n x\, dt \right\} \tag{3.70}$$

and eliminate the white noise η by performing the variable transformation $\eta(x,t) \to \phi(x,t)$. As η and ϕ are obeying the Langevin equation

$$\dot{\phi} + k\, \delta S/\delta\phi = \eta \tag{3.71}$$

the partition function becomes then

$$Z = N \int D\phi\, P[\phi, 0] \det \frac{\delta\eta}{\delta\phi} \exp\left\{ -\frac{1}{4k} \int_0^t \left(\dot{\phi} + \frac{\delta S}{\delta\phi} \right)^2 d^n x\, d\tau \right\}. \tag{3.72}$$

Here the scalar field measure $D\phi$ is a product of the usual n-dimensional path integral measures over all fictitious times. Actually, having finite fictitious time correlations (3.10) in mind, we limited the fictitious time interval to a finite range $0 < \tau_i < t$, allowing for the possibility of letting $t \to \infty$ at the end. N denotes a normalization constant and we have introduced some general initial probability distribution $P[\phi, 0]$. In the last sections we used

$$P[\phi, 0] = \prod_x \delta^n(\phi(x,0) - \phi_0(x)). \tag{3.73}$$

Further evaluation of Z requires care when calculating the determinant of $\delta\eta/\delta\phi$. There arise in fact divergencies for systems with infinitely many degrees of freedom (as, for example, in the most interesting case of field theories). These divergencies are proportional to $\delta^n(0)$ (as a consequence of twofold functional differentiation) and act, when properly regularized like counterterms to cancel some of the divergencies of the associated perturbation theory. We will, however, for simplicity, restrict ourselves in the following to a formal derivation (as in zero dimensional field theory) and refer the reader to the literature [3.7, 3.9] where specific examples of the cancellation of such volume divergencies (and further discussion about this problem) can be found.

A further non-trivial issue, which we would like to avoid discussing in this introductory section, lies in the appropriate choice of boundary conditions. We refer to section 7 on stochastic processes and supersymmetry where these problems are explored.

With all these caveats in mind we proceed now to the evaluation of the determinant. We have [recall the constant kernel of eq. (3.70)]

$$\det \frac{\delta\eta}{\delta\phi} = \det\left[\left(\partial_\tau + k \frac{\delta^2 S}{\delta\phi(\tau)\, \delta\phi(\tau')} \right) \delta(\tau - \tau') \right]$$

$$= \exp\left\{\operatorname{tr}\ln\left[\left(\partial_\tau + k\,\frac{\delta^2 S}{\delta\phi(\tau)\,\delta\phi(\tau')}\right)\delta(\tau-\tau')\right]\right\} \tag{3.74}$$

$$= \exp\left\{\operatorname{tr}\ln\left[\partial_\tau\left(\delta(\tau-\tau') + k\,\theta(\tau-\tau')\,\frac{\delta^2 S}{\delta\phi(\tau)\,\delta\phi(\tau')}\right)\right]\right\}$$

where in the second term of the last line we choose forward propagation in fictitious time. By this we mean that the system evolves from some given initial configuration only in the future fictitious time direction. Mathematically, this is realized by specifying the inverse $g(t-t')$

$$\partial_t g(t-t') = \delta(t-t') \tag{3.75}$$

of the time derivative operator ∂_t as

$$g(t-t') = \theta(t-t')\,. \tag{3.76}$$

Conversely, the choice

$$g(t-t') = -\theta(t'-t) \tag{3.77}$$

corresponds to propagation backwards in fictitious time. The causality condition on the stochastic process requires the choice of forward propagation.

Dropping an infinite overall constant we are left with

$$\exp\{\operatorname{tr}\ln[\delta(\tau-\tau') + k\,\theta(\tau-\tau')\,\delta^2 S/\delta\phi(\tau)\,\delta\phi(\tau')]\}\,. \tag{3.78}$$

Next we expand the logarithm, where when taking the trace, all terms vanish except the first one. Remembering that we choose the midpoint prescription $\theta(0) = \frac{1}{2}$ (see section 2) it gives finally (and formally, as mentioned above)

$$\det\frac{\delta\eta}{\delta\phi} = \exp\frac{k}{2}\int_0^t \mathrm{d}\tau \int \mathrm{d}^n x\, \frac{\delta^2 S}{\delta\phi(x,\tau)\,\delta\phi(x,\tau)}\,. \tag{3.79}$$

With this result the partition function becomes

$$Z = \int \mathrm{D}\phi\; P(\phi,0)\exp\left\{-\int_0^t \mathrm{d}\tau\int \mathrm{d}^n x\left[\frac{1}{4k}\left(\dot\phi + k\,\frac{\delta S}{\delta\phi}\right)^2 - \frac{k}{2}\,\frac{\delta^2 S}{\delta\phi^2}\right]\right\}. \tag{3.80}$$

We note that the term in the exponent which is linear in $\dot\phi$ is a total time derivative and contributes only to boundary terms. Choosing for convenience $k = \frac{1}{2}$ and separating the integrations over fields with fictitious times on the boundary we obtain the final form of Z

$$Z = \int \mathrm{D}\phi(0)\; P[\phi,0]\exp\left\{+\frac{S(\phi(0))}{2}\right\}\mathrm{D}\phi(t)\exp\left\{-\frac{S(\phi(t))}{2}\right\}\widetilde{\mathrm{D}\phi}\exp\left[-\int_0^t \mathrm{d}\tau\int \mathrm{d}^n x\,\mathscr{L}_{\mathrm{FP}}\right] \tag{3.81}$$

where

$$\widetilde{\mathrm{D}\phi} = \prod_{0<\tau<t} \mathrm{D}\phi(\tau) \tag{3.82}$$

and

$$\mathscr{L}_{\mathrm{FP}} = \dot{\phi}^2/2 + \tfrac{1}{8}(\delta S/\delta\phi)^2 - \tfrac{1}{4}\,\delta^2 S/\delta\phi^2\,. \tag{3.83}$$

To get an understanding of the relation of $\mathscr{L}_{\mathrm{FP}}$ to the Fokker–Planck equation (3.13) let us recast the latter into a Schrödinger type equation (with respect to the Euclidean-like fictitious time evolution). If we set

$$\psi[\phi, t] = P[\phi, t]\,\mathrm{e}^{S/2} \tag{3.84}$$

a simple calculation gives

$$\frac{\partial}{\partial t}\,\psi[\phi, t] = -2k\int \mathrm{d}^n x\, H_{\mathrm{FP}}\Big(\phi(x), \frac{\delta}{\delta\phi(x)}\Big)\,\psi[\phi, t] \tag{3.85}$$

where we defined a formally positive semidefinite Fokker–Planck Hamiltonian

$$\begin{aligned} H_{\mathrm{FP}} &= \tfrac{1}{2}(-\delta/\delta\phi(x) + \tfrac{1}{2}\delta S/\delta\phi(x))(\delta/\delta\phi(x) + \tfrac{1}{2}\delta S/\delta\phi(x)) \\ &= -\tfrac{1}{2}\delta^2/\delta\phi^2(x) + U(\phi(x)) \end{aligned} \tag{3.86}$$

with

$$U(\phi(x)) = \tfrac{1}{8}(\delta S/\delta\phi(x))^2 - \tfrac{1}{4}\delta^2 S/\delta\phi^2(x)\,. \tag{3.87}$$

The classical Lagrangian $\mathscr{L}_{\mathrm{FP}}$ associated with H_{FP} can easily be obtained (remember that we are dealing with 'Euclidean' t, so the appropriately analytically continued relations between Hamiltonian and Lagrangian formulations have to be considered) and reads

$$\mathscr{L}_{\mathrm{FP}}(\phi(x, t), \dot{\phi}(x, t)) = \dot{\phi}^2/2 + U \tag{3.88}$$

which coincides precisely with eq. (3.83).

Given now the partition function (3.81) we can define the *generating functional*

$$\begin{aligned} Z[J] = \int &\mathrm{D}\phi(0)\, P[\phi, 0]\exp\{S(\phi(0))/2\}\,\mathrm{D}\phi(t)\exp\{-S(\phi(t))/2\} \\ &\cdot\widetilde{\mathrm{D}\phi}\exp\Big[-\int_0^t \mathrm{d}\tau\int \mathrm{d}^n x\,(\mathscr{L}_{\mathrm{FP}} + J\phi)\Big] \end{aligned} \tag{3.89}$$

from which the correlations (3.10) can be derived in the standard way

$$\langle \phi_\eta(x_1, t_1) \cdots \phi_\eta(x_L, t_L) \rangle_\eta = \frac{\delta^L Z[J]}{\delta J(x_1, t_1) \cdots \delta J(x_L, t_L)} . \tag{3.90}$$

We obtain a perturbative calculation of (3.90) by splitting $Z[J]$ in free and interaction parts, with subsequent power series expansion of the latter contribution. Due to the extra vertices contained in $\mathscr{L}_{\mathrm{FP}}$ the Feynman rules become quite involved, and we will not present here more details of this approach, see ref. [3.7].

At the end of this section let us relate once again the functional approach to the Fokker–Planck formulation, namely by considering equal time correlations. We want to compare typically

$$\langle \phi_\eta(x_1, t) \cdots \phi_\eta(x_L, t) \rangle_\eta = \int \mathrm{D}\phi(0)\, P[\phi, 0] \exp\{S(\phi(0))/2\}\, \phi(x_1, t) \cdots \phi(x_L, t)$$
$$\cdot \mathrm{D}\phi(t) \exp\{-S(\phi(t))/2\} \prod_{x'} \prod_{0<\tau'<t} \mathrm{D}\phi(x', \tau') \exp\left\{-\int_0^t \mathrm{d}\tau \int \mathrm{d}^n x\, \mathscr{L}_{\mathrm{FP}}\right\} \tag{3.91}$$

with

$$\int \prod_{x'} \mathrm{D}\phi(x')\, \phi(x_1) \cdots \phi(x_L)\, P[\phi, t] . \tag{3.92}$$

If we restrict ourselves for simplicity to a free scalar theory, $\mathscr{L}_{\mathrm{FP}}$ is given in momentum space by

$$\mathscr{L}^0_{\mathrm{FP}} = \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{8}(k^2 + m^2)^2\, \phi^2 . \tag{3.93}$$

We may perform a saddle point approximation by putting

$$\phi(k, \tau) = \phi_{\mathrm{cl}}(k, \tau) + \phi_{\mathrm{Q}}(k, \tau) \tag{3.94}$$

with ϕ_{Q} vanishing on the boundaries

$$\phi_{\mathrm{Q}}(k, 0) = \phi_{\mathrm{Q}}(k, t) = 0 \tag{3.95}$$

and put the previously-used sharp initial distribution

$$P[\phi, 0] = \prod_k \delta^n(\phi(k, 0)) . \tag{3.96}$$

ϕ_{cl} is the solution of the equation of motion corresponding to $\mathscr{L}^0_{\mathrm{FP}}$

$$[-\partial_t^2 + \tfrac{1}{4}(k^2 + m^2)^2]\, \phi_{\mathrm{cl}}(k, \tau) = 0 \tag{3.97}$$

which can be found easily as

$$\phi_{\mathrm{cl}}(k, \tau) = \phi(t) \sinh \frac{k^2 + m^2}{2}\, \tau \Big/ \sinh \frac{k^2 + m^2}{2}\, t . \tag{3.98}$$

We may subsequently integrate away the ϕ_Q contribution (which contributes just to an overall factor, taken care of by appropriate normalization), insert ϕ_{cl} in $\mathscr{L}^0_{FP}$, and perform the τ-integration. As a result of these easy manipulations eq. (3.91) becomes

$$\int \prod_{k'} D\phi(k', t)\, \phi(k_1, t) \cdots \phi(k_L, t) \exp\left\{-\tfrac{1}{2} \int d^n k\, \frac{\phi(k, t)\,(k^2 + m^2)\, \phi(-k, t)}{1 - \exp\{-(k^2 + m^2)t\}}\right\} \tag{3.99}$$

and we can identify the Fokker–Planck distribution with

$$P[\phi, t] = N \exp\left\{-\frac{1}{2} \int dk\, \phi\, \frac{k^2 + m^2}{1 - \exp\{-(k^2 + m^2)t\}}\, \phi\right\} \tag{3.100}$$

after having renamed $\phi(k, t) \equiv \phi(k)$ and inserted a normalization constant N.

We see that indeed in the $t \to \infty$ limit $P(\phi, t)$ converges to the standard Euclidean path integral density $\sim\exp(-\frac{1}{2} \int d^2k\, \phi k^2 \phi)$.

3.3. Equivalence with conventional path integral quantization

In this section we would like to prove how in the large fictitious time limit stochastic correlation functions tend to the Euclidean quantum Green functions. We will discuss two characteristic approaches, one relying on Fokker–Planck arguments, the other one using the Langevin technique. For an elegant equivalence proof within the superfield formalism of stochastic quantization we refer to section 7.3.

3.3.1. Equivalence proofs within the Fokker–Planck formulation

In our discussion on stochastic perturbation theory it was conceptually advantageous to study first the Langevin equation approach. Concerning an intuitive understanding of the large time limit of correlation functions it seems, however, more suitable to work within the Fokker–Planck formulation. As the simplest (and presumably most catching) derivation of the equivalence of stochastic quantization to conventional path integral quantization, let us restrict ourselves to a system with a finite number of degrees of freedom, described by a potential V, e^{-V} being integrable. The Fokker–Planck equation reads

$$\dot{P}(x, t) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + \frac{\partial V}{\partial x}\right) P(x, t) \tag{3.101}$$

which we recast upon the already familiar transformation (3.84)

$$P(x, t) = \psi(x, t) \exp\{-V(x)/2\} \tag{3.102}$$

into the Schrödinger type equation

$$\dot{\psi} = -2H\psi \tag{3.103}$$

where

$$H=\frac{1}{2}\left(-\frac{\partial}{\partial x}+\frac{V'}{2}\right)\left(\frac{\partial}{\partial x}+\frac{V'}{2}\right)=-\frac{1}{2}\frac{\partial^2}{\partial x^2}+\frac{1}{8}\left(\frac{\partial S}{\partial x}\right)^2-\frac{1}{4}\frac{\partial^2 S}{\partial x^2} \tag{3.104}$$

is evidently a self-adjoint operator with a non-negative spectrum. Furthermore for

$$V(x)\geqslant \mathrm{const}+\alpha x^2\,,\qquad \alpha>0 \tag{3.105}$$

the spectrum of H is purely discrete [this follows from the min-max principle (see, e.g. [3.10]) and the discrete spectrum of the harmonic oscillator].

If we denote the (complete set of) eigenstates of H by ψ_n

$$H\,\Psi_n(x)=E_n\,\Psi_n(x) \tag{3.106}$$

we may expand Ψ as follows:

$$\Psi(x,t)=\sum_{n=0}^{\infty}a_n\,\Psi_n(x)\exp(-E_n t)\,. \tag{3.107}$$

It follows immediately from the factorization of H that the eigenfunction with zero energy is (up to normalization) just

$$\Psi_0(x)=\exp\{-V(x)/2\} \tag{3.108}$$

so that

$$\Psi(x,t)=a_0\exp\{-V(x)/2\}+\sum_{n=1}^{\infty}a_n\,\Psi_n(x)\exp(-E_n t)\,. \tag{3.109}$$

The eigenstate with zero energy has no nodes, and it is therefore the ground state. All other eigenvalues are strictly above zero. So, it follows that

$$\lim_{t\to\infty}\Psi(x,t)=a_0\exp\{-V(x)/2\} \tag{3.110}$$

and

$$\lim_{t\to\infty}P(x,t)=a_0\exp\{-V(x)\}\,. \tag{3.111}$$

We fix the normalization by

$$\lim_{t\to\infty}\int \mathrm{d}x\,P(x,t)=1 \tag{3.112}$$

and arrive finally at

$$\lim_{t\to\infty}P(x,t)=\frac{\exp\{-V(x)\}}{\int \mathrm{d}x\,\exp\{-V(x)\}} \tag{3.113}$$

as was to be shown.

The generalization of this non-perturbative proof to Euclidean scalar field theory can in a formal way be achieved similarly. Closer inspection reveals, unfortunately, the need for a more rigorous treatment [3.11] which goes beyond the scope of this review.

Let us phrase instead another perturbative equivalence proof that relies on the Fokker–Planck technique as well: the idea is to explicitly separate the action into a free S_0 and an interacting part S_{I}, expand the Fokker–Planck distribution in a power series of the scalar self-coupling constant and perform the equivalence proof inductively, order by order,

$$P[\phi, t] = \sum_{k=0}^{\infty} \lambda^k P_k[\phi, t] . \tag{3.114}$$

The Fokker–Planck equation then becomes

$$\dot{P}_k[\phi, t] = \mathrm{d}^n x \, \frac{\delta}{\delta\phi(x)} \left[\frac{\delta}{\delta\phi(x)} + \frac{\delta S_0}{\delta\phi(x)} \right] P_k[\phi, t] + \int \mathrm{d}^n x \, \frac{\delta}{\delta\phi(x)} \, \frac{\delta S_{\mathrm{I}}}{\delta\phi(x)} \, P_{k-1}[\phi, t] . \tag{3.115}$$

The proof proceeds inductively showing

$$\lim_{t\to\infty} P_k[\phi, t] = P_k^{\mathrm{eq}}[\phi] \qquad \text{for all } k \tag{3.116}$$

where

$$P^{\mathrm{eq}} = \sum_{k=0}^{\infty} \lambda^k P_k^{\mathrm{eq}} = \frac{\exp\{-S(\phi)\}}{\int \mathrm{D}\phi \, \exp\{-S(\phi)\}} . \tag{3.117}$$

We will not perform the details of this proof [3.12] but rather recall the example of the last section, where we calculated explicitly $P_0[\phi, t]$.

3.3.2. Equivalence proof within the Langevin formulation

We are now going to use Langevin equation techniques to show the equivalence of stochastic quantization to the standard one [3.4], see also [3.13]. Specifically, we will prove by induction on the number of vertices N that in the case of a real scalar field theory (3.27), the sum of all stochastic diagrams (with the topology of a given Feynman diagram) yields just this Feynman diagram in the limit $t\to\infty$.

The induction hypothesis requires explicit verification for $N=1$, in which case we only have to consider the lowest order 3-point vertex. According to the simplest application of the time integration rules (3.48), we get in the $t\to\infty$ limit for the sum of the diagrams of fig. 3.4:

$$-\frac{1}{2!} \lambda \cdot 2! \, \frac{1}{k_1^2 + k_2^2 + k_3^2} \left(\frac{1}{k_1^2 k_2^2} + \frac{1}{k_2^2 k_3^2} + \frac{1}{k_1^2 k_3^2} \right) = -\lambda \, \frac{1}{k_1^2 k_2^2 k_3^2} \tag{3.118}$$

where 2! is the number of possibilities of joining together the two η's. We see that we exactly obtained the corresponding Feynman diagram (all Green functions we consider are non-truncated).

Assume, then, that we have proved this equivalence for an arbitrary Feynman diagram F in all

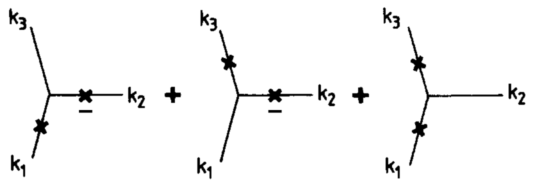


Fig. 3.4. All stochastic diagrams contributing to the 3-point vertex to lowest order.

numbers of vertices smaller than N with any number of external lines L. Consider now one of the stochastic diagrams S_F associated with F having N vertices and L external lines, as well as having a fixed-time ordering of its vertex times assigned.

We note that owing to the fictitious time θ-functions in G [eq. (3.33)] the largest time τ_{i_N} has to be at a vertex with at least one external line. In our case there are two such topological possibilities shown in fig. 3.5. [For simplicity of the presentation we just discuss case (a).]

Let us next define a stochastic diagram $S_{F'}$ which is obtained from S_F by dropping this τ_{i_N} vertex with its attached line(s); similarly we define the Feynman diagram F'.

The essential ingredient of the induction proof, allowing the step from $(N-1)$ to N vertices to be performed, is the result of (3.56) in section 3.2.1: it implies that the N-fold time integration of S_F equals (up to a given combinatorial factor) the $(N-1)$ fold time integration of $S_{F'}$, times a factor C which is given by

$$C = -\frac{\lambda}{2} \frac{1}{\sum\limits_{\substack{\text{external}\\ \text{momenta}}} k^2} . \tag{3.119}$$

As $S_{F'}$ contains only $N-1$ vertices the induction assumption may be used, so that upon summing and integrating all time-ordered stochastic diagrams S_F (keeping τ_{i_N} fixed) we obtain in the equilibrium

$$CF' = -\frac{\lambda}{2!} \frac{1}{k_1^2} F' k_1^2 \Big/ \sum_{i=1}^{L} k_i^2 = F k_1^2 \Big/ \sum_{i=1}^{L} k_i^2 . \tag{3.120}$$

Summing finally over all possible places for the longest time τ_{i_N} all the contributions evaluate to F.

We remark that the right combinatorial factors of the Feynman diagrams emerge from the iterative expansions of the fields and that case (b) can be discussed similarly. This completes the diagrammatic proof.

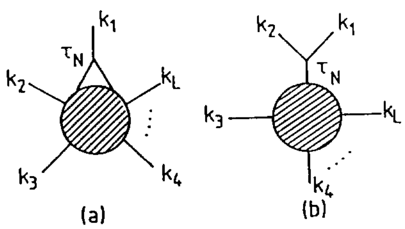


Fig. 3.5. The topological possibilities for an external line in a scalar field theory with cubic selfinteraction.

To close, let us add the references [3.14] to [3.20] where equivalence proofs are given, using different methods than those so far outlined. Concerning the inherent supersymmetry structure of stochastic processes we will come back to this issue in section 7.

4. Gauge fields

In section 4.1 we will discuss the stochastic quantization of gauge fields, and we point out the remarkable fact that gauge invariant quantities can be calculated without fixing the gauge [3.1].

Let us recall that in the standard path integral quantization procedure for gauge theories it is necessary to associate a gauge fixing term S_{GF} with the gauge invariant action S of the gauge theory under study, as otherwise the path integral density e^{-S} is not normalizable. This follows, as is well known, from the fact that with S being invariant under gauge transformations the integration over all gauge field configurations of e^{-S} gives rise to divergences due to the infinite volume of the gauge group.

The procedure of introducing a gauge fixing term has, in general, to be supplemented with Faddeev–Popov fields as well. Unfortunately, one can show that in a non-perturbative treatment the gauge fixing conditions do not generally fix the gauge uniquely, which in turn leads to ambiguities for the path integral. This is known as the Gribov problem. We recognize with this immediately one of the virtues of the stochastic quantization method, namely that the absence of the gauge fixing procedure implies a lack of the Gribov ambiguity, and an absence of ghost fields.

For reasons of simplicity we will discuss first the main features of the stochastic quantization of gauge theories for the (free) Maxwell field. Subsequently, we will calculate a typical one-loop quantity in scalar QED, and finally we will discuss Yang–Mills theories.

In section 4.2 we will present stochastic quantization supplemented with a new concept, commonly referred to as stochastic gauge fixing. Within this approach a functional formulation, analogous to the one in the scalar case, can be found. This will be discussed in some detail.

4.1. *Quantization without gauge fixing*

4.1.1. *The free Maxwell field*

The action of the free Maxwell field is given by

$$S=\int \mathrm{d}^n x\, \tfrac{1}{4} F_{\mu\nu} F_{\mu\nu} \tag{4.1}$$

where

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\,. \tag{4.2}$$

The classical field equation reads

$$\delta S/\delta A_\mu=-\partial_\nu F_{\nu\mu}=0 \tag{4.3}$$

which implies the Langevin equation

$$\dot{A}_\mu(x,t)=\partial_\nu F_{\nu\mu}(x,t)+\eta_\mu(x,t)\,. \tag{4.4}$$

Here the white noise has correlations which are straightforward generalizations of the scalar ones, namely

$$\langle \eta_\mu(x,t)\rangle = 0 \tag{4.5}$$

$$\langle \eta_\mu(x,t)\,\eta_\nu(x',t')\rangle = 2\,\delta_{\mu\nu}\,\delta^n(x-x')\,\delta(t-t')\,. \tag{4.6}$$

Performing, as in the scalar case (3.29), a Fourier transformation to momentum space, we rewrite the action as

$$S = \int \mathrm{d}^n k\, \tfrac{1}{2} A_\mu(k)\, k^2\, T_{\mu\nu}\, A_\nu(-k)\,. \tag{4.7}$$

The Langevin equation now reads

$$\dot{A}_\mu(k,t) = -k^2\, T_{\mu\nu} A_\nu + \eta_\mu\,. \tag{4.8}$$

In the above two equations we introduced the transverse projection operator $T_{\mu\nu}$

$$T_{\mu\nu} = \delta_{\mu\nu} - k_\mu k_\nu / k^2\,, \qquad T_{\mu\nu} T_{\nu\rho} = T_{\mu\rho}\,. \tag{4.9}$$

We define similarly a longitudinal projection operator $L_{\mu\nu}$

$$L_{\mu\nu} = \delta_{\mu\nu} - T_{\mu\nu} = k_\mu k_\nu / k^2\,, \qquad L_{\mu\nu} L_{\nu\rho} = L_{\mu\rho} \tag{4.10}$$

which is orthogonal to $T_{\mu\nu}$,

$$T_{\mu\nu} L_{\nu\rho} = 0\,. \tag{4.11}$$

In order to solve eq. (4.8) one studies, just as in the scalar case, the Green function $G_{\mu\nu}$ which obeys

$$\left(\delta_{\mu\nu}\,\frac{\partial}{\partial t} + T_{\mu\nu} k^2\right) G_{\nu\rho}(k,t) = \delta_{\mu\rho}\,\delta(t)\,. \tag{4.12}$$

This equation is immediately solved, and one finds

$$\begin{aligned} G_{\mu\nu} &= \theta(t)\,(\exp(-k^2 T t))_{\mu\nu} = \theta(t)\left[\delta_{\mu\nu} - k^2 t T_{\mu\nu} + \frac{(k^2 t)^2}{2}\, T_{\mu\nu} + \cdots\right] \\ &= \theta(t)\left[T_{\mu\nu} - k^2 t T_{\mu\nu} + \frac{(k^2 t)^2}{2}\, T_{\mu\nu} + \cdots\right] + \theta(t)\,(\delta_{\mu\nu} - T_{\mu\nu}) = \theta(t)\,[T_{\mu\nu} \exp(-k^2 t) + L_{\mu\nu}]\,. \end{aligned} \tag{4.13}$$

The general solution for A follows:

$$A_\mu(k,t) = \int_{-\infty}^{t} \mathrm{d}\tau\, [\exp\{-k^2(t-\tau)\} T_{\mu\nu} + L_{\mu\nu}]\, \eta_\nu(k,\tau) + C\,(\exp\{-k^2 t\}\, T_{\mu\nu} + L_{\mu\nu}) \tag{4.14}$$

where we have added the general homogeneous solution $C\,\exp(-k^2Tt)$, C being arbitrary as well. Writing as the initial condition

$$A_\mu(k, t_0) = A^0_\mu(k) \tag{4.15}$$

we finally arrive at

$$A_\mu(k, t) = \int_{t_0}^{t} \mathrm{d}\tau\, [\exp\{-k^2(t-\tau)\}\, T_{\mu\nu} + L_{\mu\nu}]\, \eta_\nu(k, \tau) + [T_{\mu\nu} \exp\{-k^2(t-t_0)\} + L_{\mu\nu}]\, A^0_\nu(k)\,. \tag{4.16}$$

Here it becomes necessary to point out an important difference between the scalar-field and the gauge-field solution. In the scalar case we could explicitly see that any dependence on the initial condition vanished exponentially for large times. This is no longer the case for a gauge theory: projecting on the longitudinal projector $L_{\mu\nu}$ the solution A_μ, we recognize that the corresponding part of the initial value is never damped. This phenomenon is typical for any gauge-type theory which exhibits a projection operator in its action. This follows because of the absence of a damping drift term in the 'longitudinal' component of the Langevin equation. Specifically, we have

$$T_{\mu\nu}\dot{A}_\nu = -k^2 T_{\mu\nu} A_\nu + T_{\mu\nu}\, \eta_\nu \tag{4.17}$$

exhibiting a damping as in the scalar-field case. However, the longitudinal part

$$L_{\mu\nu}\dot{A}_\nu = L_{\mu\nu}\, \eta_\nu \tag{4.18}$$

describes an unbounded diffusion. Out of this unboundedness we may expect trouble when calculating Green functions. This, indeed, becomes apparent already in the simplest example, the free equal-time propagator.

Let us first put the initial condition to zero, multiply together the solutions (4.16) and perform the η averages. We find immediately

$$\begin{aligned}\langle A_\mu(k, t)\, A_\nu(k', t)\rangle &= (2\pi)^n \Big[T_{\mu\nu} \int_0^t \exp\{-2k^2(t-\tau)\}\, \mathrm{d}\tau + L_{\mu\nu} 2 \int_0^t \mathrm{d}\tau \Big]\, \delta^n(k+k') \\ &= (2\pi)^n \Big[T_{\mu\nu}\, \frac{1}{k^2}\, (1 - \exp(-2k^2 t)) + 2 L_{\mu\nu} t \Big]\, \delta^n(k+k')\end{aligned} \tag{4.19}$$

which does not converge to equilibrium for $t \to \infty$, owing to the linear t-dependence in the longitudinal part of the propagator. This is, of course, a reflection of the unbounded diffusion of longitudinal gauge-field components.

We conclude at the moment that Green functions generally will exhibit (polynomial) divergences in t, and that, in general, they will not converge to some equilibrium value for large t. This is at least the case order-by-order in perturbation theory.

Our next remark concerns the choice of initial conditions. We may, of course, set the initial conditions to zero. However, a gauge transformation can always create longitudinal, persisting

dependences on the initial conditions. Let us calculate now for completeness the two-point correlation again, this time allowing for non-zero initial conditions $A^0_\mu(k)$ [4.1]. We further assume that A^0_μ may be put as

$$A^0_\mu = (k_\mu/k^2)\,\phi(k^2) \tag{4.20}$$

and allow for a distribution of the initial condition (which we will average over in our correlation functions) as well.

Specifically, let us assume that ϕ has a Gaussian distribution

$$\frac{\exp\left\{-\tfrac{1}{2}\alpha\int \mathrm{d}^n k\,\phi^2\right\}}{\int \mathrm{D}\phi\,\exp\left\{-\tfrac{1}{2}\alpha\int \mathrm{d}^n k\,\phi^2\right\}} \tag{4.21}$$

where α is an arbitrary parameter, $\alpha > 0$. For large times (when we can neglect the contribution of the transverse part of the initial condition) we arrive at

$$\lim_{t\to\infty}\langle A_\mu(k,t)\,A_\nu(k',t)\rangle = \frac{k_\mu k'_\nu}{k^2 k'^2}\,\phi(k)\,\phi(k') + \left(T_{\mu\nu}\,\frac{1}{k^2} + 2L_{\mu\nu}t\right)\delta^n(k+k')\,(2\pi)^n\,. \tag{4.22}$$

Performing an average over the initial condition with distributions (4.20) and (4.21) the first term on the r.h.s. of eq. (4.22) becomes

$$\alpha\,\frac{k_\mu k_\nu}{(k^2)^2}\,\delta^n(k+k')$$

so that the correlation function is equal to

$$\lim_{t\to\infty}\overline{\langle A_\mu(k,t)\,A_\nu(k',t)\rangle} = \left\{\frac{1}{k^2}\left[\delta_{\mu\nu} - (1-\alpha)\,\frac{k_\mu k_\nu}{k^2}\right] + 2t\,\frac{k_\mu k_\nu}{k^2}\right\}\delta^n(k+k')\,(2\pi)^n\,. \tag{4.23}$$

We recognize in the finite part of this the familiar gauge-field propagator, obtained by conventional gauge fixing in the so-called α-gauge. But we would like to stress that by *no means* can we identify the two-point function (4.23) with the conventional propagator in α-gauge. This is due to the longitudinal contribution (linear in t) which, as we will explain in section 4.1.3, is of crucial importance in the Yang–Mills case.

We have now reached a point in our discussion where the reader deserves further clarification on the (non-)convergence of Green functions for large t.

The answer is found by noting that any gauge-invariant quantity involves $F_{\mu\nu}$ or its derivatives; specifically, non-trivial gauge-invariant expectation values can be built from

$$\langle F_{\mu\nu}(x)\,F_{\rho\sigma}(y)\rangle\,. \tag{4.24}$$

It follows now immediately from antisymmetry of indices that the symmetric longitudinal contribu-

tions drop out, and that (4.24) has a well-defined limit $t\to\infty$. Explicitly, we have

$$
\begin{aligned}
\lim_{t\to\infty}\langle F_{\mu\nu}(x,t)\,F_{\rho\sigma}(x',t)\rangle &= \lim_{t\to\infty} -\frac{1}{(2\pi)^{2n}}\int \mathrm{d}^n p\,\mathrm{d}^n p' \exp\{-\mathrm{i}(px+p'x')\} \\
&\quad\cdot\{p_\mu p'_\rho\langle A_\nu(p,t)\,A_\sigma(p',t)\rangle-(\mu\leftrightarrow\nu)-(\rho\leftrightarrow\sigma)+(\mu\leftrightarrow\nu,\,\rho\leftrightarrow\sigma)\} \\
&= -\frac{1}{(2\pi)^{2n}}\int \mathrm{d}^n p\,\exp\{-\mathrm{i}p(x-x')\}\left[\frac{p_\mu p_\rho}{p^2}\,\delta_{\nu\sigma}-(\mu\leftrightarrow\nu)-(\rho\leftrightarrow\sigma)+(\mu\leftrightarrow\nu,\,\rho\leftrightarrow\sigma)\right]
\end{aligned}
\tag{4.25}
$$

which is the familiar result.

In the case of more complicated theories such as ordinary QED and Yang–Mills theories the cancellation of diverging contributions when calculating gauge-invariant generalities has been confirmed explicitly (see also sections 4.1.2 and 4.1.3) in several examples.

Let us now discuss a *functional formulation* of the stochastic quantization of Maxwell fields (see also ref. [4.2]). Splitting the gauge field into transverse and longitudinal parts (and similarly for the noise), we proceed as in the scalar case. Let us denote a gauge invariant functional by F_{GI}

$$
\begin{aligned}
&\int \mathrm{D}\eta^{\mathrm{T}}\,\mathrm{D}\eta^{\mathrm{L}}\exp\Big\{-\int \mathrm{d}^n k\,\mathrm{d}\tau\,(\eta^{\mathrm{T}2}+\eta^{\mathrm{L}2})\Big\}\,F_{\mathrm{GI}}[A] \\
&\quad=\int \mathrm{D}A^{\mathrm{T}}\exp\Big\{-\tfrac{1}{2}\int \mathrm{d}^n k\,\frac{A^{\mathrm{T}}\,k^2 A^{\mathrm{T}}}{1-\exp(-k^2 t)}\Big\}\,\mathrm{D}A^{\mathrm{L}}(0)\,P[A^{\mathrm{L}},0]\widetilde{\mathrm{D}A}^{\mathrm{L}}\exp\Big\{-\int \mathrm{d}^n k\,\mathrm{d}\tau\,\frac{\dot{A}^{\mathrm{L}}}{2}\Big\}\,F_{\mathrm{GI}}[A^{\mathrm{T}}] \\
&\quad=\int \mathrm{D}A^{\mathrm{T}}\exp\Big\{-\tfrac{1}{2}\int \mathrm{d}^n k\,\frac{A^{\mathrm{T}}k^2A^{\mathrm{T}}}{1-\exp(-k^2 t)}\Big\}\,F_{\mathrm{GI}}[A^{\mathrm{T}}]\mathrm{D}A^{\mathrm{L}}(0)\,P[A^{\mathrm{L}},0]\,\widetilde{\mathrm{D}A}^{\mathrm{L}}(t)\exp\Big\{-\int \mathrm{d}^n k\,\frac{A^{\mathrm{L}}(t)^2}{t}\Big\} \\
&\quad=N\int \mathrm{D}A^{\mathrm{T}}\exp\Big\{-\tfrac{1}{2}\int \mathrm{d}^n k\,\frac{A^{\mathrm{T}}k^2A^{\mathrm{T}}}{1-\exp(-k^2 t)}\Big\}\,F_{\mathrm{GI}}[A^{\mathrm{T}}]
\end{aligned}
\tag{4.26}
$$

where we could integrate away the (irrelevant) longitudinal components A^{L}, since F_{GI} would only depend on the transverse components. With the further identification

$$F_{\nu\mu}=k_\nu A^{\mathrm{T}}_\mu-k_\mu A^{\mathrm{T}}_\nu \tag{4.27}$$

we can reproduce very easily the previous example of $\langle FF\rangle$ as well.

It should be stressed at this point that the limit $t\to\infty$ has to be taken *after* having performed the functional integrations, i.e. in the weak sense.

We would like to close this section with a comment. It concerns the explanation of why the usual gauge-fixing procedure is not necessary in the above approach. This can most easily be seen from the fact that in the Langevin equation (4.8) there appears the operator

$$\delta_{\mu\nu}\partial/\partial t+k^2\delta_{\mu\nu}-k_\mu k_\nu\,. \tag{4.28}$$

Contrary to the ordinary Maxwell field equation, which exhibits the non-invertible transverse projection operator, it contains the additional $\delta_{\mu\nu}\,\partial/\partial t$ part. This makes it invertible.

Although we will come back to the following issue later, let us now just explicitly check the *gauge*

covariance of the Langevin equation in the Maxwell case. It can indeed be seen trivially that the Langevin equation remains unchanged upon letting

$$A_\mu(x,t) \to A_\mu(x,t) + \partial_\mu \chi(x) \tag{4.29}$$

where χ is an arbitrary function.

4.1.2. Scalar QED

We proceed now to consider scalar QED with the Euclidean action given by

$$S = \int d^n x \, (D_\mu \phi)^* (D_\mu \phi) + \tfrac{1}{4} F_{\mu\nu} F_{\mu\nu} \tag{4.30}$$

where the covariant derivative is defined as usual by

$$D_\mu \phi = (\partial_\mu - ieA_\mu)\phi \,. \tag{4.31}$$

Using the well-known classical field equations associated with eq. (4.30) we find immediately the associated Langevin equations

$$\dot{A}_\mu = \partial_\nu F_{\nu\mu} - ie[\phi^* \partial_\mu \phi - (\partial_\mu \phi^*)\phi - 2ieA_\mu \phi^* \phi] + \eta_\mu \tag{4.32}$$

$$\dot{\phi} = D^2 \phi + \eta \tag{4.33}$$

$$\dot{\phi}^* = (D^2 \phi)^* + \eta^* \tag{4.34}$$

with the noise correlations

$$\langle \eta_\mu \rangle = \langle \eta \rangle = \langle \eta^* \rangle = 0 \tag{4.35}$$

$$\langle \eta_\mu(x,t)\, \eta_\nu(x',t') \rangle = 2\, \delta_{\mu\nu}\, \delta(x-x')\, \delta(t-t') \tag{4.36}$$

$$\langle \eta(x,t)\, \eta^*(x',t') \rangle = 2\, \delta(x-x')\, \delta(t-t') \,. \tag{4.37}$$

It is convenient to write out explicitly the Langevin equations for the scalar fields:

$$\dot{\phi} = \partial^2 \phi - ieA_\mu\, \partial_\mu \phi - ie\, \partial_\mu (A_\mu \phi) - e^2 A^2 \phi + \eta \tag{4.38}$$

$$\dot{\phi}^* = \partial^2 \phi^* + ieA_\mu\, \partial_\mu \phi^* + ie\, \partial_\mu (A_\mu \phi^*) - e^2 A^2 \phi^* + \eta^* \,. \tag{4.39}$$

Going to Fourier space, with the additional conventions

$$\begin{aligned} \phi(x,t) &= \int \frac{d^n k}{(2\pi)^n} \exp(-ikx)\, \phi(k,t) \\ \phi^*(x,t) &= \int \frac{d^n k}{(2\pi)^n} \exp(+ikx)\, \phi^*(k,t) \end{aligned} \tag{4.40}$$

we find (with by now standard manipulations) the following set of integral equations:

$$\phi(k,t)=\int_0^t \mathrm{d}\tau\,\exp\{-k^2(t-\tau)\}\Big[\eta(k,\tau)-e\int\frac{\mathrm{d}^n p\,\mathrm{d}^n q}{(2\pi)^n}\,\delta^n(k-p-q)\,A_\mu(p,\tau)$$
$$\cdot\,\phi(q,\tau)\,(k_\mu+q_\mu)-e^2\int\frac{\mathrm{d}^n p\,\mathrm{d}^n q\,\mathrm{d}^n r}{(2\pi)^{2n}}\,\delta^n(k-p-q-r)\,A_\mu(p,\tau)\,A_\mu(q,\tau)\,\phi(r,\tau)\Big]\,; \tag{4.41}$$

$$\phi^*(k,t)=\int_0^t \mathrm{d}\tau\,\exp\{-k^2(t-\tau)\}\Big[\eta^*(k,\tau)-e\int\frac{\mathrm{d}^n p\,\mathrm{d}^n q}{(2\pi)^n}\,\delta^n(-k-p+q)\,A_\mu(p,\tau)$$
$$\cdot\,\phi^*(q,\tau)\,(k_\mu+q_\mu)-e^2\int\frac{\mathrm{d}^n p\,\mathrm{d}^n q\,\mathrm{d}^n r}{(2\pi)^{2n}}\,\delta^n(-k-p-q+r)\,A_\mu(p,\tau)\,A_\mu(q,\tau)\,\phi^*(r,\tau)\Big] \tag{4.42}$$

as well as

$$A_\mu(k,t)=\int_0^t [T_{\mu\nu}\exp\{-k^2(t-\tau)\}+L_{\mu\nu}]\Big[\eta_\nu(k,\tau)-e\int\frac{\mathrm{d}^n p\,\mathrm{d}^n q}{(2\pi)^n}\,\phi^*(p,\tau)\phi(q,\tau)\,(q_\mu+p_\mu)$$
$$-2e^2\int\frac{\mathrm{d}^n p\,\mathrm{d}^n q\,\mathrm{d}^n r}{(2\pi)^{2n}}\,\delta^n(k+p-q+r)\,A_\nu(p,\tau)\,\phi^*(q,\tau)\,\phi(r,\tau)\Big]\,. \tag{4.43}$$

For the assignment of momenta in diagrams, let us explicitly draw the vertices, and indicate the flow of momenta by arrows as in figs. 4.1 to 4.6.

We note further that with conventions as in eqs. (4.40), the two-point correlations of the scalar fields read

$$\langle\eta(k,t)\eta^*(k',t')\rangle=2(2\pi)^n\,\delta^n(k-k')\,\delta(t-t')\,. \tag{4.44}$$

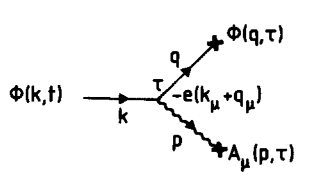


Fig. 4.1. Vertices in stochastic perturbation theory of scalar QED.

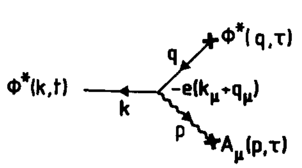


Fig. 4.2. Vertices in stochastic perturbation theory of scalar QED.

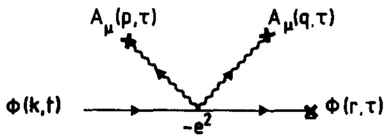


Fig. 4.3. Vertices in stochastic perturbation theory of scalar QED.

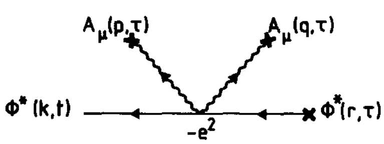


Fig. 4.4. Vertices in stochastic perturbation theory of scalar QED.

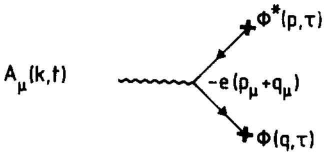


Fig. 4.5. Vertices in stochastic perturbation theory of scalar QED.

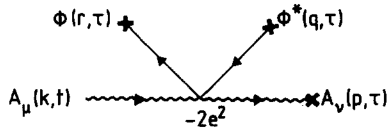


Fig. 4.6. Vertices in stochastic perturbation theory of scalar QED.

Let us now proceed to a first non-trivial exercise, namely calculating the one-loop correction to the scalar propagator $\langle \phi(k,t)\,\phi^*(k',t)\rangle$ [3.1].

To do so we list first all contributing diagrams in fig. 4.7(a) to (g).

For the sake of demonstration, we choose this gauge *variant* quantity on purpose so that we are facing divergences linear in t when performing the equilibrium limit. We would like to split the calculations into two parts, and first obtain the well-converging expression with just transverse gauge field contributions. Subsequently, we will extract the linearly diverging contribution and at the end of this exercise take the limits

$$\lim_{t\to\infty}\lim_{x\to y}\langle \phi(x,t)\,\phi(y,t)\rangle$$

upon which the diverging terms disappear explicitly, as generally expected from gauge invariant quantities.

Perhaps to the surprise of the reader, we have already acquired all the knowledge to obtain the large t contribution of the diagrams (a) to (e) *immediately*, avoiding any lengthy calculation.

Starting by considering the gauge field as purely transverse, the large t contribution is given by multiplying together vertex factors and projection operators from gauge field lines (at the moment just transverse projection operators); dividing by crossed-line contributions (i.e. the momentum^2 of the

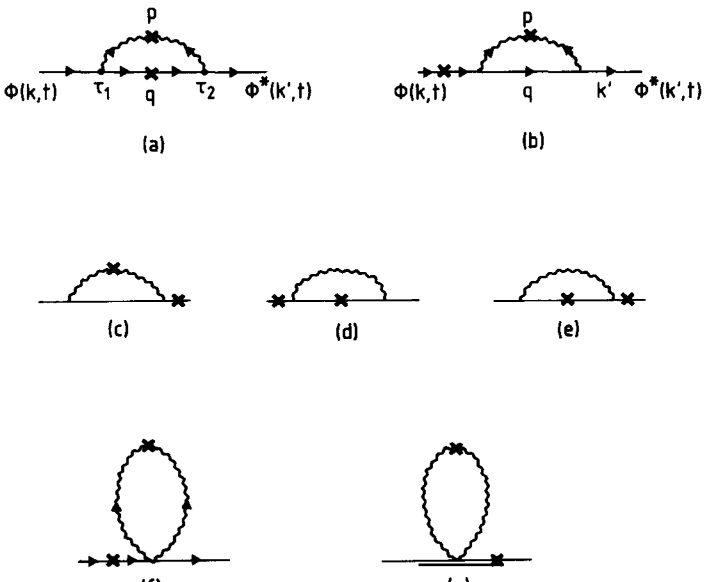


Fig. 4.7. Stochastic diagrams contributing to the one-loop correction to the $\phi^*\phi$ propagator in scalar QED.

corresponding line); multiplying with the fictitious time integration factors (which are identical to the ones discussed in the scalar-field case, as the transverse gauge field has the same exponential fictitious time dependence).

Diagram (a) has two different time orderings. The diagram being completely symmetric, it suffices to take twice the contribution of one of these, say when $\tau_1 < \tau_2$.

Apart from the overall $\int d^4p\, d^4q\, \delta(k-p-q)\, \delta(k-k')$ we then get

$$
\begin{aligned}
(\mathrm{a}) &= -2e^2 \frac{(k_\mu + q_\mu)T_{\mu\nu}(p)\,(k_\nu + q_\nu)}{p^2q^2} \frac{1}{p^2+q^2+k^2} \frac{1}{2k^2} \\
(\mathrm{b}) = (\mathrm{c}) &= -e^2 \frac{(k_\mu + q_\mu)T_{\mu\nu}(p)\,(k_\nu + q_\nu)}{p^2k^2} \frac{1}{p^2+q^2+k^2} \frac{1}{2k^2} \\
(\mathrm{d}) = (\mathrm{e}) &= -e^2 \frac{(k_\mu + q_\mu)T_{\mu\nu}(p)\,(k_\nu + q_\nu)}{k^2q^2} \frac{1}{p^2+q^2+k^2} \frac{1}{2k^2}
\end{aligned}
\tag{4.45}
$$

and (not unexpectedly, in fact) the sum gives the conventional result in Landau gauge

$$
-2e^2 \int d^np\, d^nq\, \delta^n(k-p-q)\, \delta(k-k')\, \frac{(k_\mu + q_\mu)T_{\mu\nu}(p)\,(k_\nu + q_\nu)}{k^4p^2q^2} . \tag{4.46}
$$

The remaining two diagrams can easily be evaluated in a similar way (note that the loop momentum does not contribute in the fictitious time integrations, as its line is connected to just identical fictitious times),

$$
-2e^2 \int d^np\, \frac{\mathrm{tr}\, T_{\mu\nu}}{k^2p^2} \frac{1}{2k^2} . \tag{4.47}
$$

Generalizing the results of this little exercise, one concludes that the equivalence proof for scalar fields carries over directly to scalar QED with transverse gauge bosons, having identical fictitious time dependence. It therefore follows that the sum of stochastic diagrams with transverse gauge field contributions sum up to the corresponding Feynman diagram in Landau gauge.

Next let us study the contributions of longitudinal lines. Both the G's and D's [see eqs. (4.13) and (4.19)] have non-exponential t-behaviour and general rules for the evaluation of stochastic diagrams are difficult to find. It seems an interesting challenge to work out further details along these lines.

Given the difficulties for a general discussion, we would like to restrict ourselves to the example above, and for the sake of simplicity concentrate on the leading t-contributions.

In scalar QED a general vertex involves always at least one scalar field line which carries exponential time dependence. Given some fixed time ordering, fictitious time integrations have the general structure

$$
\int_0^{\tau_{i+1}} d\tau_i\, \tau_i^n \exp\Big(\sum p^2\tau_i\Big) = \frac{1}{\sum p^2}\, \tau_{i+1}^n \exp\Big(\sum p^2\tau_{i+1}\Big) + \cdots \tag{4.48}
$$

where the neglected terms (remember that we are looking just for the dominant t-contributions) are of $O[\tau_{i+1}^{n-1} \exp(\Sigma\, p^2\tau_{i+1})]$. The general result is that any polynomial fictitious time dependence from crossed longitudinal gauge boson lines just factors out of the time integrals.

The second observation is that as every longitudinal gauge boson line does not carry exponential time dependence, no further time dependence, and consequently no further contribution from fictitious time integrations, arises.

As a direct result of these considerations the leading contribution of the longitudinal terms for the one-loop self-energy diagrams is just the corresponding Feynman diagram, where the gauge propagators are given by $2tL_{\mu\nu}$, namely

$$\int \mathrm{d}^n p\, \mathrm{d}^n q\, e^2\, 2t\, \frac{\delta^n(k-p-q)}{k^4 q^2}\, (q+k)_\mu\, L_{\mu\nu}(p)\, (q+k)_\nu - \int \mathrm{d}^n p\, e^2\, 2t\, \frac{\operatorname{tr} L_{\mu\nu}}{k^4}\,. \tag{4.49}$$

If we now allow the above quantities to combine into gauge-invariant Green functions, for example by considering

$$\lim_{t\to\infty} \lim_{x\to y} \langle \phi(x,t)\, \phi^*(y,t)\rangle$$

we pick up an additional $\int \mathrm{d}^n k$ integration in momentum space. Whereas the transverse contribution again corresponds to the standard Landau gauge part the linear t part vanishes by symmetry considerations as can be seen directly from

$$\int \mathrm{d}^n k\, \mathrm{d}^n q \left[\frac{q^4 - 2q^2k^2 + k^4}{k^4(q-k)^2 q^2} - \frac{1}{k^4}\right] = \int \mathrm{d}^n k\, \mathrm{d}^n q \left[\frac{k^2 - q^2}{k^2(q-k)^2 q^2} + \frac{2k(q-k)}{k^4(q-k)^2}\right] \equiv 0\,. \tag{4.50}$$

With a little algebra one can also check the disappearance of all the other finite longitudinal terms. We are not aware of a general proof for this cancellation.

Summarizing this section, we have learned that scalar QED can be nicely quantized without fixing the gauge. We worked out an example in detail, admitting that so far the equivalence to conventional quantization has not been shown in general.

The breakdown of the induction argument of section 3.3 is clearly caused by the encounter of non-gauge-invariant subdiagrams. We have not been able to set up a consistent induction procedure to deal with this. Ideally, one should try (especially in the case of Yang–Mills theories) to build up a formulation that involves only gauge-invariant quantities such as Wilson loops [3.18] and [4.20] but this seems to be quite difficult.

4.1.3. Yang–Mills theories

For a non-Abelian gauge theory with Lagrangian

$$\mathcal{L} = -\tfrac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} \tag{4.60}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu \tag{4.61}$$

the Langevin equation reads

$$\dot{A}_\mu(x,t) = D^{ab}_\nu F^b_{\nu\mu} + \eta_\mu\,. \tag{4.62}$$

Here D^{ab}_ν is the covariant derivative

$$D^{ab}_\nu = \partial_\nu \delta^{ab} - g f^{abc} A^c_\nu \tag{4.63}$$

and the white noise obeys

$$\langle \eta^a_\mu(x,t)\, \eta^b_\nu(x',t') \rangle = 2\, \delta^{ab}\, \delta_{\mu\nu}\, \delta^n(x-y)\, \delta(t-t')\,. \tag{4.64}$$

In momentum space (4.62) takes the form

$$\dot{A}^a_\lambda(p,t) = -k^2\, T_{\lambda\mu}\, A^a_\mu(p,t) + I^a_\lambda + \tilde{I}^a_\lambda + \eta^a_\lambda \tag{4.65}$$

where I^a_λ and $\tilde{I}^a_\lambda$ denote the (usual) three-gluon and four-gluon interaction terms.

Specifically, I^a_λ, corresponding to fig. 4.8, is given by

$$I^a_\lambda(p,t) = \frac{\mathrm{i} g f^{abc}}{2(2\pi)^n} \int \mathrm{d}^n q\, \mathrm{d}^n r\, \delta^n(p+q+r)\, A^b_\mu(-q,t)\, A^c_\nu(-r,t)\, v^I_{\lambda\mu\nu}(p,q,r) \tag{4.66}$$

with

$$v^I_{\lambda\mu\nu}(p,q,r) = \delta_{\lambda\mu}(p-q)_\nu + \text{cyclic perm}\,. \tag{4.67}$$

Similarly, one has

$$\tilde{I}^a_\lambda(p,t) = \frac{g^2}{6(2\pi)^{2n}} \int \mathrm{d}^n q\, \mathrm{d}^n r\, \mathrm{d}^n s\, \delta^n(p+q+r+s) A^b_\mu(-q,t)\, A^c_\nu(-r,t)\, A^d_\rho(-s,t)\, v^{\tilde{I}abcd}_{\lambda\mu\nu\rho} \tag{4.68}$$

with

$$v^{\tilde{I}abcd}_{\lambda\mu\nu\rho} = f^{abe} f^{cde} (\delta_{\lambda\nu}\delta_{\mu\rho} - \delta_{\lambda\rho}\delta_{\mu\nu}) + \text{symm. perm}\,. \tag{4.69}$$

which corresponds to fig. 4.9.

In the non-Abelian case the integral solution to eq. (4.65) exhibits as in the Maxwell case a longitudinal component, which does not have an exponential fictitious time dependence.

Calculating Green functions, longitudinal lines will give rise to various finite and potentially infinite terms, just as in sections 4.1.1 and 4.1.2. A non-trivial expectation is [3.1] that for gauge-invariant quantities not only the infinite terms drop out but also that the finite parts arrange themselves to give just those contributions which are conventionally associated with Faddeev–Popov ghost effects.

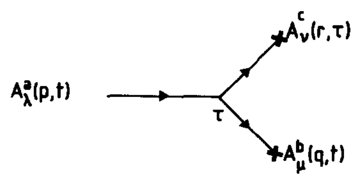


Fig. 4.8. Vertices in stochastic perturbation theory of Yang–Mills theories.

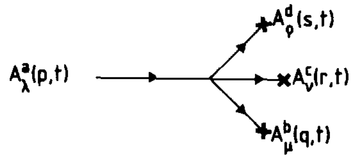


Fig. 4.9. Vertices in stochastic perturbation theory of Yang–Mills theories.

Again we have a somewhat embarrassing lack of general proofs of the above expectations.

In fact, one of the few explicit verifications of the above statements can be found in the example of calculating [4.1]

$$\lim_{t\to\infty}\lim_{x\to y}\langle F^a_{\mu\nu}(x,t)\,F^b_{\rho\sigma}(y,t)\rangle\,. \tag{4.70}$$

It has been shown that contributions from longitudinal internal lines automatically sum up to the expected Faddeev–Popov ghost contributions, whereas the transverse gluon lines render the standard Landau gauge result.

To bypass the general difficulties outlined, let us mention that one can write down [3.6] a Langevin equation which in the equilibrium limit yields ordinary perturbation theory in the axial gauge. We define

$$N_{\mu\nu}(p)=\delta_{\mu\nu}-p_\mu n_\nu/np\,. \tag{4.71}$$

This has the properties

$$N_{\mu\rho}(p)\,N_{\rho\nu}(p)=N_{\mu\nu}(p) \tag{4.72}$$

and

$$N_{\mu\rho}(p)\,N_{\nu\rho}(p)=\delta_{\mu\nu}-\frac{p_\mu n_\nu+n_\mu p_\nu}{np}+n^2\frac{p_\mu n_\nu}{(np)^2}\,. \tag{4.73}$$

Our Langevin equation is

$$\dot A^a_\mu=-p^2A^a_\mu+N_{\mu\nu}[N_{\rho\nu}(I^a_\rho+\tilde I^a_\rho)+\eta^a_\nu] \tag{4.74}$$

where I, $\tilde I$ and η are defined as before in (4.66), (4.68) and (4.64), respectively.

It is straightforward to work through the analysis of section 3 using this Langevin equation. Because (4.73) is the numerator of the usual axial gauge propagator, the stochastic perturbation theory based on (4.74) reproduces the usual axial gauge perturbation theory in the equilibrium limit. For other approaches to a stochastic formulation of gauge theories in the axial gauge see ref. [4.3]. We do not intend to discuss here problems connected with the zeros of the denominator in (4.73). Whatever is the appropriate prescription [4.4] in the usual theory must also be applied here.

We close this section by observing that though there is convincing evidence, it would be interesting to prove rigorously for gauge theories the equivalence of the Parisi–Wu approach to the conventional one.

4.2. Stochastic gauge fixing

4.2.1. Gauge transformations depending on the fictitious time

We would now like to introduce a new concept in the Parisi–Wu scheme, commonly referred to as stochastic gauge fixing [4.5–4.11, 3.12, 4.21, 4.22]. In order to expose the new concept most clearly, it will be useful to consider the Maxwell field first.

Although stochastic quantization based on

$$\dot{A}_\mu(k,t) = -k^2\, T_{\mu\nu}\, A_\nu(k,t) + \eta_\mu(k,t) \tag{4.75}$$

is, as outlined in section 4.1, perfectly consistent without gauge fixing, an alternative formulation is possible. This seems to offer various advantages, both conceptually and technically. In fact, a detailed analysis of the stochastic evolution of the gauge field in 'gauge parameter space' can be performed (in contrast to, for example, the unbounded diffusion of the gauge modes in the Parisi–Wu approach of the previous section). It can, furthermore, be shown that all *individual* stochastic diagrams contributing to some gauge-invariant quantity may remain finite for $t \to \infty$.

The basic idea [3.1] (for another interpretation see [4.22], where stochastic gauge fixing is related to the Faddeev–Popov procedure for a gauge theory in one higher dimension) is to transform the field $A_\mu(x,t)$ into a new field $B_\mu(x,t)$ so that in the Langevin equation of this new field, the previous transverse projection operator $T_{\mu\nu}$ is replaced by an invertible matrix. The transformation is required to leave unchanged all gauge-invariant quantities, and so must be simply a gauge transformation.

Let us recall then that t-independent gauge transformations leave the form of the Langevin equation invariant. From this we conclude that the required gauge transformations (let us call them from now on *generalized gauge transformations*) are necessarily dependent on fictitious time.

In momentum space we then have generally

$$B_\mu(k,t) = A_\mu(k,t) + \mathrm{i}\, k_\mu \Lambda(k,t) \tag{4.76}$$

so that

$$\dot{B}_\mu(k,t) = -k^2\, T_{\mu\nu}\, B_\nu + \mathrm{i}\, k_\mu \dot{\Lambda}(k,t) + \eta(k,t)\,. \tag{4.77}$$

Calculating a gauge-invariant Green function we may, as already outlined above, choose whatever $\dot{\Lambda}$ (or Λ, respectively) we like. We will see below that we can easily find a Λ so that the free part of the new drift term becomes *invertible*, and the ensuing propagator finite for $t \to \infty$. A systematic study of the possible generalized gauge transformations yielding finite propagators may be based on the concept of stochastic gauge constraints [4.12]. This is a generalization of the ordinary notion of gauge conditions, involving fictitious time derivatives as well as the noise field. A stochastic gauge constraint is a stochastic differential equation whose solution defines a (stochastic) constraint surface in the configuration space of the $B_\mu(x,t)$. The generalized gauge transformation is hence the mapping of the original gauge field A_μ onto the stochastic constraint surface.

It should be strongly stressed, however, that the only role in the construction of a finite propagator is placed by the choice of a generalized gauge transformation. Whether we define it from the notion of a stochastic gauge constraint, whether it may arise from different choices of stochastic gauge constraints, or finally, whether it may be defined without taking recourse to a stochastic gauge constraint at all, is completely irrelevant.

This contrasts sharply with the situation in ordinary gauge-fixing, where the uniqueness of gauge conditions is crucial.

In the present framework the so-called 'Gribov problem' takes on a new meaning. For a chosen stochastic gauge-fixing term the Langevin *dynamics* will determine whether one is confined within one Gribov horizon only, or whether one can drift through different sectors. We will analyse this in some depth in section 4.2.2.

We illustrate now the above concepts by the example of a Lorentz gauge condition and its stochastic generalizations.

As already discussed lengthily earlier, the Langevin equation (4.8) implies the absence of a damping drift term for the longitudinal part of A

$$k_\mu \dot{A}_\mu = k_\mu \eta_\mu \,. \tag{4.78}$$

We now transform A by a generalized gauge transformation into a stochastic field B, whose longitudinal components also approach an equilibrium distribution. A convenient choice turns out to be

$$\Lambda(k, t) = \alpha^{-1}\,\mathrm{i} \int_0^t \mathrm{d}\tau \exp\{-\alpha^{-1} k^2 (t-\tau)\}\; k_\nu\; A_\nu(k, \tau) \tag{4.79}$$

where α is an arbitrary constant. This we are able to derive very easily from the covariant stochastic gauge constraint

$$\alpha k_\mu \dot{B}_\mu = -k^2 k_\mu B_\mu + \alpha k_\mu \eta_\mu \tag{4.80}$$

which generalizes the Lorentz gauge condition $kB = 0$.

Inserting eq. (4.76) we have

$$\alpha k_\mu \dot{A}_\mu + \alpha \mathrm{i} k^2 \dot{\Lambda} = -k^2 k_\mu B_\mu + \alpha k_\mu \eta_\mu \tag{4.81}$$

which on account of eq. (4.8) implies

$$\dot{\Lambda} = \alpha^{-1} \mathrm{i} k_\nu B_\nu \tag{4.82}$$

i.e.

$$\Lambda = \alpha^{-1} \mathrm{i} \int_{t_0}^t \mathrm{d}\tau\; k_\nu\; B_\nu(k, \tau)\,. \tag{4.83}$$

The new Langevin equation then reads

$$\dot{B}_\mu = -k^2 (T_{\mu\nu} + \alpha^{-1} L_{\mu\nu}) B_\nu + \eta_\mu \tag{4.84}$$

corresponding to the propagator

$$\lim_{t\to\infty} \langle B_\mu(k,t)\, B_\nu(k',t)\rangle = \frac{\delta^n(k+k')}{k^2}\,(T_{\mu\nu} + \alpha L_{\mu\nu})(2\pi)^n \tag{4.85}$$

which we recognize as the familiar propagator in a covariant α-gauge.

We continue in solving for Λ in terms of A: from eqs. (4.76) and (4.82) it follows that

$$\dot{\Lambda} = \alpha^{-1}(\mathrm{i}k_\nu A_\nu - k^2\Lambda) \tag{4.86}$$

which may readily be solved:

$$\Lambda(k,t) = \alpha^{-1}\,\mathrm{i}\int_{t_0}^{t} \exp\{-\alpha^{-1}k^2(t-\tau)\}\,k_\nu\, A_\nu(k,\tau)\,\mathrm{d}\tau \tag{4.87}$$

as put forward in eq. (4.79) for the choice $t_0 = 0$. The indeterminacy of Λ (by the choice of t_0) may be interpreted by the notion of a generalized gauge orbit (see [4.12]); for our discussion we may put simply $t_0 = 0$. We also set $\alpha = 1$.

An interesting question which arises is whether the Langevin equation (4.84) for the B field remains invariant under t-independent gauge transformations. To answer this, let us extract first the gauge transform of B. Given that

$$A^{\mathrm{g}}_\mu(k,t) = A_\mu(k,t) + \mathrm{i}\,k_\mu\,\chi(k) \tag{4.88}$$

we find

$$\Lambda^{\mathrm{g}} = \Lambda - \chi(1 - \exp(-k^2 t)) \tag{4.89}$$

and hence

$$B^{\mathrm{g}}_\mu = B_\mu + \mathrm{i}\,k_\mu\,\chi\exp(-k^2 t)\,. \tag{4.90}$$

We see that the B field becomes a gauge-independent field for large times. Let us next proceed to calculate the change of the form of the Langevin equation under a gauge transformation:

$$\begin{aligned}\dot{B}^{\mathrm{g}}_\mu &= \dot{B}_\mu - k^2\,\mathrm{i}\,k_\mu\,\chi\exp(-k^2 t) = -k^2 B_\mu + \eta_\mu - \mathrm{i}\,k^2\,k_\mu\,\chi\exp(-k^2 t)\\ &= -k^2\,B^{\mathrm{g}}_\mu + \eta_\mu\,.\end{aligned} \tag{4.91}$$

We have hence proved that the form remains invariant.

It should be noted that for the choice $\alpha = 0$ in eq. (4.80) the stochastic gauge constraint is reduced to

$$k_\mu B_\mu = k_\mu A_\mu + \mathrm{i}k^2\Lambda = 0 \tag{4.92}$$

so that

$$\Lambda(k,t) = -k_\nu A_\nu / k^2 \tag{4.93}$$

and

$$B_\mu(k,t) = T_{\mu\nu}\, A_\nu(k,t) \tag{4.94}$$

which implies that the B field is the gauge-invariant projection of A. It is therefore gauge independent for all fictitious times.

Let us now discuss again the functional formulation of the Maxwell field, this time showing how the usual gauge fixed action emerges as the corresponding equilibrium distribution. We are evaluating the expectation value of a gauge-invariant functional $F_{\mathrm{GI}}(A)$:

$$\begin{aligned}\langle F_{\mathrm{GI}}(A(t))\rangle = \int & \mathrm{D}A^{\mathrm{T}}(0)\, P[A^{\mathrm{T}},0] \exp\{S(A^{\mathrm{T}}(0))/2\} \\ & \cdot \mathrm{D}A^{\mathrm{T}}(t) \exp\{-S(A^{\mathrm{T}}(t))/2\} \prod_{0<\tau<t} \mathrm{D}A^{\mathrm{T}}(\tau) \exp\Big\{-\int_0^t \mathrm{d}\tau \int \mathrm{d}^n k\, \mathscr{L}_{\mathrm{FP}}(A^{\mathrm{T}}(\tau))\Big\} \\ & \cdot \mathrm{D}A^{\mathrm{L}}(0)\, P[A^{\mathrm{L}},0]\, \mathrm{D}A^{\mathrm{L}}(t) \prod_{0<\tau<t} \mathrm{D}A^{\mathrm{L}}(\tau) \exp\Big\{-\int_0^t \mathrm{d}\tau \int \mathrm{d}^n k\, \frac{(\dot{A}^{\mathrm{L}})^2}{4}\Big\}\, F_{\mathrm{GI}}[A^{\mathrm{T}}(t)]\,.\end{aligned} \tag{4.95}$$

Transforming A now into B

$$A_\mu = B_\mu + k_\mu \int_0^t \mathrm{d}\tau\, k_\nu\, B_\nu^{\mathrm{L}}(k,\tau) \tag{4.96}$$

leaves the measure, F_{GI} and S invariant and we get (after having chosen 'sharp' initial distributions)

$$\begin{aligned}\langle F_{\mathrm{GI}}(A)\rangle = \int & \mathrm{D}B^{\mathrm{T}}(t) \exp\{-S(B^{\mathrm{T}}(t)/2\} \prod_{0<\tau<t} \mathrm{D}B^{\mathrm{T}}(\tau) \\ & \cdot \exp\Big\{-\int_0^t \mathrm{d}\tau \int \mathrm{d}^n k\, \mathscr{L}_{\mathrm{FP}}(B^{\mathrm{T}}(\tau))\Big\}\, \mathrm{D}B^{\mathrm{L}}(t) \prod_{0<\tau<t} \mathrm{D}B^{\mathrm{L}}(\tau) \\ & \cdot \exp\Big\{-\tfrac{1}{4}\int_0^t \mathrm{d}\tau\, (\dot{B}_\mu^{\mathrm{L}} + k_\mu k_\nu B_\nu^{\mathrm{L}})(\dot{B}_\mu^{\mathrm{L}} + k_\mu k_\nu B_\nu^{\mathrm{L}})\Big\}\, F_{\mathrm{GI}}[B]\,.\end{aligned} \tag{4.97}$$

Following the general procedure of the scalar case, it is obvious that in a saddlepoint approximation (which is exact for the Maxwell field) we finally end up with

$$\begin{aligned}\langle F_{\mathrm{GI}}(A)\rangle = \int & \mathrm{d}B^{\mathrm{T}}(t) \exp\Big\{-\tfrac{1}{2}\int \mathrm{d}^n k\, \frac{B^{\mathrm{T}}(k,t)\, k^2\, B^{\mathrm{T}}(-k,t)}{1-\exp(-k^2 t)}\Big\} \\ & \cdot F_{\mathrm{GI}}[B]\, \mathrm{D}B^{\mathrm{L}}(t) \exp\Big\{-\tfrac{1}{2}\int \mathrm{d}^n k\, \frac{B^{\mathrm{L}}(k,t)\, k^2\, B^{\mathrm{L}}(-k,t)}{1-\exp(-k^2 t)}\Big\}\end{aligned} \tag{4.98}$$

which in the $t \to \infty$ limit goes into

$$\int \mathrm{d}B \exp\left\{-\tfrac{1}{2}\int \mathrm{d}^n k\, B k^2 B\right\} F_{\mathrm{GI}}[B] = \int \mathrm{D}B \exp\left\{-S(B) - \tfrac{1}{2}\int \mathrm{d}^n k\,(kB)^2\right\} F_{\mathrm{GI}}[B]. \tag{4.99}$$

This is the conventional path integral expression in Feynman gauge.

In fact, a Fokker–Planck analysis can immediately give this conventional path-integral expression as its equilibrium distribution. This is so because for the evaluation of gauge-invariant Green functions we are allowed to consider the transformed field B as the relevant dynamical variable. We then study the Fokker–Planck equation associated with the Langevin equation (4.84) for B

$$\dot{P} = \int \mathrm{d}^n x \frac{\delta}{\delta B_\mu} [(\partial^2 \delta_{\mu\nu} - \partial_\mu \partial_\nu) B_\nu + \tfrac{1}{2}\,\partial_\mu \partial_\nu B_\nu] P \tag{4.100}$$

which has in fact the conventional path-integral density as its equilibrium distribution P^{eq}

$$P^{\mathrm{eq}} = N \int \mathrm{D}B \exp\left\{-S + \frac{1}{2\alpha}\int \mathrm{d}^n x\,(\partial B)^2\right\}. \tag{4.101}$$

Let us now proceed to the case of stochastically gauge-fixed scalar QED.

Corresponding to the transformation (4.76), the scalar field ϕ is transformed into ϕ^B

$$\phi^B(x,t) = \exp\left\{\mathrm{i}e \int_0^t \mathrm{d}\tau\, \partial_\nu B_\nu(x,\tau)\right\} \phi(x,t) \tag{4.102}$$

and the new coupled equations read

$$\dot{B}_\mu(x,t) = \partial^2 B_\mu - \mathrm{i}e[\phi^{*B}\,\partial_\mu \phi^B - (\partial_\mu \phi^{*B})\,\phi^B - 2\mathrm{i}eB_\mu \phi^{*B}\phi^B] + \eta_\mu \tag{4.103}$$

together with

$$\begin{aligned} \dot{\phi}^B &= D^2(B)\,\phi^B + \mathrm{i}e(\partial_\nu B_\nu)\,\phi^B + \eta^B \\ \dot{\phi}^{*B} &= D^{2*}(B)\,\phi^{*B} - \mathrm{i}e(\partial_\nu B_\nu)\,\phi^{*B} + \eta^{*B} \end{aligned} \tag{4.104}$$

where the new noise fields, defined by

$$\eta^B(x,t) = \exp\left\{\mathrm{i}e \int_0^t \mathrm{d}\tau\, \partial_\nu B_\nu(x,\tau)\right\} \eta(x,t) \tag{4.105}$$

still satisfy

$$\langle \eta^B(x,t)\,\eta^{*B}(x',t')\rangle = 2\,\delta^n(x-x')\,\delta(t-t'). \tag{4.106}$$

It is another check that under an ordinary gauge transformation the field B^g transforms as before [eq.

(4.90)], and ϕ^B transforms like

$$\phi^{Bg} = \phi^B \exp\left\{-\mathrm{i}e \int \mathrm{d}k \exp(-\mathrm{i}kx)\, \chi(k) \exp(-k^2 t)\right\}. \tag{4.107}$$

As a consequence all Langevin equations remain form invariant.

Let us now perform a Fokker–Planck analysis and study the equilibrium distribution. The Fokker–Planck equation corresponding to the action S from (4.30) and the gauge-transformed variables ϕ^B, ϕ^{*B} and B reads

$$\begin{aligned}\dot{P}(\phi^B, \phi^{*B}, B, t) = \int \mathrm{d}^n x \biggl\{ & \frac{\delta}{\delta\phi^B}\left(\frac{\delta S}{\delta\phi^{*B}} + \mathrm{i}e(\partial_\nu B_\nu)\phi^B + \frac{\delta}{\delta\phi^{*B}}\right) \\ & + \frac{\delta}{\delta\phi^{*B}}\left(\frac{\delta S}{\delta\phi^B} - \mathrm{i}e(\partial_\nu B_\nu)\phi^{*B} + \frac{\delta}{\delta\phi^B}\right) \\ & + \frac{\delta}{\delta B_\mu}\left(\frac{\delta S}{\delta B_\mu} - \partial_\mu\partial_\nu B_\nu + \frac{\delta}{\delta B_\mu}\right)\biggr\} P(\phi^B, \phi^{*B}, B, t).\end{aligned} \tag{4.108}$$

It is instructive to see whether or not the standard gauge fixed path-integral density $\exp[-S - \frac{1}{2}\int(\partial B)^2\, \mathrm{d}^n x]$ is an equilibrium distribution of eq. (4.108).

One finds that the presence of the extra drift terms, $-\mathrm{i}e(\partial B)\phi^{*B}$ and $\mathrm{i}e(\partial B)\phi^B$ leads to the remaining term

$$\mathrm{i}e \int \mathrm{d}^n x(\partial_\nu B_\nu)\left[\phi \frac{\delta S}{\delta\phi^{*B}} - \phi^* \frac{\delta S}{\delta\phi^B}\right] \exp\left\{-S + \tfrac{1}{2}\int \mathrm{d}^n x\, (\partial B)^2\right\}. \tag{4.109}$$

We have to conclude that the choice $\dot{A} = \partial B$ does *not* imply the standard gauge fixed path-integral density in the equilibrium. Of course, it we calculate a gauge-invariant quantity for all choices of $\dot{A}$ we obtain the same result. It is possible that there exists a (presumably non-local) choice of $\dot{A}(\phi^B, \phi^{*B}, B)$, which gives $\exp\{-S - \frac{1}{2}\int(\partial B)^2 \mathrm{d}^n x\}$ as the equilibrium distribution.

The next issue we are going to discuss is the concept of stochastic gauge fixing for non-Abelian gauge theories [4.5–4.11]. We will again consider a generalized gauge transformation and try to obtain the new drift terms for the transformed Langevin equation. We define

$$B_\mu = B_\mu^a \tau^a \qquad \text{etc}. \tag{4.110}$$

with

$$\operatorname{tr} \tau^a \tau^b = \tfrac{1}{2}\delta^{ab}. \tag{4.111}$$

The generalized gauge transformation is defined by $U = U(t)$ and

$$B_\mu = U^{-1} A_\mu U - g^{-1} U^{-1}\, \partial_\mu U \tag{4.112}$$

so that

$$\dot{B}_\mu = \dot{U}^{-1} A_\mu U + U^{-1} \dot{A}_\mu U + U^{-1} A_\mu \dot{U} - g^{-1} \dot{U}^{-1}\, \partial_\mu U - g^{-1} U^{-1}\, \partial_\mu \dot{U}\,. \tag{4.113}$$

We introduce $\Lambda(x, t)$ by

$$\dot{U} = g\dot{\Lambda} U\,. \tag{4.114}$$

It satisfies

$$\dot{U}^{-1} = -U^{-1} g\dot{\Lambda} \tag{4.115}$$

which can be shown easily by writing out explicitly $\partial_t(UU^{-1}) = \partial_t(1) = 0$. With this notation we then arrive at

$$\dot{B}_\mu = -U^{-1} g\dot{\Lambda} A_\mu U + U^{-1} \dot{A}_\mu U + U^{-1} A_\mu g\dot{\Lambda} U - U^{-1} (\partial_\mu \dot{\Lambda})\, U \tag{4.116}$$

or

$$\dot{B}_\mu = U^{-1} \{\delta S/\delta A_\mu - \partial_\mu \dot{\Lambda} - g[A_\mu, \dot{\Lambda}]\} U\,. \tag{4.117}$$

We identify a covariant derivative acting on $\dot{\Lambda}$ so that, as a consequence of the covariant transformation property of $\delta S/\delta A$ we finally find

$$\dot{B}_\mu = -\,\delta S/\delta B_\mu - D_\mu(B)\, \dot{\Lambda} + \eta_\mu^B \tag{4.118}$$

where the noise η^B is defined by

$$\eta_\mu^B = U^{-1}\, \eta_\mu U\,. \tag{4.119}$$

This transformed noise field satisfies again

$$\langle \eta_\mu^{aB}(x, t)\, \eta_\nu^{bB}(x', t') \rangle = 2\delta^{ab} \delta_{\mu\nu} \delta^n(x - x')\, \delta(t - t')\,. \tag{4.120}$$

Let us now discuss different choices of $\dot{\Lambda}$.

The simplest one is the already well-known choice

$$\dot{\Lambda} = -\partial_\nu B_\nu \tag{4.121}$$

which implies the Langevin equation

$$B_\lambda^a(p, t) = -p^2 B_\lambda^a + I_\lambda^a + \tilde{I}_\lambda^a + J_\lambda^a + \eta_\lambda^a \tag{4.122}$$

where besides the usual interaction terms from the action, also a new induced interaction term J_λ^a has appeared. It is a three-gluon coupling (see fig. 4.10) and it reads explicitly

$$J_\lambda^a(p, t) = \frac{\mathrm{i} g f^{abc}}{2(2\pi)^n} \int \mathrm{d}^n q\, \mathrm{d}^n r\, \delta^n(p + q + r)\, A_\mu^b(-q, t)\, A_\nu^c(-r, t)\, v_{\lambda\mu\nu}^J(p, q, r) \tag{4.123}$$

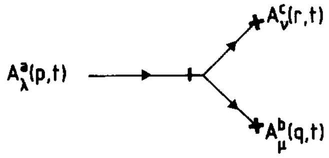


Fig. 4.10. Extra three-gluon coupling in stochastically gauge fixed Yang–Mills theories.

with

$$v^J_{\lambda\mu\nu}(p,q,r) = \delta_{\lambda\mu} r_\nu - \delta_{\lambda\nu} q_\mu \,. \tag{4.124}$$

Unlike the usual three-gluon interaction contained in I (4.66), it is a directed vertex [4.13, 4.14] because it is symmetric under the interchange of two lines only. The directedness complicates the analysis of this new interaction, and makes the relation between stochastic diagrams and Feynman diagrams non trivial. Stochastic diagrams of a given topology that contain only vertices of the ordinary type I sum to the corresponding Feynman diagram; this follows directly from the work of section 3. Because of our choice (4.121) for $\dot{A}$, this Feynman diagram is calculated in Feynman gauge.

However, for stochastic diagrams that contain one or more vertices of the new type J the situation is much more complicated.

If we sum all such diagrams, and use them to calculate a gauge-invariant quantity, we expect that they should just reproduce the usual ghost contribution.

As an example, one might consider the one-loop contribution to the gauge boson propagator $\langle A^a_\mu(p,t)\, A^b_\nu(p',t)\rangle$. A straightforward calculation of the contribution of the new vertices gives [3.6]

$$-\delta^n(p+p')\, g^2 f^{acd} f^{bcd} \int \mathrm{d}^n q\, \mathrm{d}^n r\, \delta^n(p+q+r)\, \frac{1}{p^2 q^2 r^2} \left[\frac{q_\lambda r_{\lambda'}}{p^2} - \delta_{\lambda\lambda'}\, \frac{2p^2 - q^2 - r^2}{p^2 + q^2 + r^2} \right] + \cdots \tag{4.125}$$

where the terms not explicitly written are proportional to the external momenta $p_\lambda p'_\lambda$. We recognize in the first term of the square bracket the usual ghost term, but the remaining contribution is not immediately recognizable. We may use (4.125) to obtain the contribution of the two-point functions to $\lim_{y\to x} \langle F^a_{\mu\nu}(x)\, F^{a'}_{\rho\sigma}(y)\rangle$. We find that the 'unwanted' terms result in a quantity which is exactly cancelled [3.6] by the three-point contributions to $\langle FF\rangle$ which involve the new vertex J.

It is tempting to modify the Langevin equation after applying stochastic gauge fixing by including an additional kernel operator K. With the choice $K = 1/p^2$ things apparently become very much simpler; however we are lacking a general proof to show that the sum of stochastic diagrams is independent of K. If we repeat the calculation of $\langle FF\rangle$ we do find that the answer is the same. But this is no longer true for, for example, quark–quark scattering [3.6] where the unrecognizable terms stemming from the new drift term no longer cancel. We conclude that it is not generally valid to introduce a kernel once the Langevin equation has been modified by stochastic gauge fixing. Also, one should presumably always restrict oneself to gauge covariant kernels, even in the stochastically gauge fixed case.

It is a natural task to recast the Langevin formulation of the stochastic gauge fixing procedure for Yang–Mills theories in a functional formulation as well [4.5, 4.6, 3.12, 4.9, 4.10, 4.13, 4.15]. In particular we may, in direct generalization of section 3.2, construct a generating functional for

stochastic correlation functions. It reads [4.15]

$$Z[J]=N\int\prod_{\tau}DB(\tau)\exp\left\{\int_{-\infty}^{\infty}d\tau\,d^nx\left[-\tfrac{1}{4}\left(\dot B^a_\mu+\frac{\delta S}{\delta B^a_\mu}-D^{ab}_\mu\,\partial_\nu B^b_\nu\right)^2\right.\right.$$
$$\left.\left.+\frac{1}{2}\,\frac{\delta}{\delta B^a_\mu}\left(\frac{\delta S}{\delta B^a_\mu}-D^{ab}_\mu\,\partial_\nu B^b_\nu\right)+J^a_\mu(x,\tau)\,B^a_\mu(x,\tau)\right]\right\}. \tag{4.126}$$

Let us point out that the presence of the stochastic gauge fixing term allows us to perform safely the limit $t\to\infty$ right in the generating functional. One can then derive the Feynman rules, implied by $\mathscr{L}_{FP}$ and develop a perturbation expansion. It should be noted, however, that the general structure of $\mathscr{L}_{FP}$, in interplay with the contributions of the stochastic gauge fixing term and volume divergences (see section 3) leads to quite involved diagrammatics. It seems that contrary to the Langevin approach so far no systematic simplification for summing these diagrams has been found.

Apart from these technical problems, however, several nice results have been obtained as, for example, the calculation of the β-function for Yang–Mills theory to one loop [4.15]. We refer the reader to the literature for further reference.

Parallel to the construction of the generating functional in terms of a Fokker–Planck action one may adhere to a direct study of the Fokker–Planck equation [4.5, 3.12, 4.13, 4.15] and of its equilibrium distribution. In general (see also next section) we cannot expect an exponential solution to the Fokker–Planck equation.

One may instead try to solve the Fokker–Planck equation

$$\dot P=\int d^nx\,\frac{\delta}{\delta B^a_\mu}\left(\frac{\delta S}{\delta B^a_\mu}+D^{ab}_\mu\,\dot A^b+\frac{\delta}{\delta B^a_\mu}\right)P \tag{4.127}$$

perturbatively [3.12] by setting, as in the discussion of the scalar case,

$$P=\sum_{k=0}^{\infty}g^kP_k\,. \tag{4.128}$$

We split the Yang–Mills action (4.60) into its free part and the interaction terms proportional to g and g^2

$$S=S_0+S_1+S_2 \tag{4.129}$$

so that the Fokker–Planck equation for the P_k becomes for the gauge fixing term $\dot A=\partial_\nu B_\nu$:

$$\dot P_k=\int d^nx\,\frac{\delta}{\delta B^a_\mu(x)}\left[\frac{\delta}{\delta B^a_\mu(x)}+\frac{\delta S_0}{\delta B^a_\mu(x)}-\partial_\mu\,\partial_\nu B^a_\nu\right]P_k$$
$$+\int d^nx\,\frac{\delta}{\delta B^a_\mu(x)}\left[\frac{\delta S_1}{\delta B^a_\mu(x)}+gf^{abc}B^b_\mu\,\partial_\nu B^c_\nu\right]P_{k-1}$$
$$+\int d^nx\,\frac{\delta}{\delta B^a_\mu(x)}\,\frac{\delta S_2}{\delta B^a_\mu(x)}\,P_{k-2}\,. \tag{4.130}$$

This recursive equation may then be solved order by order in perturbation theory by the methods outlined in section 3. To give just an example we have for the zeroth-order component

$$P_0 = N \exp\left[-\tfrac{1}{2} \int \mathrm{d}^4k B^a_\mu (\Delta^{ab}_{\mu\nu})^{-1} B^b_\nu\right] \tag{4.131}$$

with

$$\Delta^{ab}_{\mu\nu} = \frac{\delta^{ab}}{k^2}\left[T_{\mu\nu}(1-\exp(-2k^2t)) + L_{\mu\nu}\Big(1-\exp(-2k^2t)\Big)\right]. \tag{4.132}$$

As a further application one may perform in the $t\to\infty$ limit an expansion [4.11] of the generating functional of connected Green functions $W(J)$ defined by

$$W[J] = \ln Z[J] \tag{4.133}$$

with

$$Z[J] = \int \mathrm{D}B \exp\left\{\int \mathrm{d}^nx\, B^a_\mu(x)\, J^a_\mu(x)\right\} P[B,t]. \tag{4.134}$$

Let us finally mention that an attempt has been made to derive the above perturbation theory with its vertices directly from an ordinary action, by invoking the concept of para-statistics [4.14].

4.2.2. *Properties of the stochastic gauge fixing force*

In this section (where we follow closely refs. [4.9, 4.10]) we would like to discuss some of the properties of the new drift term $D^{ab}_\mu \dot{\Lambda}^b(x,t)$, referred to as a stochastic gauge fixing term (Zwanziger term). For the particular choice $\dot{\Lambda} = -\partial B$ we will study in which sense it actually *fixes* the gauge, and we will discuss a simple model which exhibits the basic features of the full Yang–Mills case.

Finally, we shall give a formal argument as to why stochastic quantization can 'reproduce' the Faddeev–Popov prescription by choosing a specific (non-local) stochastic gauge fixing term.

We start by considering $\dot{\Lambda} = -(1/\alpha)\partial_\nu B_\nu$. First note [4.5] that the gauge-fixing force is *non-conservative*, as follows from

$$\frac{\delta}{\delta B^a_\mu(x,t)}\, D^{bc}_\nu(y,t)\, \partial_\sigma B^c_\sigma(y,t) - \frac{\delta}{\delta B^b_\nu(y,t)}\, D^{ac}_\mu(x,t)\, \partial_\sigma B^c_\sigma(x,t) \neq 0 \tag{4.135}$$

and which, in fact, holds for a general choice of $\dot{\Lambda}$ [4.16]. It then follows that the stochastic gauge fixing term cannot be expressed as the gradient of an action, but rather can be split into a gradient *and* a divergenceless term (a pure curl)

$$D^{ab}_\mu\, \partial_\nu B^b_\nu = -\frac{\delta S_\mathrm{G}}{\delta B^a_\mu} + g f^{abc} B^b_\mu\, \partial_\nu B^c_\nu \tag{4.136}$$

with

$$S_\mathrm{G} = \frac{1}{2\alpha}\int \mathrm{d}^nx\, (\partial_\nu B_\nu)^2. \tag{4.137}$$

For the following discussion let us recall some basic notions of gauge orbits [4.17], avoiding mathematical rigour as much as tolerable.

We consider the gauge field $B_\mu(x, t) = B^a_\mu(x, t)\lambda^a$ as an element B in a Hilbert space H, with the norm given by

$$\|B\|^2 = \int \mathrm{d}^n x\, B^a_\mu(x, t)\, B^a_\mu(x, t) = (B, B)\,. \tag{4.138}$$

Note that the *distance* between two elements B_1 and B_2 is gauge invariant. We define the gauge transform B^g similar to eq. (4.112); as $g(x)$ varies over all local gauge transformations B^g describes the gauge orbit of the point B. Next we need to introduce the Faddeev–Popov operator $L(B)$

$$L(B) = -\partial_\mu D_\mu(B) \tag{4.139}$$

(where D_μ is the covariant derivative) and define a hyperplane in the Hilbert space H by $\partial B = 0$.

One defines then the Gribov region Ω as the part of the hyperplane $\partial B = 0$ where the Faddeev–Popov operator is non negative. The boundary $\partial\Omega$ is known as the Gribov horizon. It has a vanishing Faddeev–Popov operator. The shape of Ω is characterized by the fact that it is an open, convex and bounded set. It can be chosen to contain the origin.

We have now gathered enough terminology to study in general terms the effects of the gauge-fixing force (4.136). For the sake of easy argumentation let us neglect in the following the usual drift term $\delta S/\delta B_\mu$ and the random noise in eq. (4.118), which corresponds to considering a purely deterministic dynamical system (we could imagine such a situation, for example, in the limit of very small α)

$$\dot{B}_\mu = \frac{1}{\alpha}\, D^{ab}_\mu\, \partial_\nu B^b_\nu\,. \tag{4.140}$$

As a first result we can show that the gauge-fixing force is a *restoring* force, pulling back B to the origin $B = 0$. This follows as

$$\begin{aligned}\frac{\partial}{\partial t}\, \|B\|^2 &= 2(B, \dot{B}) = \frac{2}{\alpha}\, (B, \mathrm{D}\partial B) \\ &= -\frac{2}{\alpha}\, (\mathrm{D}B, \partial B) = -\frac{2}{\alpha}\, \|\partial B\|^2 \le 0\,.\end{aligned} \tag{4.141}$$

It can be shown next that the points on the hypersurface $\partial B = 0$ which lie in the Gribov region Ω are stable fixed points, whereas the points outside Ω are unstable fixed points of the gauge-fixing force. This can most easily be shown by considering a point B in the neighbourhood of the hyperplane $\partial B = 0$ and setting

$$B_\mu = B^0_\mu + \varepsilon B^\perp_\mu \tag{4.142}$$

where B^0 is in the hyperplane, $\partial B^0 = 0$, and B^1 in the orthogonal subspace. To leading order in ε we have

$$\frac{\partial}{\partial t}\, \|\partial B\|^2 = 2(\partial B, \partial\dot{B}) = -\frac{2}{\alpha}\, (\partial B, L(B)\, \partial B) = -\frac{2}{\alpha}\, (\partial B^1, L(B^0)\, \partial B^1)\varepsilon^2 \tag{4.143}$$

so that, with the properties of $L(B)$ outlined above, the stability argument is demonstrated.

The general feature that emerges [4.10] when we include in eq. (4.140) also the perturbation of a Gaussian noise term is as follows. Let us suppose that B is located near the origin and drifting arbitrarily.

Owing to the influence of the restoring force it will be dragged in the direction of the Gribov region Ω, except when being very close to it (i.e. when the restoring force is quite weak) it develops a large horizontal component along Ω by statistical fluctuations. In this way it is possible that it even may escape the vicinity of Ω, where the hyperplane $\partial B = 0$ is no longer an attractor. However, again owing to the fluctuating Brownian motion it unavoidably picks up a drift from the gauge-fixing force leading away from the $\partial B = 0$ surface and finally is strongly attracted towards the origin, instead of escaping to infinity along the hyperplane.

To understand the properties of the stochastic gauge fixing in better detail it is instructive to study a simple two-dimensional model which exemplifies impressively the above concepts [4.9].

One chooses (for $\alpha > 0$)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = -\frac{1}{\alpha} \begin{pmatrix} 2xy^2 \\ (1-x^2)y \end{pmatrix} \tag{4.144}$$

and finds easily

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(x^2+y^2) &= -\frac{2}{\alpha}(x^2+1)y^2 \\ \frac{\mathrm{d}}{\mathrm{d}t}y^2 &= -\frac{2}{\alpha}(1-x^2)y^2 \end{aligned} \tag{4.145}$$

which are the analogues of eqs. (4.141) and (4.143). Here we identify the hypersurface $\partial B = 0$ with $y = 0$, the Gribov region Ω is the interval $\Omega = \{y = 0,\ -1 \le x \le 1\}$, and the Gribov horizon $\partial\Omega$ is formed by the points $\binom{\pm 1}{0}$. The lines of flow for the force (4.144) are obtained by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1-x^2}{2xy} \tag{4.146}$$

as

$$2y^2 = \ln x^2 - x^2 + c \tag{4.147}$$

and are given in fig. 4.11.

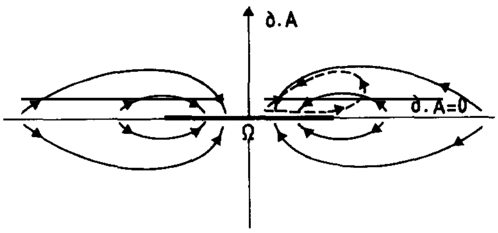


Fig. 4.11. A typical path for a particle under the combined influence of the drift and noise term.

For the sake of illustration we also drew (dotted line) a typical Brownian path for a particle under the combined influence of the drift and noise terms.

Further numerical investigation and confirmation of the above concepts for the actual Yang–Mills case can be found in ref. [4.18].

Finally, we would like to add a formal argument [4.6] that (for some arbitrary gauge-fixing condition) we can specify a stochastic gauge-fixing term in such a way that the equilibrium distribution of the corresponding Fokker–Planck equation is given by the conventional path-integral density (with gauge fixing and the Faddeev–Popov term).

To this end we conveniently use a BRS formulation of the Faddeev–Popov construction. The BRS operator is defined by

$$Q = \int \mathrm{d}^n x \left\{ (D_\mu c)^a \frac{\delta}{\delta B_\mu^a} - \frac{g}{2} f^{abc} c^b c^c \frac{\delta}{\delta c^a} + b^a \frac{\delta}{\delta \bar{c}^a} \right\} \tag{4.148}$$

where b is an auxiliary field and $c, \bar{c}$ denote (scalar) ghosts and antighosts, respectively. As is well known, this operator satisfies

$$Q^2 = 0 \tag{4.149}$$

and allows the conventional gauge-fixing part of the action to be expressed by

$$S_{\mathrm{GF}} = \int \mathrm{d}^n x \left[\frac{1}{2\alpha} b^a b^a + (\partial B^a) b^a - \bar{c}^a \partial (Dc)^a \right] = QK \tag{4.150}$$

where

$$K = \int \mathrm{d}^n x \left(\frac{1}{2\alpha} b^a + \partial B^a \right) \bar{c}^a . \tag{4.151}$$

Let us remark that b is not a dynamical field. If we integrate it out we obtain an equivalent expression for the gauge-fixing action, namely

$$S_{\mathrm{GF}} = \int \mathrm{d}^n x \left[\frac{1}{2\alpha} (\partial B)^2 - \bar{c}\, \partial \mathrm{D} c \right] \tag{4.152}$$

which (apart from the ghost term) we already used in previous discussions.

We may generalize the gauge-fixing condition and define K by an arbitrary function F

$$K = \int \mathrm{d}^n x \left(\frac{1}{2\alpha} b^a + F^a \right) \bar{c}^a . \tag{4.153}$$

What we will show now is that the Fokker–Planck equation

$$\dot{P} = \int \mathrm{d}^n x \frac{\delta}{\delta B_\mu^a} \left(\frac{\delta}{\delta B_\mu^a} + \frac{\delta S}{\delta B_\mu^a} + D^{ab} \dot{\Lambda}^b \right) P \tag{4.154}$$

has a stationary solution P^{eq} given by the conventional path-integral density

$$P^{\mathrm{eq}} = \int \mathrm{d}B\, \mathrm{D}b\, \mathrm{D}c\, \mathrm{D}\bar{c}\, \exp(-S - S_{\mathrm{GF}}) \tag{4.155}$$

provided we choose $\dot{\Lambda}$ as

$$\dot{\Lambda}^a = -\frac{1}{P^{\mathrm{eq}}} \int \mathrm{D}b\, \mathrm{D}c\, \mathrm{D}\bar{c} \int \mathrm{d}^n y\, C^a(x) \frac{\delta}{\delta B^b_\mu(y)} \left[\frac{\delta K}{\delta B^b_\mu(y)} \exp(-S - S_{\mathrm{GF}})\right]. \tag{4.156}$$

Though $\dot{\Lambda}$ is an involved quantity, the proof of the above statement can quite elegantly [4.8] be performed using the properties of the BRS generator. Before working out the details, let us note that for $F^a = \partial B^a$

$$\dot{\Lambda}^a = -\partial_\nu B_\nu \tag{4.157}$$

in the Abelian case.

We insert now eq. (4.156) in eq. (4.154) and get for the stationary Fokker–Planck equation (we denote $\int \mathrm{D}b\, \mathrm{D}c\, \mathrm{D}\bar{c}$ by $⨍$):

$$\begin{aligned} &-⨍ \int \mathrm{d}^n x \frac{\delta}{\delta B^a_\mu(x)} \frac{\delta(QK)}{\delta B^a_\mu(x)} \exp(-S - S_{\mathrm{GF}}) \\ &= ⨍ \int \mathrm{d}^n x\, \mathrm{d}^n y \frac{\delta}{\delta B^a_\mu(x)} \left\{(D^{ab}_\mu C^b(x)) \frac{\delta}{\delta B^b_\nu(y)} \left(\frac{\delta K}{\delta B^b_\nu(y)} \exp(-S - S_{\mathrm{GF}})\right)\right\}. \end{aligned} \tag{4.158}$$

For later convenience two identities should be noted, namely

$$\frac{\delta}{\delta B^a_\mu(x)} (QK) = Q \frac{\delta K}{\delta B^a_\mu(x)} - g f^{abc} \bar{c}^c \frac{\delta K}{\delta B^b_\mu(x)} \tag{4.159}$$

and

$$\begin{aligned} \frac{\delta}{\delta B^a_\mu(x)} (D^{ad}_\mu(x)\, C^d(x)) \frac{\delta}{\delta B^b_\nu(y)} &= \frac{\delta}{\delta B^b_\nu(y)} (D^{ad}_\mu(x)\, C^d(x)) \frac{\delta}{\delta B^a_\mu(x)} \\ &\quad - \mathrm{i} f^{abd}\, C^d(x) \frac{\delta}{\delta B^a_\mu(x)} \delta^n(x-y). \end{aligned} \tag{4.160}$$

Inserting the first identity in the l.h.s. of eq. (4.158) we get

$$-⨍ \int \mathrm{d}^n x \frac{\delta}{\delta B^a_\mu(x)} \left[Q\left(\frac{\delta K}{\delta B^a_\mu(x)}\right) - g f^{abc} C^c \frac{\delta K}{\delta B^b_\mu(x)}\right] \exp(-S - S_{\mathrm{GF}}). \tag{4.161}$$

Conversely, with the second identity the r.h.s. of (4.158) reads

$$\begin{aligned} &-⨍ \int \mathrm{d}^n x\, \mathrm{d}^n y \frac{\delta}{\delta B^b_\nu(y)} \left\{(D_\mu C(x))^a \frac{\delta}{\delta B^a_\mu(x)} \left(\frac{\delta K}{\delta B^b_\nu(y)} \exp(-S - S_{\mathrm{GF}})\right)\right\} \\ &+ ⨍ \int \mathrm{d}^n x \frac{\delta}{\delta B^a_\mu(x)} \left\{g f^{abc} c^c \frac{\delta K}{\delta B^b_\mu(x)} \exp(-S - S_{\mathrm{GF}})\right\}. \end{aligned} \tag{4.162}$$

With the definition (4.148) of the BRS charge this then becomes

$$
\begin{aligned}
&-\oint\int \mathrm{d}^n x\, \frac{\delta}{\delta B^b_\nu(y)} \left\{ Q\left(\frac{\delta K}{\delta B^b_\nu(y)} \exp(-S - S_{\mathrm{GF}}) \right) \right\} \\
&+\oint\int \mathrm{d}^n x\, \mathrm{d}^n y\, \frac{\delta}{\delta B^b_\nu(y)} \left\{ \frac{g}{2}\, f^{acd} C^c(x)\, C^d(x)\, \frac{\delta}{\delta C^a(x)} + b^a\, \frac{\delta}{\delta c^a} \right\} \frac{\delta K}{\delta B^b_\nu(y)} \exp(-S - S_{\mathrm{GF}}) \\
&-\oint\int \mathrm{d}^n x\, \frac{\delta}{\delta B^a_\mu(x)} \left\{ g f^{abc} c^c\, \frac{\delta K}{\delta B^b_\mu} \exp(-S - S_{\mathrm{GF}}) \right\}.
\end{aligned}
\tag{4.163}
$$

Given the BRS invariance of S and S_{GF}

$$
Q(S + S_{\mathrm{GF}}) = 0 \tag{4.164}
$$

the first and the third integral just add to the l.h.s. of eq. (4.158) given by eq. (4.161). The second integral vanishes after a few straightforward manipulations by partial integration [4.6, 4.8]. We have thus recovered the standard Faddeev–Popov formula.

At the end of this formal equivalence proof for gauge theories we would like to add a word of caution, namely that the investigation of the dynamics of the implied stochastic gauge-fixing term seems to be more involved due to the non-locality of $\dot{\Lambda}$ (4.156). In a recent paper [4.19] it has, in fact, been claimed that confinement within one Gribov region seems unlikely. But in our opinion this issue still requires investigation.

5. Tensor and string field theories

In this section we wish to discuss the possible generalizations of the Parisi–Wu scheme to fields of higher rank. Of most interest, perhaps, is the possibility of a stochastic quantization of theories involving gravity. One such theory, Einstein gravity, will be discussed in section 5.2, and in section 5.3 we present a brief discussion of stochastic quantization in connection with string field theories. We begin, however, with a much simpler system: a theory involving one antisymmetric tensor field $B_{\mu\nu}$. We restrict ourselves to the case where this tensor field is Abelian, but there are no fundamental obstacles to generalizing it to the non-Abelian case as well.

5.1. Abelian antisymmetric tensor fields

We first define a totally antisymmetric field strength $F_{\mu\nu\sigma}(x)$ associated with the field $B_{\mu\nu}(x) = -B_{\nu\mu}(x)$:

$$
F_{\mu\nu\sigma} = \partial_\mu B_{\nu\sigma} - \partial_\nu B_{\sigma\mu} + \partial_\sigma B_{\mu\nu} \tag{5.1}
$$

and a corresponding Lagrangian

$$
\mathcal{L} = -\tfrac{1}{12} F_{\mu\nu\sigma} F_{\mu\nu\sigma}\,. \tag{5.2}
$$

It is clear from (5.1) that the field strength is invariant under the following gauge transformation:

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \tag{5.3}$$

where Λ_μ is an arbitrary vector field. We can therefore view eq. (5.2) as the tensor generalization of an Abelian gauge theory. The antisymmetric tensor field $B_{\mu\nu}$ plays the role of a gauge potential.

However, there is one important and interesting difference between an ordinary field theory of a Maxwell field, and the theory defined by eq. (5.2). The tensor field theory has a much larger gauge invariance [5.1]. This is clear already from eq. (5.3) since obviously the gauge transformation *itself* is invariant under

$$\Lambda_\mu \to \Lambda_\mu + \partial_\mu \chi \tag{5.4}$$

with χ an arbitrary scalar function. This 'hidden' gauge symmetry has important consequences for the quantization of the theory (5.2) if one relies on standard methods. As we shall see, in a stochastic quantization scheme the effects of gauge invariance are automatically taken into account.

In a standard Faddeev–Popov procedure for this theory the method for introducing ghosts appears to break down. This is clear, because the Faddeev–Popov determinant associated with the transformation (5.3) is not properly defined on account of the Faddeev–Popov operator being non-invertible. This again is simply a consequence of the extra gauge invariance (5.4), which represents the new invariance associated with the first ghost fields themselves. In such a standard formulation the cure to this problem lies in the introduction of 'ghosts for ghosts' [5.2–5.5]. (For tensors of higher rank this scheme of introducing more and more ghost fields to remove unphysical degrees of freedom can continue further.)

In stochastic quantization we do not have to fix the gauge, and ghosts are therefore not necessary. The same holds even in situations where the ghosts themselves would require new ghosts. Let us now see [5.6] how the stochastic quantization prescription works for the tensor field $B_{\mu\nu}(x)$ of the Lagrangian (5.2).

In a second-order formalism, corresponding to (5.2), we have just one Langevin equation:

$$\frac{\partial}{\partial t} B_{\mu\nu}(x, t) = -\frac{\delta S}{\delta B_{\mu\nu}(x, t)} + \eta_{\mu\nu}(x, t) \tag{5.5}$$

i.e.

$$\frac{\partial}{\partial t} B_{\mu\nu}(x, t) = \partial^2 B_{\mu\nu}(x, t) + \partial_\sigma \partial_\mu B_{\nu\sigma}(x, t) + \partial_\nu \partial_\sigma B_{\sigma\mu}(x, t) + \eta_{\mu\nu}(x, t)\,. \tag{5.6}$$

We mention in passing that this is only *one* possible way of quantizing the theory (5.2). An alternative second-order formalism is based on the Lagrangian

$$\mathscr{L} = \frac{\alpha}{4}\, \varepsilon_{\mu\nu\lambda\rho} F_{\mu\nu}[H]\, B_{\lambda\rho}(x) - \frac{\alpha^2}{2}\, H_\mu(x)\, H_\mu(x) \tag{5.7}$$

with

$$F_{\mu\nu}[H] = \partial_\mu H_\nu(x) - \partial_\nu H_\mu(x)\,. \tag{5.8}$$

The action associated with (5.7) is, both classically and quantum mechanically, equivalent to the action associated with (5.2). This is readily checked by first performing partial integrations on the $H_\mu(x)$ field. The $H_\mu(x)$ field then appears as a non-dynamical auxiliary field which only enters quadratically in the action. Performing the exact Gaussian integration H_μ then yields the action corresponding to eq. (5.2). In particular, note that the (real) parameter α in eq. (5.7) is completely arbitrary since this parameter disappears upon integrating out the H_μ field; all gauge invariant quantities will hence be independent of this parameter.

Associated with the first-order formalism are *two* coupled Langevin equations:

$$\frac{\partial}{\partial t} B_{\mu\nu}(x,t) = -\frac{\alpha}{4}\, \varepsilon_{\mu\nu\rho\sigma}(\partial_\rho H_\sigma(x,t) - \partial_\sigma H_\rho(x,t)) + \eta_{\mu\nu}(x,t) \tag{5.9a}$$

$$\frac{\partial}{\partial t} H_\mu(x,t) = \alpha^2\, H_\mu(x,t) - \frac{\alpha}{2}\, \varepsilon_{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma}(x,t) + \eta_\mu(x,t)\,. \tag{5.9b}$$

One can readily check that the stochastic quantization based on these two coupled equations yields results identical to those obtained from the first-order formalism. Let us therefore restrict ourselves to the Langevin equation (5.6) of the second-order formalism.

The stochastic noise field of eq. (5.6) is of course itself an antisymmetric tensor field, $\eta_{\mu\nu}(x,t) = -\eta_{\nu\mu}(x,t)$. Its expectation values are determined by the measure of the partition function

$$Z = \int \mathrm{D}\eta \exp\left\{-\tfrac{1}{2}\int \mathrm{d}^4x\, \mathrm{d}t\, \eta_{\mu\nu}(x,t)\, \eta_{\mu\nu}(x,t)\right\}. \tag{5.10}$$

This means that any functional $F[B_{\mu\nu}]$ [and the $B_{\mu\nu}(x,t)$ field is now implicitly a functional of the noise field $\eta_{\mu\nu}(x,t)$ via the Langevin equation (5.6)] has a stochastic expectation value $\langle F[B]\rangle_\eta$ given by

$$\langle F[B]\rangle_\eta = \frac{1}{Z}\int \mathrm{D}\eta\, F[B] \exp\left\{-\tfrac{1}{2}\int \mathrm{d}^4x\, \mathrm{d}t\, \eta_{\mu\nu}(x,t)\, \eta_{\mu\nu}(x,t)\right\} \tag{5.11}$$

with $B_{\mu\nu}(x,t)$ determined from eq. (5.6). Z is the normalization constant given in (5.10).

In particular,

$$\langle \eta_{\mu\nu}(x,t)\, \eta_{\rho\sigma}(x',t')\rangle_\eta = 2(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\nu\rho}\delta_{\mu\sigma})\, \delta^4(x-x')\, \delta(t-t') \tag{5.12}$$

and similarly for higher $2n$-point functions.

Now as $t \to \infty$ these stochastic expectation values reduce to ordinary vacuum expectation values, for any *gauge-invariant* functional. In order to see how this reduction to ordinary vacuum expectation values works in detail, and how the physical degrees of freedom contribute to gauge invariant quantities, let us try to solve the Langevin equation (5.6) directly.

To this end, let us first introduce a convenient and compact notation. We start with a 'unit' 1 defined by

$$\mathbb{1}_{\mu\nu,\rho\sigma} = \tfrac{1}{2}(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\nu\rho}\delta_{\mu\sigma}) \tag{5.13}$$

which is antisymmetric in each set of indices separately, but symmetric between $(\mu\nu) \leftrightarrow (\rho\sigma)$.

Also, it is convenient to introduce the standard momentum projection operator for massless vector fields,

$$T_{\mu\nu} = \delta_{\mu\nu} - k_\mu k_\nu / k^2 \tag{5.14}$$

and define

$$\mathbb{P}_{\alpha\beta,\gamma\delta} = \tfrac{1}{2}(T_{\alpha\gamma} T_{\beta\delta} - T_{\alpha\delta} T_{\beta\gamma}) . \tag{5.15}$$

Note that $\mathbb{P}$ constructed in this way is also a projection operator:

$$\mathbb{P}^2 = \mathbb{P} , \qquad \mathbb{P}(\mathbb{1} - \mathbb{P}) = 0 , \qquad (\mathbb{1} - \mathbb{P})^2 = (\mathbb{1} - \mathbb{P}) . \tag{5.16}$$

Having introduced this notation it is trivial to solve the Langevin equation (5.6). We find

$$\mathbb{G}(k, t) = \theta(t)\, [\mathbb{P}(k) \exp(-k^2 t) + (\mathbb{1} - \mathbb{P})] \tag{5.17}$$

in momentum space, and the general solution to eq. (5.6) is hence

$$B_{\mu\nu}(k, t) = \int_0^t \mathrm{d}\tau\, \mathbb{G}_{\mu\nu,\alpha\beta}(k, t-\tau)\, \eta_{\alpha\beta}(k, \tau) + B_{\alpha\beta}(k, 0)\, [\mathbb{P} \exp(-k^2 t) + \mathbb{1} - \mathbb{P}]_{\alpha\beta\mu\nu} \tag{5.18}$$

with $B_{\mu\nu}(k, 0)$ determined by the initial conditions.

Note that the $B_{\mu\nu}$ field naturally divides into two parts:

$$B_{\mu\nu}(k, t) = B^{\mathrm{T}}_{\mu\nu}(k, t) + B^{\mathrm{L}}_{\mu\nu}(k, t) \tag{5.19}$$

with

$$B^{\mathrm{T}} = \mathbb{P} B , \qquad B^{L} = (\mathbb{1} - \mathbb{P}) B . \tag{5.20}$$

[The multiplication rule being the natural one: $(A \cdot B)_{\alpha\beta} = A_{\alpha\beta,\gamma\delta} B_{\gamma\delta}$.]

The fields $B^{\mathrm{T}}_{\mu\nu}$ and $B^{\mathrm{L}}_{\mu\nu}$ are solutions of two *decoupled* Langevin equations:

$$\frac{\partial}{\partial\tau} B^{\mathrm{T}}_{\mu\nu}(k, t) = -k^2\, B^{\mathrm{T}}_{\mu\nu}(k, t) + \eta^{\mathrm{T}}_{\mu\nu}(k, t) \tag{5.21a}$$

$$\frac{\partial}{\partial\tau} B^{\mathrm{L}}_{\mu\nu}(k, t) = \eta^{\mathrm{L}}_{\mu\nu}(k, t) \tag{5.21b}$$

in complete analogy to the Abelian gauge field case [see section 4]. As can be seen from eq. (5.21), the field $B^{\mathrm{L}}_{\mu\nu}(k, t)$ simply performs random walk around its initial configuration.

It is $B^{\mathrm{T}}_{\mu\nu}(k, t)$ which is responsible for the propagation of the physical degrees of freedom. Since $\mathbb{P}$ satisfies $\mathbb{P}^2 = \mathbb{P}$, its eigenvalues will be 0 or 1. We can therefore count the degrees of freedom associated with the propagation of $B^{\mathrm{T}} = \mathbb{P} B$ by taking the trace:

$$\mathrm{Tr}\,\mathbb{P}(k) = \mathbb{1}_{\mu\nu,\rho\sigma}\,\mathbb{P}_{\rho\sigma,\mu\nu}(k) = 3 \tag{5.22}$$

in $d=4$ dimensions.

In general, in d dimensions $B_{\mu\nu}$ will propagate $(d-1)(d-2)/2$ degrees of freedom. This is readily seen from a different argument [5.6]. Since $B_{\mu\nu}$ is antisymmetric, it has $d(d-1)/2$ different components. Since only $B^{\mathrm{T}}_{\mu\nu}$ propagates physical degrees of freedom, we must impose the transversality condition, which leads to $(d-1)$ constraints. This leaves $d(d-1)/2-(d-1)=(d-1)(d-2)/2$ degrees of freedom. Of course, for a physical scattering amplitude one must also impose the on-shell condition $k^2=0$. This is done by Wick rotating back into Minkowski space where S-matrix elements can be computed. It can be shown [5.6] that the on-shell condition $k^2=0$ reduces the number of degrees of freedom to $(d-2)(d-3)/2$, and $B_{\mu\nu}$ carries therefore in the 4-dimensional case only *one* physical degree of freedom.

We should point out that this result is in agreement with the conventional ghosts-for-ghosts approach [5.2], and reflects one of the surprising properties of antisymmetric tensor fields in 4 space-time dimensions. A simple argument can actually be given for this result: introduce a scalar field ϕ through

$$F_{\mu\nu\sigma}(x) = \varepsilon_{\mu\nu\sigma\rho}\,\partial^{\rho}\phi(x) \tag{5.23}$$

in terms of which the Lagrangian (5.2) takes the form

$$\mathscr{L} = -\tfrac{1}{2}\partial_{\mu}\phi\,\partial^{\mu}\phi \tag{5.24}$$

i.e., the action density of a free, massless, scalar field! Of course, such manipulations are strictly speaking valid only at the 'classical' level of equations of motion. As we have shown above, this actually remains valid upon quantization, but only on the mass shell.

As an example, let us now compute the transverse propagator. Using eqs. (5.18) and (5.21) one finds [5.6]

$$D^{\mathrm{T}}_{\mu\nu,\rho\sigma} = \lim_{t\to\infty}\langle B^{\mathrm{T}}_{\mu\nu}(k,t)\,B^{\mathrm{T}}_{\rho\sigma}(k',t)\rangle = \mathbb{P}_{\mu\nu,\rho\sigma}(k)(2\pi)^d\,\delta^d(k+k')\,\frac{1}{k^2}\,. \tag{5.25}$$

Similarly, for a gauge-invariant quantity such as the $F_{\nu\mu\sigma}-F_{\rho\gamma\delta}$ correlation function, one finds, for example [5.6]:

$$\lim_{t\to\infty}\langle F_{\alpha\beta\gamma}(k,t)\,F_{\alpha\beta\gamma}(k',t)\rangle = -(2\pi)^d\,\delta^d(k+k')\,\frac{3(d-1)(d-2)}{2} \tag{5.26}$$

in d dimensions. This, of course, agrees with the result of standard quantization procedures. In this Abelian case, the crucial rôle played by the ghosts-for-ghosts could in any case only be ascertained if one computed, for example, finite temperature effects [5.5], or used a gauge-fixing condition containing covariant coupling to other fields such as gravity [5.3].

5.2. Einstein gravity

5.2.1. Linearized gravity and the indefiniteness problem

In this section we would like to apply the stochastic scheme to the gravitational field. Given the many

known problems which arise when one attempts to quantize the gravitational field by conventional methods [5.7] we should not be surprised when encountering difficulties within the stochastic approach as well. Indeed, let us recall one of the obstacles for the quantization of Euclidean gravity, namely the fact that the Einstein Hilbert action S_{EH} is not positive definite [5.8]. As a consequence $\exp(-S_{\mathrm{EH}})$ will eventually increase exponentially so that a straightforward path integral quantization in Euclidean space is not possible and modifications of the standard path integral procedure are necessary [5.8]. At the beginning of this section, we will outline the standard gauge-fixing procedure for linearized gravity and sketch the approach of ref. [5.8] thereafter.

We will subsequently see that a straightforward application of the stochastic quantization scheme is not possible [5.9, 5.10], as the stochastic process never will relax to equilibrium. This is a direct consequence of the indefiniteness of the Euclidean action. We will explain that this non-relaxation is intrinsically different from the one encountered in ordinary gauge theories (see section 4.1.2), persisting as well when calculating gauge invariant quantities.

A solution to this problem may possibly be found when allowing for a more general formulation of stochastic quantization than so far considered: guided by the principle of general covariance with respect to field redefinitions a one-parameter family of Langevin equations for the metric tensor field has been proposed [5.11] (for an earlier attempt within a related context see also ref. [5.12]), and a preferred parameter value has been pointed out.

The indefiniteness problem manifests itself when one tries to give a probabilistic interpretation of the noise. We discuss a rather natural precription for how to deal with this and study the equilibrium limit of the corresponding Fokker–Planck distribution. Remarkably we find an equilibrium distribution which is just the same as that corresponding to the approach of ref. [5.8].

Before closing this section we will put several other stochastic approaches to quantum gravity [5.9, 5.10, 5.13, 5.14] into perspective (see also [5.24]) and end our discussion with a few remarks concerning the stochastic approach to gravity beyond the linear approximation [5.11].

We start from the Einstein–Hilbert action (for standard textbooks on gravity see, for example, ref. [5.15]) in Euclidean (or more precisely Riemannian) space-time

$$S_{\mathrm{EH}}[g_{ab}] = -\frac{1}{2x}\int \mathrm{d}^4x\,\sqrt{g}\,R[g_{ab}] \tag{5.27}$$

where

$$x = 8\pi G/c^3 \tag{5.28}$$

$$g = -\det g_{ab} \tag{5.29}$$

and R is the curvature scalar of the positive-definite metric $g_{ab}(x)$. Expanding the metric around flat space-time

$$g_{ab} = \delta_{ab} + 2\sqrt{x}\,h_{ab} \tag{5.30}$$

one obtains in lowest order (up to surface terms) the quadratic action [5.15]

$$S^{(0)}_{\mathrm{EH}}(h_{ab}) = \tfrac{1}{2}\int \mathrm{d}^4x\,h_{ab}V_{abcd}h_{cd} \tag{5.31}$$

describing a helicity-2 field in Euclidean space-time, with V_{abcd} being given in momentum space by

$$V_{abcd} = -\frac{k^2}{2}(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} - 2\delta_{ab}\delta_{cd}) - (k_a k_b \delta_{cd} + \delta_{ab} k_c k_d) + \tfrac{1}{2}(k_a k_c \delta_{bd} + k_a k_d \delta_{bc} + k_b k_c \delta_{ad} + k_b k_a \delta_{cd}) . \tag{5.32}$$

From (5.32) it follows that

$$V_{abcd} = V_{(ab)(cd)} = V_{cdab} \tag{5.33}$$

(where we denoted symmetrization of indices by round brackets), which implies that V defines a self-adjoint operator on the linear space of symmetric tensor fields. A more compact representation of V may be obtained upon the introduction of the following complete set of projection operators [5.16]

$$P^2_{abcd} = \tfrac{1}{2}(T_{ac}T_{bd} + T_{ad}T_{bc}) - \tfrac{1}{3}T_{ab}T_{cd} \tag{5.34}$$

$$P^1_{abcd} = \tfrac{1}{2}(T_{ac}L_{bd} + T_{ad}L_{bc} + T_{bc}L_{ad} + T_{bd}L_{ac}) \tag{5.35}$$

$$P^0_{abcd} = L_{ab}L_{cd} \tag{5.36}$$

$$P^{0\prime}_{abcd} = \tfrac{1}{3}T_{ab}T_{cd} \tag{5.37}$$

where again as in section 4

$$T_{ab} = \delta_{ab} - k_a k_b / k^2 \tag{5.38}$$

$$L_{ab} = k_a k_b / k^2 . \tag{5.39}$$

Denoting the unit operator on the space of symmetric tensor fields by

$$\mathbb{1}_{abcd} = \tfrac{1}{2}(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) \tag{5.40}$$

one can easily check the completeness relation

$$P^2 + P^1 + P^0 + P^{0\prime} = \mathbb{1} . \tag{5.41}$$

For convenience of later use, let us note two useful relations, namely

$$(P^1 + 2P^0)_{abcd} = \tfrac{1}{2}(\delta_{ac}L_{bd} + \delta_{ad}L_{bc} + \delta_{bc}L_{ad} + \delta_{bd}L_{ac}) \tag{5.42}$$

$$(3P^{0\prime} - P^0)_{abcd} = \delta_{ab}\delta_{cd} - \delta_{ab}L_{cd} - L_{ab}\delta_{cd} \tag{5.43}$$

which follow directly from the definitions (5.34)–(5.37). In terms of the above spin projection operators V is simply proportional to the difference of two of them

$$V = -k^2(P^2 - 2P^{0\prime}) \tag{5.44}$$

which clearly shows the indefiniteness of the Euclidean action already at the linearized level: $P^{0\prime}$ (which comprises the conformal degrees of freedom as it contains contributions of the trace part of h_{ab}) appears in association with a crucial minus sign and will imply the action to be unbounded from below. Let us, therefore, review in the following one proposed modification [5.8] of the standard path integral procedure for linearized gravity. Using this prescription we will, as an example, calculate the free graviton propagator and compare it with the expressions obtained within the stochastic scheme.

Owing to the gauge invariance of the action (5.31) under the gauge transformations

$$h_{ab} \to h_{ab} + \partial_a \Lambda_b + \partial_b \Lambda_a \tag{5.45}$$

where Λ_a corresponds to an infinitesimal coordinate transformation, we have to impose a gauge fixing condition on h_{ab}.

We consider the linear, covariant and local gauge fixing condition

$$C_a^{(\lambda)}(h) = \partial_c h_{ac} - \lambda \partial_a h_{cc} = 0\,, \qquad \lambda \neq 1 \tag{5.46}$$

which is implemented by adding the gauge fixing term

$$S_{\mathrm{GF}}^{(\lambda,\alpha)}(h) = \int \mathrm{d}^4 x\, \alpha^{-1} C_a^{(\lambda)}(h) C_a^{(\lambda)}(h) \tag{5.47}$$

to the action (5.31). We do not have to introduce in our case Faddeev–Popov ghost fields, as they decouple in the linear approximation. In momentum space we obtain, using the projection operators (5.34)–(5.37) as well as relations (5.42), (5.43),

$$\begin{aligned}\alpha^{-1} C_a^{(\lambda)} C_a^{(\lambda)} &= -\alpha^{-1} h_{ab}[\tfrac{1}{2}(k_b \delta_{ea} + k_a \delta_{eb}) - \lambda k_e \delta_{ab}][\tfrac{1}{2}(k_c \delta_{ed} + k_d \delta_{ec}) - \lambda k_e \delta_{cd}] h_{cd} \\ &= -\alpha^{-1} h_{ab} k^2 [\tfrac{1}{2}(P^1 + 2P^0)_{abcd} + \lambda(3P^{0\prime} - P^0 - \delta\delta)_{abcd} + \lambda^2 \delta_{ab}\delta_{cd}] h_{cd}\,.\end{aligned} \tag{5.48}$$

The simplest expression for the gauge fixed action is obtained for $\lambda = \frac{1}{2}$, $\alpha = \frac{1}{2}$ (the 'harmonic' gauge) in which case it becomes, using the completeness relation (5.41),

$$S = S_{\mathrm{EH}}^{(0)}(h) + S_{\mathrm{GF}}^{(1/2,1/2)}(h) = \int \mathrm{d}^4 k\, h k^2 (\mathbb{1} - \tfrac{1}{2}\delta\delta) h\,. \tag{5.49}$$

It should be noted that the gauge fixing procedure did not change the indefiniteness property of the action. This can most easily be seen by studying the eigenstates h_{E} of the operator $\mathbb{1} - \frac{1}{2}\delta\delta$

$$(\mathbb{1} - \tfrac{1}{2}\delta\delta) h_{\mathrm{E}} = E h_{\mathrm{E}}\,. \tag{5.50}$$

If we make the general ansatz for h_{E}

$$h_{\mathrm{E}} = a\mathbb{1} + b\delta\delta \tag{5.51}$$

we find two solutions, namely

$$h_1 = \mathbb{1} - \delta\delta/4\,, \qquad h_{-1} = \delta\delta \tag{5.52}$$

which clearly shows the indefiniteness of the gauge fixed action (5.49).

In order to obtain instead a positive definite action it was proposed in ref. [5.8] to continue h in the complex plane in a specific way, namely to require the trace part (the conformal mode) of h_{ab} to be purely imaginary. Specifically,

$$h = \left(\mathbb{1} - \frac{\delta\delta}{4}\right) h_{\mathrm{Re}} + \mathrm{i}\,\frac{\delta\delta}{4}\, h_{\mathrm{Im}} \tag{5.53}$$

so that one obtains instead of (5.49)

$$\tilde{S} = \tfrac{1}{2} \int \mathrm{d}^4k\, k^2 \left[h_{\mathrm{Re}} \left(\mathbb{1} - \frac{\delta\delta}{4}\right) h_{\mathrm{Re}} + h_{\mathrm{Im}} \frac{\delta\delta}{4} h_{\mathrm{Im}} \right] = \tfrac{1}{2} \int \mathrm{d}^4k\, k^2\, h\, h^* \tag{5.54}$$

which clearly is positive definite. It is now easy to calculate the free graviton propagator in the harmonic gauge, which becomes

$$\begin{aligned} &\int \mathrm{D}\left[\left(\mathbb{1} - \frac{\delta\delta}{4}\right) h_{\mathrm{Re}}\right] \mathrm{D}\left[\frac{\delta\delta}{4} h_{\mathrm{Im}}\right] \exp\left\{-\tfrac{1}{2} \int \mathrm{d}^4k\, k^2 \left[h_{\mathrm{Re}} \left(\mathbb{1} - \frac{\delta\delta}{4}\right) h_{\mathrm{Re}} + h_{\mathrm{Im}} \frac{\delta\delta}{4} h_{\mathrm{Im}} \right]\right\} \\ &\quad \cdot \left[h_{\mathrm{Re}} \left(\mathbb{1} - \frac{\delta\delta}{4}\right) h_{\mathrm{Re}} - h_{\mathrm{Im}} \frac{\delta\delta}{4} h_{\mathrm{Im}} \right] \Big/ \left\{ \int \mathrm{D}\left[\left(\mathbb{1} - \frac{\delta\delta}{4}\right) h_{\mathrm{Re}}\right] \mathrm{D}\left[\frac{\delta\delta}{4} h_{\mathrm{Im}}\right] \right. \\ &\quad \left. \cdot \exp\left\{-\tfrac{1}{2} \int \mathrm{d}^4k\, k^2 \left[h_{\mathrm{Re}} \left(\mathbb{1} - \frac{\delta\delta}{4}\right) h_{\mathrm{Re}} + h_{\mathrm{Im}} \frac{\delta\delta}{4} h_{\mathrm{Im}} \right]\right\}\right\}. \end{aligned} \tag{5.55}$$

Using (5.53) and the fact that

$$h^2 = h_{\mathrm{Re}} \left(\mathbb{1} - \frac{\delta\delta}{4}\right) h_{\mathrm{Re}} - h_{\mathrm{Im}} \frac{\delta\delta}{4} h_{\mathrm{Im}} = h\left(\mathbb{1} - \frac{\delta\delta}{2}\right) h^* \tag{5.56}$$

we obtain, finally,

$$\frac{\int \mathrm{D}h\, \mathrm{D}h^* \exp\left(-\tfrac{1}{2} \int \mathrm{d}^4k\, k^2 h h^*\right) h\left(\mathbb{1} - \frac{\delta\delta}{2}\right) h^*}{\int \mathrm{D}h\, \mathrm{D}h^* \exp\left(-\tfrac{1}{2} \int \mathrm{d}^4k\, k^2 h h^*\right)} = \left(\mathbb{1} - \frac{\delta\delta}{2}\right) \frac{1}{k^2} \tag{5.57}$$

which ends our short discussion on the method proposed in ref. [5.8].

5.2.2. Breakdown of naïve stochastic quantization of gravity

In this section we would like to study the consequences of the indefinite action (5.31) for stochastic quantization, using a straightforward (however as will be seen, too naïve) application of the stochastic methods presented in the previous sections. The Langevin equation reads in this case

$$\frac{\partial}{\partial t} h_{ab}(k,t) = V_{abcd} h_{cd} + \xi_{ab} \tag{5.58}$$

with the white noise fulfilling

$$\langle \xi_{ab}(k,t) \rangle = 0 \tag{5.59}$$

$$\langle \xi_{ab}(k,t)\, \xi_{cd}(k',t') \rangle = 2\,(2\pi)^4\, \mathbb{1}\, \delta^4(k+k')\, \delta(t-t')\,, \qquad \text{etc}\,. \tag{5.60}$$

We can solve eq. (5.58) by finding first the solution $G(k,t)$ of the deterministic part of this equation, i.e. setting η to zero,

$$G(k,t) = \theta(t)\, \mathrm{e}^{tV} \tag{5.61}$$

subject to the boundary condition

$$G(k,0) = \mathbb{1}\,. \tag{5.62}$$

Using the fact that P^2 and $P^{0\prime}$ are orthogonal projectors satisfying (5.41), we easily obtain for (5.61)

$$\begin{aligned} G(k,t) &= \theta(t) \exp(-k^2 t P^2) \exp(2k^2 t P^{0\prime}) \\ &= (\exp(-k^2 t)\, P^2 + \mathbb{1} - P^2)\, (\exp(2k^2 t)\, P^{0\prime} + \mathbb{1} - P^{0\prime})\, \theta(t) \\ &= (\exp(-k^2 t)\, P^2 + \exp(2k^2 t)\, P^{0\prime} + P^1 + P^0)\, \theta(t) \end{aligned} \tag{5.63}$$

which shows, as a direct consequence of the crucial minus sign in (5.44), a 'runaway' behaviour in the $P^{0\prime}$ contribution. Continuing, we find the unique solution of (5.58) with the initial condition

$$h_{ab}(k,0) = 0 \tag{5.64}$$

as

$$h_{ab}(k,t) = \int_0^t G_{abcd}(k, t-\tau)\, \xi_{cd}(k,\tau)\, \mathrm{d}\tau \tag{5.65}$$

and evaluate the equal time two-point function to be

$$\begin{aligned} &\langle h_{ab}(k,t)\, h_{cd}(k',t) \rangle \\ &\quad = 2\,(2\pi)^4\, \delta^4(h+h') \int_0^t \mathrm{d}\tau\, G_{abem}(k,t-\tau)\, G_{cdem}(k,t-\tau) \\ &\quad = (2\pi)^4\, \delta^4(k+k') \left[\frac{1}{k^2}\, P^2 (1 - \exp(-2k^2 t)) - \frac{1}{2k^2}\, P^{0\prime}(1 - \exp(+4k^2 t)) + 2t(P^1 + P^0) \right]. \end{aligned} \tag{5.66}$$

In the Parisi–Wu approach the equal time-correlation functions are supposed to yield the Euclidean Green functions in the limit $t \to \infty$. Obviously this limit does not exist at all in the gravitational case as the right-hand side of (5.66) diverges both linearly and exponentially for $t \to \infty$.

Let us recall that the stochastic quantization of the Maxwell field yields a divergent propagator, too. But there the divergence is only linear and – being a pure gauge term proportional to $L_{\mu\nu}$ – reflects just the random walk behaviour of the gauge modes (see section 4); the divergences do not contribute to gauge invariant quantities.

In the gravity case, however, any gauge invariant quantity involves the linearized Riemann tensor [5.15]

$$R_{abcd} = 4k_{[a} \; h_{b][c} \; k_{d]} \tag{5.67}$$

(where we have indicated antisymmetrization of index pairs with square brackets) and all non-trivial gauge invariant expectation values can be constructed from

$$\langle R_{abcd}(k, t) \, R_{ijkl}(k', t') \rangle = 16 k_{[a} \; k_{[i} \langle h_{b][c}(k, t) \, h_{j][k}(k', t') \rangle \, k_{d]} \; k_{l]} \, . \tag{5.68}$$

We see easily from (5.68) and the definitions (5.34)–(5.37) that only P^2 and P^{0_w} contribute to gauge invariant quantities, as only they contain momentum independent terms

$$P^{2(\mathrm{m.ind.})} \sim \mathbb{1} - \tfrac{1}{3}\delta\delta \, , \qquad P^{0_w(\mathrm{m.ind.})} \sim \tfrac{1}{3}\delta\delta \tag{5.69}$$

whereas P^1 and P^0 are pure gauge modes. As a consequence, we can identify in the large time limit of the 2-point correlation (5.66) linear divergences corresponding to the gauge modes as well as exponential divergences for the gauge invariant part.

Generalizing this result to arbitrary Green functions we have to conclude that we failed in our first attempt to stochastically quantize linearized gravity. Apart from the (harmless) polynomial divergences in t (which drop out in gauge invariant quantities) we generally have to face exponentially increasing terms as well. In the following we are going to discuss several approaches in order to overcome this problem.

5.2.3. Generally covariant stochastic formulation

In this section we would like to show that stochastic quantization of gravity should be formulated in more general terms than so far considered. The concept [5.11] that we are going to use is manifest covariance of the Langevin equation with respect to redefinitions implied by general coordinate transformations. This necessitates the introduction of a metric tensor functional $G^{abcd}(x, x')$ (which depends on the metric tensor field g_{ab} and which should not be confused with it) in field configuration space. The most general field metric G^{abcd} that is local, and with respect to which the actions of general coordinate transformations on g_{ab} are isometries, is given [5.17] by the one-parameter (λ)-family of field metrics

$$G^{abcd}(x, x') = \frac{C}{2} \, g^{1/2} (g^{ac} g^{bd} + g^{ad} g^{bc} + \lambda g^{ab} g^{cd}) \, \delta^4(x - x') \tag{5.70}$$

$$\lambda \neq -\tfrac{1}{2} \tag{5.71}$$

where the (irrelevant) constant C may be chosen for convenience as $C = 1$. We are actually interested in the inverse field metric G_{abcd}, which will enter into the covariant Langevin equation for g_{ab}, and which reads

$$G_{abcd} = \frac{1}{2Cg^{1/2}} \left(g_{ac} g_{bd} + g_{ad} g_{bc} + \mu g_{ab} g_{cd} \right) \delta^4(x - x') \tag{5.72}$$

where

$$\mu = -\lambda/(2\lambda + 1) . \tag{5.73}$$

We also note that in 4 space-time dimensions the determinant of G^{abcd} is independent of g_{ab} [5.17]. In the linearized case G_{abcd} becomes

$$G^{(0)}_{abcd}(x, x') = \left(\mathbb{1}_{abcd} + \frac{\mu}{2}\, \delta_{ab} \delta_{cd} \right) \delta^4(x - x') \tag{5.74}$$

and the manifestly covariant Langevin equation for h_{ab} is given by

$$\frac{\partial h_{ab}}{\partial t} = - \int \mathrm{d}x'\, G^{(0)}_{abcd}(x, x')\, \frac{\delta S^{(0)}_{\mathrm{EH}}(h)}{\delta h_{cd}} + \xi_{ab} \tag{5.75}$$

or

$$\dot{h} = \left(\mathbb{1} + \frac{\mu}{2}\, \delta\delta \right) V h + \xi \tag{5.76}$$

and

$$\dot{h} = -k^2 \left(\mathbb{1} + \frac{\mu}{2}\, \delta\delta \right) (P^2 - 2P^{0\prime}) h + \xi . \tag{5.77}$$

In the above linearized equations the white noise ξ involves, by the principle of general coordinate invariance, in its Gaussian distribution the field metric G^{abcd} again

$$\mathrm{D}\xi \exp\left\{ -\tfrac{1}{4} \int \mathrm{d}^4x\, \mathrm{d}^4x'\, \mathrm{d}t\, \xi(x, t)_{ab}\, G^{(0)abcd}(x, x')\, \xi(x', t)_{cd} \right\} \tag{5.78}$$

so that the 2-point correlation of the noise reads in Fourier space

$$\langle \xi(k, t)\, \xi(k', t) \rangle = \left(\mathbb{1} + \frac{\mu}{2}\, \delta\delta \right) 2\, (2\pi)^4\, \delta^4(k + k')\, \delta(t - t') . \tag{5.79}$$

In the previous sections we have interpreted stochastic differential equations in the sense of Stratanovich (see the discussion in section 2). It can, however, be argued (see [5.11]) that in the gravity case a physically meaningful formulation preferably requires an Ito interpretation. Unfortunately, in this case the manifest covariance of (5.75)–(5.77) is lost owing to the Ito rules for variable transformations and a Langevin equation which manifestly transforms covariantly has to be constructed (see

[5.23]). The covariantized Langevin equation then contains an extra term involving the Laplace–Beltrami operator with respect to the field metric G^{abcd}

$$\Delta_G g_{ab} = \int \mathrm{d}^4x'\, G^{-1/2} \frac{\delta}{\delta g_{cd}(x')} (G^{1/2}\, G_{abcd}(x, x')) \tag{5.80}$$

which, given the independence of G from g_{ab} and taking care of symmetrization, reads

$$\Delta_G g_{ab} = \int \mathrm{d}^4x'\, \tfrac{1}{2} \left(\frac{\delta}{\delta g_{cd}(x')} + \frac{\delta}{\delta g_{dc}(x')} \right) G_{abcd}(x, x')\,. \tag{5.81}$$

It is readily evaluated as follows:

$$\Delta_G g_{ab} = \frac{g}{2}\, g^{-1/2} g_{ab}\, (1 + \mu)\, \delta^4(0) \tag{5.82}$$

where we used, for example (see ref. [5.15]),

$$\delta g^{-1/2}/\delta g_{ab} = -\tfrac{1}{2} g^{-1/2} g^{ab}\,. \tag{5.83}$$

As in this section we are concerned only with the linear approximation of gravity we should extract the corresponding contribution from (5.82). Referring the reader for more details to section 5.2.4 we, however, find that the Ito term contributes to the linearized Langevin equation (5.75)–(5.78) at least to order $\sqrt{k}$ and can therefore be omitted. As an independent argument for the non-appearance of the Ito term to lowest order we observe that the considered field transformations are the gauge transformations (5.45). These being linear they do not contribute to the Ito term, which involves twofold functional derivatives.

Though the parameter λ (or μ) was *a priori* unrestricted, it is clear that a preferred value can be deduced. In fact the value

$$\lambda = \mu = -1 \tag{5.84}$$

is favoured by the following two remarkable properties:

i) The divergent Ito term (5.82) vanishes, leaving a well-defined Langevin equation. (As stated above, by formally counting powers in k the Ito term does not arise in the linear approximation; it should, however, vanish in higher orders for consistency.) This is the parameter value for which the field g_{ab} is harmonic with respect to the metric G^{abcd}, i.e.

$$\Delta_G g_{ab} = 0\,. \tag{5.85}$$

ii) Only for this value of λ the drift term in the Langevin equation (5.77) exhibits a projection operator. With

$$P^{\mathrm{R}} = \left(\mathbb{1} - \frac{\delta\delta}{2}\right)(P^2 - 2P^{0\prime}) = P^2 - 2P^{0\prime} + \delta(\delta - L) \tag{5.86}$$

it holds indeed, using (5.34)–(5.37) and (5.41)–(5.43), that

$$(P^{\mathrm{R}})^2 = P^{\mathrm{R}} . \tag{5.87}$$

In the following we restrict our discussion to the value $\lambda = \mu = -1$. The presence of the projection operator allows us to solve easily the Langevin equation

$$\dot{h} = -k^2 P^{\mathrm{R}} h + \xi \tag{5.88}$$

namely

$$h = \int_0^t \mathrm{d}\tau \,[\exp\{-k^2(t-\tau)\}\, P^{\mathrm{R}} + \mathbb{1} - P^{\mathrm{R}}]\, \eta(\tau) . \tag{5.89}$$

It is important to observe that, owing to the presence of the projection operator P_{R} (in complete analogy to the case of the Maxwell field), gauge invariant quantities do not have linear (or, generally, polynomial) divergences in t. To see this we have to show that the momentum independent part of $\mathbb{1} - P_{\mathrm{R}}$ vanishes, so that all possible polynomial t-dependences (which are all associated with $\mathbb{1} - P_{\mathrm{R}}$) drop out in gauge invariant quantities. The above assertion is easily checked by extracting the momentum-independent part of P_{R}

$$P^{\mathrm{R(m.ind.)}} \sim \mathbb{1} - \tfrac{1}{3}\delta\delta - \tfrac{2}{3}\delta\delta + \delta\delta = \mathbb{1} \tag{5.90}$$

so that

$$\mathbb{1} - P^{\mathrm{R(m.ind.)}} \sim 0 . \tag{5.91}$$

Given that (5.79) is maintained we can calculate, as an example, the free propagator

$$\langle hh \rangle = P^{\mathrm{R}} \left(\mathbb{1} - \frac{\delta\delta}{2} \right) P^{\mathrm{R}}\, \delta^4(k+k')\, \frac{(2\pi)^4}{k^2} + \cdots \tag{5.92}$$

where the terms not explicitly written are proportional to $\mathbb{1} - P_{\mathrm{R}}$. From the above discussion we finally extract the momentum independent part

$$\langle hh \rangle^{\mathrm{(m.ind.)}} = \left(\mathbb{1} - \frac{\delta\delta}{2} \right) \delta^4(k+k')\, \frac{1}{k^2}\, (2\pi)^4 \tag{5.93}$$

which is in agreement with the result (5.57) that we obtained in the modified path integral approach [5.8].

5.2.4. Resolving the indefiniteness problem

The careful reader might have wondered by the end of the last section, whether or how, the indefiniteness problem actually was solved. In fact, a closer inspection tells us that the problem has just been hidden so far, manifesting itself as the non-positive-definite noise correlation (5.79) and the analysis (5.50)–(5.52) (remember we put $\lambda = \mu = -1$).

The simplest prescription we can think of is to set

$$\xi = \left(\mathbb{1} - \frac{\delta\delta}{4}\right)\eta + \mathrm{i}\,\frac{\delta\delta}{4}\,\eta \tag{5.94}$$

where

$$\langle \eta\eta \rangle = \mathbb{1}\, 2(2\pi)^4\, \delta^4(k+k')\, \delta(t-t')\,. \tag{5.95}$$

This construction leaves (5.79) unchanged

$$\begin{aligned}\langle \xi\xi \rangle &= \left\langle \eta\left(\mathbb{1} - \frac{\delta\delta}{4}\right)\eta\right\rangle - \left\langle \eta\,\frac{\delta\delta}{4}\,\eta\right\rangle \\ &= \left(\mathbb{1} - \frac{\delta\delta}{2}\right) 2\,(2\pi)^4\, \delta^4(k+k')\, \delta(t-t')\end{aligned} \tag{5.96}$$

but gives a probabilistic interpretation of the 'bad' noise ξ in terms of the Gaussian one η.

The price one has to pay is that the field h becomes complex in general, though this does not affect Green functions, as noises are always contracted pairwise. Before continuing our discussion by splitting the Langevin equation into real and imaginary parts, we would like to formulate the concept of stochastic gauge fixing for linearized gravity. Given the gauge transformation (5.45) it follows from general principles (see section 4) that the stochastically gauge fixed Langevin equation is given by

$$\dot{h}_{ab} = -k^2 P^{\mathrm{R}}_{abcd} h_{cd} - \mathrm{i}(k_a \Lambda_b + k_b \Lambda_a) + \xi_{ab} \tag{5.97}$$

where the most general local ansatz for Λ_a, which is linear in h, is given by

$$\Lambda_a = -\mathrm{i}\alpha^{-1}[\tfrac{1}{2}(k_d \delta_{ac} + k_c \delta_{ad}) + \tfrac{1}{2}\beta k_a \delta_{cd}] h_{cd} \tag{5.98}$$

with α and β being arbitrary constants. In an analysis similar to (5.48) we then obtain

$$\begin{aligned}&-\mathrm{i}(k_a \Lambda_b + k_b \Lambda_a) \\ &\quad = -2^{-1}[\tfrac{1}{2}(k_b k_d \delta_{ac} + k_b k_c \delta_{ad} + k_a k_d \delta_{bc} + k_a k_c \delta_{bd}) + \beta k_a k_b \delta_{cd}] h_{cd} \\ &\quad = -k^2 \alpha^{-1}[P^1 + 2P^0 + \beta L\delta]_{abcd} h_{cd}\end{aligned} \tag{5.99}$$

and the stochastically gauge fixed Langevin equation becomes

$$h = -k^2[P^2 + P^1\alpha^{-1} + P^0(2\alpha^{-1} - 1) + P^{0'} + L\,\delta(1+\beta)]h + \xi\,. \tag{5.100}$$

With the choice $\alpha = 1$, $\beta = -1$, and using the completeness relation (5.12), the remarkably simple equation follows:

$$h = -k^2 h + \xi \tag{5.101}$$

where still

$$\langle \xi\xi \rangle = 2(2\pi)^4\left(\mathbb{1} - \frac{\delta\delta}{2}\right) \delta^4(k+k')\, \delta(t-t') . \tag{5.102}$$

It is completely trivial to solve (5.101):

$$h = \int_0^t \mathrm{d}\tau \exp\{-k^2(t-\tau)\}\ \xi(\tau) \tag{5.103}$$

and calculate the free propagator as

$$\langle hh \rangle = \left(\mathbb{1} - \frac{\delta\delta}{2}\right) \delta^4(k+k')\, \frac{1}{k^2}\, (2\pi)^4 \tag{5.104}$$

in agreement with (5.57) and (5.93). If we distinguish between the real and imaginary parts of the field h

$$h = h_{\mathrm{Re}} + \mathrm{i}h_{\mathrm{Im}} \tag{5.105}$$

and the noise ξ as in (5.94) we obtain

$$\dot{h}_{\mathrm{Re}} = -k^2 h_{\mathrm{Re}} + \left(\mathbb{1} - \frac{\delta\delta}{4}\right) \eta \tag{5.106}$$

$$\dot{h}_{\mathrm{Im}} = -k^2 h_{\mathrm{Im}} + \frac{\delta\delta}{4}\, \eta . \tag{5.107}$$

With the initial condition $h(0, k) = 0$ we read off immediately that

$$\frac{\delta\delta}{4}\, h_{\mathrm{Re}}(t) = 0 , \qquad \left(\mathbb{1} - \frac{\delta\delta}{4}\right) h_{\mathrm{Im}}(t) = 0 , \qquad \forall t . \tag{5.108}$$

Finally let us discuss the Fokker–Planck formulation of the stochastic process (5.106), (5.107). The corresponding Fokker–Planck equation (see sections 2 and 3) reads

$$\dot{P}_{\mathrm{FP}} = \int \mathrm{d}^4x \left\{ \frac{\delta}{\delta h_{\mathrm{Re}}} \left[-\partial^2 h_{\mathrm{Re}} + \left(\mathbb{1} - \frac{\delta\delta}{4}\right) \frac{\delta}{\delta h_{\mathrm{Re}}} \right] + \frac{\delta}{\delta h_{\mathrm{Im}}} \left[-\partial^2 h_{\mathrm{Im}} + \frac{\delta\delta}{4}\, \frac{\delta}{\delta h_{\mathrm{Im}}} \right] \right\} P_{\mathrm{FP}} \tag{5.109}$$

and has the simple equilibrium distribution

$$\begin{aligned} P_{\mathrm{FP}}^{\mathrm{eq}} = {}& \delta\left[\frac{\delta\delta}{4}\, h_{\mathrm{Re}} \right] \delta\left[\left(\mathbb{1} - \frac{\delta\delta}{4}\right) h_{\mathrm{Im}} \right] \\ & \cdot \exp\left\{ -\tfrac{1}{2} \int \mathrm{d}^4k\, k^2 \left[h_{\mathrm{Re}} \left(\mathbb{1} - \frac{\delta\delta}{4}\right) h_{\mathrm{Re}} + h_{\mathrm{Im}} \frac{\delta\delta}{4}\, h_{\mathrm{Im}} \right] \right\} \end{aligned} \tag{5.110}$$

which, upon comparison with (5.53), (5.54) is just equivalent to the modified path integral prescription [5.8], discussed before.

For the sake of completeness let us finally discuss some other approaches to overcoming the indefiniteness problem. As an attempt one might reformulate the unsuccessful approach of (5.58)–(5.60) (corresponding to setting $\lambda = 0$ in (5.72)) by using the modified S from (5.54) as a starting point. Being not primarily interested in a gauge fixed formulation, some generalization of (5.53) has to be looked for in that case. A prescription can be found in ref. [5.10] where, instead of the operator V given in (5.44) it was proposed to consider

$$\tilde{V} = -k^2(P^2 + 2(\gamma - 1)P^{0\prime}) . \tag{5.111}$$

Here γ is a constant which is adjusted so that $\gamma > 1$ at first, but which is set to zero *after* having performed the integrations over the internal fictitious times and *after* having taken the limit $t \to \infty$. Redoing, as an example, the calculation of the 2-point function we obtain

$$\begin{aligned}\langle h_{ab}(k, t)\, h_{cd}(k', t)\rangle = (2\pi)^4\, \delta^4(k + k') \cdot \Big[\frac{1}{k^2}\, P^2\, (1 - \exp(-2k^2 t)) \\ + \frac{1}{2(\gamma - 1)}\, P^{0\prime}(1 - \exp\{-4(\gamma - 1)k^2 t\}) + 2(P^1 + P^0)t \Big] .\end{aligned} \tag{5.112}$$

We see that the gauge invariant (i.e. momentum independent) part of (5.112) relaxes to equilibrium as well and finally obtain in the limit $\gamma \to 0$

$$\lim_{\gamma\to 0} (\lim_{t\to\infty} \langle h_{ab}(k, t)\, h_{cd}(h', t)\rangle^{(\mathrm{m,ind.})}) = \left(\mathbb{1} - \frac{\delta\delta}{2}\right) \delta^4(k + k')\, \frac{1}{k^2}\, (2\pi)^4 \tag{5.113}$$

in accordance with (5.57). The prescription of inserting the γ-factor actually amounts to redefining the divergent integral for the $P^{0\prime}$ mode as

$$\frac{\int \mathrm{d}z\, z^2 \exp(z^2/2)}{\int \mathrm{d}z \exp(z^2/2)} \to \lim_{\gamma\to 0} \frac{\int \mathrm{d}z\, z^2 \exp\{-(\gamma - 1)z^2/2\}}{\int \mathrm{d}z \exp\{-(\gamma - 1)z^2/2\}} = \lim_{\gamma\to 0} (\gamma - 1)^{-1} = -1 \tag{5.114}$$

and gives the same as if one had rotated the z-axis in the imaginary direction.

Another approach to deal with bottomless Euclidean actions has been given in ref. [5.13], where the ground state of the (generally) positive definite Fokker–Planck Hamilton operator (see section 3) serves to define the Euclidean path integral. Although this method might be of interest in various non-gravity models, it does not reproduce the free propagator as given in (5.57); its applicability beyond the linearized level is unclear.

A further proposal, still reverting to $\lambda = 0$, can be found in ref. [5.9] (and was also used in ref. [5.14] in a somewhat different context of a bimetric formulation of gravity) to rotate the fictitious time axis in the imaginary direction. If the correlation function is interpreted as a tempered distribution on Euclidean space-time we have

$$\lim_{t\to\infty} \exp(\pm \mathrm{i} k^2 t) = \pm \mathrm{i}\pi\, \delta(k^2) = 0 \tag{5.115}$$

where the second equation holds in Euclidean space of dimension $d \geq 3$ (for more details concerning the distributional interpretation see section 9). Thus the exponentially increasing terms in (5.66) effectively vanish, leaving again the result (5.57) as its momentum independent contribution.

We conclude this section by observing that several stochastic approaches to linearized Euclidean gravity have been formulated, implying essentially the same free propagators. It seems clearly desirable to apply these methods to the full non-linear theory and explore their (eventually different) implications. Let us remark that one of the methods presented, the generally covariant stochastic formulation of ref. [5.11], will however still receive further confirmation at the free level, namely when discussing the stochastic quantization of string field theories (see section 5.3).

5.2.5. Beyond the linear approximation in Einstein gravity

So far we have formulated stochastic quantization of Einstein gravity at the linearized level. In the following we want to point out some of the difficulties one encounters when going beyond, using mainly the formulation of ref. [5.11].

One immediate problem that arises in higher orders in x is that generally the noise field ξ_{ab} is no longer Gaussian. One can understand this from the fact that the field metric G^{abcd} depends on the metric tensor field g_{ab}, which on its own depends (being a solution of the Langevin equation) on the noise ξ_{ab}. This implies generally that the noise distribution ceases to be Gaussian and most of the techniques, we acquainted ourselves with during the last sections, cannot be applied any longer.

In ref. [5.14] a bimetric formulation has been adopted. Another proposal can be found in ref. [5.11] to relate the inconvenient noise ξ_{ab} to a Gaussian noise field $\xi^{(0)}_{ab}$ by

$$\xi_{ab}(x) = \int \mathrm{d}^4x'\, E^{cd}_{ab}(x, x')\, \xi^{(0)}_{cd}(x') \tag{5.116}$$

where $\xi^{(0)}_{ab}(x)$ is distributed as given in the last section, eq. (5.102),

$$\langle \xi^{(0)}_{ab}(x, t)\, \xi^{(0)}_{cd}(x', t') \rangle = G^{0}_{abcd}(x, x')\, 2\, \delta(t - t') \tag{5.117}$$

and a stochastic vielbein functional $E^{cd}_{ab}(x, x')$ is introduced. $E^{cd}_{ab}(x, x')$ depends on g_{ab} and is required to obey

$$\int \mathrm{d}^4x\, \mathrm{d}^4x'\, G^{(0)}_{abcd}(x, x')\, E^{ab}_{kl}(x, y)\, E^{cd}_{mn}(x', y') = G_{klmn}(y, y') \tag{5.118}$$

so that the noise distribution is transformed into a Gaussian one.

We then observe that only in the Ito interpretation spurious terms with possibly vielbein $E^{cd}_{ab}(x, x')$ dependence (which is unphysical) can be avoided. For this reason we adopt an Ito interpretation. As a consequence a covariantized Langevin equation has to be considered (see section 5.2.3) so that as a generalization of (5.77) we obtain for the full non-linear theory

$$\dot{g}_{ab} - \tfrac{9}{2}(1 + \mu)\, \delta^4(0)\, g^{-1/2} g_{ab} = \frac{1}{2x}\left(R_{ab} - \frac{\mu + 1}{2}\, g_{ab} R\right) + \xi_{ab}\,. \tag{5.119}$$

Here we used, starting from (5.27), that

$$\frac{\delta S_{\mathrm{EH}}}{\delta g_{ab}} = -\frac{1}{2x}\, g^{1/2}\, (R^{ab} - \tfrac{1}{2} g^{ab} R) \tag{5.120}$$

where R_{ab} is the Einstein tensor and the noise ξ_{ab} is given by (5.116). We remark that, in order to allow for the noise to have its canonical dimension L^{-3} and in order to directly relate (5.119) to the linearized version (5.77), we may insert in (5.119) a constant kernel $4x$ so that we get

$$\dot{g}_{ab} - 18\,(1+\mu)\,\delta^4(0)\,x\,g^{-1/2}\,g_{ab} = 2\Big(R_{ab} - \frac{\mu+1}{2}\,g_{ab}R\Big) + 2x^{1/2}\xi_{ab}\,. \tag{5.121}$$

As in the last section the favoured value $\lambda = \mu = -1$ should be adopted and the indefiniteness problem, represented by the indefinite $\xi^{(0)}$ noise distribution, see (5.116), can be overcome by setting again

$$\xi^{(0)} = \Big(\mathbb{1} - \frac{\delta\delta}{4}\Big)\,\eta + \mathrm{i}\,\frac{\delta\delta}{4}\,\eta \tag{5.122}$$

where η is purely Gaussian as in (5.59), (5.60). One may split the metric tensor field g_{ab} into real and imaginary parts and consider similarly (5.116) and (5.122). As a consequence the corresponding Fokker–Planck equation gets a quite involved structure and no equilibrium distribution has been found so far.

Let us finally indicate a perturbative treatment of the full Einstein Langevin equation (5.119) by setting

$$E^{cd}_{ab}(x, x') = |g|^{-1/4} e^c_a(x)\, e^d_b(x')\, \delta^4(x - x') \tag{5.123}$$

and

$$g_{ab}(x) = e^c_a(x)\, e^d_b(x)\, \delta_{cd} \tag{5.124}$$

which fulfils (5.118) as can easily be checked. Given the expansion

$$g_{ab} = \delta_{ab} + 2x^{1/2} h_{ab} \tag{5.125}$$

a corresponding expansion for $e^a_b(x)$ can be formulated and the drift term as well as the noise term (5.116) can be expanded in a power series in x. It remains an open question whether this perturbation theory is equivalent to the standard one (supplemented, for example, with the prescription [5.8]) and whether some of the most disturbing other features of gravity, as for example non-renormalizability, can be overcome in this approach.

5.3. *Free bosonic string field theory*

In this section we would like to discuss briefly the implementation of stochastic quantization concepts in the field theory of free bosonic strings [5.18]. We do not intend to present here an introduction to string field theories (for a recent review see, for example, ref. [5.19]) and we will try to avoid technical details on string field theory, supplying with our analysis the appropriate references. However, we should remind the reader that bosonic string theories are self-consistent only in an unphysical number of dimensions, $D = 26$, and that even in 26 dimensions such theories still contain a tachyonic sector, which we will simply disregard in the following.

We consider our discussion as a first step towards the application of stochastic concepts to interacting

string field theories, possibly including superstrings as well. We will not comment on the possibility of a compactification of the extra dimensions to $D=4$.

Let us point out that string field theories incorporate in a non-trivial fashion Einstein gravity and antisymmetric tensor fields. Treating string theories in stochastic quantization therefore nicely puts the discussion of the two previous sections into perspective.

The gauge invariant classical action for *free* bosonic string fields that we use is given in ref. [5.20] (see also ref. [5.21]):

$$S = \tfrac{1}{2} \int \mathrm{d}^{26}x \, \langle \Psi(x) | H - qT_3^{-1}q | \Psi(x) \rangle \,. \tag{5.126}$$

Let us first discuss the open string case. Here $|\Psi(x)\rangle$ is a vector in the Fock space spanned by the harmonic oscillators a_n^+, b_n^+, $\bar{b}_n^+$. (The reader should not confuse the Fock-space brackets with stochastic averages, see later.) Here the a_n^+ operators correspond to the vibrational excitations of the string, while b_n^+ and $\bar{b}_n^+$ are fermionic ghosts and anti-ghosts of the first quantized theory. $|\Psi(x)\rangle$ is subject to two constraints: one which ensures that the shifted first quantized ghost number is zero, and a symmetrization constraint which makes the operator T_3 invertible [5.20].

The operators in (5.126) are defined by expanding the first quantized BRS charge of the string in terms of ghost zero modes [5.22]:

$$Q = b_0 H - \bar{b}_0 T_3 + q \,. \tag{5.127}$$

The fact that $Q^2 = 0$ (in 26 dimensions) implies

$$[H, T_3] = [H, q] = [T_3, q] = 0 \,, \qquad q^2 = T_3 H \,. \tag{5.128}$$

H is a positive definite operator (except for the tachyonic sector which we are disregarding).

If one explicitly performs the expectation value with respect to the ghost oscillators in (5.126), one finds that the action (5.126) is not positive definite in Euclidean space. This is a consequence of the ghost oscillators being Grassmannian, and it was in fact to be expected since string field theories contain fields of arbitrarily high spin. (We already saw in the last section that Euclidean linearized Einstein gravity has a negative contribution in the action. Indeed, all suggested theories of fields with spin ≥ 2 are plagued by the indefiniteness problem.)

For closed strings the action is still given by (5.126), but the interpretation differs. $|\Psi(x)\rangle$ is now a vector in the Fock space spanned by $a_n^{+\mathrm{R}}$, $a_n^{+\mathrm{L}}$, $b_n^{+\mathrm{R}}$, ... etc., i.e. both the right and left movers of the first quantized string. For the closed string the fields obey another further constraint and (5.128) remains valid modulo terms which vanish when they act on $|\Psi\rangle$.

Varying the action (5.126) with respect to $\langle \Psi(x)|$ we obtain the classical equation of motion

$$0 = (H - qT_3^{-1}q)\, |\Psi\rangle = HP|\Psi\rangle \tag{5.129}$$

where

$$P = \mathbb{1} - H^{-1}qT_3^{-1}q \tag{5.130}$$

is a projection operator ($P^2 = P$), on account of (5.128). Thus the equation of motion contains a single projector. This is true for both open and closed strings; only the interpretation of (5.130) differs.

According to the standard procedure of stochastic quantization, we now go one step further and supplement the Euclidean string field with an additional fictitious time coordinate t. With (5.130) we associate the Langevin equation:

$$\frac{\partial}{\partial t} |\Psi(x,t)\rangle = -HP\,|\Psi(x,t)\rangle + |\eta(x,t)\rangle \,. \tag{5.131}$$

Denoting stochastic averages by round brackets, we require

$$\begin{aligned} &(|\eta\rangle) = (\langle\eta|) = 0 \\ &(|\eta(x,t)\rangle\,\langle\eta(x',t')|) = 2\,\delta^{26}(x-x')\,\delta(t-t')\,\mathbb{1} \end{aligned} \tag{5.132}$$

and similarly for higher n-point functions.

Given the fact that $|\Psi(x,t)\rangle$ and $|\eta(x,t)\rangle$ carry no indices, eqs. (5.131) and (5.132) appear the most natural starting point for a stochastic treatment of string field theory.

Since a projector appears in the drift term of the Langevin equation (5.131), the general solution is easily found. In momentum space it reads

$$|\Psi(k,t)\rangle = \int_0^t d\tau\,[e^{-H(t-\tau)}P + (\mathbb{1} - P)]\,|\eta(k,\tau)\rangle \,. \tag{5.133}$$

As a first example we compute the two-point correlation:

$$(|\Psi(k,t)\rangle\,\langle\Psi(k',t')|) = [H^{-1}P(1 - e^{-2Ht}) + 2t(\mathbb{1} - P)]\,\delta^{26}(k+k')(2\pi)^{26} \tag{5.134}$$

which exhibits in its 'longitudinal part' the well-known linear t-factor characteristic of the random walk behaviour in gauge parameter space (see section 4.1). Calculating a gauge invariant quantity as, for example, the expectation value of $\langle\Psi|HP|\Psi\rangle$, we obtain from eq. (5.134):

$$\begin{aligned} (\langle\Psi(k,t)|HP|\Psi(k',t)\rangle) &= \mathrm{Tr}[HP|\Psi(k',t)\rangle\langle\Psi(k,t)|] \\ &= \mathrm{Tr}[P(1 - e^{-2Ht})]\,\delta^{26}(k+k')(2\pi)^{26} \end{aligned} \tag{5.135}$$

where the trace is defined with respect to a complete set of states in the first quantized Fock space. Note that the 'longitudinal' modes have been projected away in (5.135), as is expected from general principles for all gauge invariant quantities. Let us emphasize at this point that the stochastic scheme is perfectly consistent without having added gauge fixing terms to the action and no second quantized ghost fields are needed. Given that H is a positive definite operator, strict relaxation to equilibrium is obtained for $t \to \infty$, and the quantum expectation value of $\langle\Psi|HP|\Psi\rangle$ reads

$$\lim_{t\to\infty} (\langle\Psi(k,t)|HP|\Psi(k',t)\rangle) = \mathrm{Tr}\,P\,\delta^{26}(k+k')(2\pi)^{26} \,. \tag{5.136}$$

For further clarification we now study the massless sector of the closed string, which contains

antisymmetric tensor fields and linearized gravity, in component form. Expanding the string fields as [5.20]

$$|\Psi\rangle = \left[\cdots + \tfrac{1}{2}(h_{\mu\nu} + \mathrm{i}A_{\mu\nu})\, a_{1\mu}^{+\mathrm{L}}\, a_{1\nu}^{+\mathrm{R}} + \frac{\mathrm{i}}{\sqrt{2}}\, \phi\, b_1^{+(\mathrm{R}}\bar{b}_1^{\mathrm{L})+} + \cdots \right] |0\rangle \tag{5.137}$$

where ϕ, $h_{\mu\nu}$ and $A_{\mu\nu}$ are scalar, symmetric and antisymmetric tensor fields, respectively, and using the explicit mode expansions [5.20] for H, ϕ, T_3^{-1}, the action for the massless sector becomes

$$S = -\int \mathrm{d}^{26}x\, X^T \Gamma \Box P X \tag{5.138}$$

where

$$X = \begin{pmatrix} A_{\mu\nu} \\ h_{\mu\nu} \\ \phi \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} \mathbb{1}^{\mathrm{a}} & & \\ & \mathbb{1}^{\mathrm{s}} & \\ & & -1 \end{pmatrix} \tag{5.139}$$

and

$$P = \begin{pmatrix} P^{\mathrm{A}} & 0 & 0 \\ 0 & P^2 - P^0 + P^{0\prime} & -\sqrt{2}\,L \\ 0 & \sqrt{2}\,L & 2 \end{pmatrix}. \tag{5.140}$$

Here P^2, P^0, $P^{0\prime}$ and P^{A} denote the projection operators on the space of symmetric and antisymmetric tensor fields, which were introduced in the preceding two subsections (and which have to be trivially generalized to $D = 26$ dimensions); similarly $\mathbb{1}^{\mathrm{S}}$ and $\mathbb{1}^{\mathrm{A}}$ denote the corresponding unit operators, $L = \partial\partial/\partial^2$, and δ denotes the Euclidean metric.

The action can be diagonalized using

$$X = M\hat{X}\,, \qquad M = \begin{pmatrix} \mathbb{1}^{\mathrm{A}} & 0 & 0 \\ 0 & \mathbb{1}^{\mathrm{S}} & -\dfrac{1}{\sqrt{D-2}}\,\delta \\ 0 & -\dfrac{1}{\sqrt{2}}\,\delta & \sqrt{\dfrac{2}{D-2}} \end{pmatrix} \tag{5.141}$$

so that

$$S = -\int \mathrm{d}^D x\, \hat{X}^{\mathrm{T}} \Box \Pi\, \hat{X} \tag{5.142}$$

where

$$\Pi = M^{\mathrm{T}} \Gamma P M = \begin{pmatrix} P^{\mathrm{A}} & 0 & 0 \\ 0 & P^2 - (D-2)P^{0\prime} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.143}$$

We see, as anticipated, that for the massless sector (in the symmetric tensor field part) the action is indeed indefinite. Note also that our manipulations were done in D dimensions (actually $D = 26$) and that Π is no projector, although P, eq. (5.140), is.

Expanding the noise string field in a similar way as (5.137)

$$|\eta\rangle = \left[\cdots + \tfrac{1}{2}(\eta^{\mathrm{S}}_{\mu\nu} + \mathrm{i}\eta^{\mathrm{A}}_{\mu\nu}) \cdot a^{+\mathrm{L}}_{1\mu}\, a^{+\mathrm{R}}_{1\nu} + \frac{1}{\sqrt{2}}\, \eta^{\phi} \cdot b^{+(\mathrm{R}}_{1} \bar{b}^{\mathrm{L})+} + \cdots \right] |0\rangle \tag{5.144}$$

we extract from (5.131) the component field Langevin equation

$$\frac{\partial}{\partial t} X = \Box P X + \xi \tag{5.145}$$

where

$$\xi = \begin{pmatrix} \eta^{\mathrm{A}} \\ \eta^{\mathrm{S}} \\ \eta^{\phi} \end{pmatrix} . \tag{5.146}$$

The two-point correlation of ξ can be read off from (5.132):

$$(\xi, \xi^{\mathrm{T}}) = \begin{pmatrix} \mathbb{1}^{\mathrm{A}} & & \\ & \mathbb{1}^{\mathrm{S}} & \\ & & -1 \end{pmatrix} 2\delta^{D}(x - x')\, \delta(t - t') . \tag{5.147}$$

We note as a general feature that owing to the ghost oscillator structure in (5.144) the noise component fields are Gaussian only up to signs, i.e. they are correlated as plus minus the appropriate unit operators. This is how the indefiniteness problem appears within the stochastic approach to string theory.

The diagonalized version of the Langevin equation (5.145) is

$$\frac{\partial}{\partial t} \hat{X} = M^{-1} \frac{\partial}{\partial t} X = \Box M^{-1} P X + M^{-1} \xi . \tag{5.148}$$

Defining

$$\hat{P} = M^{-1} P M \tag{5.149}$$

and

$$\hat{\xi} = M^{-1} \xi . \tag{5.150}$$

We thus get

$$\frac{\partial}{\partial t} \hat{X} = \Box \hat{P} \hat{X} + \hat{\eta} . \tag{5.151}$$

Using M from (5.141) we calculate straightforwardly

$$M^{-1}=\begin{pmatrix} \mathbb{1}^{\mathrm{A}} & 0 & 0 \\ 0 & \mathbb{1}^{\mathrm{S}}-\dfrac{\delta\delta}{D-2} & -\dfrac{\sqrt{2}}{D-2}\,\delta \\ 0 & -\dfrac{1}{\sqrt{D-2}}\,\delta & -\sqrt{\dfrac{2}{D-2}} \end{pmatrix} \tag{5.152}$$

and find $\hat{P}$ as

$$\hat{P}=\begin{pmatrix} P^{\mathrm{A}} & & \\ & P^{\mathrm{R}} & \\ & & 1 \end{pmatrix} \tag{5.153}$$

where

$$\begin{aligned} &P^{\mathrm{R}}=P^{2}-(D-2)P^{0\prime}+\delta(\delta-L) \\ &(P^{\mathrm{R}})^{2}=P^{\mathrm{R}}\,. \end{aligned} \tag{5.154}$$

For the physical noise correlations we find straightforwardly, using M^{-1},

$$\langle\hat{\xi},\hat{\xi}^{\mathrm{T}}\rangle=\begin{pmatrix} \mathbb{1}^{\mathrm{A}} & & \\ & \mathbb{1}^{\mathrm{S}}-\dfrac{1}{D-2}\,\delta\delta & \\ & & 1 \end{pmatrix} 2\,\delta^{D}(x-x')\,\delta(t-t')\,. \tag{5.155}$$

We remark that for $D=4$ the symmetric tensor point in (5.151), (5.154) and (5.155), corresponding to linearized Einstein gravity, is identical to the generally covariant approach of [5.11]; it is remarkable that the string approach directly gives the preferred parameter value $\lambda=-1$.

Now let us return to the problem of the non-Gaussian noise. In a similar spirit as in the last section we might set $\eta^{\phi}=\mathrm{i}\eta_{p}^{\phi}$, where η_{p}^{ϕ} is real. Then η_{p}^{ϕ} is Gaussian and a probabilistic interpretation is possible. The generalization to all mass levels is obvious.

In order to discuss a Fokker–Planck formulation we have to implement the stochastic gauge fixing procedure (see section 4) for the string field case. With the gauge transformation being given by

$$\delta|\Psi\rangle=q|\Lambda\rangle \tag{5.156}$$

the stochastically gauge fixed Langevin equation then becomes

$$\frac{\partial}{\partial t}|\Psi\rangle=-HP|\Psi\rangle+q|\Lambda\rangle+|\eta\rangle \tag{5.157}$$

where $|\Lambda\rangle$ is – apart from the constraints mentioned before – an arbitrary string field. Choosing, for example, $|\Lambda\rangle=T_{3}^{-1}q|\Psi\rangle$ we obtain the simple Langevin equation

$$\frac{\partial}{\partial t}|\Psi\rangle = -H|\Psi\rangle + |\eta\rangle$$

where, studying the massless sector, we find the same symmetric tensor part as in (5.101). Distinguishing between the real and imaginary parts of the fields we find that the Fokker–Planck equation corresponding to the massless sector of (5.157) becomes

$$\dot{P}_{\mathrm{FP}} = \int \mathrm{d}^{26}x \left\{ \frac{\delta}{\delta h_{\mathrm{RE}}} \left(-\Box h_{\mathrm{Re}} + \frac{\delta}{\delta h_{\mathrm{Re}}} \right) + \frac{\delta}{\delta h_{\mathrm{Im}}} (-\Box h_{\mathrm{Im}}) + \frac{\delta}{\delta \phi_{\mathrm{Re}}} (-\Box \phi_{\mathrm{Re}}) + \frac{\delta}{\delta \phi_{\mathrm{Im}}} \left(-\Box \phi_{\mathrm{Im}} + \frac{\delta}{\delta \phi_{\mathrm{Im}}} \right) \right\} P_{\mathrm{FP}} \tag{5.158}$$

and the equilibrium distribution is easily found as

$$P_{\mathrm{FP}}^{\mathrm{eq}} = \delta[\phi_{\mathrm{Re}}]\,\delta[h_{\mathrm{Im}}] \exp\left\{ -\int \mathrm{d}^{26}x\, [(\partial h_{\mathrm{Re}})^2 + (\partial \phi_{\mathrm{Im}})^2] \right\}. \tag{5.159}$$

Thus our prescription is equivalent to performing the path integral along the imaginary axis for the troublesome directions in field configuration space.

The above calculation can be repeated straightforwardly for the diagonalized fields. One finds an equilibrium distribution which is (with respect to the symmetric tensor field part) not quite the same as that given in the last section; however, it implies the same propagators. We can understand this ambiguity in the equilibrium distribution by noting that a probabilistic reinterpretation of the noise η^h by continuation in the complex plane is not unique. As, however, the noise *correlations* always remain unchanged, this does not lead to different Green functions.

6. Fermions

So far our discussion has been restricted to theories containing *bosonic* fields only. Even though, as we saw, the formalism became increasingly more complex as we went from fields transforming as scalars under the Lorentz group to fields of spin one and two, the main ideas nevertheless carried through intact. The pure 'Brownian motion' interpretation of the Langevin equation for scalar fields obviously lost some of its immediate meaning as we went to higher spins, partly as a consequence of the gauge and reparametrization invariances typically contained in such theories, but also because the random noise fields themselves had to transform as spin-one and spin-two fields under the Lorentz group. In spite of these complications, convergence of stochastic $[(D+1)$-dimensional$]$ expectation values to standard D-dimensional vacuum expectation values as $t \to \infty$ could nevertheless be assured, as it basically corresponded to the convergence towards thermodynamic equilibrium in the $(D+1)$-dimensional system.

When we introduce theories containing fermion fields this convergence towards thermodynamic equilibrium can really be jeopardized. The root of this problem lies in the fact that there exists no classical analogue of fermion fields. In an operator language this manifests itself in the appearance of operators that are not positive definite. Not surprisingly, therefore, the solution to this (first) problem lies in properly choosing a bosonized version of the Langevin equation. The purpose of this

bosonization is precisely to transform all relevant operators into operators which are positive definite with respect to the set of states under consideration.

One further complication associated with the treatment of theories with (massless) fermions is the appearance of a new set of global invariances, those associated with chiral-flavour symmetries. In some instances it appears imperative that the Langevin equations transform covariantly under all local and global symmetries. This requires some special care.

6.1. The naïve fermionic Langevin equation

Let us, for simplicity, first consider a theory described solely in terms of spin-1/2 fermion fields, i.e. defined by an action $S = S[\psi, \bar{\psi}]$. Working in Euclidean space, the fields $\psi(x)$ and $\bar{\psi}(x)$ have to be treated as *independent* Grassmann variables.

The immediate generalization of the boson field Langevin equation (3.3) leads to the following two equations for $\psi(x, t)$ and $\bar{\psi}(x, t)$:

$$\frac{\partial}{\partial t}\, \psi(x, t) = -\frac{\delta S}{\delta \bar{\psi}(x, t)} + \eta(x, t) \tag{6.1}$$

and*

$$\frac{\partial}{\partial t}\, \bar{\psi}(x, t) = \frac{\delta S}{\delta \psi(x, t)} + \bar{\eta}(x, t)\,. \tag{6.2}$$

[Note the change of sign in eq. (6.2) due to the complete anticommutativity of the fields $\psi(x, t)$ and $\bar{\psi}(x, t)$.]

Obviously, eqs. (6.1) and (6.2) force the noise fields $\eta(x, t)$ and $\bar{\eta}(x, t)$ themselves to become anticommuting spin-1/2 fields.

By analogy with the bosonic case one would define stochastic expectation values

$$\langle \eta \rangle = \langle \bar{\eta} \rangle = 0\,, \qquad \langle \eta_\alpha(x, t)\, \bar{\eta}_\beta(x', t') \rangle = 2\delta_{\alpha\beta}\, \delta^D(x - x')\, \delta(t - t') \tag{6.3}$$

and similarly for higher n-point functions, taking into account the anticommuting nature of the noise fields $\eta(x, t)$ and $\bar{\eta}(x, t)$.

The Langevin equations (6.1) and (6.2) can, for a given action $S(\psi, \bar{\psi})$, be solved by essentially the same methods we have already discussed for bosonic fields. Perturbatively, the simplest way of solving the equations is by writing them as two coupled integral equations, and then solve them by iteration. This expansion will, just as in the bosonic case, formally generate the standard perturbation theory known from ordinary Feynman-diagram techniques, once all the fermionic noise fields have been contracted away.

However, in the general case the choice (6.1) and (6.2) will *not* generate the proper equilibrium configurations at $t \to \infty$. We only have to consider a free fermionic theory to see this.

Choosing to work with a Euclidean gamma matrix representation normalized by $\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}$, our free Euclidean fermion action takes the form

* In the following the conjugate equation will not be written down unless explicitly required.

$$S[\psi, \bar{\psi}] = -\mathrm{i} \int \mathrm{d}^D x\, \bar{\psi}(x)\, (\not{\partial} + \mathrm{i}m)\, \psi(x)\,. \tag{6.4}$$

The Langevin equation (6.1) then becomes

$$\frac{\partial}{\partial t}\, \psi(x, t) = (\mathrm{i}\not{\partial} - m)\, \psi(x, t) + \eta(x, t) \tag{6.5}$$

which is readily solved by the introduction of the Green function $g(x, t)$ satisfying

$$\begin{gathered}\left(\frac{\partial}{\partial t} - \mathrm{i}\not{\partial} + m\right)_{\alpha\sigma} g_{\sigma\beta}(x, t) = \delta_{\alpha\beta}\, \delta^D(x)\, \delta(t) \\ g(x, t) = 0\,, \qquad \text{for } t < 0\,.\end{gathered} \tag{6.6}$$

Note that although the left-hand side of eq. (6.6) of course is diagonal in the spinor indices ($\propto \delta_{\alpha\beta}$), $g(x, t) = g_{\alpha\beta}(x, t)$ itself is not. Formally,

$$g(x, t) = \theta(t) \int \frac{\mathrm{d}^D p}{(2\pi)^D} \exp\{-(\not{p} + m)t + \mathrm{i}px\} \tag{6.7}$$

which solves for $\psi(x, t)$

$$\psi(x, t) = \int_0^\infty \mathrm{d}t' \int \mathrm{d}^D x'\; g(x' - x, t' - t)\, \eta(x', t')\,. \tag{6.8}$$

From eq. (6.7) it looks superficially as if $g(x, t) \to 0$ as $t \to \infty$, as is required for convergence towards equilibrium. However, as mentioned, $g_{\alpha\beta}(x, t)$ is not diagonal in the spinor indices, and in order to check convergence we must first diagonalize this matrix. This is done by noting [6.1] that the matrix $(\not{p} + m)$ can be diagonalized by a unitary matrix $U(p)$:

$$\not{p} + m = U^+(p) \begin{pmatrix} \mathrm{i}\sqrt{p^2} + m & & & 0 \\ & \mathrm{i}\sqrt{p^2} + m & & \\ & & -\mathrm{i}\sqrt{p^2} + m & \\ 0 & & & -\mathrm{i}\sqrt{p^2} + m \end{pmatrix} U(p)\,. \tag{6.9}$$

The two (doubly degenerate) eigenvalues are hence $\pm \mathrm{i}\sqrt{p^2} + m$. The Green function $g(x, t)$ itself is therefore also diagonalized by $U(p)$:

$$g(x, t) = \theta(t) \int \frac{\mathrm{d}^D p}{(2\pi)^D}\, U^+(p) \exp\{-\tilde{D}(p)\, t\}\, U(p) \exp\{-mt + \mathrm{i}px\} \tag{6.10}$$

where $\tilde{D}(p)$ is the diagonal matrix of eq. (6.9). As can be seen, the only convergence factor arises from the exp $\{-mt\}$ factor. This means that in the massless limit $m = 0$, the Green function $g(x, t)$ becomes ill-defined, and the formal solution (6.8) meaningless. This is clearly an unwanted situation, in particular if one is interested in the chiral properties of the theory.

6.2. Introduction of kernels

The remedy of the problem of convergence discussed in the last subsection consists in noting [6.2, 6.3] the freedom we have in choosing our Langevin equations. In particular, we can make use of our freedom in introducing 'kernels' into the Langevin equations. It should be kept in mind, though, that just as in ordinary quantization, zero modes may cause difficulties for the massless fermion theory, and of course a kernel cannot remove such problems. We already discussed this issue in the case of scalar fields (see section 3), but there the introduction of possible kernels was mostly a matter of convenience. For fermionic Langevin equations kernels become crucial.

Let us recall that one possible generalization of eq. (6.1) is to insert a kernel $V(x, y)$ in front of the action-derivative:

$$\frac{\partial}{\partial t}\,\phi(x, t) = -\int \mathrm{d}^D y\, V(x, y)\,\frac{\delta S}{\delta\phi(y, t)} + \eta(x, t) \tag{6.11}$$

provided we simultaneously change the stochastic expectation values to

$$\langle \eta(x, t)\,\eta(x', t')\rangle = 2\,V(x, x')\,\delta(t - t')\,. \tag{6.12}$$

The kernel $V(x, y)$ cannot be chosen completely arbitrarily. For example, for this simple scalar case it is obvious that $V(x, y)$ must be positive definite, since otherwise we would operate with negative stochastic correlations and naively a complete lack of convergence as $t\to\infty$. For theories with fermions this is no longer the case. In fact, the kernel must now be chosen in such a way as to precisely cancel negative eigenvalues from $\delta S/\delta\bar{\psi}(x, t)$. Such a kernel will, in general, not be positive definite.

One choice of such a kernel is [6.2, 6.3]:

$$K(x, y) = (\mathrm{i}\,\partial\!\!\!/_x + m)\,\delta(x - y) = \mathrm{i}\,\delta\!\!\!/'(x - y) + m\,\delta(x - y) \tag{6.13}$$

where we have defined

$$\delta\!\!\!/'(x - y) = \gamma_\mu\,\frac{\partial}{\partial x_\mu}\,\delta^D(x - y)\,. \tag{6.14}$$

As in the bosonic case, we introduce a random noise field $\theta(x, t)$; here it is fermionic and we assign to it [and its 'conjugate' field $\bar{\theta}(x, t)$] the following stochastic expectation values:

$$\langle \theta(x, t)\rangle = 0 = \langle \bar{\theta}(x, t)\rangle \tag{6.15a}$$

$$\langle \theta_\alpha(x, t)\,\bar{\theta}_\beta(x', t')\rangle = 2[\mathrm{i}\,\delta\!\!\!/'(x - x') + m\,\delta(x - x')]_{\alpha\beta}\,\delta(t - t')\,. \tag{6.15b}$$

For example, for a *free* fermionic theory the generalized Langevin equation

$$\frac{\partial}{\partial t}\,\psi(x, t) = -\int \mathrm{d}^D y\, K(x, y)\,\frac{\delta S}{\delta\bar{\psi}(y, t)} + \theta(x, t) \tag{6.16}$$

becomes simply

$$\frac{\partial}{\partial t}\,\psi(x,t) = (\partial^2 - m^2)\,\psi(x,t) + \theta(x,t) \tag{6.17}$$

which is just like the *bosonic* Langevin equation: all knowledge of the fermionic nature of the theory is contained in the stochastic expectation values of the θ's.

For an arbitrary functional of the θ's we have the stochastic expectation values

$$\langle F[\theta,\bar\theta]\rangle = \frac{\int \mathrm{D}\theta\,\mathrm{D}\bar\theta\, F[\theta,\bar\theta]\exp\left\{-\frac{1}{2}\int \mathrm{d}^Dx\,\mathrm{d}t\,\bar\theta(x,t)(\mathrm{i}\not{\partial}+m)^{-1}\,\theta(x,t)\right\}}{\int \mathrm{D}\theta\,\mathrm{D}\bar\theta\,\exp\left\{-\frac{1}{2}\int \mathrm{d}^Dx\,\mathrm{d}t\,\bar\theta(x,t)(\mathrm{i}\not{\partial}+m)^{-1}\,\theta(x,t)\right\}}\,. \tag{6.18}$$

In particular, for the $2n$-point functions

$$\langle \theta(x_1,t_1)\cdots\bar\theta(x_n,t_n)\rangle = \sum_{\text{perm}} \varepsilon_{\mathrm{p}} \prod_{\text{pairs}} \langle \theta(x_i,t_i)\,\bar\theta(x_j,t_j)\rangle \tag{6.19}$$

where ε_{p} indicates the sign of the permutation, obtained from the complete anti-commutativity relations of the θ's:

$$\begin{aligned}
&\{\theta(x,t),\theta(y,t')\} = 0\\
&\{\theta(x,t),\bar\theta(y,t')\} = 0\\
&\{\bar\theta(x,t),\bar\theta(y,t')\} = 0\,.
\end{aligned} \tag{6.20}$$

For the free theory, eq. (6.17) is readily solved explicitly by the introduction of the corresponding *diagonal* Green function

$$\begin{aligned}
&\left\{\frac{\partial}{\partial t} - \partial^2 + m^2\right\} G(x,t) = \delta^D(x)\,\delta(t)\\
&\quad G(x,t) = 0\,, \qquad \text{for } t<0
\end{aligned} \tag{6.21}$$

i.e.

$$G(x,t) = \theta(t)\int \frac{\mathrm{d}^Dp}{(2\pi)^D}\exp\{-(p^2+m^2)t + \mathrm{i}px\}\,. \tag{6.22}$$

[The corresponding Green function $\bar G(x,t)$ for the conjugate Langevin is simply $\bar G(x,t) = G(x,t)^*$.]

The free fermion theory therefore has the explicit solution

$$\psi(x,t) = \int_0^t \mathrm{d}t' \int \mathrm{d}^Dx'\; G(x-x',t-t')\,\theta(x',t')\,. \tag{6.23}$$

This expression has exactly the same convergence properties as the corresponding expression for scalar fields. In particular, it converges even in the massless case (with a damping factor $\sim \exp(-p^2 t)$).

If convergence is assured in the case of interacting fermion fields as well, the corresponding integral equation for $\psi(x, t)$ takes the form

$$\psi(x, t) = \int_0^\infty \mathrm{d}t' \int \mathrm{d}^D x' \, G(x - x', t - t') \left\{ \theta(x', t') - \frac{\delta S_{\mathrm{int}}[\psi, \bar{\psi}]}{\delta \bar{\psi}(x', t')} \right\} \tag{6.24}$$

which, when coupled with its conjugate equation, uniquely determines $\psi(x, t)$.

6.3. Perturbation theory

Although so far we have not yet demonstrated that the kernel (6.13) is sufficient to ensure convergence towards equilibrium in the case of *interacting* fermion fields, it is nevertheless fairly straightforward to see that at least in weak coupling perturbation theory the kernel prescription of eq. (6.13) has exactly the right properties to provide convergence as $t \to \infty$. In this subsection we shall demonstrate this in some detail.

To start, consider first a purely fermionic theory. For definiteness, choose

$$S_{\mathrm{int}}[\psi, \bar{\psi}] = -g \int \mathrm{d}^D x \, \bar{\psi}(x) \, \psi(x) \, \bar{\psi}(x) \, \psi(x) \, . \tag{6.25}$$

To simplify combinatorical factors, etc., one can think of this as the interaction of two types of fermions, each carrying some internal quantum number that distinguishes it from the other kind. Furthermore, although we will keep an arbitrary dimensionality D throughout, we can always imagine performing the calculation in two space-time dimensions, in which case this theory is renormalizable.

Substituting the interaction term (6.25) into the exact integral equation (6.24), we obtain a highly non-linear equation for the solution $\psi(x, t)$ of the Langevin equation. This equation is not explicitly solvable. However, as in the case of boson fields, treating the coupling constant g as a small parameter we can solve the equation by iteration. As compared with the case of boson fields, the introduction of a kernel $K(x, y)$ only complicates the perturbative expansion slightly. This is illustrated in fig. 6.1. Green functions are indicated by lines ('stochastic progagators'); dashed lines correspond to the bosonic Green

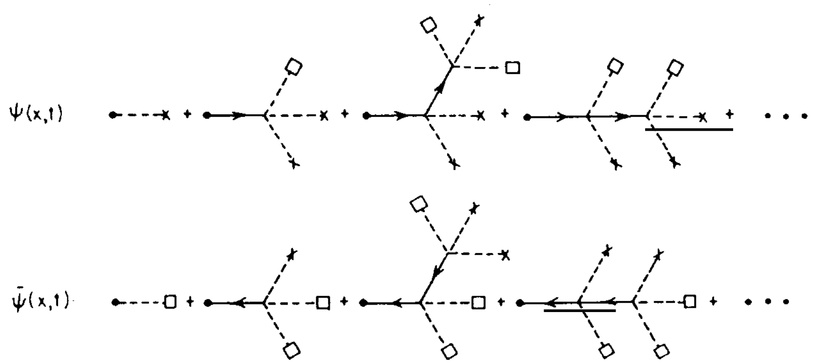


Fig. 6.1. The perturbation expansion for the fermion fields; crosses and boxes represent η and $\bar{\eta}$ fields respectively; solid lines indicate Γ and broken lines G.

functions $G(x, t)$, whereas fully drawn lines are the *fermionic* Green functions corresponding to

$$\Gamma(x, t) = \int d^D y\, K(x, y)\, G(y, t) \tag{6.26}$$

i.e. the convolution of the bosonic Green function with the kernel $K(x, y)$.

Uncontracted noise fields $\theta(x, t)$ and $\bar{\theta}(x, t)$ are indicated by crosses and boxes, respectively.

Let us first consider the 2-point function $\langle \psi\bar{\psi}\rangle$. To lowest order this is simply given by

$$\langle \psi(x, t)\, \bar{\psi}(x', t')\rangle = \int_0^\infty d\tau_1\, d\tau_2 \int d^D x_1\, d^D x_2\, G(x - x_1, t - \tau_1)\, G(x' - x_2, t' - \tau_2)\, \langle \theta(x_1, \tau_1)\, \bar{\theta}(x_2, \tau_2)\rangle\,. \tag{6.27}$$

Using the stochastic expectation values (6.15) and the momentum-space representation (6.22) of the Green functions, this immediately leads to the momentum-space propagator

$$\begin{aligned} \langle \psi(p, t)\, \bar{\psi}(p', t')\rangle &= (2\pi)^D\, \delta^D(p - p')\, \Delta(p; t, t') \\ &= -(2\pi)^D\, \delta^D(p - p') \frac{\not{p} - m}{p^2 + m^2} \exp\{-(p^2 + m^2)(t + t')\}\, [\exp\{2(p^2 + m^2)\tilde{t}\} - 1] \end{aligned} \tag{6.28}$$

where $\tilde{t} = \min\{t_1, t_2\}$. For practical purposes the 'time ordered' propagator $t < t'$ is useful:

$$\Delta(p; t, t') = -\frac{\not{p} - m}{p^2 + m^2} \exp\{-(p^2 + m^2)(t - t')\}\, [1 - \exp\{-2(p^2 + m^2)t'\}]\,. \tag{6.29}$$

Note that for equal times this simply reduces to the standard Euclidean fermion propagator:

$$\Delta(p; t, t') \to \frac{1}{\not{p} + m} \qquad \text{as } t = t' \to \infty \tag{6.30}$$

as required.

Just as in the case of boson fields, all of the standard perturbation theory results can be generated in the way described above. One uses the expansion represented in fig. 6.1 [obtained by iteration of eq. (6.24)] and contracts all stochastic noise fields away by means of the rules (6.15) and (6.19). As an example, consider the one-loop corrections to the 2-point function (6.27). These corrections can be represented by the two diagrams of fig. 6.2. The two diagrams are actually identical and equal to

$$(a) = g \int_0^\infty d\tau \int \frac{d^D p}{(2\pi)^D}\, \Gamma(k; t_1, \tau)\, \mathrm{Tr}[\Delta(p; \tau, \tau)]\, \Delta(k; t_2, \tau)\,. \tag{6.31}$$

In the limit $t_1 = t_2 \to \infty$ we obtain

$$(a) + (b) = g \int \frac{d^D p}{(2\pi)^D} \frac{1}{\not{k} + m} \mathrm{Tr}\Big(\frac{1}{\not{p} + m}\Big) \frac{1}{\not{k} + m} \tag{6.32}$$

the usual result.

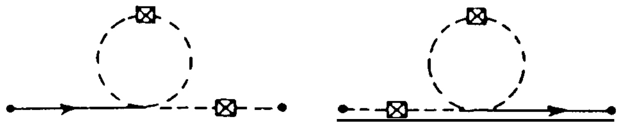


Fig. 6.2. The one-loop corrections to the 2-point function in a $(\bar{\psi}\psi)^2$ theory.

The computation of the 4-point function is equally simple. Figure 6.3a shows the stochastic graphs contributing to $\Gamma^{(4)}$ and for comparison fig. 6.3b shows the corresponding graphs of ordinary perturbation theory. To 'zeroth' order we have

$$\begin{aligned}\Gamma^{(4)} &= \int_0^\infty \mathrm{d}\tau\, \Gamma(p_1,t_1-\tau)\, \Delta(p_2,t_2-\tau)\, \Delta(p_4,t_4-\tau)\, \Delta(p_3,t_3-\tau) + \text{permutations} \\ &= (\not{p}_1 - m)\, \frac{(\not{p}_2 - m)}{p_2^2+m^2}\, \frac{(\not{p}_4 - m)}{p_4^2+m^2}\, \frac{(\not{p}_3 - m)}{p_3^2+m^2}\, \frac{1}{\sum_i p_i^2 + 4m^2} + \text{permutations}\end{aligned} \tag{6.33}$$

as $t \to \infty$. Adding the four permutations we clearly get what corresponds to the single Feynman diagram of ordinary perturbation theory. It is not difficult to see how this works to higher orders in perturbation theory. For example, the one-loop 4-point function is given by ('s-channel')

$$\begin{aligned}\Gamma^{(4)} = \int \frac{\mathrm{d}^D q}{(2\pi)^D} \int \mathrm{d}\tau_1 \int \mathrm{d}\tau_2\, \{&\Gamma(p_1;t_1,\tau_1)\, \Delta(p_4;t_4,\tau_1)\, \mathrm{Tr}[\Delta(q;\tau_1,\tau_2)\, \Delta(k;\tau_2,\tau_1)] \\ &\cdot \Gamma(p_3;t_4,\tau_2)\, \Delta(p_2;t_2,\tau_2) + \Gamma(p_1;t_1,\tau_1)\, \Delta(p_4;t_4,\tau_1) \\ &\cdot \mathrm{Tr}[\Gamma(q;\tau_1,\tau_2)\, \Delta(k;\tau_2,\tau_1)]\, \Delta(p_3;t_3,\tau_2)\, \Delta(p_2;t_2,\tau_2)\} + \text{terms with } q \leftrightarrow k\,,\end{aligned} \tag{6.34}$$

where for abbreviation we have put $k = p_1 + p_4 - q$. It is a straightforward calculation to show that in the limit $t_1 = t_2 = t_3 = t_4 \to \infty$ one indeed recovers the usual Feynman diagram result.

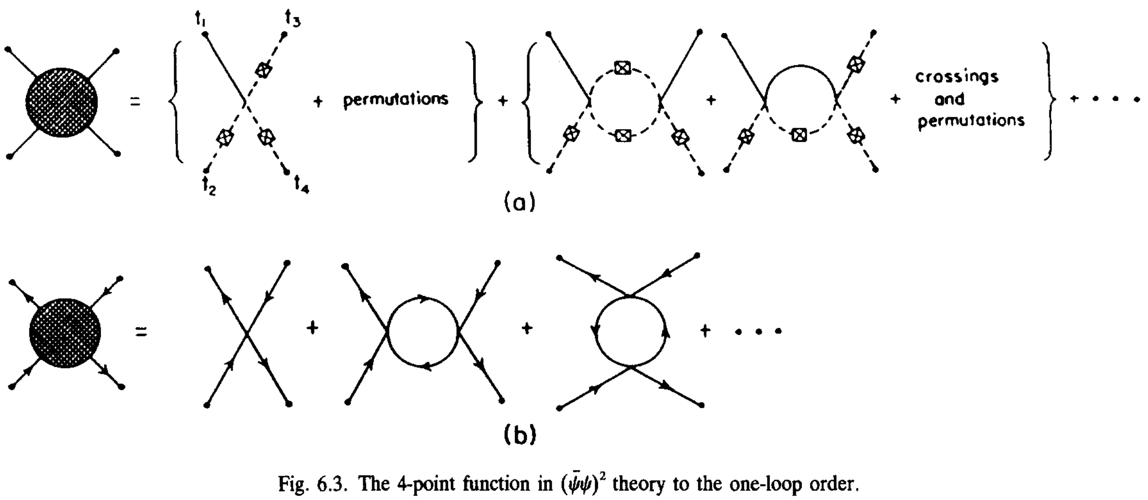


Fig. 6.3. The 4-point function in $(\bar{\psi}\psi)^2$ theory to the one-loop order.

The formalism easily carries over to the case of fermions coupled to other fields. In fact, *without* the introduction of a kernel [as in eqs. (6.1) and (6.2)] we are faced with the curious situation that the time parameter t in the fermionic case has dimension one, whereas in, say, the bosonic case the time parameter is of dimension two. In that case we are therefore forced to introduce yet another mass parameter if we wish to couple fermions to bosons and identify the two fictitious times. *With* the kernel (6.13) this is no longer necessary: t now has dimension two even in the case of fermions, and bosons can now immediately be coupled to fermions.

Of most interest, of course, is the coupling of fermions to gauge fields. Let us therefore now briefly discuss one possible set of stochastic relaxation equations, those based on the kernel (6.13). We will illustrate how the method works in perturbation theory by calculating explicitly the one-loop vacuum polarization graph of QED.

The Langevin equations for the general non-Abelian case look as follows:

$$\frac{\partial}{\partial t} A^a_\lambda(x,t) = D_\nu F^a_{\nu\lambda}(x,t) + J^a_\lambda(x,t) + \eta^a_\lambda(x,t) \tag{6.35a}$$

$$\frac{\partial}{\partial t} \psi(x,t) = (\partial^2 - m^2)\, \psi(x,t) + g \int \mathrm{d}^D y\, K(x,y)\, \not{A}(y,t)\, \psi(y,t) + \theta(x,t) \tag{6.35b}$$

with $J^a_\lambda = \frac{1}{2} g\, \bar{\psi}(x,t)\, \gamma_\lambda \lambda^a\, \psi(x,t)$. This corresponds to the Euclidean action

$$S[A,\psi,\bar{\psi}] = \int \mathrm{d}^D x (\tfrac{1}{4}\, F^a_{\lambda\nu}\, F^a_{\lambda\nu} - \bar{\psi}(\mathrm{i}\not{D} - m)\psi) \tag{6.36}$$

with the standard notation (see section 4):

$$F^a_{\lambda\nu} = \partial_\lambda A^a_\nu - \partial_\nu A^a_\lambda - g f^{abc} A^b_\lambda A^c_\nu\,, \qquad \not{D} = \not{\partial} - \mathrm{i}g\not{A}\,.$$

The stochastic expectation values for the η fields are, as in section 4.1 given by

$$\langle \eta^a_\lambda(x,t)\, \eta^b_\nu(x',t')\rangle = 2\,\delta^{ab}\,\delta_{\lambda\nu}\delta^D(x-x')\,\delta(t-t') \tag{6.37}$$

whereas the fermionic fields θ (and $\bar{\theta}$) have expectation values as indicated in eq. (6.12), with $V(x,x') = K(x,x')$.

In the Abelian case we can simplify the situation considerably by choosing to work in a specific gauge, say the Feynman gauge. As we saw in section 4, this is, in the Abelian case, equivalent to adding a Zwanziger term. The equivalent of eq. (6.35a) then reads

$$\frac{\partial}{\partial t} A_\lambda(x,t) = \partial_\nu F_{\lambda\nu}(x,t) + \partial_\lambda \partial_\nu A_\nu(x,t) + J_\lambda(x,t) + \eta_\lambda(x,t) \tag{6.38}$$

with $J_\mu(x,t) = g\, \bar{\psi}(x,t)\, \gamma_\mu\, \psi(x,t)$.

Introducing the Green function $G_{\mu\nu}(x,t)$ for eq. (6.38),

$$G_{\mu\nu}(x,t) = \delta_{\mu\nu}\, \theta(t) \int \frac{\mathrm{d}^D p}{(2\pi)^D} \exp\{-p^2 t + \mathrm{i}px\} \tag{6.39}$$

we find for the 'time-ordered' 2-point function,

$$\langle A_\mu(p,t)\,A_\nu(p',t')\rangle = (2\pi)^D\,\delta^D(p+p')\,D_{\mu\nu}(p;t,t')$$
$$= (2\pi)^D\,\delta^D(p+p')\,\delta_{\mu\nu}\,\frac{1}{p^2}\exp\{-p^2(t-t')\}\,(1-\exp\{-2p^2t'\}) \tag{6.40}$$

in momentum space. This is the stochastic propagator corresponding to the standard Feynman gauge propagator of ordinary perturbation theory.

The iterative solution of the Langevin equations (6.35b) and (6.38) can now be illustrated diagrammatically as shown in fig. 6.4, the η's being indicated by open circles as in section 3. Using this expansion we can easily calculate what corresponds to the single-loop vacuum polarization diagram in ordinary field-theory language. The stochastic diagrams for this quantity are shown in fig. 6.5.

With the notation indicated in the figure and in the previous sections we have

$$(\mathrm{a}) = \int \frac{\mathrm{d}^Dq}{(2\pi)^D}\int_0^\infty \mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\mathrm{Tr}\{\Delta(q_1;\tau_1,\tau_2)\,\gamma_\alpha\,\Delta(q_2;\tau_1,\tau_2)\,\gamma_\beta\}\,G_{\lambda\alpha}(p,t_1-\tau_1)\,G_{\beta\nu}(p,t_2-\tau_2) \tag{6.41a}$$

$$(\mathrm{b}) = \int \frac{\mathrm{d}^Dq}{(2\pi)^D}\int_0^\infty \mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\mathrm{Tr}\{\Gamma(q_1;\tau_1-\tau_2)\gamma_\alpha\,\Delta(q_2;\tau_1,\tau_2)\gamma_\beta\}\,D_{\lambda\alpha}(p;t_1,\tau_1)\,G_{\beta\nu}(p,t_2-\tau_2) \tag{6.41b}$$

and similarly for (c).

Adding the crossed diagrams and taking the limit $t_1 = t_2 \to \infty$ we are left with

$$(\mathrm{a}) = \int \frac{\mathrm{d}^Dq}{(2\pi)^D}\,\tilde{\Pi}_{\lambda\nu}(p,q)\,\frac{1}{p^2}\,\frac{1}{p^2+q_1^2+q_2^2+2m^2}\,\frac{1}{q_1^2+m^2}\,\frac{1}{q_2^2+m^2} \tag{6.42a}$$

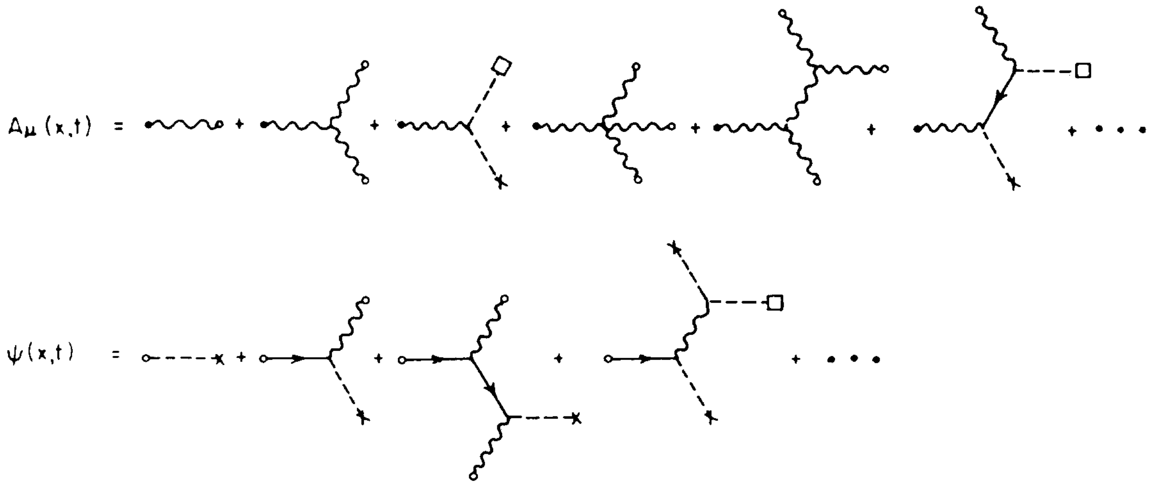


Fig. 6.4. Perturbative expansion of fields in a gauge theory: open circles denote noise fields $\eta_\mu(x,t)$.

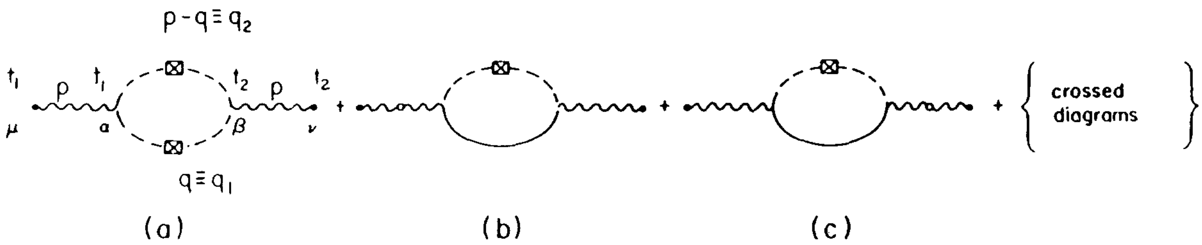


Fig. 6.5. The one-loop vacuum polarization in QED.

$$(\mathrm{b}) = \int \frac{\mathrm{d}^D q}{(2\pi)^D}\, \tilde{\Pi}_{\lambda\nu}(p, q) \left(\frac{1}{p^2}\right)^2 \frac{1}{p^2 + q_1^2 + q_2^2 + 2m^2} \left(\frac{1}{q_1^2 + m^2}\, \frac{1}{q_2^2 + m^2}\right) \tag{6.42b}$$

$$(\mathrm{c}) = (\mathrm{b}) \tag{6.42c}$$

where $\tilde{\Pi}_{\lambda\nu}(p, q) = \mathrm{Tr}\{(\not{q} - m)\gamma_\lambda(\not{q}_2 - m)\gamma_\nu\}$. Finally, adding eqs. (6.42a), (6.42b) and (6.42c) we are left with the standard vacuum polarization tensor of QED.

6.4. A fermionic Fokker–Planck equation

In the previous section we demonstrated, in some specific examples, how the kernel $K(x, y)$ of eq. (6.13) ensured convergence for $t \to \infty$ within the context of weak-coupling perturbation theory. Such a demonstration is, of course, necessary, but we should keep in mind that some of the most interesting applications of stochastic quantization are likely to be found outside the perturbative regime.

We shall here try to see if the kernel prescription of eq. (6.13) will ensure convergence for $t \to \infty$ *beyond* the weak-coupling perturbative region. At the same time it will become clear why the method will work correctly to any order in perturbation theory for a sufficiently small coupling constant g.

Our approach will, as in the bosonic case, be that of constructing the master equation for the stochastic probability distribution: the Fokker–Planck equation. In this case, however, the 'probability distribution' will involve Grassman variables, and its immediate interpretation in terms of probabilities will be lost. Nevertheless, an analysis very similar to that of the bosonic case can be carried out. Let us now see how.

We start by introducing a distribution functional $P[\psi, \bar{\psi}, t]$ by

$$P[\psi, \bar{\psi}, t] = \frac{\int \mathrm{D}\theta\, \mathrm{D}\bar{\theta}\, \delta(\psi - \psi[\theta])\, \delta(\bar{\psi} - \bar{\psi}[\theta]) \exp\left\{-\frac{1}{2}\int \mathrm{d}^D x\, \mathrm{d}t\, \bar{\theta}(\mathrm{i}\not{\partial} + m)^{-1}\theta\right\}}{\int \mathrm{D}\theta\, \mathrm{D}\bar{\theta} \exp\left\{-\frac{1}{2}\int \mathrm{d}^D x\, \mathrm{d}t\, \bar{\theta}(\mathrm{i}\not{\partial} + m)^{-1}\theta\right\}} \tag{6.43}$$

so that *any* stochastic θ-expectation value can be written

$$\langle F[\psi(x, t), \bar{\psi}(y, t')]\rangle = \int \mathrm{D}\psi\, \mathrm{D}\bar{\psi}\, F[\psi(x), \bar{\psi}(y)]\, P[\psi, \bar{\psi}, t]\,. \tag{6.44}$$

In eq. (6.43) we have distinguished between the original fermion fields ψ, $\bar{\psi}$ and those of the solutions $\psi[\theta]$, $\bar{\psi}[\theta]$ to the Langevin equation (6.16) and its 'conjugate' equation.

Treating ψ and $\bar{\psi}$ as independent variables (since we are working in Euclidean space), and going through the same steps as in the bosonic case, we find the fermionic Fokker–Planck equation.

$$\frac{\partial}{\partial t} P[\psi, \bar{\psi}, t] = \int \mathrm{d}^D x\, \mathrm{d}^D y\, K(x, y) \left\{ \frac{\delta}{\delta \bar{\psi}} \left(\frac{\delta S}{\delta \psi} P \right) - \frac{\delta}{\delta \psi} \left(\frac{\delta S}{\delta \bar{\psi}} P \right) + 2 \frac{\delta P}{\delta \bar{\psi}\, \delta \psi} \right) \right\} \tag{6.45}$$

with the kernel $K(x, y)$. We are here suppressing all spinor indices in order to make the formulas more readable.

To see formally if this Fokker–Planck equation will drive the system correctly towards the equilibrium distribution $\exp\{-S\}$, we introduce a wave functional $\Psi[\psi, \bar{\psi}, t]$ through the definition

$$P[\psi, \bar{\psi}, t] = \exp\{ -\tfrac{1}{2} S[\psi, \bar{\psi}]\}\, \Psi[\psi, \bar{\psi}, t] . \tag{6.46}$$

This wave functional $\Psi[\psi, \bar{\psi}, t]$ is readily seen to be a solution to the Schrödinger-like equation:

$$\frac{\partial}{\partial t} \Psi[\psi, \bar{\psi}, t] = -\tilde{H}\, \Psi[\psi, \bar{\psi}, t] \tag{6.47}$$

where the 'Hamiltonian' is given by

$$\tilde{H} = \int \mathrm{d}^D x\, \mathrm{d}^D y\, K(x, y) \left\{ \frac{1}{2} \frac{\delta S}{\delta \bar{\psi}} \frac{\delta S}{\delta \psi} - \frac{\delta^2 S}{\delta \bar{\psi}\, \delta \psi} - 2 \frac{\delta^2}{\delta \bar{\psi}\, \delta \psi} \right\} . \tag{6.48}$$

Elements $\Psi_{(1)}$ and $\Psi_{(2)}$ of the extended Hilbert space associated with this Hamiltonian can be given the inner product

$$\langle \Psi_{(1)} | \Psi_{(2)} \rangle = \int \mathrm{D}\psi\, \mathrm{D}\bar{\psi}\, \Psi^*_{(1)}[\psi, \bar{\psi}] \Psi_{(2)}[\psi, \bar{\psi}] . \tag{6.49}$$

With this definition the correctly normalized zero-energy eigenstate Ψ_0 of the Hamiltonian $\tilde{H}$ is given by

$$\Psi_0[\psi, \bar{\psi}] = \frac{\exp\{ -\frac{1}{2} S[\psi, \bar{\psi}]\}}{\int \mathrm{D}\psi\, \mathrm{D}\bar{\psi} \exp\{ -\frac{1}{2} S[\psi, \bar{\psi}]\}} \tag{6.50}$$

as can readily be checked from eq. (6.48). As this wave functional formally has no nodes, it is also the ground state of $\tilde{H}$.

In general, we expect that any solution $\Psi[\psi, \bar{\psi}, t]$ can be expanded into a complete set of eigenstates,

$$\Psi[\psi, \bar{\psi}, t] = \sum_{n=0}^{\infty} a_n\, \Psi_n[\psi, \bar{\psi}] \exp(-E_n t) . \tag{6.51}$$

Now, if the Hamiltonian $\tilde{H}$ corresponding to the action S has a mass gap, i.e. if $E_n > 0$ for all $n > 0$, then as $t \to \infty$ all higher excitations are suppressed, and one has

$$\Psi[\psi, \bar{\psi}, t \to \infty] = \Psi_0[\psi, \bar{\psi}]\,. \tag{6.52}$$

Hence, from eq. (6.46)

$$P[\psi, \bar{\psi}, t \to \infty] = \frac{\exp\{-S[\psi, \bar{\psi}]\}}{\int \overline{\mathrm{D}\psi}\, \overline{\mathrm{D}\bar{\psi}} \exp\{-\overline{S[\psi, \bar{\psi}]}\}} \tag{6.53}$$

so that

$$\langle F[\psi(\theta), \bar{\psi}(\theta)\rangle \to \langle F[\psi, \bar{\psi}]\rangle \qquad \text{as } t \to \infty \tag{6.54}$$

for any functional $F[\psi, \bar{\psi}]$, as required.

This non-perturbative 'proof' that stochastic expectation values reduce to ordinary vacuum expectation values as $t \to \infty$ is, obviously, even less rigorous than the corresponding proof for the bosonic case. Recall that in the bosonic case we could write the Fokker-Planck 'Hamiltonian' in an at least formally positive definite form: $H = Q^+Q$. [As rather ill-defined functional quantities are involved, a real check of positivity requires a much more detailed analysis.]

Even though we therefore are unable to rigorously prove convergence with the kernel $K(x, y)$ of eq. (6.13), it was obvious in perturbation theory how this kernel ensured convergence towards equilibrium. Let us now see how we can understand this fact from the point of view of the fermionic Fokker–Planck equation.

To this end, let us go back to eq. (6.45) and expand the distribution amplitude $P[\psi, \bar{\psi}, t]$ into eigenstates χ_n of H_p, the operator appearing on the right-hand side of eq. (6.45):

$$P[\psi, \bar{\psi}, t] = \sum_{n=0}^{\infty} \chi_n[\psi, \bar{\psi}]\, c_n \exp(-\lambda_n t) \tag{6.55}$$

where the eigenvalues λ_n are determined by

$$H_\mathrm{p}\chi_n = -\lambda_n \chi_n\,. \tag{6.56}$$

It is easy to see that $\chi_0 = \exp\{-S[\psi, \bar{\psi}]\} / \int \mathrm{D}\psi\, \mathrm{D}\bar{\psi} \exp\{-S[\bar{\psi}, \psi]\}$ is a solution of eq. (6.56) with $\lambda_0 = 0$. If (as in the case of the regular Fokker–Planck Hamiltonian $\tilde{H}$) we can show that all other eigenvalues λ_n, $n \geq 1$, have positive real parts, then in the limit $t \to \infty$, $P[\psi, \bar{\psi}] \to \chi_0[\psi, \bar{\psi}] = \exp\{-S[\psi, \bar{\psi}]\} / \int \mathrm{D}\psi\, \mathrm{D}\bar{\psi} \exp\{-S[\psi, \bar{\psi}]\}$, so that the fermionic Fokker–Planck equation drives the system to the correct equilibrium distribution. Thus the problem is reduced to considering the non-zero eigenvalues of H_p. Note that in the bosonic case the 'Hamiltonian' H_p has positive eigenvalues *before* introducing the kernel $V(x, y)$; thus V itself must be positive. This is no longer true in the fermionic case: The 'Hamiltonian' H_p without a kernel is not even Hermitian, and its eigenvalues are therefore not constrained to be real. Thus the kernel which has to be introduced should in general not be positive. This was, of course, apparent already from our discussion of the Langevin equation (6.16).

With the choice of kernel $K(x, x')$ of eq. (6.13) it can be shown, for example by employing the coherent state representation of H_p, that the eigenvalues of H_p are always positive if $S[\psi, \bar{\psi}]$ is a free fermionic action. In the interacting case we may write the eigenvalues as $\lambda_n = |\lambda_n^0| + g\mu_n + \mathrm{O}(g^2)$,

where g is the coupling constant of the theory, $|\lambda_n^0|$ is the (positive) free eigenvalue, and μ_n is arbitrary. It follows that as long as we restrict ourselves to weak coupling ($g \ll 1$), the $t \to \infty$ limit of eq. (6.46) is $P[\psi, \bar{\psi}, t \to \infty] = \chi_0[\psi, \bar{\psi}]$, and therefore

$$\langle F[\psi[\theta], \bar{\psi}[\theta]\rangle \to \langle F[\psi, \bar{\psi}]\rangle$$

as required.

What happens beyond weak coupling? Although nothing indicates it, it is *conceivable* that the kernel prescription (6.13) fails for coupling constants beyond a certain g^*, at which point one could envisage, for instance, a phase transition. This is, however, pure speculation; no indications of such a possible phase transition have been found.

6.5. Other kernels

From our discussion in the last section it is clear that the kernel $K(x, x')$ of eq. (6.13) is far from being unique. Other kernels can be introduced, depending on the circumstances.

A particular kernel $K(x, x')$ has been suggested [6.4, 6.5], which is restricted to theories where the fermion fields appear in the form of bilinears. This is actually not a very strong restriction: it incorporates, for example, the case of gauge theories with minimal coupling. Also, it is very often possible to reduce other fermion interactions to bilinear forms through the introduction of auxiliary fields and extra interactions associated with them. For example, the interaction Lagrangian (6.25) can be put in a bilinear form by the so-called Hubbard–Stratanovich transformation:

$$\begin{aligned}\exp\left\{+g\int \mathrm{d}^D x\, \bar{\psi}(x)\,\psi(x)\,\bar{\psi}(x)\,\psi(x)\right\} &= \int \mathrm{D}\sigma \exp\Big(-g\int \mathrm{d}^D x\, \{[\sigma(x) - \bar{\psi}(x)\,\psi(x)]^2 \\ &\qquad - \bar{\psi}(x)\,\psi(x)\,\bar{\psi}(x)\,\psi(x)\}\Big) \\ &= \int \mathrm{D}\sigma \exp\Big(-g\int \mathrm{d}^D x\, [\sigma(x)^2 - 2\,\bar{\psi}(x)\,\sigma(x)\,\psi(x)]\Big)\end{aligned} \tag{6.57}$$

which involves just one auxiliary field $\sigma(x)$. [In deriving eq. (6.57) we have simply inserted a factor of unity:

$$1 = \int \mathrm{D}\sigma \exp\left\{-g\int \mathrm{d}^D x\, [\sigma(x) - \bar{\psi}(x)\,\psi(x)]^2\right\} \tag{6.58}$$

with the measure $\mathrm{D}\sigma$ normalized suitably.]

Having made the restriction to actions bilinear in the Fermi fields, we can write the interaction Lagrangian in the form

$$S = \int \mathrm{d}^D x\, \bar{\psi}(x)\, G(x)\, \psi(x)\,. \tag{6.59}$$

It has been shown [6.4] that an appropriate Fokker–Planck equation for an action of this form is

$$\frac{\partial}{\partial t} P[\psi, \bar{\psi}, t] = \int \mathrm{d}^D x\, H_{\mathrm{p}}\, P[\psi, \bar{\psi}, t] \tag{6.60}$$

with

$$H_{\mathrm{p}} = \int \mathrm{d}^D x \left[\frac{\delta}{\delta\psi(x)}\, G^\dagger(x) \left(\frac{\delta}{\delta\bar{\psi}(x)} + \frac{\delta S}{\delta\bar{\psi}(x)} \right) - \frac{\delta}{\delta\bar{\psi}(x)}\, (G^{\mathrm{T}}(x))^\dagger \left(\frac{\delta}{\delta\psi(x)} + \frac{\delta S}{\delta\psi(x)} \right) \right]. \tag{6.61}$$

In particular, using eq. (6.59),

$$H_{\mathrm{p}} = \int \mathrm{d}^D x \left[\frac{\delta}{\delta\psi(x)}\, G^\dagger(x) \left(\frac{\delta}{\delta\bar{\psi}(x)} + G(x)\, \psi(x) \right) - \frac{\delta}{\delta\bar{\psi}(x)}\, (G^{\mathrm{T}}(x))^\dagger \left(\frac{\delta}{\delta\psi(x)} - \bar{\psi}(x)\, \hat{G}(x) \right) \right]. \tag{6.62}$$

Langevin equations corresponding to eq. (6.60) can be written in the form [6.4, 6.5].

$$\frac{\delta}{\delta t}\, \psi(x,t) = -G^\dagger(x)\, G(x)\, \psi(x,t) + \tfrac{1}{2} G^\dagger(x)\, \theta_1(x,t) + \theta_2(x,t) \tag{6.63}$$

$$\frac{\partial}{\partial t}\, \bar{\psi}(x,t) = -(G(x)\, G^\dagger(x))^{\mathrm{T}}\, \bar{\psi}(x,t) + \bar{\theta}_1(x,t) + \tfrac{1}{2} G^\dagger(x)\, \bar{\theta}_2(x,t) \tag{6.64}$$

where now *two* stochastic noise fields $\theta_1(x,t)$ and $\theta_2(x,t)$ have been introduced (plus their two 'conjugate' counterparts). These noise fields must have stochastic correlations given as follows:

$$\langle \theta_i(x,t) \rangle = \langle \bar{\theta}_i(x,t) \rangle = 0 \tag{6.65}$$

$$\langle \theta_i(x,t)\, \bar{\theta}_j(x',t') \rangle = -\langle \bar{\theta}_j(x',t')\, \theta_i(x,t) \rangle = 2\, \delta_{ij}\, \delta^D(x-x')\, \delta(t-t'). \tag{6.66}$$

It has also been argued [6.4, 6.6] that, equivalently, the Langevin equations (6.63) and (6.64) can be expressed (in an asymmetric fashion) by just *one* stochastic field $\theta(x,t)$ [and its associated $\bar{\theta}(x,t)$]:

$$\frac{\partial}{\partial t}\, \psi(x,t) = -G^\dagger(x)\, G(x)\, \psi(x,t) + G^\dagger(x)\, \theta(x,t) \tag{6.67}$$

$$\frac{\partial}{\partial t}\, \bar{\psi}(x,t) = -(G(x)\, G^\dagger(x))^{\mathrm{T}}\, \bar{\psi}(x,t) + \bar{\theta}(x,t) \tag{6.68}$$

with the θ's having ordinary δ-function correlations.

The advantage of the Langevin equations (6.63) and (6.64) is that formally one can demonstrate positive definiteness of the associated 'Hamiltonian' H_{p} of eq. (6.61). To see this, transform to a coherent state representation [6.1] in which

$$\bar{\tilde{\psi}}^\dagger \sim \delta/\delta\bar{\psi}\,, \qquad \bar{\tilde{\psi}} \sim \bar{\psi}\,, \qquad \tilde{\psi}^\dagger \sim \delta/\delta\psi \qquad \text{and } \tilde{\psi} \sim \psi$$

and where consequently the 'Hamiltonian' H_{p} of eq. (6.61) takes the form

$$H_{\mathrm{p}} = \int \mathrm{d}^D x [\tilde{\psi}^\dagger(x)\, G^\dagger(x)\, (\bar{\tilde{\psi}}^\dagger(x) + G(x)\, \tilde{\psi}(x)) - \bar{\tilde{\psi}}^\dagger (G^{\mathrm{T}}(x))^\dagger\, (\tilde{\psi}^\dagger(x) - \bar{\tilde{\psi}}(x)\, \hat{G}(x))]\,. \tag{6.69}$$

By a similarity transformation

$$\tilde{H}_{\mathrm{p}} = \exp\left\{-\int \mathrm{d}^D x\, \tilde{\psi}^\dagger(x)\, G^{-1}(x)\, \tilde{\tilde{\bar{\psi}}}{}^\dagger(x)\right\} H_{\mathrm{p}} \exp\left\{\int \mathrm{d}^D x\, \tilde{\psi}^\dagger(x)\, G^{-1}(x) \tilde{\tilde{\bar{\psi}}}{}^\dagger(x)\right\} \tag{6.70}$$

this gets a formally positive definite expression:

$$\tilde{H}_{\mathrm{p}} = \int \mathrm{d}^D x\, [\tilde{\psi}^\dagger(x)\, G^\dagger(x)\, G(x)\, \tilde{\psi}(x) + \tilde{\tilde{\bar{\psi}}}{}^\dagger(x)\, (G(x)\, G^\dagger(x))^{\mathrm{T}}\, \tilde{\tilde{\bar{\psi}}}] \tag{6.71}$$

and one would therefore expect the 'Hamiltonian' (6.61) to have only real, positive (or zero) eigenvalues. Of course, one would still have to demonstrate the existence of a 'mass gap', i.e. that the eigenvalues $\lambda_n > 0$ for all $n \geqslant 1$. This appears even more difficult to prove in the case of Grassmann variables than in the bosonic case.

As the diagrammatic techniques for the perturbation theory associated with the Langevin equations (6.63) and (6.64) are completely analogous to the expansions in the bosonic case and the kernel prescription of the previous subsection, we shall omit a discussion of this perturbation expansion here. We note in passing that the Langevin equations (6.63) and (6.64) do not strictly speaking correspond to the use of a kernel prescription, although the resulting form is very similar.

Another kernel prescription which attempts to overcome damping problems by effectively letting the stochastic time variable evolve in two different directions has also been proposed [6.7]. However, as soon as interactions are introduced only one time direction is admissible, in which case one must return to the original kernel prescription.

Kernels will also turn out to play an important role in numerical simulations of lattice-regularized fermion systems. We will return to this point in section 12.

6.6. *The fermionic Langevin equations and their symmetries*

One question we have not raised so far is the problem of *symmetries* of the Langevin equations. For ordinary stochastic *quantization* of regularized or finite theories this turns out to be of no importance. However, when we turn to stochastic *regularization* of theories with infinities, it becomes imperative that the whole formalism is covariant with respect to the symmetries of the underlying theory. For this reason it is important to check the invariances of the Langevin equations.

As we will have in mind applications to a QED- or QCD-like theory, we will concentrate on gauge invariance and global chiral invariance. For simplicity of the discussion, let us restrict ourselves to an Abelian gauge theory only. The generalization to the non-Abelian theory is completely straightforward.

With the kernel $K(x, x')$ of eq. (6.13) the Langevin equations are

$$\frac{\partial}{\partial t} A_\mu(x, t) = \partial_\nu F_{\mu\nu}(x, t) + g\, \bar{\psi}(x, t)\, \gamma_\mu\, \psi(x, t) + \eta_\mu(x, t) \tag{6.72}$$

and

$$\frac{\partial}{\partial t}\, \psi(x, t) = (\partial^2 - m^2)\, \psi(x, t) + g \int \mathrm{d}^D y\, K(x, y)\, \not{A}(y, t)\, \psi(y, t) + \theta(x, t)\,. \tag{6.73}$$

These equations are, in the limit $m = 0$, invariant under global *chiral transformations*,

$$\psi(x,t) \to e^{i\alpha\gamma_5}\psi(x,t)\,, \qquad \bar\psi(x,t) \to \bar\psi(x,t)\, e^{i\alpha\gamma_5} \tag{6.74a}$$

$$\theta(x,t) \to e^{i\alpha\gamma_5}\theta(x,t)\,, \qquad \bar\theta(x,t) \to \bar\theta(x,t)\, e^{i\alpha\gamma_5} \tag{6.74b}$$

$$A_\mu(x,t) \to A_\mu(x,t)\,, \qquad \eta_\mu(x,t) \to \eta_\mu(x,t)\,. \tag{6.74c}$$

Equations (6.72) and (6.73) are also invariant under global gauge transformations:

$$\psi(x,t) \to e^{i\alpha}\psi(x,t)\,, \qquad \bar\psi(x,t) \to \bar\psi(x,t)\, e^{-i\alpha} \tag{6.75a}$$

$$\theta(x,t) \to e^{i\alpha}\theta(x,t)\,, \qquad \bar\theta(x,t) \to \bar\theta(x,t)\, e^{-i\alpha} \tag{6.75b}$$

$$A_\mu(x,t) \to A_\mu(x,t)\,, \qquad \eta_\mu(x,t) \to \eta_\mu(x,t)\,. \tag{6.75c}$$

Note that this global gauge transformation simply reflects charge conservation.

However, eqs. (6.72) and (6.73) are *not* invariant under *local* gauge transformations. This is an obvious consequence of our non-covariant choice of kernel $K(x, x')$ of eq. (6.13).

Nevertheless, we know that the answers obtained from the non-covariant equations (6.72) and (6.73) are completely in agreement with the standard covariant Feynman graph method. How is this possible? Two effects actually work to compensate each other: the Langevin equations (6.72) and (6.73) are not invariant under local gauge transformations, but at the same time the fermionic noise measure transforms in exactly such a way as to compensate the lack of gauge invariance in the Langevin equations themselves.

For purposes of stochastic *regularization* the manifestly gauge invariant equations of the type (6.63) and (6.64) are needed. These equations are, in the case of gauge theories, where $G(x) = D(x)$, invariant under local gauge transformations. They are also, in the limit $m = 0$, invariant under global chiral transformations. Amusingly, the two noise fields now have to transform under chirality with opposite signs:

$$\theta_1(x,t) \to e^{-i\alpha\gamma_5}\theta_1(x,t) \qquad \text{when} \qquad \theta_2(x,t) \to e^{i\alpha\gamma_5}\theta_2(x,t)\,.$$

This means that even though the Langevin equations are formally chiral, they mix right-handed noise fields with left-handed noise fields and vice versa. This seems to be an inescapable feature.

7. Stochastic processes and supersymmetry

Supersymmetry appears in the context of stochastic quantization from many different angles. First of all, simply from the field theoretic point of view it would be of interest to see if any problems arise when one attempts to quantize supersymmetric theories stochastically. Clearly the main obstacle would appear to be the inclusion of fermions in this scheme, but as we have just discussed in the previous section, this problem seems to be solved. A second question one can ask is whether supersymmetry is preserved in such a stochastic quantization procedure. As it turns out, this question can be answered in

the positive; in fact, for supersymmetric *gauge* theories stochastic quantization appears to have one definite advantage: since a gauge fixing term is unnecessary, supersymmetry will not be broken at *any* step. This holds both for the Abelian and non-Abelian case. The issue of stochastic quantization of supersymmetric theories will be discussed towards the end of this section.

Perhaps a far more promising feature of supersymmetric theories quantized stochastically is the possibility of using stochastic *regularization* as a regularization scheme for such theories. Indeed, it appears at the moment as if stochastic regularization is the *only* viable candidate for a regularization scheme which manifestly conserves both supersymmetry, chiral symmetry and gauge invariance. This will be discussed in greater detail in section 8.

However, supersymmetry is related to stochastic quantization also at a much deeper level. As an example, even purely scalar field theories will, when quantized stochastically, display a 'hidden' supersymmetry. This issue, which is intimately connected with the existence of a so-called 'Nicolai map' for supersymmetric field theories [7.1], will be the main subject of this section.

We start with a seemingly unrelated problem, that of Parisi-Sourlas 'dimensional reduction' of scalar field theories in external random fields [7.2, 7.3]. In fact, this Parisi-Sourlas reduction *is* closely related to both supersymmetry and stochastic quantization. This becomes apparent when one establishes the connection to the Nicolai map, which is the subject of the following subsection. In section 7.3 we discuss the superspace formulation and the dimensional reduction procedure for stochastic quantization. It will give an elegant equivalence proof, in addition to those already outlined in section 3. We conclude with a brief description of how supersymmetric field theories can be quantized by stochastic methods.

7.1. Parisi–Sourlas reduction

The phenomenon of dynamical 'dimensional reduction' was first noted within the context of critical phenomena associated with spin systems in random external fields. (This may sound like a rather academic problem, but systems very close to such a situation can in fact be created and studied in the laboratory.) As has been learned from renormalization group theory (for an early review see, for example, ref. [7.4], the detailed long-distance behaviour of, for example, Ising spin systems can, sufficiently close to a critical point, be understood from the behaviour of a scalar field theory

$$S[\phi] = \int \mathrm{d}^D x\, \mathscr{L}[\phi(x)] = \int \mathrm{d}^D x\, \{-\tfrac{1}{2}\phi(x)\,\partial^2\phi(x) + \tfrac{1}{2}m^2\,\phi(x)^2 + V[\phi]\} \tag{7.1}$$

with a potential

$$V[\phi] = a\,\phi(x)^3 + b\,\phi(x)^4\,. \tag{7.2}$$

In this field theory language the coupling to a random external magnetic field can simply be represented by adding to $S[\phi]$ a source term:

$$S[\phi] \to S[\phi] - h(x)\,\phi(x)\,. \tag{7.3}$$

One must, of course, also simulate the effect of randomness in the external field. This can be accomplished by computing all physical quantities of interest for *fixed* $h(x)$, and subsequently performing the desired averaging over $h(x)$. The resulting quantities will then be obtained in what is called a

quenched average. (We shall return to the idea of quenched averages in section 8. It there appears within the context of large-N field theory, but in fact there is a rather close connection to the present discussion.)

To be specific, we can consider an averaging with the simplest possible weight: the Gaussian distribution. The introduction of an external field with Gaussian distribution should remind us of the rôle played by the noise field $\eta(x, t)$ in stochastic quantization, and of course there is a close parallel. This will become more clear below.

For example, for the free energy of the system subjected to an external field $h(x)$ we would have

$$\mathscr{F}[h] = -\ln \int \mathrm{D}\phi \, \exp\Big\{-\int \mathrm{d}^D x \, [\mathscr{L}[\phi(x)] - h(x)\, \phi(x)]\Big\} \tag{7.4}$$

and a subsequent averaging over the random field $h(x)$ then gives

$$\overline{\mathscr{F}[h]} = \int \mathrm{D}h \, \mathscr{F}[h] \exp\Big\{-\frac{1}{2c} \int \mathrm{d}^D x \, h(x)^2\Big\} . \tag{7.5}$$

In general, we will now have two kinds of correlation functions [7.6]: the 'standard' type

$$X(x, y) = \overline{\langle \phi(x)\, \phi(y)\rangle - \langle \phi(x)\rangle \langle \phi(y)\rangle} \tag{7.6}$$

and one induced by the random magnetic field $h(x)$:

$$G(x, y) = \overline{\langle \phi(x)\rangle \langle \phi(y)\rangle} . \tag{7.7}$$

Note that this last Green function $G(x, y)$ will be non-vanishing even if $\overline{\langle \phi(x)\rangle} = 0$. Note also that with the Gaussian distribution considered above, we have

$$\overline{h(x)} = 0 , \qquad \overline{h(x)\, h(y)} = c\, \delta^D(x - y) , \qquad \text{etc.} \tag{7.8}$$

with the higher $2n$-point functions of $h(x)$ obtained from the 2-point function above through Wick's theorem.

For definiteness, let us in the following consider the Lagrangian

$$\mathscr{L}[\phi(x)] = \tfrac{1}{2}(\partial_\mu \phi(x))^2 + \tfrac{1}{2}m^2\, \phi(x)^2 + \tfrac{1}{4}\lambda\, \phi(x)^4 . \tag{7.9}$$

In a saddle point approximation, valid when $c \to \infty$ (i.e. when the strength of the random magnetic field becomes large compared with thermal fluctuations), it is necessary to rescale the $\phi(x)^4$ coupling $\lambda \to \lambda/c$ and the fields $\phi(x) \to \phi(x)\sqrt{c}$, $h(x) \to h(x)\sqrt{c}$. This brings out a common factor of c in front of the action:

$$Z[h] = \int \mathrm{D}\phi \, \exp\Big\{-c \int \mathrm{d}^D x \, [\tfrac{1}{2}(\partial_\mu \phi)^2 + \tfrac{1}{2}m^2\phi^2 + \tfrac{1}{4}\lambda\phi^4 - h\phi]\Big\} \tag{7.10}$$

and we are now ready to analyse the system by saddle point methods.

In such a saddle point (or steepest descent) approach one has simply, for example,

$$\langle \phi(x) \rangle = \phi^{(1)}(h, x) \tag{7.11}$$

where $\phi^{(1)}(h, x)$ is the solution of the classical differential equation:

$$-\partial^2\phi(x) + m^2\, \phi(x) + \lambda\, \phi(x)^3 = h(x) \tag{7.12}$$

which has least action. (In general there will be a whole set of solutions, and this will unfortunately [7.5] turn out to have some serious consequences. More about this later.)

Similarly, one has in the saddle point approximation

$$c^{-1}\, G(x, y) = \overline{\langle \phi(x) \rangle\, \langle \phi(y) \rangle} = \overline{\phi^{(1)}(h, x)\, \phi^{(1)}(h, y)} \tag{7.13}$$

for the Green function of eq. (7.7). The solution $\phi^{(1)}(h, x)$ of eq. (7.12) can be found, for example, by iterating the equation in λ. This corresponds to ordinary perturbation theory in λ, and is of course completely analogous to the perturbative treatment of the Langevin equation for a scalar field, as we already encountered it in section 3.

Computing Green functions directly in such a diagrammatic way [7.5, 7.7] one finds, remarkably, that these Green functions are *identical* to the Green functions computed for the same theory *without* the random magnetic field, but in 2 dimensions lower. It was this 'dimensional reduction' from D to $(D-2)$ which was shown in a beautiful paper by Parisi and Sourlas [7.2] to be a consequence of a hidden supersymmetry in the problem. It is, in that language, the fermionic coordinates which are responsible for the reduction from D to $(D-2)$ dimensions.

In order to reveal this hidden supersymmetry one can go through the following manipulations. Consider the expectation value of, for example, the correlation function $G(x, y)$ of eq. (7.7). In the above approximation we have

$$\begin{aligned} c^{-1}\, G(x, y) &= \overline{\langle \phi(x) \rangle\, \langle \phi(y) \rangle} = \overline{\phi^{(1)}(h, x)\, \phi^{(1)}(h, y)} \\ &= \int \mathrm{D}h\, \phi^{(1)}(h, x)\, \phi^{(1)}(h, y) \exp\left\{-\tfrac{1}{2} \int \mathrm{d}^D x\, h(x)^2\right\} \end{aligned} \tag{7.14}$$

i.e.

$$\begin{aligned} c^{-1}\, G(x, y) = &\int \mathrm{D}h\, \mathrm{D}\phi \prod_{\lambda} \delta(-\partial^2\phi(x) + m^2\, \phi(x) + \lambda\, \phi(x)^3 - h(x)) \det[-\partial^2 + m^2 + 3\lambda\, \phi(x)^2] \\ &\cdot \phi^{(1)}(h, x)\, \phi^{(1)}(h, y) \exp\left\{-\tfrac{1}{2} \int \mathrm{d}^D x\, h(x)^2\right\} \\ = &\int \mathrm{D}\phi\, \det[-\partial^2 + m^2 + 3\lambda\, \phi(x)^2]\, \phi(x)\, \phi(y) \\ &\cdot \exp\left\{-\tfrac{1}{2} \int \mathrm{d}^D x\, [-\partial^2\phi(x) + m^2\, \phi(x) + \lambda\, \phi(x)^3]\right\}. \end{aligned} \tag{7.15}$$

The determinant appearing in eq. (7.15) can now be exponentiated into the action by the introduction of two anticommuting (but spinless) fields $\bar{\psi}(x)$ and $\psi(x)$. This is possible because of the identity

$$\det[-\partial^2 + m^2 + 3\lambda\,\phi(x)^2] = \int \mathrm{D}\bar{\psi}\,\mathrm{D}\psi\,\exp\left\{-\int \mathrm{d}^D x\,\bar{\psi}(x)\,[-\partial^2 + m^2 + 3\lambda\,\phi(x)^2]\,\psi(x)\right\}. \tag{7.16}$$

The resulting effective action is supersymmetric. This is perhaps most easily seen by the further introduction of a scalar auxiliary field $\omega(x)$ [7.2], which also serves to close the supersymmetry algebra. The final result is

$$c^{-1}\,G(x,y) = \int \mathrm{D}\phi\,\mathrm{D}\omega\,\mathrm{D}\bar{\psi}\,\mathrm{D}\psi\,\phi(x)\,\phi(y)\,\exp\left\{-\int \mathrm{d}^D x \mathscr{L}_{\mathrm{eff}}(\phi,\bar{\psi},\psi)\right\} \tag{7.17}$$

with

$$\mathscr{L}_{\mathrm{eff}} = -\tfrac{1}{2}\omega(x)^2 + \omega(x)\,(-\partial^2\,\phi(x) + m^2\,\phi(x) + \lambda\,\phi(x)^3) + \bar{\psi}(x)\,(-\partial^2 + m^2 + 3\lambda\,\phi(x)^2)\,\psi(x)\,. \tag{7.18}$$

This Lagrangian is invariant under the following supersymmetry transformations [7.2]:

$$\begin{aligned} &\delta\phi = -\bar{\varepsilon}_\mu x_\mu \psi\,, \qquad && \delta\psi = 0\,, \\ &\delta\omega = 2\bar{\varepsilon}_\mu \partial_\mu \psi\,, \qquad && \delta\bar{\psi} = \bar{\varepsilon}_\mu x^\mu \omega + 2\bar{\varepsilon}_\mu\,\partial^\mu \phi \end{aligned} \tag{7.19}$$

where $\bar{\varepsilon}_\mu$ is an infinitesimal anticommuting vector. (We encourage the reader to verify this supersymmetry invariance explicitly! It is a simple exercise which only involves a few partial differentiations, plus use of the anticommutativity of ψ and $\bar{\psi}$.)

Parisi and Sourlas have offered a compelling argument why this supersymmetry is responsible for the dimensional reduction [7.2]. First note that on account of the supersymmetry the action of eq. (7.17) can be expressed directly in superspace:

$$S_{\mathrm{eff}} = \int \mathrm{d}^D x\,\mathrm{d}\bar{\theta}\,\mathrm{d}\theta\,[\tfrac{1}{2}\Phi(-\nabla^2_{\mathrm{SS}})\Phi + V[\Phi]] \tag{7.20}$$

where Φ is an appropriate superfield. The measure integrates over all space coordinates $\mathrm{d}^D x$ and over the Grassmann variables $\mathrm{d}\theta$ and $\mathrm{d}\bar{\theta}$. As θ and $\bar{\theta}$ are completely anticommuting variables, the most general form of ϕ is

$$\Phi(x,\theta,\bar{\theta}) = \phi(x) + \bar{\theta}\,\psi(x) + \bar{\psi}(x)\,\theta + \theta\bar{\theta}\,\omega(x) \tag{7.21}$$

[the variables ϕ, ψ, $\bar{\psi}$ and ω are precisely the variables of the 'component field' action (7.17)]. The superspace Laplacian ∇^2_{SS} is defined by

$$\nabla^2_{\mathrm{SS}} = \partial^2 + 4\,\partial^2/\partial\bar{\theta}\,\partial\theta \tag{7.22}$$

and it is convenient to use a non-standard normalization for the Grassmann integrations:

$$\int \mathrm{d}\theta = \int \mathrm{d}\bar{\theta} = 0; \qquad \int \mathrm{d}\bar{\theta}\,\mathrm{d}\theta\,\bar{\theta}\theta = -1/\pi \tag{7.23}$$

(the quantity $-1/\pi$ on the right-hand side is related to the surface area of a unit sphere in $D = -2$ dimensions, as we shall see soon).

Parisi and Sourlas have argued that a space with anticommuting variables θ and $\bar{\theta}$ is equivalent to a space of *commuting* variables of dimensions $D = -2$. As an illustration of this, first recall that the supersymmetry transformations (7.19) in the compact superspace formulation simply express invariance under superspace rotations which leave the metric $x^2 + \bar{\theta}\theta$ invariant. Then, if f is a supersymmetrically invariant function, $f = f(x^2 + \bar{\theta}\theta)$, we have

$$f(x^2 + \bar{\theta}\theta) = f(x^2) + \bar{\theta}\theta\,\mathrm{d}f(x^2)/\mathrm{d}x^2 \tag{7.24}$$

and hence, using the integration rules (7.23):

$$\int \mathrm{d}^D x\,\mathrm{d}\bar{\theta}\,\mathrm{d}\theta\, f(x^2 + \bar{\theta}\theta) = -\frac{1}{\pi}\int \mathrm{d}^D x\, f'(x^2)\,. \tag{7.25}$$

On the other hand, in general we have for a D-dimensional integral of commuting coordinates:

$$\int \mathrm{d}^D x\, f(x^2) = A(D)\int \mathrm{d}x\, x^{D-1}\, f(x^2) \tag{7.26}$$

where $A(D) = 2\pi^{D/2}/\Gamma(D/2)$ is the surface area of the unit sphere in D dimensions.

Consider now the following simple manipulations:

$$\begin{aligned}\int \mathrm{d}^{D-2}x\, f(x^2) &= A(D-2)\int \mathrm{d}x\, x^{D-2}\, f(x^2) = -A(D-2)\int \mathrm{d}x \left(\frac{1}{D-2}\right) 2\,\frac{\mathrm{d}f(x^2)}{\mathrm{d}x^2}\, x^{D-1} \\ &= -\frac{2}{D-2}\,\frac{A(D-2)}{A(D)}\int \mathrm{d}^D x\, f'(x^2) = -\frac{1}{\pi}\int \mathrm{d}^D x\, f'(x^2)\end{aligned} \tag{7.27}$$

and hence, by eq. (7.25):

$$\int \mathrm{d}^D x\,\mathrm{d}\bar{\theta}\,\mathrm{d}\theta\, f(x^2 + \bar{\theta}\theta) = \int \mathrm{d}^{D-2}x\, f(x^2)\,. \tag{7.28}$$

The above formula (7.28) is the magic relation needed to derive the dimensional reduction of Green functions such as the one of eq. (7.13). Parisi and Sourlas [7.2] gave an all-order proof in perturbation theory, and this proof was a little later nicely completed in full non-perturbative fashion [7.8, 7.9]. We will come back in detail to the non-perturbative reduction argument of [7.8] when discussing the superfield formulation of stochastic quantization in section 7.3.

A prediction of dimensional reduction from D to $D-2$ for spin systems such as those discussed in the beginning of this section have drastic experimental consequences. For one thing, the lower critical dimension should now be the most physical one of 3 dimensions, instead of 1 dimension! Also, the

critical exponents should be radically changed in dimensions 3 or higher [7.10], although this may not be of much physical relevance.

Unfortunately, it appears as if the $D \to (D-2)$ dimensional reduction prediction has been disproved by experiments. (For a recent review of the experimental situation, see, for example, ref. [7.11].) Thus, long range order has been observed even in 3-dimensional systems, and it seems as if the lower critical dimensionality will be $D=2$ instead of $D=3$, as predicted from the $D \to (D-2)$ dimensional reduction.

How can these *experimental* facts be reconciled with the previous theoretical analysis? Parisi [7.5] has pointed out how the above arguments can fail. To understand this, let us return to eq. (7.15). The manipulations used there are valid as long as a unique solution $\phi^{(1)}(h, x)$ of the saddle point equation (7.12) exists, which then minimizes the action. This is in fact the case when $m^2 > 0$. However, when multiple solutions exist (as for $m^2 < 0$), one needs to modify eq. (7.15) accordingly. This is clear, since in the solution for $G(x, y)$ we want the contribution only from the one solution $\phi^{(1)}(h, x)$ which minimizes the action, but if multiple solutions exist, then these extra solutions will appear in the functional integral (7.15) also. These extra solutions contribute to the supersymmetric form of the effective action, and are hence partly responsible for the dimensional reduction $D \to (D-2)$. Properly removing these solutions will therefore destroy the supersymmetry, and very likely also the dimensional reduction prediction.

To make these remarks more precise, let us now assume that we in fact are in a situation where a whole set of solutions $\phi^{(1)}(h, x)$, $\phi^{(2)}(h, x)$, $\phi^{(3)}(h, x), \ldots, \phi^{(n)}(h, x)$ exists of eq. (7.12), and $\phi^{(1)}(h, x)$ again is the solution which minimizes the action.

The solution of $G(x, y)$ from eq. (7.17) then reads

$$c^{-1} G(x, y) = \int \mathrm{D}h \sum_{i=1}^{n} \phi^{(i)}(h, x)\, \phi^{(i)}(h, y)\, \mathrm{sign}\{\det[-\partial^2 + m^2 + 3\lambda\, \phi^{(i)}(h, x)^2]\} \exp\left\{-\tfrac{1}{2} \int \mathrm{d}^D x\, h(x)^2\right\} \tag{7.29}$$

but this does obviously not coincide with the correct expression (7.14)! One therefore needs to introduce a function $C[\phi]$ [7.6] with the following property

$$C[\phi] = \begin{cases} 1, & \text{if } \phi = \phi^{(1)}(h, x) \\ 0, & \text{if } \phi = \phi^{(i)}(h, x), \qquad i = 2, \ldots, n \end{cases} \tag{7.30}$$

with the help of which

$$c^{-1} G(x, y) = \int \mathrm{D}\phi\, \mathrm{D}\omega\, \mathrm{D}\bar\psi\, \mathrm{D}\psi\, \phi(x)\, \phi(y)\, C[\phi] \exp\left\{-\int \mathrm{d}^D x\, \mathscr{L}_{\mathrm{eff}}(\phi, \bar\psi, \psi)\right\} \tag{7.31}$$

is the correct expression for the correlation function. Here

$$S_{\mathrm{eff}} = \int \mathrm{d}^D x\, \mathscr{L}_{\mathrm{eff}}(\phi, \bar\psi, \psi) \tag{7.32}$$

is the supersymmetric effective action corresponding to (7.18). However, due to the insertion of $C[\phi]$ in (7.31), the correlation function will no longer coincide with the corresponding supersymmetric one,

and the $D \to (D-2)$ prediction is lost. The expression (7.31) is of course still a useful starting point if one can determine $C[\phi]$. Parisi has found that $C[\phi] \propto \theta[m^2 + \lambda\,\phi(x)^2]$ in a simple zero-dimensional example [7.5], but it appears as if the general solution will be difficult to find.

7.2. *The Nicolai map*

The somewhat mysterious appearance of a supersymmetry in the stochastic problem discussed in section 7.1 finds its natural explanation within the context of the Nicolai map [7.1]. The literature list connected with this subject is rather long (see, for example, refs. [7.12–7.17]), and we shall not enter into a detailed discussion of all the results (or problems) connected with this map. For very readable reviews of most of the more recent developments we refer the reader to ref. [7.18].

Before discussing the content of the Nicolai map, let us first see how manipulations very similar to the ones used in the previous section (for a problem without any fictitious time parameter) can be used to reveal a similar 'hidden supersymmetry' associated with the Langevin equation itself.

We start in the simplest possible way by considering not a *field theory*, but just the Langevin equation associated with a point particle being subjected to random background noise. This corresponds to the very real physical problem of the Brownian motion of a (classical) particle in a heat bath. Surprisingly, this problem turns out to be *equivalent* to a supersymmetric *quantum mechanical* problem. Let us now see why.

The Langevin equation for the particle reads

$$\mathrm{d}x/\mathrm{d}t = -\delta S/\delta x + \eta(t) \tag{7.33}$$

where x represents the space coordinate of the particle. (Note, incidentally, that for this particular system we can identify the time parameter t with ordinary time!)

Expectation values are, as usual, evaluated as the path integral

$$\langle x(t_1)\cdots x(t_n)\rangle = \int \mathrm{D}\eta\, x(t_n)\cdots x(t_n)\exp\left\{-\tfrac{1}{4}\int \mathrm{d}t\,\eta(t)^2\right\} \tag{7.34}$$

over a Gaussian noise, i.e. $\langle \eta(t_1)\,\eta(t_2)\rangle_\eta = 2\delta(t_1 - t_2)$ etc.

Repeating the same kinds of manipulations as in the Parisi–Sourlas case, we now attempt to make a change of variables: $\eta \to x$. From eq. (7.33) this is seen to involve the Jacobian

$$\det|\delta\eta(t)/\delta x(t')| = \det[(\mathrm{d}/\mathrm{d}t + V')\,\delta(t-t')] \tag{7.35}$$

where we have introduced $V = \delta S/\delta x$.

For the partition function Z itself this shift of variables implies

$$\begin{aligned} Z &= \int \mathrm{D}\eta\,\exp\left\{-\tfrac{1}{4}\int \mathrm{d}t\,\eta(t)^2\right\} \\ &= \int \mathrm{D}x\,\mathrm{D}\eta\,\det[\mathrm{d}/\mathrm{d}t + V']\,\delta(\mathrm{d}x/\mathrm{d}t + V - \eta)\exp\left\{-\tfrac{1}{4}\int \mathrm{d}t\,\eta(t)^2\right\} \\ &= \int \mathrm{D}x\,\det[\mathrm{d}/\mathrm{d}t + V']\exp\left\{-\tfrac{1}{4}\int \mathrm{d}t\,(\dot{x} + V)^2\right\}. \end{aligned} \tag{7.36}$$

As in the Parisi–Sourlas case we can use a fermionic path integral representation of the determinant:

$$\det[\mathrm{d}/\mathrm{d}t + V'] = \int \mathrm{D}\bar{\psi}\, \mathrm{D}\psi \exp\left\{\int \mathrm{d}t\, \bar{\psi}(\mathrm{d}/\mathrm{d}t + V')\, \psi\right\} \tag{7.37}$$

with the help of which we can extract a local action density,

$$\mathscr{L}_{\mathrm{eff}} = \tfrac{1}{4}(\mathrm{d}x/\mathrm{d}t + V)^2 - \bar{\psi}\,(\mathrm{d}/\mathrm{d}t + V')\, \psi \tag{7.38}$$

and

$$Z = \int \mathrm{D}x\, \mathrm{D}\bar{\psi}\, \mathrm{D}\psi \exp\left\{-\int \mathrm{d}t\, \mathscr{L}_{\mathrm{eff}}(x, \bar{\psi}, \psi)\right\}. \tag{7.39}$$

This system is recognized as Witten's example of supersymmetric quantum mechanics [7.12, 7.19, 7.20]. Before proceeding, let us remind the reader that we started with a purely *classical* (although stochastic) problem of Brownian motion. As the simple manipulations above have shown, this system is completely equivalent to a (supersymmetric) quantum mechanical problem!

The hidden supersymmetry of the Langevin equation (7.33) does have some interesting consequences. This stems from the fact that with a (global or local) symmetry (and an action principle) one can rather easily derive non-trivial relations among various sets of Green functions. These relations, collectively known as 'Ward Identities', follow because an action symmetry implies a certain amount of redundancy: with an appropriate shift of variables the whole path integral is left invariant. This must necessarily imply a constraint on the Green functions, unless of course the symmetry is spontaneously broken.

In order to derive these Ward Identities [7.21], let us first for convenience do a rescaling of the time variable $t \to t/2$; then

$$\mathscr{L}(x, \bar{\psi}, \psi) = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\dot{x}V + \tfrac{1}{8}V^2 - \bar{\psi}(\mathrm{d}/\mathrm{d}t + \tfrac{1}{2}V')\psi\,. \tag{7.40}$$

The term $\dot{x}V/2$ does not contribute to the action since it is just a surface term:

$$\tfrac{1}{2}\int \mathrm{d}t\, \dot{x}V = \tfrac{1}{2}\int \mathrm{d}x\, \delta S/\delta x = 0\,. \tag{7.41}$$

Finally, in order to have a supersymmetry algebra which closes, we introduce an auxiliary field D:

$$\mathscr{L} = \tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}D^2 - \tfrac{1}{2}DV - \bar{\psi}(\mathrm{d}/\mathrm{d}t + \tfrac{1}{2}V')\psi\,. \tag{7.42}$$

The action corresponding to (7.42) is invariant under the following supersymmetry transformations,

$$\begin{aligned} &\delta x = \bar{\varepsilon}\psi - \bar{\psi}\varepsilon\,, \qquad \delta D = \bar{\varepsilon}\dot{\psi} + \dot{\bar{\psi}}\varepsilon\,, \\ &\delta\psi = (\dot{x} + D)\varepsilon\,, \quad \delta\bar{\psi} = \bar{\varepsilon}(\dot{x} - D) \end{aligned} \tag{7.43}$$

as is easily checked. It is also straightforward to show that two successive transformations yield one time

translation,

$$[\delta_1, \delta_2] = -4\,\mathrm{d}/\mathrm{d}t \tag{7.44}$$

and that the rest of the supersymmetry algebra closes as well.

In order to derive the Ward Identities, we now introduce a set of sources $\{J_x, J_D, \zeta, \bar{\zeta};\}$ which form a scalar super-multiplet under (7.43). Their transformation properties are required to be [7.21]:

$$\begin{aligned} \delta J_x &= \bar{\varepsilon}\dot{\zeta} + \dot{\bar{\zeta}}\varepsilon\,, \qquad & \delta J_D &= \bar{\varepsilon}\zeta - \bar{\zeta}\varepsilon\,, \\ \delta\zeta &= \varepsilon J_x + \varepsilon \dot{J}_D\,, \qquad & \delta\bar{\zeta} &= -\bar{\varepsilon}J_x + \bar{\varepsilon}J_D\,. \end{aligned} \tag{7.45}$$

Thus, with

$$\mathscr{L} \to \mathscr{L} + xJ_x + DJ_D + \bar{\zeta}\psi + \bar{\psi}\zeta \tag{7.46}$$

we have $\delta\mathscr{L} = 0$ under the transformations (7.43) and (7.45). This invariance of the Lagrangian obviously implies $\delta Z = 0$ (with Z being the partition function) as well. But then, by using the functional chain rule,

$$0 = \delta Z = \frac{\delta Z}{\delta J_x}\,\delta J_x + \frac{\delta Z}{\delta J_D}\,\delta J_D - \frac{\delta Z}{\delta\zeta}\,\delta\zeta + \delta\bar{\zeta}\,\frac{\delta Z}{\delta\bar{\zeta}} \tag{7.47}$$

which, on account of (7.45), implies

$$\frac{\delta Z}{\delta J_x}\,(\bar{\varepsilon}\dot{\zeta} + \dot{\bar{\zeta}}\varepsilon) + \frac{\delta Z}{\delta J_D}\,(\bar{\varepsilon}\zeta - \bar{\zeta}\varepsilon) - \frac{\delta Z}{\delta\zeta}\,(\varepsilon J_x + \varepsilon\dot{J}_D) + (-\bar{\varepsilon}J_x + \bar{\varepsilon}\dot{J}_D)\,\frac{\delta Z}{\delta\bar{\zeta}} = 0\,. \tag{7.48}$$

This is the master equation for all Ward Identities of the theory. Specific identities, which are all non-trivial relations among Green functions of the theory, are obtained by differentiating eq. (7.48) with respect to one or several of the sources of the set $\{J_x, J_D, \zeta, \bar{\zeta}\}$, and then letting all sources equal to zero.

As an example, let us first differentiate (7.48) with respect to $\delta/\delta J_D$ and $\delta/\delta\zeta$. From eq. (7.48) only 3 terms can contribute:

$$-\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\delta^2 Z}{\delta J_D\,\delta J_x}\,\bar{\varepsilon} + \frac{\delta^2 Z}{\delta J_D^2}\,\bar{\varepsilon} + \bar{\varepsilon}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\delta^2 Z}{\delta\zeta\,\delta\bar{\zeta}} = 0 \tag{7.49}$$

i.e.

$$-\frac{\mathrm{d}}{\mathrm{d}t}\,\langle xD\rangle + \tfrac{1}{4}\langle v^2\rangle + \frac{\mathrm{d}}{\mathrm{d}t}\,\langle\psi\bar{\psi}\rangle = 0\,. \tag{7.50}$$

Similarly, if we differentiate with respect to $\delta/\delta J_x$ and $\delta/\delta\zeta$:

$$-\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\delta^2 Z}{\delta J_x^2}\,\bar{\varepsilon} + \frac{\delta^2 Z}{\delta J_x\,\delta J_D}\,\bar{\varepsilon} - \bar{\varepsilon}\,\frac{\delta^2 Z}{\delta\zeta\,\delta\bar{\zeta}} = 0 \tag{7.51}$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x^2\rangle - \tfrac{1}{2}\langle xV\rangle + \langle \psi\bar{\psi}\rangle = 0 \tag{7.52}$$

and so on. These are all non-trivial identities enforced on the Green functions by the supersymmetry of the Lagrangian (7.42)

It is natural to ask at this point about the meaning and significance of the hidden supersymmetry in the classical stochastic system (7.33), and in particular about the consequences of Ward Identities such as (7.50) and (7.52) on this classical system of Brownian motion. To get some insight into this, let us return to the path integral formulation (7.38), (7.39) of the problem, and try to evaluate directly the fermionic determinant appearing there.

Clearly,

$$\det[\mathrm{d}/\mathrm{d}t + V'] = \prod_n \lambda_n \tag{7.53}$$

where λ_n are the eigenvalues of the operator $(\mathrm{d}/\mathrm{d}t) + V'$. However, this operator has a continuous spectrum, and in order to give meaning to the right-hand side of eq. (7.53), some regularization procedure must be introduced. This is not necessarily an unambiguous procedure, and particular care is often required when some symmetries are present, as in this case, where we certainly wish to preserve the supersymmetry throughout. One way to do this is to simply enclose the system in a finite 'volume' of extent T. To preserve the supersymmetry we must employ *periodic* boundary conditions for both the bosons and fermions. The determinant is now calculable, e.g. by a zeta-function regularization technique [7.21]. We first solve the eigenvalue problem

$$(\mathrm{d}/\mathrm{d}t + V')\psi_n = \lambda_n\psi_n \tag{7.54}$$

with the trivial solution

$$\psi_n(t) = \psi_n(0)\exp\left\{\int_0^t \mathrm{d}\tau\,(\lambda_n - V')\right\} \tag{7.55}$$

$$\lambda_n = \frac{2\pi i n}{T} + \frac{1}{T}\int_0^T \mathrm{d}\tau\, V'\,, \qquad |n| = 0, 1, 2, \ldots . \tag{7.56}$$

Instead of attempting to compute $\Pi_n\lambda_n$ directly (the product is clearly diverging, and some procedure must be applied in order to remove this divergence), one can – as mentioned above – compute the determinant by means of zeta-function regularization. This technique is easily described. One first forms the zeta-function of the operator,

$$\zeta(s) = \sum_n \lambda_n^{-s}\,. \tag{7.57}$$

Then, if all involved quantities are well defined, one has

$$\frac{\mathrm{d}}{\mathrm{d}s}\zeta(s)\bigg|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}\sum_n \exp(-s\ln\lambda_n)\bigg|_{s=0} = -\sum_n \ln\lambda_n \tag{7.58}$$

i.e. formally

$$\ln \det[\text{operator}] = -\frac{\mathrm{d}}{\mathrm{d}s}\,\zeta(s)\bigg|_{s=0} \tag{7.59}$$

if these are convergent quantities. In the general case one can view eq. (7.59) as one particular *definition* of the determinant [7.22].

For the case at hand, where the differential operator is of first order, we form instead the zeta-function of the absolute values of the eigenvalues,

$$\zeta(s) = \sum_n |\lambda_n|^{-s} \tag{7.60}$$

[this defines one particular way to avoid the branch cut of the logarithm, as follows from eq. (7.58)], and compute [7.21]

$$\frac{\mathrm{d}}{\mathrm{d}s}\,\zeta(s)|_{s=0} = \zeta'_{\mathrm{R}}(0, a) + \zeta'_{\mathrm{R}}(0, -a) + \ln\left|\frac{T}{2\pi i a}\right|(\zeta_{\mathrm{R}}(0, a) + \zeta_{\mathrm{R}}(0, -a)) - \ln\left|\frac{T}{2\pi i a}\right| \tag{7.61}$$

with

$$a = \frac{1}{2\pi\mathrm{i}}\int_0^T \mathrm{d}\tau\, V' \tag{7.62}$$

and $\zeta_{\mathrm{R}}(s, a)$ being the generalized Riemann zeta function:

$$\zeta_{\mathrm{R}}(s, a) = \sum_n \left(\frac{1}{n+a}\right)^s . \tag{7.63}$$

Miraculously, the unwieldy expression (7.61) reduces to an extremely simple form if we make use of the identities

$$\zeta'_{\mathrm{R}}(0, a) = \ln \Gamma(a) - \tfrac{1}{2}\ln(2\pi) \tag{7.64a}$$

$$\zeta_{\mathrm{R}}(0, a) = \tfrac{1}{2} - a \tag{7.64b}$$

$$\Gamma(a)\,\Gamma(-a) = -\frac{1}{a}\,\frac{\pi}{\sin(\pi a)} \tag{7.64c}$$

and we finally end up with [7.21]

$$\det\left[\frac{\mathrm{d}}{\mathrm{d}t} + V'\right] = -2\mathrm{i}\sinh\left\{\tfrac{1}{2}\int_0^T \mathrm{d}\tau\, V'\right\} . \tag{7.65}$$

(Note, incidentally, that had we used *antiperiodic* boundary conditions for the fermions, which explicitly breaks the supersymmetry at the boundary, the sinh function in eq. (7.65) would have been replaced by a cosh.)

At this final stage we are allowed to let the cut-off T go to infinity again, since no divergences are encountered. We can then substitute (7.65) into the partition function of eq. (7.39) and associate (7.65) with a term in the action. Since the determinant in (7.65) is a *difference* of two exponentials, we see that we actually end up with two different effective actions, $S^{\pm}$:

$$Z = \int \mathrm{D}x\, \mathrm{D}\bar{\psi}\, \mathrm{D}\psi\, \mathrm{e}^{-S} = \int \mathrm{D}x\, \mathrm{e}^{-S^+} - \int \mathrm{D}x\, \mathrm{e}^{-S^-} \tag{7.66}$$

i.e. $Z = Z^+ - Z^-$.

Surprisingly, if we compute the two Hamiltonians $H^{\pm}$ associated with the action densities $\mathscr{L}^{\pm}$ we recover two Fokker–Planck Hamiltonians [7.21, 7.23]:

$$H^{\pm} = -\tfrac{1}{2}\delta^2/\delta x^2 + \tfrac{1}{8}(\delta S/\delta x)^2 \pm \tfrac{1}{4}\delta^2 S/\delta x^2 . \tag{7.67}$$

The first Fokker–Planck Hamiltonian H^- is not new to us; it is precisely the one we derived in section 3, and used there to motivate in a non-perturbative way why stochastic quantization leads to ordinary vacuum expectation values in the equilibrium limit. The fact that here *two* Fokker–Planck Hamiltonians appear is a direct consequence of the underlying supersymmetry. In the literature of statistical mechanics H^- and H^+ are known as forward and backward Fokker–Planck Hamiltonians, respectively. The forward Hamiltonian H^- corresponds to the (forward) Fokker–Planck Lagrangian which we obtained in section 3 by choosing propagation forward in time, see eqs. (3.75), (3.76). One can verify similarly that H^- corresponds to the choice of propagation backwards in time, eqs. (3.75), (3.77).

The careful reader might have wondered whether the causal interpretation of the Langevin equation, which requires propagation forward in time, does not contradict the appearance of both forward and backward Fokker–Planck dynamics. Clarification of this issue can be obtained [7.23] by recalling basic features of supersymmetric theories [7.19, 7.20, 7.24]. Due to unbroken supersymmetry one of the Hamiltonians, H^+, has necessarily strictly positive eigenvalues so that, in the equilibrium limit $t \to \infty$, only the forward dynamics survives. This can straightforwardly be understood in an expansion of Z^+ and Z^- in eigenstates of H^+ and H^-; in the $t \to \infty$ limit only the zero ground state of H^- contributes. We summarize that in the physically relevant equilibrium limit the stochastic and supersymmetric formulations are equivalent.

We can now return to the question of the implication of the previously derived Ward Identities. Re-introducing the set of sources $\{J_X, J_D, \zeta, \bar{\zeta}\}$, we obviously also have for the generating functional $Z[J] \equiv Z[J_X, J_D, \zeta, \bar{\zeta}]$:

$$Z[J] = Z^+[J] - Z^-[J] . \tag{7.68}$$

All Green functions can be computed by differentiating $Z[J]$ with an appropriate set of sources.

Specifically, consider now one particular set of Green functions [e.g. eqs. (7.50) and (7.52)] which fulfil the supersymmetric Ward Identities, whose general form follows from eq. (7.48). Let $F[x, V(x)]$ stand for such a generic combination of functions. Then, by definition,

$$\langle F[x, V(x)] \rangle_Z = 0 \tag{7.69}$$

and hence, by eq. (7.68),

$$\langle F[x, V(x)]\rangle_{Z+} = \langle F[x, V(x)]\rangle_{Z-} \,. \tag{7.70}$$

Thus, for the particular combination of functions which satisfies one of the infinite set of Ward Identities (7.48), expectation values with respect to H^+ and H^- are *identical.*

This symmetry between the two classical Fokker–Planck Hamiltonians H^+ and H^- is, as we have seen, a direct consequence of the supersymmetry in the equivalent quantum mechanical problem. An intriguing connection!

The relation between the supersymmetry connected with the Langevin equation and (forward, backward) Fokker–Planck Hamiltonians has been discussed in further detail by Gozzi [7.23]; an explanation of the supersymmetry by the Onsager principle of microscopic reversibility can be found in [7.25].

In a more general framework, the Langevin eq. (7.33) gives one particular *local* realization of a Nicolai map [7.1]. Such a map is characterized by the following theorem: supersymmetric theories can be described by a transformation T_g of the bosonic fields of the theory,

$$\mathrm{T}_g : \phi(x) \rightarrow \phi'(x, g, \phi(x)) \tag{7.71}$$

with g indicating generically a set of coupling constants. For a supersymmetric theory such a map exists with the following remarkable properties [7.1]:

i) T_g is invertible (at least in perturbation theory).

ii) $S(g, \phi) = S_0[\phi'(g, \phi)]$ where $S(g, \phi)$ is the part of the action left over after integrating out the fermions. S_0 is a *free* (quadratic, or of 'covariance one') action.

iii) The Jacobian of the transformation T_g equals the Matthews–Salam determinant from the fermion integrations.

In a loose way we can phrase this as follows: take a supersymmetric theory and integrate out the fermion fields from the path integral. This produces a non-trivial determinant inside the (bosonic) path integral. There now exists a transformation T of the bosonic fields whose Jacobian determinant exactly *cancels* the determinant of the fermion integrations, and which simultaneously transforms the left-over bosonic action into a *free* bosonic action.

This miraculous property of supersymmetric theories is of course intimately related to the well-known cancellations between bosonic and fermionic degrees of freedom in such theories, and in particular to the vanishing vacuum energy.

To give another simple, but illustrative example [7.26], consider a free supersymmetric Lagrangian in 2 dimensions,

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 + \tfrac{1}{2}m^2\phi^2 + \tfrac{1}{2}\mathrm{i}\bar{\psi}(\not{\partial} + m)\psi \tag{7.72}$$

where $\bar{\psi}$ and ψ are Majorana spinors. Performing the Grassmann integrations, we obtain $\Delta = \det(\not{\partial} + m)^{1/2}$. Now, in 2 dimensions,

$$\det[\not{\partial} + m] = \det[\gamma_5(\not{\partial} + m)\gamma_5] = \det[-\not{\partial} + m] = \det[-\not{\partial}^2 + m^2]^{1/2} = \det[\partial^2 + m^2] \,. \tag{7.73}$$

Consider then the bosonic transformation

$$\mathrm{T}:\quad \phi(x)\to(\partial^2+m^2)^{-1/2}\phi_0(x)$$

which has the Jacobian $\det[(\partial^2+m^2)]^{-1/2}$. Performing this transformation inside the partition function, we have, successively,

$$\begin{aligned} Z &= \int \mathrm{D}\phi\, \mathrm{D}\bar{\psi}\, \mathrm{D}\psi \exp\{-S[\phi,\bar{\psi},\psi]\} \\ &= \int \mathrm{D}\phi_0 \det[\partial^2+m^2]^{-1/2} \det[\partial^2+m^2]^{1/2} \exp\Big\{-\tfrac{1}{2}\int \mathrm{d}^2x\, \phi_0(\partial^2+m^2)^{-1/2} \\ &\qquad \times(\partial^2+m^2)\,(\partial^2+m^2)^{-1/2}\,\phi_0\Big\} \\ &= \int \mathrm{D}\phi_0 \exp\Big\{-\tfrac{1}{2}\int \mathrm{d}^2x\,\phi_0(x)^2\Big\} \end{aligned} \tag{7.74}$$

i.e. a manifestly free bosonic theory!

The last line in eq. (7.74) should remind us of the noise measure in stochastic quantization. In fact, reading eqs. (7.36)–(7.39) backwards we see that this *is* precisely a Nicolai map construction if we identify [7.12] the Nicolai map T with the transformation $x\leftrightarrow\eta$. Thus, the Langevin equation (7.33) is the explicit analytic form of the Nicolai map of supersymmetric quantum mechanics. The Nicolai map will in general not be of a simple local form as in this case.

Possible problems associated with an analysis as above could arise from the fact that we have tacitly assumed the fermion determinant to always have the same sign. (Recall that the Jacobian actually should involve the *modulus*.) Further relationships between supersymmetric and Fokker–Planck Hamiltonians, an account of other Nicolai maps of the Langevin type, as well as a discussion of the above problem, have been given by Claudson and Halpern [7.27]. See also Bern and Chan [7.28].

Returning to the Parisi–Sourlas scheme of section 7.1 we see from eqs. (7.14) and (7.17)–(7.18) that again a free bosonic action [in $h(x)$] is related to a supersymmetric field theory. In this case the Nicolai map is given by the classical field equation

$$-\partial^2\phi(x)+m^2\,\phi(x)+\lambda\,\phi(x)^3=h(x)\,. \tag{7.75}$$

We recognize a great similarity between the correspondence of stochastic processes to supersymmetric field theories in the examples of the Parisi–Sourlas scheme and of the Langevin equation. In fact an even further generality has been discussed and exhibited in [7.29]: It has been shown that for any local constraint equation F

$$F(\phi)=\eta \tag{7.76}$$

expressing a field ϕ in terms of a noise field η (for which some probability distribution is provided) the generating functional exhibits a BRS invariance. In special cases (as e.g. discussed above) even a supersymmetry emerges.

We close this subsection by remarking that the concept of Parisi–Sourlas reduction has been exploited for an alternative stochastic quantization scheme [7.30], which is based on 'dimensional reduction'.

7.3. The superfield formulation of stochastic quantization

In this subsection we will give the superfield formulation of stochastic quantization and show as an interesting consequence an elegant non-perturbative proof for the equivalence of stochastic quantization to the conventional path-integral quantization. For simplicity we restrict ourselves to scalar field theories [7.31–7.33], see also [7.34–7.37]; for attempts to formulate the superfield approach for gauge theories, see [7.38–7.39], and for fermions see [7.40].

Most of the formalism we need has already been worked out in the sections above. We consider the functional formulation of stochastic quantization and study the Fokker–Planck action (3.83)

$$S_{\mathrm{FP}} = \int \mathrm{d}^n x\, \mathrm{d}t [\tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{8}(\delta S/\delta\phi)^2 - \bar{\psi}(\partial_t + \tfrac{1}{2}\delta S/\delta\phi)\psi] . \tag{7.77}$$

Repeating the steps of the last section we construct a supersymmetric action which we now recast in the form

$$S_{\mathrm{FP}} = \int \mathrm{d}^n x\, \mathrm{d}t\, \mathrm{d}\bar{\theta}\, \mathrm{d}\theta\, [\Phi \bar{D} D \Phi + \mathscr{L}(\Phi)] . \tag{7.78}$$

Here we introduced the superfield Φ

$$\Phi = \phi + \bar{\theta}\psi + \bar{\psi}\theta + \bar{\theta}\theta F \tag{7.79}$$

with F as auxiliary field. The covariant derivatives D, $\bar{D}$ are defined by

$$D = \partial/\partial\bar{\theta} + \theta\, \partial/\partial t , \qquad \bar{D} = \partial/\partial\theta + \bar{\theta}\, \partial/\partial t ; \tag{7.80}$$

$\mathscr{L}(\Phi)$ denotes the original Lagrangian in terms of the superfield Φ. The supersymmetry transformations read

$$\begin{aligned} &\delta\phi = \bar{\varepsilon}\psi - \bar{\psi}\varepsilon , \qquad \delta\bar{\psi} = \bar{\varepsilon}(\dot{\phi} - F) , \\ &\delta\psi = \varepsilon(\dot{\phi} + F) , \qquad \delta F = \bar{\varepsilon}\dot{\psi} + \dot{\bar{\psi}}\varepsilon , \end{aligned} \tag{7.81}$$

and correspond to the following translation in superspace

$$\delta t = -\bar{\varepsilon}\theta + \bar{\theta}\varepsilon , \qquad \delta\theta = \varepsilon , \qquad \delta\bar{\theta} = \bar{\varepsilon} . \tag{7.82}$$

Let us remark that this superspace formulation is different from the usual one [7.41] insofar as supersymmetry transformations involve $\bar{\theta}$, θ and only the fictitious time coordinate.

Invariance of the action (7.78) under supersymmetry transformations is obvious by construction with covariant derivatives (and can be checked explicitly).

Choosing periodic boundary conditions the generating functional (3.89) for stochastic correlations can finally be recast into

$$
\begin{aligned}
&Z[J] = \int \tilde{\mathrm{D}}\Phi \exp\left\{-\int \mathrm{d}^4x\, \mathrm{d}\tau\, [\Phi \bar{D} D \Phi + \mathscr{L}(\Phi) + J\Phi]\right\} \\
&\tilde{\mathrm{D}}\Phi = \prod_x \prod_t \prod_\theta \prod_{\bar\theta} \mathrm{D}\Phi(x, t, \theta, \bar\theta)\,.
\end{aligned}
\tag{7.83}
$$

Here we restrict the superfield source to the form

$$
J(x, t, \theta, \bar\theta) = J(x)\, \delta(t)\, \delta(\bar\theta)\, \delta(\theta) \tag{7.84}
$$

as eventually we are interested only in equal fictitious time correlations of the lowest component field ϕ (we choose initial conditions at $t_0 \to -\infty$ so that at $t = 0$ the system could already relax to equilibrium).

We now follow the idea of Cardy [7.8] and present an equivalence proof of stochastic quantization to the usual path integral quantization, see also [7.32, 7.33, 7.37]. Cardy's idea, adapted to the Parisi–Wu scheme, consists in introducing an 'interpolating' generating functional $Z_\lambda(J)$,

$$
Z_\lambda[J] = \int \tilde{\mathrm{D}}\Phi \exp\left\{-\int \mathrm{d}^4x\, \mathrm{d}\tau\, \mathrm{d}\bar\theta\, \mathrm{d}\theta\, [\mathscr{L}_\lambda(\Phi) + J\Phi]\right\} \tag{7.85}
$$

where

$$
\mathscr{L}_\lambda(\Phi) = [\lambda + (1-\lambda)\, \delta(t)\, \delta(\bar\theta)\, \delta(\theta)]\, \mathscr{L}(\Phi) + \Phi \bar{D} D \Phi\,. \tag{7.86}
$$

It is clear to see that for $\lambda = 1$ we obtain the supersymmetric generating functional (7.83). Furthermore, it follows for $\lambda = 0$ that all component fields with fictitious times τ less or equal to $t = 0$, except for $\phi(x, 0) \equiv \phi(x)$, can be integrated out (compare also with section 3). As a consequence, we obtain for this parameter value $\lambda = 0$

$$
Z_{\lambda=0}[J] = \int \mathrm{D}\phi \exp\left\{-\int \mathrm{d}^4x\, [\mathscr{L}(\phi) + J(x)\, \phi(x)]\right\} \tag{7.87}
$$

the usual path integral for Euclidean field theory. The essential step is to show that correlation functions derived from $Z_\lambda[J]$ are independent of λ. To do so we observe that $Z_\lambda[J]$ is *not* invariant under the full supersymmetry transformation (7.81) and (7.82) respectively. This follows from

$$
\int \mathrm{d}t\, \mathrm{d}\bar\theta\, \mathrm{d}\theta\, \delta(t)\, \delta(\bar\theta)\, \delta(\theta) = \int \mathrm{d}t\, \mathrm{d}\bar\theta\, \mathrm{d}\theta\, \theta(t - \bar\theta\theta)\,. \tag{7.88}
$$

However, $Z_\lambda[J]$ is invariant under [7.39]

$$
\begin{aligned}
&\delta\phi = -\bar\psi\varepsilon\,, \qquad && \delta\bar\psi = 0\,, \\
&\delta\psi = \varepsilon(\dot\phi + F)\,, \qquad && \delta F = \dot{\bar\psi}\varepsilon
\end{aligned}
\tag{7.89}
$$

or

$$
\delta t = \bar\theta\varepsilon\,, \qquad \delta\theta = \varepsilon\,, \qquad \delta\bar\theta = 0\,. \tag{7.90}
$$

We then have

$$\frac{\partial}{\partial\lambda}\ln Z_\lambda[J]=\int \mathrm{d}^n x\int_{-\infty}^{0}\mathrm{d}t\,\mathrm{d}\bar{\theta}\,\mathrm{d}\theta\,\langle\mathscr{L}(\Phi)\rangle_{J,\lambda}\,[1-\delta(t)\,\delta(\bar{\theta})\,\delta(\theta)]\,. \tag{7.91}$$

Because of the symmetry (7.90) the integrand depends on t, $\bar{\theta}$, θ only via some function f of the invariant $t-\bar{\theta}\theta$

$$\langle\mathscr{L}(\Phi)\rangle_{J,\lambda}=f_{J,\lambda}(x,t-\bar{\theta}\theta)=f_{J,\lambda}(x,t)-\bar{\theta}\theta\,\partial_t f_{J,\lambda}(x,t) \tag{7.92}$$

and we find easily

$$\begin{aligned}\frac{\partial}{\partial\lambda}\ln Z_\lambda[J]&=\int \mathrm{d}^n x\int_{-\infty}^{0}\mathrm{d}t\,\mathrm{d}\bar{\theta}\,\mathrm{d}\theta\,[-\bar{\theta}\theta\partial_t f_{J,\lambda}(x,t)-\delta(t)\,\delta(\bar{\theta})\,\delta(\theta)\,f_{J,\lambda}(x,t)]\\&=-\lim_{t\to-\infty}-\int \mathrm{d}^n x\,f_{J,\lambda}(x,t)=-\lim_{t\to\infty}-\int \mathrm{d}^n x\,f_\lambda(x,t)\,.\end{aligned} \tag{7.93}$$

The last manipulation followed because the source term acted at an infinite time distance away from $t=0$ so that f became independent of J. We thus have shown λ-independence of Green functions derived from Z_λ, as functional derivatives with respect to J on eq. (7.93) vanish.

This finishes the non-perturbative proof of equivalence of stochastic quantization to the standard Euclidean path integral prescription.

7.4. Supersymmetric field theories

We now finally address ourselves to a short discussion on how to stochastically quantize supersymmetric field theories. As a simple example [7.42, 7.43], see also [7.44], let us explain the procedure in the case of a free vector superfield V, which constitutes the supersymmetric generalization of the free Maxwell field. We follow the conventions of ref. [7.45]. For convenience of notation we prefer to formulate our expressions in Minkowski space and implicitly understand an appropriate continuation to Euclidean space. The action

$$S=\tfrac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,V\,\partial^2 PV \tag{7.94}$$

contains as in the Maxwell case of a projection operator P

$$P=\frac{1}{8\partial^2}D\bar{D}^2D\,,\qquad P^2=P \tag{7.95}$$

which is defined in terms of the covariant derivatives D_α, $\bar{D}_{\dot{\alpha}}$

$$\begin{aligned}D_\alpha&=\partial/\partial\theta^\alpha-\mathrm{i}\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_m\\ \bar{D}_{\dot{\alpha}}&=\frac{-\partial}{\partial\bar{\theta}^{\dot{\alpha}}}+\mathrm{i}\theta^\alpha\sigma^m_{\alpha\dot{\alpha}}\partial_m\end{aligned} \tag{7.96}$$

$$D^2=D^\alpha D_\alpha\,,\qquad \bar{D}^2=\bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}\,,\qquad \partial^2=\partial^2/\partial t^2-\vec{\nabla}^2\,. \tag{7.97}$$

The vector superfield V obeys the hermicity condition

$$V^{\dagger} = V \tag{7.98}$$

and has the following component field decomposition

$$\begin{aligned} V = {} & C + \mathrm{i}\theta X - \mathrm{i}\bar{\theta}\bar{X} + \tfrac{1}{2}\mathrm{i}\theta^2(M + \mathrm{i}N) - \tfrac{1}{2}\mathrm{i}\bar{\theta}^2(M - \mathrm{i}N) \\ & - \theta\sigma\bar{\theta}v + \mathrm{i}\theta^2\bar{\theta}(\bar{\lambda} - \tfrac{1}{2}\mathrm{i}\bar{\sigma}\,\partial X) - \mathrm{i}\bar{\theta}^2\theta(\lambda - \tfrac{1}{2}\mathrm{i}\sigma\,\partial\bar{X}) + \tfrac{1}{2}\theta^2\bar{\theta}^2(D - \tfrac{1}{2}\partial^2 C)\,. \end{aligned} \tag{7.99}$$

In analogy with the Maxwell field case the Langevin equation for V reads

$$\dot{V} = -\partial^2 PV + \underline{\eta} \tag{7.100}$$

where the noise is a superfield as well, with component fields defined by

$$\underline{\eta} = \eta_0 + \theta^{\alpha}\eta_{1\alpha} + \bar{\theta}_{\dot{\alpha}}\eta_2^{\dot{\alpha}} + \theta^2\eta_3 + \bar{\theta}^2\eta_4 + \theta\sigma^n\bar{\theta}\eta_{5n} + \bar{\theta}^2\theta^{\alpha}\eta_{6\alpha} + \theta^2\bar{\theta}_{\dot{\alpha}}\eta_7^{\dot{\alpha}} + \theta^2\bar{\theta}^2\eta_8\,. \tag{7.101}$$

The solution of (7.100) then follows simply (now having converted to Euclidean space)

$$V(k, t) = \int_0^t \mathrm{d}\tau\,[\exp\{-k^2(t - \tau)\}\,P + 1 - P]\,\underline{\eta}(k, \tau) \tag{7.102}$$

and allows to calculate propagators etc., as discussed in the preceding sections. We note without surprise (for more details see section 4) that in the equilibrium limit generally arise divergent contributions from the gauge modes; they will drop out when calculating gauge invariant expressions.

A further remark concerns the Langevin equations for specific component fields of V. As an example we obtain for the vector field v_n

$$\dot{v}_m = -\partial^2 v_m + \partial_m\partial^n v_n + \eta_{5m} \tag{7.103}$$

which coincides with (4.4). As the supersymmetric drift term is strictly positive we expect to find Langevin equations for the fermionic fields where the problem with negative fermionic modes has been overcome. This is indeed the case as can be seen e.g. from the coupled equations

$$\begin{aligned} &\dot{\bar{X}}_{\dot{\alpha}} = -\mathrm{i}\partial_m\lambda^{\alpha}\sigma^m_{\alpha\dot{\alpha}} + \mathrm{i}\eta_{2\dot{\alpha}} \\ &\dot{\lambda} - \tfrac{1}{2}\mathrm{i}\sigma^m_{\alpha\dot{\alpha}}\,\partial_m\dot{\bar{X}}^{\dot{\alpha}} = -\tfrac{1}{2}\partial^2\lambda_{\alpha} + \mathrm{i}\eta_{6\alpha} \end{aligned} \tag{7.104}$$

which combine to

$$\dot{\lambda} = -\partial^2\lambda + \mathrm{i}\eta_{6\alpha} - \tfrac{1}{2}\sigma^m_{\alpha\dot{\alpha}}\,\partial_m\eta_2^{\dot{\alpha}}\,. \tag{7.105}$$

Hence the supersymmetric formalism leads naturally to the kernel prescription discussed in the last section. Let us remark finally that (7.104) and (7.105) are covariant equations as in the Abelian case the covariant derivative for λ is just ∂_m.

It is of further interest to study chiral superfields [7.42, 7.43] and their coupling to vector superfields [7.43]. In a study of supersymmetric QED covariant component field Langevin equations have been shown again to emerge; the electron Langevin equation exhibits the covariant kernel $\partial_m - \mathrm{i}eA_m$ and a two noise structure [7.43], as discussed in section 6.

8. The large-N limit

One of the surprising applications of stochastic quantization has been in the field of large-N physics. 'Large-N theories' is a rather loosely defined class of field theory models in which some number of internal degrees of freedom N is sent to infinity; as examples one can think of a scalar vector field $\Phi^{(a)}(x)$, $(a = 1, 2, \ldots N)$, a Hermitian matrix field $M_{ab}(x)$, $(a, b = 1, 2, \ldots N)$ and, in particular, a non-Abelian gauge field $A_\mu^{ab}(x)$ in the limit of infinitely many colours. This last theory, QCD in the limit of an infinite number of colours, is of course of most interest, since it is hoped to have at least most qualitative features in common with ordinary 3-colour QCD. However, in general, field theories become far more tractable in the limit $N \to \infty$, and as such can be studied in their own right. Thus, although we shall have in mind a QCD-like theory with an infinite number of colours, it is often more convenient to describe the techniques in terms of more simple models.

The role which can be played by stochastic quantization in large-N physics became apparent after some rather spectacular developments in lattice gauge theories for $N = \infty$, starting with a paper by Eguchi and Kawai [8.1]. In this paper it was argued that full $N = \infty$ lattice gauge theories could be formulated on just one single hypercube on the lattice, instead of using the whole infinite volume. The proof of this was based on a $\mathrm{U}(1)^D$ symmetry (D = number of space-time dimensions), which, although conserved at strong coupling, was shown to be spontaneously broken at sufficiently weak coupling [8.2]. At the same time a modified scheme, based on the 'quenching' of a certain set of variables, was proposed [8.2]. The quenching was introduced in order to preserve the $\mathrm{U}(1)^D$ invariance for all values of the coupling. It was soon realized that the 'quenched momentum prescription', as it came to be called, was a completely general and derivable consequence of the $N \to \infty$ limit [8.3, 8.4, 8.5], and in fact could be generalized to continuum theories as well. Although usually derived from weak coupling perturbation theory, the quenched momentum prescription has been checked in non-perturbative regimes as well [8.2].

Stochastic quantization enters this field by offering an extremely simple derivation of the quenched momentum prescription [8.6, 8.7]. Furthermore, if the quenching prescription is extended to incorporate the fictitious time direction also, then a novel interpretation of a 'Master Field' in $N = \infty$ theories emerges [8.7]. This 'Quenched Master Field' is the solution of an algebraic matrix equation. We shall briefly discuss both of these topics in what follows, but we start with a short review of the basic notation in the limit $N \to \infty$. For more details of $N = \infty$ theories we refer the reader to, for example, the excellent and readable review by Coleman [8.8]

8.1. The double-line notation, planar diagrams

To begin, let us consider a pure SU(N) Yang–Mills gauge theory with an infinite number of colours:

$N \to \infty$. We shall later see how one can incorporate dynamical quark fields, but let us first simply view quarks as external sources.

It was first observed by 't Hooft [8.9] that Feynman diagrams for $N = \infty$ Yang–Mills theory can be represented very conveniently in terms of double lines which keep track of the colour indices. Actually, before illustrating this, it is worthwhile to notice [8.10] that a double-line notation is useful at *any* N if one wants to compute the colour factor of any given Feynman diagram in an SU(N) gauge theory. For example, if we consider the (Feynman gauge) gluon propagator,

$$\langle A_\mu^{ab}(k)\, A_\nu^{cd}(-k)\rangle = \left[\delta^{ad}\delta^{cb} - \frac{1}{N}\,\delta^{ab}\delta^{cd}\right]\frac{\delta_{\mu\nu}}{k^2} \tag{8.1}$$

then this is easily represented in terms of a double-line notation. Consider, for example, lowest order quark–antiquark scattering (fig. 8.1). In fig. 8.1a we have the ordinary Feynman diagram. As far as the colour flow is concerned we can use eq. (8.1) to represent this diagram as shown in fig. 8.1b. Similarly, quark–quark scattering takes the form shown in fig. 8.2. This double-line notation for gluon propagators is extremely useful if one wants to compute the colour factor of a given diagram [8.10]. Clearly, as $N \to \infty$ the last pieces in figs. 8.1 and 8.2 can be dropped, and we can simply represent the gluon propagator as two antiparallel lines:

$$\langle A_\mu^{ab}\; A_\nu^{cd}\rangle = \mu \;\begin{matrix} a \longrightarrow d \\ b \longleftarrow c \end{matrix}\; \nu\,.$$

Similarly, the quark-gluon vertex, the 3-gluon vertex and the 4-gluon vertex can conveniently be drawn in double-line notation at $N = \infty$. This is shown in fig. 8.5.

The limit $N \to \infty$ cannot be taken in a meaningful way without a simultaneous rescaling of the coupling constants of the theory. This should not come as a surprise; we can for an SU(N) gauge theory think of the asymptotic freedom formula

$$g^2(\Lambda) = 12\pi^2/(11N \ln \Lambda^2) \tag{8.2}$$

which relates the bare coupling g to the cut-off Λ. If we want a meaningful $N \to \infty$ limit of the renormalization group equation (8.2) we must demand that g^2N is kept fixed as $N \to \infty$, i.e. rescale

$$g \to g/\sqrt{N}\,. \tag{8.3}$$

This rescaling can, of course, also be understood on a much more simple diagrammatic basis. Since the number of colours goes to infinity, it is clear that all Feynman diagrams naively will grow without

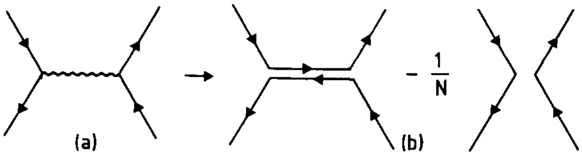


Fig. 8.1. Double-line notation for quark–antiquark scattering in an Su(N) gauge theory.

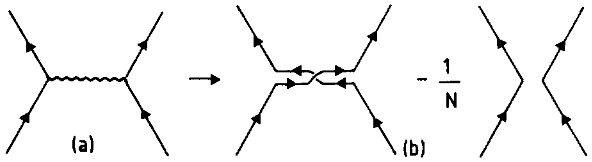


Fig. 8.2. Double-line notation for quark–quark scattering.

bounds as $N\to\infty$, simply because the number of separate 'colour diagrams' goes to infinity. As an example, consider the one-loop vacuum polarization diagrams in fig. 8.3, which define $I^{ab}_{\mu\nu}(k)$.

A simple calculation gives [where $f(k)$ is some regularized integral over the internal-loop momentum]

$$I^{ab}_{\mu\nu}(k) = -g^2\, C_2(\mathrm{G})\, \delta_{ab}\, f(k) \tag{8.4}$$

with the 'Casimir'

$$C_2(\mathrm{G}) = N \qquad \text{for } \mathrm{G} = \mathrm{SU}(N)\,. \tag{8.5}$$

In order for eq. (8.4) to remain finite as $N\to\infty$ it is clear that a rescaling as in eq. (8.3) is required. These two ways of understanding the rescaling of the coupling constant are of course equivalent: the one-loop β-function result (8.2) follows precisely by evaluation of the counterterms to diagrams such as the one in fig. 8.3.

Interestingly, in the limit $N\to\infty$ only a distinct topological subset of all Feynman diagrams survives. This is the set of *planar diagrams* [8.9]. Instead of attempting to give a general proof of this statement, we encourage the reader to check its validity by choosing a few examples. Consider, for instance, the two Feynman diagrams in fig. 8.4, which both correspond to higher order corrections to the vacuum polarization tensor. The diagram (a) in fig. 8.4 is of the order of g^6, whereas the diagram (b) is of the order of g^8. Thus naively diagram (b) is of higher order than (a). However, only diagram (b) is *planar*, and it is actually only this diagram of the two which remains finite as $N\to\infty$. The non-planar diagram (a) *vanishes* as $1/N^2$ when $N\to\infty$.

A quick way to see this is to make use of the double-line notation and redraw the diagrams as shown in fig. 8.5. The diagram (a) is of the order of g^6 and has just one closed loop; it is therefore of the order

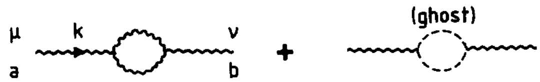


Fig. 8.3. The one-loop graphs contributing to the vacuum polarization in an SU(N) gauge theory.

Fig. 8.4. Examples of non-planar (a) and planar (b) Feynman diagrams. Only the planar diagram survives in the large-N limit.

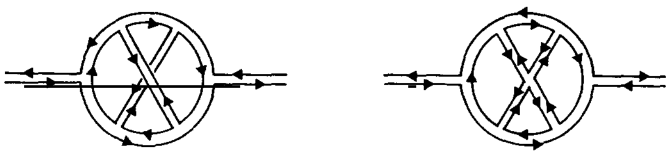


Fig. 8.5. Double-line notation for the diagrams of fig. 8.4. This illustrates the large-N counting.

of $(g/\sqrt{N})^6 \cdot N = g^6/N^2$. Even though diagram (b) is of the order of g^8, it has *four* closed loops and it is therefore of the order of $(g/\sqrt{N})^8 \cdot N^4 = g^8$. Thus, as $N \to \infty$ diagram (a) vanishes as $1/N^2$, whereas diagram (b) remains finite.

By considering such examples it is clear that the only diagrams which are not suppressed are those with the largest number of closed loops in the double-line notation; these are precisely the planar diagrams. For a more rigorous discussion we refer the reader to the original paper by 't Hooft [8.9]. Needless to say, planar diagrams form a gauge-invariant subset of all Feynman diagrams.

So far our discussion has been restricted to perturbation theory. The large-N limit implies another interesting simplification which is valid beyond perturbation theory: factorization. The factorization property at large N was first emphasized by Migdal [8.11] and Witten [8.12]. It implies that vacuum expectation values become free of fluctuations, i.e.

$$\langle AB \rangle = \langle A \rangle \langle B \rangle \tag{8.6}$$

and hence, in general,

$$\langle ABCD \dots \rangle = \langle A \rangle \langle B \rangle \langle C \rangle \langle D \rangle \dots \tag{8.7}$$

for gauge-invariant operators A, B, C, D, ...

The factorization property is easy to check in perturbation theory, again making use of the simplifying double-line notation. For simple matrix models at large N the factorization property is easy to understand also on a non-perturbative level, since it simply corresponds to the saddle-point solution becoming exact as $N \to \infty$ (owing to factors of N^2 in front of the action integral). The saddle-point solution is of course free of fluctuations, i.e. it fulfils the factorization property (8.6).

This analogy with simple matrix models, which could be solved exactly at the saddle point, i.e. solved by just one ('classical') matrix-field configuration at $N = \infty$, led Witten to suggest [8.12] that large-N QCD could be exactly solved in terms of classical 'Master Field' $A_\mu^{\mathrm{cl}}(x)$. Of course, because of gauge invariance such a Master Field will by necessity not be unique. However, since we must demand that $A_\mu^{\mathrm{cl}}(x)$ is invariant under translations, modulo gauge transformations, it follows that if such a field should exist, we should be able to choose a gauge such that it is space-time independent, $A_\mu^{\mathrm{cl}}(x) = A_\mu^{\mathrm{cl}}(0)$.

The prospect of finding such a Master Field for $N = \infty$ Yang–Mills theory, through which basically the whole theory could be solved, started to look rather bleak when it was realized [8.13] that even simple two-matrix models did not possess a Master Field at $N = \infty$. The breakdown in the argument for the existence of a Master Field [8.13] occurs because actually not all invariant operators will fulfill the factorization property (8.7). Similarly, such two-matrix models [8.14] are not solved by the saddle point even at $N = \infty$.

However, the concept of a $N=\infty$ Master Field for all matrix theories, including $N=\infty$ gauge theories, could get a new interpretation within the framework of stochastic quantization [8.7]. This will be the subject of section 8.3, but we first turn to a discussion of the quenched momentum prescription for $N=\infty$ theories

8.2. The quenched momentum prescription

Following Gross and Kitazawa [8.4], let us first consider a 'scalar' large-N matrix model given by the action

$$S=\int \mathrm{d}^D x\, \mathrm{Tr}\left\{\tfrac{1}{2}(\partial_\mu \Phi(x))^2+\tfrac{1}{2}m^2\,\Phi^2(x)+\frac{\lambda_3}{3!\sqrt{N}}\,\Phi^3(x)+\frac{\lambda_4}{4!N}\,\Phi^4(x)\right\}. \tag{8.8}$$

Here Φ represents a Hermitian $N\times N$ matrix with scalar entries ϕ_{ij}. This model has a global U(N) invariance,

$$\Phi(x)\rightarrow \mathrm{U}^\dagger\,\Phi(x)\,\mathrm{U} \tag{8.9}$$

and we will therefore restrict ourselves to the evaluation of U(N) invariant Green functions.

As in the case of $N=\infty$ Yang–Mills theories, discussed in the previous section, the $N\rightarrow\infty$ limit of the theory (8.8) corresponds to *planar* Feynman diagrams. The double-line notation is straightforward also in this case, and simply corresponds to having a line for each of the indices in ϕ_{ij}. The propagator, for instance, can be drawn

$$\langle \phi_{ij}(k)\phi_{kl}(-k)\rangle=\frac{\delta_{il}\delta_{jk}}{k^2+m^2}=\begin{matrix} i\longrightarrow l\\ j\longleftarrow k\end{matrix} \tag{8.10}$$

and similarly for the Φ^3 and Φ^4 vertices.

Because of the identification $i=l$ and $j=k$ in the propagator (8.10) we can assign to the momentum k^μ flowing through the propagator indices i and j in colour space $k^\mu_{ij}=k^\mu$. If we now take the double-line notation (8.10) seriously, we can, furthermore, imagine that directed momenta k^μ_i and k^μ_j flow through *each* of the lines. Since obviously the *total* momentum of the two lines must match that of the propagator itself, we have

$$k^\mu_{ij}=k^\mu_i-k^\mu_j\,. \tag{8.11}$$

It is clear that such an assignment of momenta is self-consistent and compatible with momentum conservation at all vertices as well. The propagator (8.10) can then be written

$$\langle \phi_{ij}(k)\,\phi_{kl}(-k)\rangle=\frac{\delta_{il}\delta_{jk}}{(k^\mu_i-k^\mu_j)^2+m^2}\,. \tag{8.12}$$

So far we have not achieved anything; we have simply introduced a new set of variables. But the analysis can be carried further [8.4]: we can imagine the propagator (8.12) as being derived from an 'action' of the form

$$\tilde{S}=\tfrac{1}{2}\mathrm{Tr}\{m^2\Phi^2-[P_\mu,\Phi]^2\} \tag{8.13}$$

(this action is not dimensionally correct, a detail which will soon be remedied). Here the momentum operator P_μ is a matrix defined by

$$(P^\mu)_{ij} = \delta_{ij} p_i^\mu \,. \tag{8.14}$$

The matrix Φ in the action $\tilde{S}$ of eq. (8.13) is not a function of the space-time coordinates x_μ and is, in this sense, *reduced*: the theory is only defined on a single point! In spite of this, the full 'action'

$$\tilde{S} = \tfrac{1}{2} \operatorname{Tr}\{m^2\Phi^2 - [P_\mu, \Phi]^2\} + \frac{\lambda_3}{3!\sqrt{N}} \operatorname{Tr} \Phi^3 + \frac{\lambda_4}{4!N} \operatorname{Tr} \Phi^4 \tag{8.15}$$

does correctly reproduce the integrands of all planar Feynman diagrams associated with the starting action (8.8). To get a complete equivalence to the full $N = \infty$ theory we must therefore supplement the action (8.15) with a set of 'Feynman rules' which tells us how to integrate over internal loop momenta. This is accomplished by the 'quenched momentum prescription' [8.2–8.5]. For a matrix field $\Phi(x)$ it tells us that $\Phi(x)$ can be related to $\Phi(0)$ by the 'translation operator' P_μ of eq. (8.14):

$$\Phi(x) = \exp(\mathrm{i}P^\mu x_\mu)\, \Phi(0) \exp(-\mathrm{i}P^\mu x_\mu)\,. \tag{8.16}$$

Letting $\Phi \equiv \Phi(0)$ we have for the components,

$$\phi_{ij}(x) = \exp\{\mathrm{i}(p_i^\mu - p_j^\mu)x_\mu\}\, \phi_{ij}\,. \tag{8.17}$$

In this way all Green functions can be computed from the reduced action

$$\tilde{S} = \left(\frac{2\pi}{\Lambda}\right)^D \operatorname{Tr}\left\{\tfrac{1}{2}m^2\Phi^2 - \tfrac{1}{2}[P_\mu, \Phi]^2 + \frac{\lambda_3}{3!\sqrt{N}}\, \Phi^3 + \frac{\lambda_4}{4!N}\, \Phi^4\right\} \tag{8.18}$$

which is defined on just one point. [Note the factor of $(2\pi/\Lambda)^D$ in front of the action density, which makes the action dimensionally correct. The ultraviolet cut-off Λ will be taken to infinity at the end of the calculations. See below.]

Loop momenta are correctly integrated over by the following prescription: integrate over all p_i^μ 'momenta' over a normalized measure, such that the integral of 1 gives 1. This is readily accomplished by means of the ultraviolet cut-off Λ, and there is a large amount of arbitrariness here. One can, for instance, choose a measure [8.4]

$$\int^{\Lambda} \mathrm{d}P_\mu = \int_{-\Lambda/2}^{\Lambda/2} \prod_{i=1}^{N} \left[\frac{\mathrm{d}^D p_i}{\Lambda^D}\right] \left(\frac{\Lambda}{2\pi}\right)^D . \tag{8.19}$$

In this way the dimensionful parameter Λ serves two purposes simultaneously: it regularizes ultraviolet divergencies (at least as long as they are only logarithmic) and it defines a normalized measure for the momenta integrations.

For future reference it is important to realize that instead of integrating over the momenta as in eq. (8.19) (or any other normalized prescription), we could instead choose the 'quenched momenta' p_i^μ as random numbers with a uniform distribution on the D-dimensional hypercube of volume Λ^D. This

follows from the identity [8.4]

$$\lim_{N\to\infty}\left\{\frac{1}{N}\left(\frac{\Lambda}{2\pi}\right)^{D}\sum_{i=1}^{N} f[p_i^{\mu}(\Lambda)]\right\} = \int_{-\Lambda/2}^{\Lambda/2}\prod_{i=1}^{N}\frac{\mathrm{d}^D p_i}{\cancel{(2\pi)^D}}\, f(p_j^{\mu}) \tag{8.20}$$

for an arbitrary function $f(x)$. Here the hypercube has been divided into N subcubes of volume Λ^D/N, and p_i^{μ} is a chosen 'momentum' from each subcube. In the limit where these subcubes become infinitesimally small (i.e. when $N\to\infty$) we recover the continuum integration measure as in eq. (8.20). (This is simply the classical Monte Carlo representation of a multidimensional integral.) It can be shown that for finite N the standard deviation of this approximation falls as $1/\sqrt{N}$, so it unfortunately does not decrease very rapidly as N increases.

This quenched momentum prescription, which we have here only sketched loosely, was originally derived from a rather involved analysis of the planar Feynman diagrams of the $N\to\infty$ theory. Instead of pursuing this path, we shall here review how simple and elegant the derivation of the quenched momentum prescription becomes when one makes use of stochastic quantization. Our presentation will follow rather closely the original paper of Alfaro and Sakita [8.6].

Let us again for simplicity consider the action of a matrix field theory such as the one described by the action (8.8).

The Langevin equation corresponding to this action reads

$$\frac{\partial}{\partial t}\phi_{ij}(x,t) = (\partial^2 - m^2)\,\phi_{ij}(x,t) - \frac{\lambda_3}{2\sqrt{N}}\,\phi_{ij}(x,t)^2 - \frac{\lambda_4}{6N}\,\phi_{ij}(x,t)^3 + \eta_{ij}(x,t) \tag{8.21}$$

with a standard Hermitian $N\times N$ matrix noise field $\eta_{ij}(x,t)$ having correlations

$$\langle\eta_{ij}(x,t)\,\eta_{kl}(x',t')\rangle = 2\delta_{il}\delta_{jk}\delta^D(x-x')\,\delta(t-t') \tag{8.22}$$

and similarly for the higher n-point functions.

The advantage of using stochastic quntization to derive the quenched momentum prescription lies in the fact that the Boltzmann factor associated with the η-field is a simple Gaussian distribution. The proof then goes through two steps; first one shows that a reduced noise field η_{ij} given according to eq. (8.17) can reproduce the Gaussian distribution; next one shows that the solution $\phi_{ij}(x,t)$ of the Langevin equation (8.21) precisely equals the solution to eq. (8.17) for the field ϕ_{ij} itself.

The first step is rather simple. We assume that $\eta_{ij}(x,t)$ can be written in the form

$$\eta_{ij}(x,t) = \exp\{\mathrm{i}(p_i - p_j)\,x\}\,\eta_{ij}(t) \tag{8.23a}$$

where $\eta_{ij}(t)$ is a reduced noise field with distribution

$$\mathrm{D}\eta\,\exp\left\{-\tfrac{1}{4}(2\pi/\Lambda)^D\int\mathrm{d}t\,\mathrm{Tr}[\eta(t)^2]\right\}. \tag{8.23b}$$

To see that the reduction formula (8.23), together with the distribution above, correctly reproduces the Gaussian properties, consider first the U(N) invariant 2-point function:

$$\langle \mathrm{Tr}[\eta(x,t)\,\eta(x',t')]\rangle = \sum_{i,j}\int \prod_{k=1}^{N} \frac{\mathrm{d}^D p_k}{\Lambda^D} \exp\{\mathrm{i}(p_i-p_j)\cdot(x-x')\}\,\langle \eta_{ij}(t)\,\eta_{ji}(t')\rangle$$
$$= 2\left(\frac{\Lambda}{2\pi}\right)^D \delta(t-t') \sum_{i,j}\int \prod_{k=1}^{N} \frac{\mathrm{d}^D p_k}{\Lambda^D} \exp\{\mathrm{i}(p_i-p_j)\cdot(x-x')\}$$
$$\to 2N^2\,\delta^D(x-x')\,\delta(t-t') \qquad \text{as } N\to\infty \tag{8.24}$$

which agrees with eq. (8.22).

It is not hard to see how this generalizes to any $2n$-point function of the noise field. Let us, in detail, go through the calculation of the four-point function:

$$\langle \mathrm{Tr}[\eta(x_1,t_1)\,\eta(x_2,t_2)\,\eta(x_3,t_3)\,\eta(x_4,t_4)]\rangle$$
$$= \sum_{ijkl}\int \prod_{m=1}^{N} \frac{\mathrm{d}^D p_m}{\Lambda^D} \exp\{\mathrm{i}[(p_i-p_j)x_1 + (p_j-p_k)x_2 + (p_k-p_l)x_3 + (p_l-p_i)x_4]\}$$
$$\cdot\,\langle \eta_{ij}(t_1)\,\eta_{jk}(t_2)\,\eta_{kl}(t_3)\,\eta_{li}(t_4)\rangle\,. \tag{8.25}$$

It follows from eq. (8.23b) that

$$\langle \eta_{ij}(t_1)\,\eta_{kl}(t_2)\,\eta_{mn}(t_3)\,\eta_{pq}(t_4)\rangle = 4(\Lambda/2\pi)^{2D}$$
$$\cdot\,\{\delta_{jk}\delta_{il}\delta_{mq}\delta_{np}\,\delta(t_1-t_2)\,\delta(t_3-t_4) + \delta_{jp}\delta_{iq}\delta_{lm}\delta_{kn}\,\delta(t_1-t_4)\,\delta(t_2-t_3)$$
$$+\,\delta_{jm}\delta_{in}\delta_{lp}\delta_{kq}\,\delta(t_1-t_3)\,\delta(t_2-t_4)\}\,. \tag{8.26}$$

Actually, only the first two terms will contribute in the large-N limit if we compute U(N) invariant quantities. Consider, for example,

$$\langle \mathrm{Tr}[\eta(t_1)\,\eta(t_2)\,\eta(t_3)\,\eta(t_4)]\rangle$$
$$= 4(\Lambda/2\pi)^{2D}\cdot[N^3\,\delta(t_1-t_2)\,\delta(t_3-t_4) + N^3\,\delta(t_1-t_4)\,\delta(t_2-t_3) + N\,\delta(t_1-t_3)\,\delta(t_2-t_4)] \tag{8.27}$$

from the three terms in eq. (8.26). We can graphically illustrate the Wick decomposition of eq. (8.26) as shown in fig. 8.6. Note again that it is the non-planar contribution (8.6c) which is suppressed as $N\to\infty$.

Now inserting eq. (8.26) into (8.25) we find

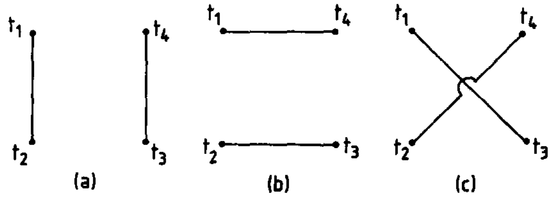


Fig. 8.6. The 3 different contractions for the 4-point function of the noise fields.

$$
\begin{aligned}
&\langle \mathrm{Tr}[\eta(x_1,t_1)\,\eta(x_2,t_2)\,\eta(x_3,t_3)\,\eta(x_4,t_4)]\rangle \\
&= 4\left(\frac{\Lambda}{2\pi}\right)^{2D} \sum_{ijkl} \Big\{ \delta(t_1-t_2)\,\delta(t_3-t_4)\,\delta_{ki} \int \prod_{m=1}^{N} \frac{\mathrm{d}^D p_m}{\Lambda^D} \exp\{\mathrm{i}[(p_i-p_j)(x_1-x_2)+(p_i-p_l)(x_3-x_4)]\} \\
&\quad + \delta(t_1-t_4)\,\delta(t_2-t_3)\,\delta_{jl} \int \prod_{m=1}^{N} \frac{\mathrm{d}^D p_m}{\Lambda^D} \exp\{\mathrm{i}[(p_i-p_j)\,(x_1-x_4)+(p_j-p_k)(x_2-x_3)]\} \\
&\quad + \delta(t_1-t_3)\,\delta(t_2-t_4)\,\delta_{ij} \int \prod_{m=1}^{N} \frac{\mathrm{d}^D p_m}{\Lambda^D} \exp\{\mathrm{i}[(p_i-p_j)\,(x_1-x_3)]\} \Big\}
\end{aligned}
\tag{8.28}
$$

and hence, as $N\to\infty$:

$$
\begin{aligned}
&\langle \mathrm{Tr}[\eta(x_1,t_1)\,\eta(x_2,t_2)\,\eta(x_3,t_3)\eta(x_4,t_4)\rangle \to 4N^3[\delta(t_1-t_2)\,\delta(t_3-t_4)\,\delta^D(x_1-x_2)\,\delta^D(x_3-x_4) \\
&\quad + \delta(t_1-t_4)\,\delta(t_2-t_3)\,\delta^D(x_1-x_4)\,\delta^D(x_2-x_3)]
\end{aligned}
\tag{8.29}
$$

which agrees exactly with what one obtains in the limit $N\to\infty$ of the full theory.

There is no need to go further and check higher $2n$-point functions; the manipulations are completely similar. This, then, demonstrates that as long as we compute U(N) invariant Green functions the reduced noise field given by eq. (8.23) serves as a valid change of variables in the $N\to\infty$ limit. We now notice that a $\phi_{ij}(t)$ determined by

$$
\phi_{ij}(x,t) = \exp\{\mathrm{i}(p_i-p_j)\,x\}\;\phi_{ij}(t)
\tag{8.30}
$$

clearly is a solution of the 'reduced' Langevin equation

$$
\frac{\partial}{\partial t}\,\phi_{ij}(t) = -[(p_i-p_j)^2+m^2]\,\phi_{ij}(t) - \frac{\lambda_3}{2\sqrt{N}}\,\phi_{ij}^2(t) - \frac{\lambda_4}{3N}\,\phi_{ij}^3(t) + \eta_{ij}(t)\,.
\tag{8.31}
$$

It is also evident that for any U(N) invariant Green function

$$
\lim_{t=t_1=\cdots t_k\to\infty} \langle \mathrm{Tr}[\Phi(t_1)\cdots\Phi(t_k)]\rangle = \langle \mathrm{Tr}[\Phi^k]\rangle
\tag{8.32}
$$

where the vacuum expectation value is determined with respect to the *reduced* action

$$
\tilde{S} = \left(\frac{2\pi}{\Lambda}\right)^D \left\{ \tfrac{1}{2}\,\mathrm{Tr}[m^2\Phi^2 - [P_\mu,\Phi]^2] + \frac{\lambda_3}{3!\sqrt{N}}\,\mathrm{Tr}\,\Phi^3 + \frac{\lambda_4}{4!N}\,\mathrm{Tr}\,\Phi^4 \right\}.
\tag{8.33}
$$

Then

$$
\begin{aligned}
&\langle \mathrm{Tr}[\Phi(x_1)\cdots\Phi(x_k)]\rangle = \lim_{t\to\infty} \langle \mathrm{Tr}[\Phi(x_1,t)\cdots\Phi(x_k,t)]\rangle \\
&= \lim_{t\to\infty} \int \prod \frac{\mathrm{d}^D p}{\Lambda^D} \sum_{i,j,\ldots} \exp\{\mathrm{i}(p_i-p_j)x_1\}\cdots\langle \phi_{ij}(t)\cdots\phi_{mi}(t)\rangle \\
&= \int \prod \left(\frac{\mathrm{d}^D p}{\Lambda^D}\right) \sum_{i,j,\ldots} \exp\{\mathrm{i}(p_i-p_j)x_1\}\cdots\langle \phi_{ij}\cdots\phi_{mi}\rangle
\end{aligned}
\tag{8.34}
$$

which, together with eqs. (8.30) and (8.33), precisely agrees with the quenched momentum prescription of Gross and Kitazawa.

The quenched momentum prescription can also be applied to a Yang–Mills theory interacting with matter fields, and in particular with fermions. Let us here focus our attention on theories with fermions, and derive the quenched momentum prescription for such fields using again stochastic quantization [8.15].

Usually matter fields are ignored within the context of large-N physics. This is due to the fact that if one assigns quark fields to the *fundamental* representation of the gauge group, then their contribution to all SU(N) invariant quantities is suppressed by powers of $1/N$. This is intuitively easy to understand, since there are $\sim N^2$ gluons at large N, but only N quarks, if the quarks are put in the fundamental representation of SU(N). Hence, in the formal $N \to \infty$ limit such dynamical fermions will be completely neglected. This, of course, also implies that one has rather heavily mutilated the original theory (QCD) by taking the $N \to \infty$ limit.

There is, however, an alternative to this standard $1/N$ expansion picture: the topological $1/N$ expansion [8.16, 8.11]. Here both the limits $N \to \infty$ and $N_f \to \infty$ (N_f being the number of flavours) are taken, the ratio $\rho = N_f/N$ being kept fixed. Furthermore, as in the standard $1/N$ expansion, also the coupling $g\sqrt{N}$ is kept fixed.

In the topological $1/N$ expansion the $N = \infty$ theory is still *planar*, but internal quark loops are no longer suppressed. In this sense dynamical fermions (quarks) are just as fully incorporated into the model as in the original SU(3) theory.

For definiteness, let us consider the simple fermion Langevin system given by the kernel $K(x, y)$ of eq. (6.13) and noise fields $\theta(x, t)$ and $\bar{\theta}(x, t)$, as discussed in section 6. The results are readily generalized to other fermionic Langevin equations.

We then have, for a general action S,

$$\frac{\partial}{\partial t} \Psi(x, t) = -\int \mathrm{d}^D y\, K(x, y) \frac{\delta S}{\delta \bar{\Psi}(x, t)} + \theta(x, t) \tag{8.35}$$

and the corresponding equation for $\bar{\Psi}(x, t)$. The derivation of the quenched momentum prescription now proceeds just as in the bosonic case. We start by introducing a set of random *numbers* p^i_μ (i is again a colour index, and μ is the Lorentz index) with a flat probability distribution on the D-dimensional hypercube $[-\Lambda/2, \Lambda/2]^D$. Λ will again simultaneously play the role of a momentum cut-off; it will eventually be sent to infinity.

Given a set of spinor fields $\varepsilon(t)$ and $\bar{\varepsilon}(t)$, and an associated measure

$$\mathrm{D}\varepsilon\, \mathrm{D}\bar{\varepsilon}\, \exp\left\{-\tfrac{1}{2} \int \mathrm{d}t \left(\frac{2\pi}{\Lambda}\right)^D \sum_i \bar{\varepsilon}^i (-\not{p}^i + m)^{-1} \varepsilon^i \right\}$$

we wish to demonstrate that (with α being the spinor index)

$$\theta^i_\alpha(x, t) = \exp(\mathrm{i}p^i x)\, \varepsilon^i_\alpha(t) \tag{8.36a}$$

$$\bar{\theta}^i_\alpha(x, t) = \bar{\varepsilon}^i_\alpha(t) \exp(-\mathrm{i}p^i x) \tag{8.36b}$$

is a valid identification in the large-N limit, provided one computes SU(N) invariant quantities.

For the 2-point function this is straightforward. We have

$$\sum_i \langle \theta^i_\alpha(x,t)\,\bar\theta^\beta_i(x',t')\rangle = \sum_i \left(\frac{\Lambda}{2\pi}\right)^D \int_{-\Lambda/2}^{\Lambda/2} \prod_{j=1}^N \frac{\mathrm{d}^D p^j}{\Lambda^D} \exp(\mathrm{i}p^i x)\exp(-\mathrm{i}p^i x')\,\langle \varepsilon^i_\alpha(t)\,\bar\varepsilon^\beta_i(t')\rangle$$

$$= 2\sum \left(\frac{\Lambda}{2\pi}\right)^D \int_{-\Lambda/2}^{\Lambda/2} \prod_{j=1}^N \frac{\mathrm{d}^D p^j}{\Lambda^D} [-\not p^i + m]_{\alpha\beta} \exp\{\mathrm{i}p^i(x-x')\}\,\delta(t-t')$$

$$= 2N[\mathrm{i}\not\partial_x + m]_{\alpha\beta}\,\delta^D(x-x')\,\delta(t-t') \tag{8.37}$$

as N, $\Lambda\to\infty$. This result agrees exactly with what one obtains from the correlation

$$\langle \theta^i_\alpha(x,t)\,\bar\theta^\beta_j(x',t')\rangle = 2\delta^{ij}K(x,x')_{\alpha\beta}\delta(t-t') \tag{8.38}$$

of the unreduced theory.

Similarly, consider a 4-point function:

$$\sum_{ij} \langle \theta^i_\alpha(x_1,t_1)\,\theta^j_\beta(x_2,t_2)\,\bar\theta^\gamma_i(x_3,t_3)\,\bar\theta^\delta_j(x_4,t_4)\rangle$$

$$= \sum_{i,j} \left(\frac{\Lambda}{2\pi}\right)^{2D} \int_{-\Lambda/2}^{\Lambda/2} \prod_{j=1}^N \frac{\mathrm{d}^D p^j}{\Lambda^D} \exp(\mathrm{i}p^i x_1)\exp(\mathrm{i}p^j x_2)\exp(-\mathrm{i}p^i x_3)\exp(-\mathrm{i}p^j x_4)$$

$$\times\langle \varepsilon^i_\alpha(t_1)\,\varepsilon^j_\beta(t_2)\,\bar\varepsilon^\gamma_i(t_3)\,\bar\varepsilon^\delta_j(t_4)\rangle$$

$$= 4\sum_{ij} \left(\frac{\Lambda}{2\pi}\right)^{2D} \int_{-\Lambda/2}^{\Lambda/2} \prod_{j=1}^N \frac{\mathrm{d}^D p^j}{\Lambda^D} \{[\mathrm{i}\not\partial_{x_1} + m]_{\alpha\delta}[\mathrm{i}\not\partial_{x_2} + m]_{\beta\gamma}\delta^{ij}\delta(t_2-t_3)\,\delta(t_1-t_4)$$

$$-[\mathrm{i}\not\partial_{x_1} + m]_{\alpha\gamma}\,[\mathrm{i}\not\partial_{x_2} + m]_{\beta\gamma}\,\delta^{ii}\,\delta^{jj}\,\delta(t_1-t_3)\,\delta(t_2-t_4)\}\exp\{\mathrm{i}p^i(x_1-x_3) + \mathrm{i}p^j(x_2-x_4)\}$$

$$\simeq -4N^2\,K(x_1,x_3)_{\alpha\gamma}\,K(x_2,x_4)_{\beta\delta}\,\delta(t_1-t_3)\,\delta(t_2-t_4)$$

$$+4N(\Lambda/2\pi)^D[\mathrm{i}\not\partial_{x_1} + m]_{\alpha\delta}\,[\mathrm{i}\not\partial_{x_2} + m]_{\beta\gamma}\,\delta^D(x_1+x_2-x_3-x_4)\,\delta(t_2-t_3)\,\delta(t_1-t_4)$$

$$\simeq -4N^2\,K(x_1,x_3)_{\alpha\gamma}\,K(x_2,x_4)_{\beta\delta}\,\delta(t_1-t_3)\,\delta(t_2-t_4) \qquad \text{as } N\to\infty \tag{8.39}$$

where we have used the fact that

$$\langle \varepsilon^i_\alpha(t_1)\,\varepsilon^j_\beta(t_2)\,\bar\varepsilon^\gamma_i(t_3)\,\bar\varepsilon^\delta_j(t_4) = 4[-\not p^i + m]_{\alpha\delta}\,[-\not p^j + m]_{\beta\gamma}\,\delta^{ij}$$

$$\cdot\,\delta(t_2-t_3)\,\delta(t_1-t_4) - 4[-\not p^i + m]_{\alpha\gamma}\,[-\not p^j + m]_{\beta\delta}\,\delta^{ii}\,\delta^{jj}\,\delta(t_1-t_3)\,\delta(t_2-t_4)\,. \tag{8.40}$$

The large-N limit of eq. (8.39) is precisely what one would obtain from the unreduced θ, $\bar\theta$-measure. Obviously, this property will persist for higher $2n$-point functions, provided that $N > n$. This is just as in the bosonic case. Note also that, as in the bosonic case, it is crucial that the limit $\Lambda\to\infty$ is taken *after* the $N\to\infty$ limit.

The rest of the derivation of the quenched momentum prescription for fermions proceeds similarly. Letting

$$\psi^i_\alpha(x, t) = \exp(\mathrm{i}p^i x)\, \psi^i_\alpha(t) \tag{8.41a}$$

$$\bar{\psi}^i_\alpha(x, t) = \bar{\psi}^i_\alpha(t) \exp(-\mathrm{i}p^i x) \tag{8.41b}$$

leads to the Langevin equation

$$\frac{\partial}{\partial t}\, \psi^i_\alpha(t) = -((p^i)^2 + m^2)\, \psi^i_\alpha(t) - \text{Int. terms} + \varepsilon^i_\alpha(t) \tag{8.42}$$

and it follows, as in the bosonic case, that one could equally well have started with the *reduced* action

$$\tilde{S}[\psi, \bar{\psi}] = \left(\frac{\Lambda}{2\pi}\right)^D \sum_i \bar{\psi}_i[\not{p}^i + m + \text{Int. terms}]\psi_i\,. \tag{8.43}$$

The identifications (8.41) and (8.43) lead precisely to the quenched momentum prescription for fermions.

8.3. *Quenched master fields*

Stochastic quantization can be used to more than just to rederive the quenched momentum prescription, which we discussed in the previous section. Greensite and Halpern [8.7] noticed that if one takes quenching one step further and introduces a 'quenched momentum prescription' also for the 'momentum' associated with the fictitious time direction, the Langevin equation reduces in the large-t limit to an algebraic equation. This completely quenched Langevin equation leads to a 'classical' stochastic solution of the large-N theory, which the authors dubbed the 'Quenched Master Field'. Although it is perhaps not a Master Field in the original sense of Witten [8.12], it does have the remarkable property of reducing vacuum expectation values to direct products of stochastic 'classical' matrix field variables. The Quenched Master Field can be obtained for simple large-N matrix models, as well as for large-N QCD [8.17], and is in this sense completely general for large-N theories.

To start, we go back to eq. (8.23) and investigate what happens if we extend the quenching prescription to the fictitious time direction as well, i.e. let

$$\eta_{ij}(x, t) = \exp\{\mathrm{i}(\tilde{p}_i - \tilde{p}_j)t + \mathrm{i}(p_i - p_j)x\}\, \eta_{ij} \tag{8.44}$$

where the reduced noise 'field' η_{ij} (in zero dimensions) has a Gaussian distribution:

$$\mathrm{D}\eta \exp\{-\tfrac{1}{4}(2\pi/\Lambda)^D(2\pi/\tilde{\Lambda})\,\mathrm{Tr}\,[\eta^2]\}\,. \tag{8.45}$$

(Note that $\tilde{\Lambda}$ is of dimension 2, whereas Λ is of dimension 1.)

It follows, completely analogously to the discussion in section 8.2, that the reduced noise field (8.44) can correctly reproduce the U(N) invariant Green functions of the η fields in the limit $N \to \infty$.

A master noise field will eventually generate the Quenched Master Field from the Langevin equation. We have already seen, in the previous section, how the noise field quenched in just the

ordinary D-dimensional space generates the quenched field of the full theory. The next step is to see which field configurations will be generated by a completely quenched noise field of the form (8.44).

Not surprisingly, one finds [8.7] that for $t\to\infty$ the solution to the Langevin equation with a noise field of the form (8.44) can be written

$$\phi_{ij}(x,t)=\exp\{\mathrm{i}(\tilde{p}_i-\tilde{p}_j)t+\mathrm{i}(p_i-p_j)x\}\,\phi_{ij} \tag{8.46}$$

as $t\to\infty$. Of course, this particular choice of $\phi_{ij}(x,t)$ is incompatible with our freedom in choosing initial conditions of $\phi_{ij}(x,0)$ [8.18]. Nevertheless, since the initial conditions can be chosen arbitrarily, the particular choice

$$\phi_{ij}(x,0)=\exp\{\mathrm{i}(p_i-p_j)x\}\,\phi_{ij} \tag{8.47}$$

should be admissible. Secondly, one can show that corrections to eq. (8.46) will be damped exponentially as $t\to\infty$ when $N=\infty$.

Then, for sufficiently large t, the Langevin equation reduces to an algebraic equation. For an action of the form (8.8) this algebraic matrix equation reads

$$\mathrm{i}(\tilde{p}_i-\tilde{p}_j)\phi_{ij}=-[(p^i_\mu-p^j_\mu)^2+m^2]\,\phi_{ij}-\frac{\lambda_3}{2\sqrt{N}}\,\phi^2_{ij}-\frac{\lambda_4}{3!N}\,\phi^3_{ij}+\eta_{ij} \tag{8.48}$$

with $\phi_{ij}(x,t)$ determined from ϕ_{ij} *via* eq. (8.46).

The first step towards determining the Quenched Master Field now consists in finding a 'master noise field' η^*_{ij} which satisfies

$$\langle\mathrm{Tr}[\eta(x_1,t_1)\cdots\eta(x_k,t_k)]\rangle=\mathrm{Tr}[\eta^*(x_1,t_1)\cdots\eta^*(x_k,t_k)] \tag{8.49}$$

in the limit $N\to\infty$. This is achieved in ref. [8.7] in the following way: First one finds a reduced master noise field (matrix) such that

$$\begin{aligned}\langle\mathrm{Tr}[\eta(x_1,t_1)\cdots\eta(x_k,t_k)]\rangle&=\sum_{i_1,\dots i_k}\int_{-\Lambda/2}^{\Lambda/2}\prod_{m=1}^{N}\left(\frac{\mathrm{d}^Dp_m}{\Lambda^D}\right)\int_{-\tilde{\Lambda}/2}^{\tilde{\Lambda}/2}\prod_{m=1}^{N}\frac{\mathrm{d}\tilde{p}_m}{\tilde{\Lambda}}\\&\quad\cdot\exp\{\mathrm{i}(\tilde{p}_{i_1}-\tilde{p}_{i_2})t+\mathrm{i}(p_{i_1}-p_{i_2})x\}\exp\{\mathrm{i}(\tilde{p}_{i_2}-\tilde{p}_{i_3})t+\mathrm{i}(p_{i_2}-p_{i_3})x\}\cdots\{\eta^*_{i_1i_2}\cdots\eta^*_{i_ki_1}\}\end{aligned} \tag{8.50}$$

where the master noise field has the property that

$$\langle\mathrm{Tr}[\eta(x_1,t_1)\cdots\eta(x_k,t_k)]\rangle=\mathrm{Tr}[\eta^*(x_1,t_1)\cdots\eta^*(x_k,t_k)] \tag{8.51}$$

when $N=\infty$.

Next one picks a set of 'master momenta' $\tilde{p}^*_i$ and p^*_i such that the master noise field of the unreduced theory is

$$\eta^*_{ij}(x,t)=\exp\{\mathrm{i}(\tilde{p}^*_i-\tilde{p}^*_j)t+\mathrm{i}(p^*_i-p^*_j)x\}\,\eta^*_{ij}\,. \tag{8.52}$$

As mentioned in section 8.2, instead of performing the integrations in eq. (8.50), we could instead have chosen a set of random numbers with uniform distribution on the hypercube and used property (8.20) to determine the integral. This is behind the proof of Greensite and Halpern [8.7], which shows that at $N=\infty$ eq. (8.51) is satisfied for the master noise field of eq. (8.52). We refer the reader to ref. [8.7] for details on how the master noise field actually can be explicitly constructed. Here let us just summarize the properties of the field $\eta^*_{ij}(x,t)$: one can choose a (reduced) matrix η^*_{ij} as a matrix of Gaussian random numbers, i.e. with a probability distribution

$$\mathrm{D}\eta^* \exp\{-\tfrac{1}{4}(2\pi/\Lambda)^D(2\pi/\tilde{\Lambda})\,\mathrm{Tr}[\eta^{*2}]\}\,.$$

Then one takes N random numbers ('momenta') distributed uniformly on the hypercube $[-\Lambda/2, \Lambda/2]^D[-\tilde{\Lambda}/2, \Lambda/2]$. From these momenta $\{\tilde{p}^*_i, p^*_i\}$ and the random matrix η^*_{ij} of Gaussian distribution, one constructs $\eta^*_{ij}(x,t)$ according to eq. (8.52).

Substituting the master momenta p^{i*}_μ and master noise fields η^*_{ij} one now finds the Master Field $\phi^*_{ij}(x,t)$ by

$$\phi^*_{ij}(x,t) = \exp\{\mathrm{i}(\tilde{p}^*_i - \tilde{p}^*_j)t + \mathrm{i}(p^*_i - p^*_j)x\}\;\phi^*_{ij}(p^*, \eta^*)\,. \tag{8.53}$$

This field has the property

$$\langle \mathrm{Tr}[\phi(x_1,t)\cdots\phi(x_k,t)]\rangle = \mathrm{Tr}[\phi^*(x_1,t)\cdots\phi^*(x_k,t)] \tag{8.54}$$

and finally taking the limit $t\to\infty$ one recovers the standard large-N vacuum expectation values.

For practical calculations the solution $\phi^*_{ij}(x,t)$ of the Langevin eq. (8.48) [combined with eq. (8.53)] is far from trivial. Explicit, or even just approximate, solutions have so far only been obtained for low dimensional matrix field models [8.19–8.22]. A variational solution [8.19], which has been quite successful for such models, works also for finite-N theories [8.23] even though, for such models, there does not exist any Master Field.

9. Stochastic quantization in Minkowski space

In the preceding sections we have always restricted ourselves to Euclidean space-time. In this section we will discuss a possible modification of the original approach, which should allow us to stochastically quantize directly in Minkowski space-time. By this we mean that we will try to obtain the Minkowski space Green functions as equilibrium limits of correlation functions with respect to specific stochastic processes, without having to define them by Wick rotation of the Euclidean averages.

We start our discussion by investigating scalar fields. The main idea [9.1, 9.2] is to replace the Langevin equation for the Euclidean scalar field ϕ

$$\dot{\phi} = -\delta S_\mathrm{E}/\delta\phi + \eta \tag{9.1}$$

by the following generalized Langevin equation for the field ϕ in Minkowski space:

$$\dot{\phi} = \mathrm{i}\,\delta S/\delta\phi + \eta\,. \tag{9.2}$$

Here S_E and S denote the actions for a self-interacting scalar field in Euclidean and Minkowski space, respectively,

$$S_E = \int d^4x \left[(\partial_\mu \phi)(\partial_\mu \phi) + m^2\phi^2 + \frac{\lambda}{4!}\,\phi^4 \right] \tag{9.3}$$

$$S = \int d^4x \left[(\partial_\mu \phi)(\partial^\mu \phi) - m^2\phi^2 - \frac{\lambda}{4!}\,\phi^4 \right] \tag{9.4}$$

where we defined covariant and contravariant indices with respect to the metric $g_{\mu\nu}$,

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{9.5}$$

and η is a Gaussian white noise with the (usual) correlations

$$\langle \eta(x, t)\, \eta(x', t') \rangle = 2\,\delta(x - x')\,\delta(t - t') \tag{9.6}$$

following from the Gaussian distribution

$$D\eta \exp\left\{ -\tfrac{1}{4} \int \eta^2(x, t)\, d^4x\, dt \right\}. \tag{9.7}$$

From the complex drift term in (9.2) it is obvious that the stochastic process ϕ is generally complex and we will show that the existence of an equilibrium limit for $t \to \infty$ constitutes a more subtle problem than in the Euclidean case.

To be more specific we study the Langevin equation corresponding to (9.2) and (9.4)

$$\dot{\phi} = i(k^2 - m^2)\phi - i\,\frac{\lambda}{3!}\,\phi^3 + \eta\,. \tag{9.8}$$

Considering first the free case $\lambda = 0$ the unique solution with $\phi(t, 0) = 0$ is simply given by

$$\phi(k, t) = \int_0^t \exp\{i(k^2 - m^2)(t - \tau)\}\, \eta(k, \tau) \tag{9.9}$$

and we evaluate straightforwardly the 2-point correlation

$$\begin{aligned} \langle \phi(k, t)\, \phi(k', t) \rangle &= 2(2\pi)^4 \int_0^t d\tau \exp\{2i(k^2 - m^2)(t - \tau)\}\, \delta^4(k + k') \\ &= (2\pi)^4\, \delta^4(k + k')\, \frac{i}{k^2 - m^2}\, [1 - \exp\{2i(k^2 - m^2)t\}]\,. \end{aligned} \tag{9.10}$$

It seems as if we have reached a fatal point in our discussion, as the strict limit $t \to \infty$ obviously does

not exist in (9.10). However, help is obtained by recalling that Minkowski-space Green functions in fact have to be interpreted in the distributional sense (see, for example, ref. [9.3]). This implies that eq. (9.10) should actually be integrated over momenta k and k', after having multiplied it by a (momentum space) test function. If we thus generalize the concept of taking the equilibrium limit we can indeed obtain relaxation of the stochastic process. There is an apparent ambiguity in this interpretation, as $1/(k^2-m^2)$ may be identified with the distribution

$$\frac{1}{k^2-m^2}=\frac{\alpha}{k^2-m^2+\mathrm{i}0}+\frac{1-\alpha}{k^2-m^2-\mathrm{i}0} \tag{9.11}$$

where α is an arbitrary constant (see, for example, [9.4]). It follows however, irrespective of the value of α, that

$$\frac{1}{k^2-m^2}\,[1-\exp\{\mathrm{i}(k^2-m^2)2t\}]=\mathrm{P}\,\frac{1}{k^2-m^2}\,[1-\exp\{\mathrm{i}(k^2-m^2)2t\}] \tag{9.12}$$

where P denotes the principal value. In the distributional interpretation it holds further that

$$\lim_{s\to\infty}\mathrm{e}^{\mathrm{i}xs}=0\,,\qquad \lim_{s\to\infty}\mathrm{P}\Big(\frac{1}{s}\Big)\,\mathrm{e}^{\mathrm{i}xs}=\mathrm{i}\pi\delta(x) \tag{9.13}$$

so that the large time limit of (9.10), interpreted in the distributional sense, is given by

$$\begin{aligned}\mathrm{d.}\lim_{t\to\infty}\langle\phi(k,t)\,\phi(k',t)\rangle&=(2\pi)^4\delta^4(k+k')\Big[\mathrm{P}\Big(\frac{1}{k^2-m^2}\Big)-\mathrm{i}\pi\,\delta(k^2-m^2)\Big]\\&\equiv(2\pi)^4\,\delta^4(k+k')\,\frac{\mathrm{i}}{k^2-m^2+\mathrm{i}0}\,.\end{aligned} \tag{9.14}$$

As promised above, we obtained directly the Feynman propagator in the generalized equilibrium limit of the stochastic process (9.8)

As the consistency of the above interpretation has not been checked to higher orders, let us point out a less sophisticated alternative way of arriving at (9.14). This is to add a negative imaginary mass term $-(\mathrm{i}/2)\varepsilon\phi^2$ to the action (9.4) and let ε tend to zero after all calculations have been performed. The Langevin equation reads in this case

$$\dot\phi=\mathrm{i}(k^2+\mathrm{i}\varepsilon-m^2)\phi-\frac{\lambda}{3!}\,\phi^3+\eta \tag{9.15}$$

and all oscillating exponentials [as, for example, the second term in (9.10)] are damped, since k^2-m^2 is replaced by $k^2-m^2+\mathrm{i}\varepsilon$, everywhere. It is actually straightforward to develop the perturbative solution to (9.15) in analogy to section 3 and repeat all the discussion on stochastic diagrams (and, for example, the perturbative equivalence proofs) for Minkowski space. We do not wish to go into the details here (see, for example, [9.1] and [9.5]).

It is of interest to study the Fokker–Planck equation corresponding to (9.2) and extract its equilibrium distribution. The situation is, however, much more complicated than in the Euclidean case, because the fields are now complex

$$\phi=\phi_{\mathrm{Re}}+\mathrm{i}\phi_{\mathrm{Im}} \tag{9.16}$$

(so that the Fokker–Planck equation becomes a partial functional differential equation, see below) and because we have to implement the generalized procedure for obtaining the equilibrium limit.

For simplicity we would therefore like to discuss the Fokker–Planck equation associated with the free scalar field Langevin equation and the iε prescription included. It reads

$$\dot{\phi}_{\mathrm{Re}} = -\varepsilon\phi_{\mathrm{Re}} - (k^2 - m^2)\phi_{\mathrm{Im}} + \eta \tag{9.17}$$

$$\dot{\phi}_{\mathrm{Im}} = (k^2 - m^2)\,\phi_{\mathrm{Re}} - \varepsilon\phi_{\mathrm{Im}} \tag{9.18}$$

where we used a real noise η. The associated Fokker–Planck equation is obtained straightforwardly as (see section 2)

$$\begin{aligned}\dot{P}[\phi_{\mathrm{Re}}, \phi_{\mathrm{Im}}, t] = \int \mathrm{d}^4x \Big\{ &\frac{\delta}{\delta\phi_{\mathrm{Re}}}\Big[\varepsilon\phi_{\mathrm{Re}} + (-\partial^2 - m^2)\,\phi_{\mathrm{Im}} + \frac{\delta}{\delta\phi_{\mathrm{Re}}}\Big] \\ &+ \frac{\delta}{\delta\phi_{\mathrm{Im}}}\big[-(-\partial^2 - m^2)\,\phi_{\mathrm{Re}} + \varepsilon\phi_{\mathrm{Im}}\big]\Big\}\, P[\phi_{\mathrm{Re}}, \phi_{\mathrm{Im}}, t]\end{aligned} \tag{9.19}$$

and can (after some efforts) be solved explicitly [9.6]. For large t the following equilibrium distribution emerges (see also [9.7, 9.8] and section 11)

$$P^{\mathrm{eq}}[\phi_{\mathrm{Re}}, \phi_{\mathrm{Im}}] = \exp\Big\{-\varepsilon \int \mathrm{d}^q k \Big[\phi_{\mathrm{Re}}^2 + \Big(1 + \frac{2\varepsilon^2}{k^2 - m^2}\Big)\,\phi_{\mathrm{Im}}^2 - \frac{2\varepsilon}{k^2 - m^2}\,\phi_{\mathrm{Re}}\phi_{\mathrm{Im}}\Big]\Big\} \tag{9.20}$$

which indeed implies the standard Feynman propagator

$$\lim_{\substack{t\to\infty \\ \varepsilon\to 0}} \langle \phi(k,t)\,\phi(k',t)\rangle = \int \mathrm{D}\phi_{\mathrm{Re}}\,\mathrm{D}\phi_{\mathrm{Im}}\, P^{\mathrm{eq}}\,\phi(k)\,\phi(k') = \frac{\mathrm{i}}{k^2 + \mathrm{i}0}\,\delta^4(k + k')\,. \tag{9.21}$$

Note that the real valued equilibrium distribution (9.20) actually allows the Minkowski-space Feynman propagator to be obtained without involving the standard complex exp(iS) path integral expression.

Let us now try to formulate stochastic quantization in Minkowski space for gauge fields. Here the corresponding actions are, respectively,

$$S_E = \int \mathrm{d}^4x\, \tfrac{1}{4} F^a_{\mu\nu}\, F^a_{\mu\nu} \tag{9.22}$$

$$S = -\int \mathrm{d}^4x\, \tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \tag{9.23}$$

and we have to find an appropriate generalization of

$$\dot{A}_\mu = -\delta S_E/\delta A_\mu + \eta_\mu \tag{9.24}$$

and

$$\langle \eta_\mu\, \eta_\nu \rangle = \delta_{\mu\nu}\, 2\delta^4(x - x')\,\delta(t - t') \tag{9.25}$$

to Minkowski space. The main observation is that we have to consider a Langevin equation which is transforming covariantly under Lorentz transformations, upon which we set [substituting the Minkowskian $-g_{\mu\nu}$ for the Euclidean $\delta_{\mu\nu}$ and inserting a factor of i as in (9.2)]

$$\dot{A}_\mu = \mathrm{i} g_{\mu\nu} \frac{\delta S}{\delta A_\nu} + \eta_\mu \tag{9.26}$$

and also

$$\langle \eta_\mu \eta_\nu \rangle = -g_{\mu\nu}\, 2\, \delta^4(x-x')\, \delta(t-t') . \tag{9.27}$$

We observe that (9.27) does not permit an immediate classical probabilistic interpretation since the metric $-g_{\mu\nu}$ is not positive definite. This situation reminds us of the indefiniteness problem in gravity and string field theories, see section 5. Indeed, it has been put forward in [9.9] that one may introduce a complex noise

$$\eta_\mu = \eta_\mu^{\mathrm{R}} + \mathrm{i}\eta_\mu^{\mathrm{I}} \tag{9.28}$$

where

$$\begin{aligned}
&\langle \eta_\mu^{\mathrm{R}}(x,t)\, \eta_\nu^{\mathrm{R}}(x',t') \rangle = 2\, R_{\mu\nu}(x-x')\, \delta(t-t') \\
&\langle \eta_\mu^{\mathrm{I}}(x,t)\, \eta_\nu^{\mathrm{I}}(x',t') \rangle = 2\, I_{\mu\nu}(x-x')\, \delta(t-t') \\
&\langle \eta_\mu^{\mathrm{I}}(x,t)\, \eta_\nu^{\mathrm{R}}(x',t') \rangle = 0
\end{aligned} \tag{9.29}$$

such that $R_{\mu\nu}$ and $I_{\mu\nu}$ have only positive eigenvalues and

$$R_{\mu\nu}(x-x') - I_{\mu\nu}(x-x') = -g_{\mu\nu}\, \delta^4(x-x') . \tag{9.30}$$

In this respect we would like to add, however, that the formulation of the stochastic quantization for gauge fields in the axial gauge (see section 4.1) allows A_0 and η_0 to be zero (when choosing $n=[1,0,0,0]$) so that in this case a classical probabilistic interpretation can be maintained.

We have emphasized the role of Feynman's iε in our analysis. If we add such an imaginary mass term to (9.26) the free part of the drift term becomes in fact invertible for finite ε. This inverse reads

$$\frac{g^{\mu\nu} + p^\mu p^\nu / \mathrm{i}\varepsilon}{p^2 + \mathrm{i}\varepsilon} . \tag{9.31}$$

We might now feel inclined to perform the $t \to \infty$ limit before letting $\varepsilon \to 0$. In this case the invertible free part of the drift term makes it possible to apply the equivalence proof of scalar field theories (see section 3): The sums of stochastic diagrams yield ordinary Feynman diagrams with the propagator (9.31). As a check one may now repeat the calculations of $\langle FF \rangle$ and quark–quark scattering in this formalism [9.5]. When finally $\varepsilon \to 0$ the negative powers of ε cancel in these gauge invariant quantities and leave behind a term which is just what is normally associated with ghost contributions. However, *it is too small by a factor of two.*

This discrepancy reflects the well-known fact [9.10] that the zero mass limit of a massive 'Yang–Mills' theory does not coincide with the massless Yang–Mills theory. In the present context the problem arises because we have taken the equilibrium limit of the stochastic process before $\varepsilon \to 0^+$ [which effectively corresponds to giving the gauge field a (complex) mass].

We conclude that we have to perform the limit $\varepsilon \to 0^+$ before allowing the approach to equilibrium $t \to \infty$.

Let us close this section by mentioning that the above procedure can be straightforwardly generalized to the stochastic quantization of Einstein gravity in pseudo-Riemannian space (see [9.11]); for further discussion on numerical studies we refer the reader to [9.12] and to section 11.

10. Stochastic regularization

An interesting generalization of the Parisi–Wu scheme of stochastic quantization suggests a new regularization method of quantum field theory. We will here mainly concentrate on two such regularization schemes (corresponding to a non-Markovian [10.1, 10.2] and a Markovian [10.3] generalization, respectively) which preserve throughout all the symmetries of the unregularized theory under study, including gauge symmetry, chiral symmetry, and supersymmetry.

We will explicitly apply the new concept of stochastic regularization by calculating the vacuum polarization tensor in QED [10.4]. We end this section by a discussion of Ward identities within the stochastic framework [10.5, 10.6].

10.1. Basic concepts

In the preceding sections (especially in sections 3 and 4) when we were discussing stochastic perturbation theory, we always tacitly assumed that individual stochastic diagrams were conveniently regularized. In most of the cases we in fact allowed for an arbitrary dimension n of physical space-time so that we could, in principle, apply dimensional regularization. In this context it is important to know how the naive degree of divergence of a stochastic diagram relates to that of the corresponding Feynman diagram: To keep the argument simple let us discuss just scalar field theory with cubic self-interaction. We recall (see section 3.2.1) that the large time-limit contribution of a time-ordered stochastic diagram is given by

$$\int \prod_{\text{loops}} d^n p \cdot \prod_{\text{crosses}} \frac{1}{p^2} \prod_{\text{vertices}} \frac{\lambda}{\sum_{W_k} p^2} . \tag{10.1}$$

Here λ is the scalar self-coupling constant and W_k belongs to a specific set of momenta [see eq. (3.54)] associated with he kth vertex. Let us assume that we consider a stochastic diagram with L loops, N_c crossed lines, and N vertices. Let us further assume that this stochastic diagram has E_0 uncrossed external lines ($E_0 \geq 1$) and E_c crossed external lines.

One can easily find from the discussion of section 3.2.1 that for this ϕ^3 theory the number of crosses in a stochastic diagram is

$$N_c = \tfrac{1}{2}(E_0 + E_c + N) \tag{10.2}$$

so that the number of crosses on internal lines is

$$N_c^I = N_c - E_c\,. \tag{10.3}$$

It then follows from the standard definition of the naive degree of divergence and from (10.1) that

$$w_{st} = nL - 2N_c^I - 2(N-1)\,. \tag{10.4}$$

Let us remark that the last fictitious time integration brings down in the denominator just (external momenta)^2, which explains the last term on the right-hand side of (10.4). We may express w_{st} by the definitions (10.2) and (10.3) further as

$$w_{st} = nL + 2 - E_0 + E_c - 3N\,. \tag{10.5}$$

On the other hand, the degree of divergence of the corresponding Feynman diagram is given by

$$w = nL + E - 3N \tag{10.6}$$

where $E = E_0 + E_c$ is the number of external lines. It then follows that

$$w_{st} = w + 2(1 - E_0)\,, \qquad E_0 \geq 1 \tag{10.7}$$

which means that the naive degree of divergence of a stochastic diagram never exceeds that of the corresponding Feynman diagram. Thus we could well be satisfied by regularizing stochastic diagrams with conventional techniques such as dimensional regularization, Pauli–Villars, etc.

We may, however, make one step further: the presence of the extra dimension of the fictitious time coordinate, with its associated dynamics, suggests a completely new regularization scheme. We will see in a short while that this regularization scheme may be constructed in such a way that it automatically preserves all the symmetries of the theory. With this property, stochastic regularization becomes a viable candidate for a field-theory regulator.

The basic idea of stochastic regularization (formulated first in a somewhat different context in ref. [10.7]) is to change the original stochastic process of eqs. (3.3), (3.7) and (3.8) by introducing a specific kernel for *either* the *drift* term *or* the noise correlation. The modified stochastic process is different from the original one insofar as drift term *and* noise correlation are not *both* changed appropriately; only in the limit where the kernel approaches unity are both stochastic processes identical.

Let us restrict ourselves in the following to a modification of only the noise term (for further discussion see ref. [10.8]).

How can we expect a new regularization scheme by modifying the noise correlation? In what sense should we modify it? We can give a quick argument as to why such a possibility exists. Suppose we decide to modify the noise correlation (3.41) by

$$\langle \eta(k,t)\,\eta(k',t')\rangle = 2(2\pi)^n \left(\frac{\Lambda^2}{\Lambda^2 + k^2}\right)^m \delta^n(k+k')\,\delta(t-t') \tag{10.8}$$

where m is some positive integer. The free propagator can then easily be derived as

$$\langle \phi(k, t)\, \phi(k', t) \rangle = (2\pi)^n \left(\frac{\Lambda^2}{\Lambda^2 + k^2} \right)^m \frac{1}{k^2} (1 - \exp(-2k^2 t))\, \delta^n(k + k') \tag{10.9}$$

and we recognize additional powers of momenta in the denominator. As any stochastic diagram of higher order necessarily contains crossed lines in its loops we can conclude that (for a sufficiently high m) all ultraviolet divergences can be regularized.

Our first candidate for a stochastic regulator suffers, however, from a serious drawback when it is applied, for example, to gauge theories: It spoils the gauge invariance. By this we mean that even when evaluating what should be a gauge-invariant Green function the regulator introduces gauge dependence. This can easily be understood from the fact that as a consequence of the choice (10.8) the noise correlations themselves are not gauge-invariant. Consider, for example, the correlation of the charged noises from scalar QED

$$\langle \eta^g(x, t)\, \eta^{*g}(x', t) \rangle = U(x)\, U^{-1}(x')\, \langle \eta(x, t)\, \eta^*(x', t) \rangle \neq \langle \eta(x, t)\, \eta^*(x', t) \rangle\,. \tag{10.10}$$

Two possibilities to overcome this gauge-invariance problem can be found. One solution is to allow only for a t-dependent kernel [10.1, 10.2]; this corresponds to choosing

$$\langle \eta(x, t)\, \eta(x', t') \rangle = 2\, \delta(x - x')\, K(t - t') \tag{10.11}$$

which we recognize as a non-Markovian generalization of the Parisi–Wu approach. As the δ-function for the physical space-time coordinates remains untouched, the noise correlations, as for example (10.10), remain generally invariant under symmetry transformations. The second possibility [10.3] is to consider gauge-covariant generalizations of (10.8) using covariant derivatives instead of ordinary ones.

Let us discuss first the non-Markovian scheme [10.1]. The following series of kernels were proposed

$$K_m(t) = \frac{1}{2m!}\, \Lambda^2 (\Lambda^2 |t|)^m \exp(-\Lambda^2 |t|) \tag{10.12}$$

where specifically

$$K_0(t) = \tfrac{1}{2} \Lambda^2 \exp(-\Lambda^2 |t|)\,. \tag{10.13}$$

We may then straightforwardly derive the propagator as

$$\lim_{t \to \infty} \langle \phi(k, t)\, \phi(k', t) \rangle = (2\pi)^n\, \frac{\delta^n(k + k')}{k^2}\, \frac{\Lambda^2}{\Lambda^2 + k^2}\,. \tag{10.14}$$

It should be remarked that one must now be careful not to apply wrongly the rules we developed in section 3.2.2 for the large t-limits of stochastic diagrams. Specifically, one can *not* simply substitute (10.14) for a crossed line and leave all other steps unchanged. This is clear, because for different fictitious times the scalar field two-point correlation is

$$\langle \phi(k, t)\, \phi(k', t') \rangle = (2\pi)^n\, \delta^n(k + k')\, \frac{\Lambda^2}{k^4 - \Lambda^4} \cdot \left[-\frac{\Lambda^2}{k^2} \exp\{-k^2 (t - t')\} + \exp\{-\Lambda^2 (t - t')\} \right] + \cdots \tag{10.15}$$

which shows an involved fictitious time dependence. We will discuss this issue to some more extent in the next section when we are calculating $\Pi_{\mu\nu}$ in QED.

It is interesting to note that if we perform a fictitious time Fourier transformation we may supplement the ultraviolet regulator with an infrared regulator as well, by simple multiplication [10.1, 10.7].

The Fourier transform of (10.13) is given by

$$K(\omega) = \Lambda^4/(\Lambda^4 + \omega^2) \tag{10.16}$$

and we can implement a convenient infrared regulator by

$$\tilde{K}(w) = \frac{\Lambda^4}{\Lambda^4 + \omega^2}\,\frac{\omega^2}{\omega^2 + \mu^4}\,. \tag{10.17}$$

The regulator exhibits two scales and by construction preserves all the symmetries of the action. For $\Lambda^2 \gg \mu^2$ the Fourier transform of $\tilde{K}(\omega)$ simplifies and we find

$$\tilde{K}(t) \simeq \tfrac{1}{2}\Lambda^2 \exp(-\Lambda^2|t|) - \mu^2 \exp(-\mu^2|t|)\,. \tag{10.18}$$

As a simple application we calculate the equilibrium limit of the free propagator

$$\lim_{\substack{t\to\infty \\ \Lambda^2 \gg \mu^2}} \langle \phi(k,t)\,\phi(k',t)\rangle \simeq \frac{\Lambda^2}{k^2(k^2+\Lambda^2)} - \frac{\mu^2}{k^2(k^2+\mu^2)} \simeq \frac{\Lambda^2}{(k^2+\Lambda^2)(k^2+\mu^2)} \tag{10.19}$$

which is indeed regulated in the ultraviolet as well as in the infrared.

Among many important issues of this new regulator remains the discussion of a functional formulation, specifically of the Fokker–Planck equation [10.1]. We may repeat the general method of section 2 in order to determine this equation. We then discover an important difference from the unregularized case, namely that in

$$\left\langle \eta(t)\,\frac{\delta F}{\delta\phi(t)}\right\rangle = \int \mathrm{d}t'\,K(t-t')\left\langle \frac{\delta^2 F}{\delta\phi(t)\,\delta\phi(t)}\,\frac{\delta\phi(t)}{\delta\eta(t')}\right\rangle \tag{10.20}$$

all intermediate times are contributing. As a consequence of the different fictitious time arguments, we cannot evaluate $\delta\phi(t)/\delta\eta(t')$ in closed form. Only in the free case do we have an explicit solution

$$\frac{\delta\phi(x,t)}{\delta\eta(y,t')} = \theta(t-t')\int \mathrm{d}^n k\,\exp\{-\mathrm{i}k(x-y)\}\,\exp\{-k^2(t-t')\}\,. \tag{10.21}$$

In the interacting case one may only establish an iterative solution, in analogy to the iterative solution of $\phi(x,t)$.

In the free case we then find

$$\dot{P} = \int \mathrm{d}^n x\,\frac{\delta}{\delta\phi(x)}\left[\frac{\delta S}{\delta\phi(x)} + \int \mathrm{d}^n y\,M(x-y,t)\,\frac{\delta}{\delta\phi(y)}\right]P \tag{10.22}$$

where

$$M(x-y,t)=\int_0^t \mathrm{d}\tau \int \mathrm{d}^n k \exp\{-\mathrm{i}k(x-y)\}\, K(t-\tau)\exp\{-k^2(t-\tau)\}\,. \tag{10.23}$$

In order to obtain the stationary Fokker–Planck equation it is convenient to work in the fictitious time Fourier space. With the choice (10.16) for $K(\omega)$ we find that the equilibrium distribution P_{eq} has to satisfy

$$\begin{aligned}0=\int \mathrm{d}^n x\, \frac{\delta}{\delta\phi(x)}\Big[&(-\partial^2+m^2)\,\phi(x)+\int \mathrm{d}^n y\int \mathrm{d}^n k\int \mathrm{d}\omega\\ &\cdot \exp\{-\mathrm{i}k(x-y)\}\,\frac{\Lambda^4}{\omega^2+\Lambda^4}\,\frac{1}{k^2+m^2-\mathrm{i}\omega}\,\frac{\delta}{\delta\phi(y)}\Big]P^{\mathrm{eq}}\end{aligned} \tag{10.24}$$

which is readily solved by

$$P^{\mathrm{eq}}=\exp\Big[-\int \mathrm{d}^n x\, \phi(x)\,(-\partial^2+m^2)\Big(1+\frac{-\partial^2+m^2}{\Lambda^2}\Big)\,\phi(x)\Big]\,. \tag{10.25}$$

We learn from this example that the non-Markovian regulator generally introduces higher derivatives in the equilibrium distribution.

Let us mention that another choice [10.2] for a non-Markovian regulator

$$K_\sigma(t)=\tfrac{1}{2}\sigma|t|^{\sigma-1}\,,\qquad \sigma>0 \tag{10.26}$$

allows us to obtain regularized stochastic diagrams with the ultraviolet divergences appearing as poles of σ. In this respect, this regularization scheme makes contact with the analytic regularization scheme of Speer [10.9]. It also shares many features with dimensional regularization.

As an interesting application of this regulator we should mention the evaluation of critical exponents [10.10], which gives an alternative to the usual ε-expansion.

Next we discuss the covariant derivative regularization scheme [10.3, 10.11–10.14] which, in contrast to the previous scheme is purely Markovian.

The basic idea is to generalize in a covariant way the noise structure of the gauge-field Langevin equation to

$$\dot A^a_\mu=-\frac{\delta S}{\delta A^a_\mu}+\int \mathrm{d}^4 y\, K^{ab}(x,y)\,\eta^b_\mu(y,t) \tag{10.27}$$

where K is a function of the covariant Laplacian Δ

$$\Delta^{ab}(x,y)=\int \mathrm{d}^4 z\, D^{ac}_\mu(x,z)\, D^{cb}_\mu(z,y) \tag{10.28}$$

with

$$D^{ab}_\mu(x,y)=D^{ab}_\mu(x)\,\delta^4(x-y)\,. \tag{10.29}$$

To maintain the connection with the original unregularized theory we require that K approaches unity as a cut-off Λ approaches infinity. This then leaves a large class of possible regulator-functions, such as [10.11]

$$K_n^{ab}(x, y) = [(1 - \Delta/\Lambda^2)^{-n}]^{ab}(x, y) \tag{10.30}$$

or [10.12]

$$K_n^{ab}(x, y) = (\exp(\Delta/\Lambda^2))^{ab}(x, y)\,. \tag{10.31}$$

We remark that the choices (10.28–10.31) guarantee gauge covariance of the Langevin equation (10.27) by general construction (note that of course we can explicitly verify this property by performing a power series expansion of K in terms of Δ/Λ^2).

For practical application of this scheme one has to perform a power series expansion of K in the Langevin equation in powers of the coupling constant. One defines regulator vertices Γ_1 and Γ_2 by

$$\Delta^{ab}(x, y)/\Lambda^2 = \delta^{ab}\,\partial^2(x, y)/\Lambda^2 + g(\Gamma_1)^{ab}(x, y) + g^2(\Gamma_2)^{ab}(x, y) \tag{10.32}$$

and

$$\begin{aligned}(\Gamma_1)^{ab}(x, y) &= f^{abc}(\partial_\mu^x A_\mu^c(x) + A_\mu^c(x)\,\partial_\mu^x)\delta^4(x-y)/\Lambda^2\\ (\Gamma_2)^{ab}(x, y) &= f^{ach}f^{cbe}A_\mu^h(x)\,A_\mu^e(x)\,\delta^4(x-y)/\Lambda^2\end{aligned} \tag{10.33}$$

where the derivatives act on everything on the right. A straightforward expansion of K follows immediately.

Let us suppose that we obtained an expansion of K in the coupling constant correct up to some order. We then solve iteratively the Langevin equation and construct stochastic diagrams. This program, supplemented with a stochastic gauge fixing term, seems quite involved technically, but it can be carried through [10.11, 10.13, 10.14]. In this scheme one can give a formulation in terms of *regularized* Schwinger–Dyson equations [10.11, 10.13].

It is important to remark that the non-Markovian stochastic regularization scheme, when applied to gauge theories, is not free of problems [10.18, 10.25, 10.26] and may even lead to inconsistencies. It is a crucial observation [10.25] that within the framework of stochastic gauge fixing the noise fields η^B, η^{*B}, η_μ^B, which depend on the gauge field in a nonlinear way (see eqs. (4.105) and (4.119), respectively) can not simply be replaced by the original noise fields η, η^*, η_μ: In contrast to the unregularized case (see eqs. (4.106) and (4.120)) the appropriate noise correlations are no longer identical, which follows from the absence of the fictitious time delta function in the noise correlations, see [10.12, 10.26]. As a verification of this argument the explicit breakdown of gauge invariance was shown, when using the untransformed noise fields η, η^*, η_μ [10.18] (see also [10.26]). It seems necessary to investigate still in more detail the non-Markovian stochastic regularization procedure relying on the gauge field depending noise fields η^B, η^{*B}, η_μ^B, or following the original Parisi–Wu approach. In this latter case, however, in Yang–Mills theories, unregularized internal longitudinal loops seem to pose severe problems as well [10.26]!

We summarize that stochastic regularization offers appealing new field-theory regularization schemes which manifestly preserve all symmetries. It remains to be seen whether the somewhat bothersome technical disadvantages of these new regularization schemes (as well as conceptual problems in the

non-Markovian case) can be handled successfully. In the following subsections we limit our discussion to the Abelian case (within the Parisi–Wu approach and with an external background gauge field, respectively), where the above problems do not arise.

10.2. The vacuum polarization tensor in QED

We will calculate in the following the divergent part of the vacuum polarization tensor $\Pi_{\mu\nu}$ from the $t\to\infty$ limit of the one-loop correction to the photon propagator [10.4]. For this purpose, it is consistent to omit the longitudinal contribution in the gauge field integral equation and, moreover, substitute $T_{\mu\nu}\to\delta_{\mu\nu}$. In fact, we will prove transversality of the divergent part of $\Pi_{\mu\nu}$ by using stochastic regularization, so that all longitudinal contributions of the attached gauge fields are projected away. A similar calculation [10.15] of $\Pi_{\mu\nu}$ was carried out for scalar QED and effectively relied on the same features, though actually the standard gauge-fixing procedure was used by choosing the Feynman gauge.

We recall from section 6 that we arrive generally at the following set of coupled Langevin equations describing QED:

$$\begin{aligned}
\dot{\psi} &= \int \mathrm{d}^4y\, K(x,y)\,[\mathrm{i}(\not{D}+\mathrm{i}m)\psi+\eta_1](y,t)+\eta_2(x,t)\\
\dot{\bar{\psi}} &= \int \mathrm{d}^4y\, K(x,y)^{\mathrm{T}}\,[\mathrm{i}(\not{D}'+\mathrm{i}m)^{\mathrm{T}}\,\bar{\psi}+\bar{\eta}_2](y,t)+\bar{\eta}_1(x,t)\\
\dot{A}_\mu &= \partial_\nu F_{\nu\mu}+e\bar{\psi}\gamma_\mu\psi+\eta_\mu
\end{aligned}\tag{10.34}$$

where we introduced a kernel operator $K(x, y)$ which effectively bosonizes the drift terms in the fermionic Langevin equations. Furthermore we define

$$D=\partial-\mathrm{i}eA\,,\qquad D'=-\partial-\mathrm{i}eA\,.\tag{10.35}$$

$\eta_1, \bar{\eta}_1, \eta_2, \bar{\eta}_2$ represent Gaussian, Grassmann noises, and stochastic regularization is implemented by

$$\begin{aligned}
&\langle\eta_i(x,t)\rangle=\langle\bar{\eta}_i(x,t)\rangle=0\,,\qquad i=1,2\\
&\langle\eta_i(x,t)\,\bar{\eta}_k(x',t')\rangle=2\delta_{ik}\delta^4(x-x')\,\alpha_\Lambda(t-t')
\end{aligned}\tag{10.36}$$

where we choose as stochastic regulator

$$\alpha_\Lambda(t)=-\Lambda^4\,\frac{\partial}{\partial\Lambda^2}\left(\frac{\alpha^0_\Lambda(t)}{\Lambda^2}\right)\tag{10.37}$$

$$\alpha^0_\Lambda(t)=\tfrac12\Lambda^2\exp(-\Lambda^2|t|)\tag{10.38}$$

which obeys

$$\lim_{\Lambda\to\infty}\alpha_\Lambda(t)=\delta(t)\,.\tag{10.39}$$

Similarly we could modify the bosonic white noise correlations of the gauge field

$$\langle \eta_\mu(x,t)\rangle = 0\,, \qquad \langle \eta_\mu(x,t)\,\eta_\nu(x',t')\rangle = 2\delta_{\mu\nu}\delta^4(x-x')\,\delta(t-t') \tag{10.40}$$

but we refrain from doing so, as we will not need it for the calculations in this section.

There is an apparent freedom in the choice of the kernel operator $K(x, y)$: the simplest one is given by

$$K(x,y) = \mathrm{i}(\not\partial_x - \mathrm{i}m)\,\delta^4(x-y)\,, \qquad \bar{K}(x,y) = \mathrm{i}(-\not\partial_x - \mathrm{i}m)\,\delta^4(x-y)\,. \tag{10.41}$$

However, gauge covariance of the fermionic Langevin equation is lost in this case. One could naively argue that this would not matter in the equilibrium limit $t \to \infty$ (where in the unregularized case the usual field theory with all its symmetries is recovered), but this is not so in the framework of stochastic *regularization*; calculating the photon vacuum polarization tensor to one loop, one indeed finds a quadratic divergence in Λ, which may be compared with the similar result obtained when using a naive momentum cut-off as regulator. Quadratic divergences in Λ and further inconsistencies of the choice (10.41) for QED have also been reported in ref. [10.16]. We conclude that the choice (10.41) is not acceptable for the purpose of stochastic regularization.

It seems immediately favourable to generalize (10.41) to the covariant choice [10.17]

$$K(x,y) = \mathrm{i}(\not{D}_x - \mathrm{i}m)\,\delta^4(x-y)\,, \qquad \bar{K}(x,y) = \mathrm{i}(\not{D}'_x - \mathrm{i}m)\,\delta^4(x-y) \tag{10.42}$$

so that QED is described by the following gauge and chiral (for $m = 0$) covariant Langevin equations

$$\begin{aligned}
\dot{\psi} &= -(\not{D} - \mathrm{i}m)(\not{D} + \mathrm{i}m)\,\psi + \mathrm{i}(\not{D} - \mathrm{i}m)\,\eta_1 + \eta_2 \\
\dot{\bar{\psi}} &= -(\not{D}' - \mathrm{i}m)^{\mathrm{T}}(\not{D}' + \mathrm{i}m)^{\mathrm{T}}\,\bar{\psi} + \bar{\eta}_1 + \mathrm{i}(\not{D}' - \mathrm{i}m)^{\mathrm{T}}\,\bar{\eta}_2 \\
\dot{A}_\mu &= \partial_\nu F_{\nu\mu} + e\bar{\psi}\gamma_\mu\psi + \eta_\mu\,.
\end{aligned} \tag{10.43}$$

In the rest of this paper, we will use this above set of coupled equations and discuss its usefulness and consistency.

After standard manipulations, with $\alpha^0_\Lambda(t)$ from (10.38), we then easily find to lowest order

$$\begin{aligned}
\langle \psi(k,t)\,\bar{\psi}(k',t')\rangle^0 &= (2\pi)^4\,\delta^4(k-k')\,(\not{k} + m)\,\frac{\Lambda^2}{(k^2+m^2)^2 - \Lambda^4} \\
&\quad\times\left[-\frac{\Lambda^2}{k^2+m^2}\exp\{-(k^2+m^2)|t-t'|\} + \exp\{-\Lambda^2|t-t'|\}\right] + \cdots
\end{aligned} \tag{10.44}$$

$$\begin{aligned}
\langle \eta_1(k,t)\,\bar{\psi}(k',t')\rangle^0 = \langle \psi(k,t)\,\bar{\eta}_2(k',t')\rangle^0 &= \tfrac{1}{2}(2\pi)^4\delta^4(k-k')\,\frac{\Lambda^2}{k^2+m^2-\Lambda^2} \\
&\quad\times\left[-\frac{2\Lambda^2}{k^2+m^2+\Lambda^2}\exp\{-(k^2+m^2)|t-t'|\} + \exp\{-\Lambda^2|t-t'|\}\right]
\end{aligned} \tag{10.45}$$

where the dots indicate exponentially suppressed contributions.

In the following, we generalize methods known from the unregularized case in order to find easily the large time limit (of sums) of stochastically regularized stochastic diagrams: we observe that $\langle \psi\bar{\psi} \rangle$ as well as the $\langle \eta_1 \bar{\psi} \rangle$ and $\langle \psi \bar{\eta}_2 \rangle$ correlations (10.44) and (10.45) each contain two terms, having different fictitious time dependence in the exponents. Let us therefore split a given stochastic diagram where (10.44) and (10.45) are involved in all parts, such that in crossed (fermionic) lines only single terms of (10.44) and (10.45) appear. Now we may introduce fictitious time orderings and construct the large time limit of the stochastically regulated diagrams as in section 3: the only difference emerging is that now there are contributions in the denominators which do not depend on (momenta)^2 alone, but also on Λ^2, corresponding to a part of (10.44) or (10.45) which involved $\exp(-\Lambda^2 t)$.

As an example, let us consider a specific contribution to the one-loop correction to $\langle A_\mu A_\nu \rangle^0$ as depicted in fig. 10.1 [here again, the index ^{0} means regularization with $\alpha^0_\Lambda(t)$]. Let us further assume that we are interested just in the transverse part of the photon contribution.

The stochastic diagram of fig. 10.1 contains one regulated (fermion) line and is split according to (10.44) into two parts. The fictitious time integrations then immediately give for $t \to \infty$ in the first case

$$\frac{1}{k^2 + p^2 + q^2} \frac{1}{2k^2} \tag{10.46}$$

whereas in the second case

$$\frac{1}{k^2 + \Lambda^2 + q^2} \frac{1}{2k^2} . \tag{10.47}$$

Thus the large time limit of the stochastic diagram of fig. 10.1 follows as

$$\int \frac{\mathrm{d}^4 p\, \mathrm{d}^4 q}{(2\pi)^4} \delta^4(k - p + q)\, T_{\mu\rho}\, T_{\nu\sigma} \operatorname{Tr}\{ \not{p} \gamma_\rho (\not{k} \gamma_\sigma - 2p_\sigma)\}$$
$$\cdot \frac{1}{k^2} \left(\frac{1}{k^2 + p^2 + q^2} \frac{1}{2k^2} \frac{(-\Lambda^4)}{(p^4 - \Lambda^4)\, p^2} + \frac{1}{k^2 + \Lambda^2 + q^2} \frac{1}{2k^2} \frac{\Lambda^2}{p^4 - \Lambda^4} \right) . \tag{10.48}$$

As mentioned in section 3, calculations can conveniently be shortened by using a recursion method of summing stochastic subdiagrams. In the case of $\Pi_{\mu\nu}$ the application of this method requires a calculation of the 'effective' stochastic vertex

$$V^0_\mu(p_1, p_2, p_3) = \lim_{t\to\infty} \langle \psi(p_1, t)\, A_\mu(p_2, t)\, \bar{\psi}(p_3, t) \rangle^0 . \tag{10.49}$$

The diagrams contributing are depicted in fig. 10.2. Let us introduce

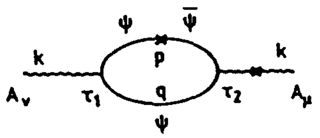


Fig. 10.1. An example of a stochastic diagram contributing to the one-loop correction of the photon propagator.

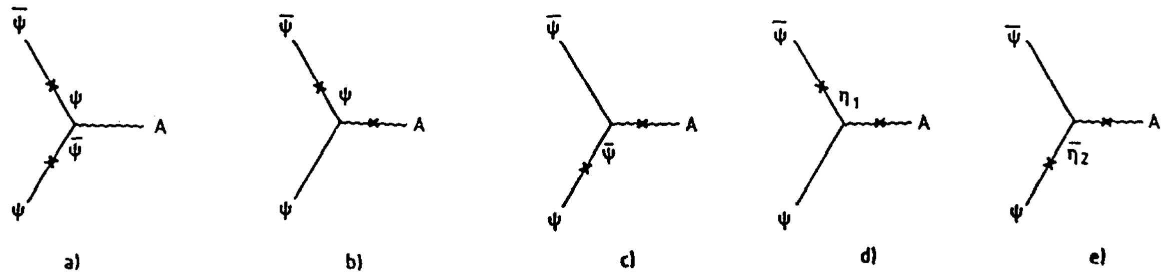


Fig. 10.2. The stochastic diagrams contributing to the effective vertex V^0_μ.

$$\begin{aligned} &\Sigma = p_1^2 + p_2^2 + p_3^2\,, \qquad \Sigma_3 = p_1^2 + p_2^2 + \Lambda_3^2\,, \\ &\Sigma_1 = \Lambda_1^2 + p_2^2 + p_3^2\,, \qquad \Sigma_{13} = \Lambda_1^2 + p_2^2 + \Lambda_3^2\,. \end{aligned} \tag{10.50}$$

With the methods which we just outlined, we calculate the $t \to \infty$ limits of these diagrams and obtain

$$\begin{aligned} \langle A_\mu A_\nu \rangle = &\int \frac{d^4 p_1\, d^4 p_3}{(2\pi)^4}\, \delta^4(p_1 + p_2 - p_3)\, \frac{1}{p_2^2} \\ &\cdot \Lambda_1^4 \Lambda_3^4\, \frac{\partial}{\partial \Lambda_1^2}\, \frac{\partial}{\partial \Lambda_3^2}\, \mathrm{tr}\left\{ \gamma_\nu V^0_\mu(p_1, p_2, p_3)\, \frac{1}{\Lambda_1^2 \Lambda_3^2} \right\}. \end{aligned} \tag{10.51}$$

Keeping just potentially (i.e. for $\Lambda \to \infty$) diverging contributions, we then identify $\Pi_{\mu\nu}$ as

$$\begin{aligned} \Pi_{\mu\nu}(k) = \int \frac{d^4 p}{(2\pi)^4} \Big\{ &\mathrm{tr}[\gamma_\nu \not p \gamma_\mu (\not p + \not k)] \cdot k^2 \Lambda^8\, \frac{\partial}{\partial \Lambda_1^2}\, \frac{\partial}{\partial \Lambda_3^2}\, \frac{1}{p^2 + \Lambda_1^2}\, \frac{1}{p^2 + \Lambda_3^2}\, \frac{1}{p^2}\, \frac{1}{(p+k)^2}\, \frac{1}{\Sigma} \Big|_{\Lambda_1 = \Lambda_3 = \Lambda} \\ &-2\mathrm{tr}[\gamma_\nu \not p (\gamma_\mu \not k - 2p_\mu)]\, \Lambda^4\, \frac{\partial}{\partial \Lambda^2}\, \frac{2p^2 + \Lambda^2}{\Sigma p^2 (p^2 + \Lambda^2)^2} \\ &-2\mathrm{tr}(\gamma_\mu \gamma_\nu)\, \Lambda^4\, \frac{\partial}{\partial \Lambda^2}\, \frac{1}{\Sigma (p^2 + \Lambda^2)} \Big\} \end{aligned} \tag{10.52}$$

where we partly performed a shift $p_1 \to -p_1 - p_2$ for simplification and finally put $p_1 = p$, $p_2 = k$. Evaluating the traces, differentiating and reinserting the mass terms explicitly (at places of relevance only), we get

$$\Pi_{\mu\nu}(k) = A_{\mu\nu} + B_{\mu\nu} + C_{\mu\nu} \tag{10.53}$$

with

$$A_{\mu\nu} = \int \frac{d^4 p}{(2\pi)^4}\, \frac{-[p^2 + (pk)]\,\delta_{\mu\nu} + 2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu}{\Sigma (p^2 + m^2)[(p+k)^2 + m^2]}\, \frac{4\Lambda^8 k^2}{(p^2 + \Lambda^2)^4}$$

$$B_{\mu\nu} = \int \frac{d^4 p}{(2\pi)^4}\, \frac{2p_\mu p_\nu - (pk)\,\delta_{\mu\nu} + p_\mu k_\nu + p_\nu k_\mu}{(p^2 + m^2)\Sigma}\, \frac{8\Lambda^4 (3p^2 + \Lambda^2)}{(p^2 + \Lambda^2)^3}$$

$$C_{\mu\nu} = -\int \frac{d^4p}{(2\pi)^4} \frac{\delta_{\mu\nu} 8\Lambda^4}{\Sigma(p^2+\Lambda^2)^2} . \tag{10.54}$$

By power counting, we expect the following types of divergences:

$$\delta_{\mu\nu} k^2 \log \Lambda^2 , \qquad k_\mu k_\nu \log \Lambda^2 \tag{10.55}$$

and

$$\delta_{\mu\nu} \Lambda^2 , \qquad \delta_{\mu\nu} m^2 \log \Lambda^2 . \tag{10.56}$$

Let us first study the k-dependent, logarithmically-divergent part of $\Pi_{\mu\nu}$. Since it is a polynomial in k of degree 2, we may distinguish three cases:

a) there is in (10.54) an explicit $\delta_{\mu\nu} k^2$ or $k_\mu k_\nu$ factor, so that we may set $k=0$ in the rest of the corresponding loop integration, as all k-dependence has already been found;

b) there is an explicit k_μ or k_ν factor in (10.54); denoting the diverging part of $\Pi_{\mu\nu}$, of which we omitted the above k_μ or k_ν factor, by $\Pi^{1,\mathrm{div}}_{\mu\nu}$; it then holds that

$$\Pi^{1,\,\mathrm{div}}_{\mu\nu} = k^\sigma \frac{\partial}{\partial k^\sigma} \Pi^{1,\,\mathrm{div}}_{\mu\nu} \tag{10.57}$$

or, effectively,

$$\frac{1}{\Sigma} \to -\frac{(pk)}{2(p^2+m^2)^2} ; \tag{10.58}$$

c) there is no explicit k-factor in (10.54); in this case the diverging part of $\Pi_{\mu\nu}$ fulfils

$$\Pi^{2,\,\mathrm{div}}_{\mu\nu} = \tfrac{1}{2} k^\sigma \frac{\partial}{\partial k^\sigma} \Pi^{2,\,\mathrm{div}}_{\mu\nu} \tag{10.59}$$

or, effectively,

$$1/\Sigma \to \tfrac{1}{2}[(pk)^2/(p^2+m^2)^3 - k^2/(p^2+m^2)^2] . \tag{10.60}$$

Using furthermore

$$\int d^4p\, f(p^2)\, p_\mu p_\nu (pk)\,(pk) = \tfrac{1}{24} \int d^4p\, f(p^2)\, p^4\, [\delta_{\mu\nu} k^2 + 2k_\mu k_\nu] \tag{10.61}$$

we easily find to leading order, with

$$I = \frac{1}{(2\pi)^4} \int d^4p \frac{\Lambda^2}{p^2(p^2+m^2)(p^2+\Lambda^2)} = \frac{1}{(2\pi)^4} \log \Lambda^2 + \cdots \tag{10.62}$$

that

$$A_{\mu\nu} = -k^2\delta_{\mu\nu}I$$

$$B_{\mu\nu} = -\tfrac{2}{3}(\partial_{\mu\nu}k^2 + 2k_\mu k_\nu)I$$

$$C_{\mu\nu} = 3k^2\delta_{\mu\nu}I\,. \tag{10.63}$$

Thus the k-dependent, divergent part of $\Pi_{\mu\nu}$ adds up to the transverse result

$$\Pi^{\mathrm{div}}_{\mu\nu}(k) = \tfrac{4}{3}(k^2\delta_{\mu\nu} - k_\mu k_\nu)I\,. \tag{10.64}$$

What still remains to be shown is that the photon does not acquire a mass at the one-loop level, which is guaranteed if the vacuum polarization tensor is vanishing for external zero momentum. This can indeed be seen from

$$\Pi_{\mu\nu}(0) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{4\delta_{\mu\nu}\Lambda^4}{(p^2+\Lambda^2+m^2)^2\,(p^2+m^2)}\left[\frac{p^2(3p^2+\Lambda^2+m^2)}{(p^2+m^2)(p^2+\Lambda^2+m^2)} - 2\right] \tag{10.65}$$

and a few straightforward manipulations (compare also with the similar result in ref. [10.15]).

10.3. Ward Identities

Up to now we were led to the conclusion that in stochastic quantization the Euclidean Green functions could be obtained as the large time limit of the corresponding correlation functions. In the course of the last sections we became aware of new stochastic regularization schemes which preserved all symmetries.

It seems an obvious challenge to discuss the issue of anomalies within this context. There exist various ways and methods to tackle this problem [10.19–10.21, 10.4, 10.14, 10.22]. Let us choose in the following a perturbative calculation [10.4], with the gauge field considered as an external background field (which does not depend on the fictitious time and which has no associated Langevin equation). We consider massive fermions and first perform the $t\to\infty$ limit in our calculations before finally allowing $m=0$. We use the non-Markovian regulator (10.37, 10.38).

To start, let us define

$$j^5_\mu(x,t) = \bar\psi(x,t)\,\gamma_\mu\gamma_5\psi(x,t) \tag{10.66}$$

so that

$$\langle\partial_\mu j^5_\mu(x,t)\rangle = \langle(\not\partial^{\mathrm{T}}\bar\psi)\gamma_5\psi\rangle - \langle\bar\psi\gamma_5\not\partial\psi\rangle\,. \tag{10.67}$$

We could now iterate straightforwardly the Langevin equation (10.43) and calculate the $t\to\infty$ limit of the (several) corresponding diagrams. It is, however, easier to use the Langevin equation to replace $\not\partial\psi$ and $\not\partial^{\mathrm{T}}\bar\psi$ by

$$\not\partial\psi = -(\not D - \mathrm{i}m)^{-1}\dot\psi - \mathrm{i}\eta_1 - (\not D - \mathrm{i}m)^{-1}\eta_2 - \mathrm{i}m\psi + \mathrm{i}e\not A\psi \tag{10.68}$$

and similarly for $\not\partial^{\mathrm{T}}\bar\psi$, to get

$$\langle \partial_\mu j^5_\mu \rangle = \langle [(\not D' - \mathrm{i}m)^{\mathrm{T}^{-1}}\dot{\bar\psi}]\gamma_5\psi\rangle + \langle \bar\psi\gamma_5(\not D - \mathrm{i}m)^{-1}\dot\psi\rangle + 2\mathrm{i}m\langle\bar\psi\gamma_5\psi\rangle$$
$$+ \langle\{\mathrm{i}\bar\eta_1 + [(\not D' - \mathrm{i}m)^{\mathrm{T}^{-1}}\bar\eta_2]\}\gamma_5\psi\rangle + \langle\bar\psi\gamma_5(\mathrm{i}\eta_1 + (\not D - \mathrm{i}m)^{-1}\eta_2)\rangle\,. \tag{10.69}$$

The first two terms on the right-hand side of the equation above vanish for $t\to\infty$ due to the appearance of fictitious time derivatives [10.17], so we get as the anomalous contribution

$$A = 2\langle\{\mathrm{i}\bar\eta_1 + (\not D' - \mathrm{i}m)^{\mathrm{T}^{-1}}\bar\eta_2\}\gamma_5\psi\rangle\,. \tag{10.70}$$

Now it is interesting to remark that in a perturbative expansion of ψ, $\bar\psi$ there is just *one* term contributing to A: if we do not iterate ψ, $\bar\psi$ often enough, the trace of γ-matrices, involving γ_5, gives zero; if we iterate too often, then the stochastic regulator leads to a vanishing result for $\Lambda\to\infty$.

Let us remind the reader that this phenomenon is very similar to what happens when using Schwinger's point-splitting method [10.23] as a regularization scheme, and where also only one diagram out of infinitely many contributed to the anomaly.

Iterating thus ψ, $\bar\psi$ to the second order in e, using

$$\int (\not D'_x - \mathrm{i}m)^{\mathrm{T}^{-1}}(\not D_{x'} - \mathrm{i}m)\,\delta(x-x')\,\mathrm{d}x = 1 \tag{10.71}$$

everything which contributes to A is given by

$$A = \frac{e^2}{(2\pi)^{12}}\,\mathrm{i}2\int \mathrm{d}^4y\,\mathrm{d}^4y'\,\mathrm{tr}\{\gamma_5\not\partial_y\not A(y)\,\not\partial_y\not A(y')\}\cdot B(\Lambda^2, x, y, y') \tag{10.72}$$

where

$$B(\Lambda^2, x, y, y') = -\int \mathrm{d}^4k\,\mathrm{d}^4k'\,\mathrm{d}^4k''$$
$$\cdot\frac{\Lambda^2}{(k^2+\Lambda^2)(k'^2+\Lambda^2)(k''^2+\Lambda^2)}\exp\{\mathrm{i}k(x-y) + \mathrm{i}k'(y-y') + \mathrm{i}k''(y'-x)\}\,. \tag{10.73}$$

Finally, it is easy to prove by Fourier transformation that

$$\lim_{\Lambda^2\to\infty} B(\Lambda^2, x, y, y') = \frac{-\pi^2}{2}(2\pi)^8\,\delta^4(x-y)\,\delta^4(x-y') \tag{10.74}$$

so that we obtain the well-known result

$$A = \frac{e^2}{16\pi^2}\,\varepsilon_{\mu\nu\rho\sigma}F_{\mu\nu}\,F_{\rho\sigma}\,. \tag{10.75}$$

In order to analyse the above result more carefully, we now construct the appropriate Ward

Identities [10.5, 10.6], by recasting the Langevin formulation in a functional integral representation (see section 3.2.2). This involves the action S_{FP}, and it introduces the usual field theory procedures.

Let us concentrate on the issue of chiral gauge symmetries and consider the following Langevin equation for fermions in an external gauge field

$$\begin{aligned}\dot{\psi} &= -(\not{D} - \mathrm{i}m)\,(\not{D} + \mathrm{i}m)\,\psi + \eta \\ \dot{\bar{\psi}} &= -(\not{D}' - \mathrm{i}m)^{\mathrm{T}}\,(\not{D}' + \mathrm{i}m)^{\mathrm{T}}\,\bar{\psi} + \bar{\eta}\,.\end{aligned} \tag{10.76}$$

We have reverted to a one-noise formulation in order to keep the Ward Identity discussion simple. Then, starting our discussion for the unregularized case,

$$\langle \eta^{\alpha}(x,t)\,\bar{\eta}^{\beta}(x',t')\rangle = 2\not{D}^{\alpha\beta}\,\delta^{4}(x-x')\,\delta(t-t') \tag{10.77}$$

it is easy to check that (10.76) transform covariantly under the chiral transformations

$$\begin{aligned}\psi' &= \exp(\mathrm{i}\alpha\gamma_5)\,\psi\,, \qquad \bar{\psi}' = \bar{\psi}\exp(\mathrm{i}\alpha\gamma_5)\,, \\ \eta' &= \exp(\mathrm{i}\alpha\gamma_5)\,\eta\,, \qquad \bar{\eta}' = \bar{\eta}\exp(\mathrm{i}\alpha\gamma_5)\,.\end{aligned} \tag{10.78}$$

Similarly, the noise correlation (10.77) remains invariant under chiral transformations, as can be checked explicitly:

$$\begin{aligned}\langle \eta'^{\alpha}\bar{\eta}'^{\beta}\rangle &= (\exp(\mathrm{i}\alpha\gamma_5))^{\alpha\gamma}\,(\exp(\mathrm{i}\alpha\gamma_5))^{\delta\beta}\,\langle \eta^{\gamma}\bar{\eta}^{\delta}\rangle \\ &= 2(\exp(\mathrm{i}\alpha\gamma_5))^{\alpha\gamma}\,\not{D}^{\gamma\delta}(\exp(\mathrm{i}\alpha\gamma_5))^{\delta\beta}\,\delta^{4}(x-x')\,\delta(t-t') \\ &= \not{D}^{\alpha\beta}\,2\delta^{4}(x-x')\,\delta(t-t') = \langle \eta^{\alpha}\bar{\eta}^{\beta}\rangle\,.\end{aligned} \tag{10.79}$$

We now transform the generating functional

$$Z[J,\bar{J}] = \int \mathrm{D}\eta\,\mathrm{D}\bar{\eta}\,\exp\Big\{-\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}t\,[\eta(\not{D}+\mathrm{i}m)^{-1}\bar{\eta} + J\bar{\psi} + \bar{J}\psi]\Big\} \tag{10.80}$$

into (see [10.5])

$$Z[J,\bar{J}] = \int \prod_{\tau} \mathrm{D}\psi(\tau)\,\mathrm{D}\bar{\psi}(\tau)\,\exp\Big\{-S[\psi(t),\bar{\psi}(t)] - S_{\mathrm{FP}} - \int (J\bar{\psi} + \bar{J}\psi)\,\mathrm{d}^4x\,\mathrm{d}t\Big\} \tag{10.81}$$

where

$$S[\psi(t),\bar{\psi}(t)] = \mathrm{i}\int \mathrm{d}^4x\,\bar{\psi}(x,t)\,\not{D}\,\psi(x,t) \tag{10.82}$$

and

$$S_{\rm FP} = \tfrac{1}{2}\mathrm{i}\int \mathrm{d}^4x \int_0^t \mathrm{d}\tau\, [\dot{\bar{\psi}}(x,\tau)\,(\not{D} - \mathrm{i}m)^{-1}\dot{\psi}(x,\tau) + \bar{\psi}(x,\tau)\,(\not{D} + \mathrm{i}m)\,(\not{D}^2 + m^2)\,\psi(x,\tau)]\,. \tag{10.83}$$

The standard procedure of obtaining the chiral Ward Identity corresponding to the variations

$$\delta\psi(x,\tau) = \mathrm{i}\,\alpha(x)\,\gamma_5\,\psi(x,\tau)\,, \qquad \delta\bar{\psi}(x,\tau) = \mathrm{i}\,\alpha(x)\,\bar{\psi}(x,\tau)\,\gamma_5 \tag{10.84}$$

results in

$$\langle \partial_\mu J^5_\mu(x,t)\rangle + 2\mathrm{i}m\langle\bar{\psi}(x,t)\,\gamma_5\,\psi(x,t)\rangle + \left\langle \int_0^t \mathrm{d}\tau\, \dot{\bar{\psi}}(x,\tau) \left\{\gamma_5, \frac{\mathrm{i}m}{\not{D}^2 + m^2}\right\} \dot{\psi}(x,\tau)\right\rangle \tag{10.85}$$

$$+ \left\langle \int_0^t \mathrm{d}\tau\, \bar{\psi}(x,\tau)\,\{\gamma_5, \mathrm{i}m(\not{D}^2 + m^2)\}\,\psi(x,\tau)\right\rangle$$

where we denote $\partial_\mu J^5_\mu(x,t)$ as

$$-\partial_\mu J^5_\mu(x,t) = \bar{\psi}(x,t)\{\gamma_5, \not{D}\}\psi(x,t) + \int_0^t \mathrm{d}\tau\, \dot{\bar{\psi}}(x,\tau)\left\{\gamma_5, \frac{\not{D}}{\not{D}^2 + m^2}\right\}\dot{\psi}(x,\tau)$$

$$+ \int_0^t \mathrm{d}\tau\, \bar{\psi}(x,\tau)\,\{\gamma_5, \not{D}(\not{D}^2 + m^2)\}\,\psi(x,\tau)\,. \tag{10.86}$$

Let us stress that the form (10.85) of the chiral Ward Identity does not allow, as it stands, a conventional field theory interpretation. For this one needs to recast it in a form where the fermionic fields appear only depending on the final time t. This has been carried out in [10.5].

Using a functional formulation of stochastic quantization in the superfield formulation (see section 7.3 and also [10.24]) a conserved super Noether current can be constructed as well, see [10.6]. The expectation value of the lowest component of its derivative then becomes the Ward Identity corresponding to (10.85).

In order to obtain regularized expressions one may now implement the stochastic regularization scheme and evaluate the correlation functions in the Ward Identities. In both approaches [10.5, 10.6] it has been argued that the usual 'anomalous' Ward Identities (now involving *vacuum expectation* values) are obtained. However, in view of the fact that stochastic regularization of Langevin systems formally respects both gauge and chiral invariances, the appearance of chirality-breaking terms remains to be elucidated.

11. Numerical applications

Until now our focus has been mainly on formal aspects of stochastic quantization: the equivalence with other more standard quantization prescriptions, details of perturbative expansions, the problem of gauge theories, fermions, stochastic regularization, etc. In most of the cases (but certainly not all!) not

many *new* results have been obtained, only *reformulations* have been found. Although such reformulations may be of a great deal of value, it is nevertheless refreshing to see that stochastic quantization and Langevin equation techniques recently have been found to be extremely useful in *numerical* solutions of such rich and complex field theories as QCD.

We shall here try to give a brief account of the numerical methods which can be used in solving field theories by Langevin dynamics. It is not by coincidence that we have put this section at the end of our review; it simply covers a field which is still in a phase of rapid development, and which holds the promise of giving us even more in the future.

Most attention is naturally paid towards the numerical simulation of (lattice) gauge theories with fermions (and perhaps scalars as well, to simulate Higgs phases). Clearly one of the most pressing issues, at the time of writing, is the development of a fast and accurate numerical algorithm for treating fermion fields. Langevin equation techniques seem especially promising here. However, other efficient and cleverly constructed algorithms have been proposed in the past few years, and we do not wish to imply that Langevin techniques are the only ones which hold some promise. In particular, other 'fifth-time' approaches to fermions, such as the one based on the microcanonical ensemble [11.1], share many features with methods based on Langevin equations. One can also imagine hybrid methods [11.2], or completely new schemes along the same lines. For the readers who would like some recent comparative studies, we refer, for example, to ref. [11.3]. We remark that also dynamical systems with constraints can be investigated [11.28] within the Parisi–Wu scheme.

11.1. Numerical solutions in the continuum: Some toy models

The numerical solutions we shall be concerned with here will almost exclusively arise from *time-discretized* Langevin equations. Furthermore, in order to control ultraviolet divergences one will almost always study the various field theories on a lattice, i.e. both t and x will be discretized. Before getting to that point, however, it may be useful to see an example of how numerical techniques can be applied directly to the continuum (in both t and x) Langevin equation. In order to avoid ultraviolet divergences we will restrict ourselves to simple toy models, models defined by 'field theories' in very low dimensions. We will show in detail an example of a theory in the ultralocal limit: zero dimensions. Here the numerical Langevin technique that we shall describe simply consists in an approximate evaluation of a very complicated integral of the ordinary kind. The same method can be used to numerically evaluate excitation energies in quantum mechanical systems [11.4]. This corresponds to a one-dimensional field theory, and we are then already at a level where the Langevin equation can help us evaluate very complicated functional integrals in an approximate manner.

To start, let us again consider a scalar field theory,

$$S[\phi] = \int d^Dx \left[\tfrac{1}{2}(\partial_\mu \phi)^2 + \tfrac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\,\phi^4 \right] \tag{11.1}$$

and the associated Langevin equation

$$\frac{\partial}{\partial t}\,\phi(x,t) = (\partial^2 - m^2)\,\phi(x,t) - \frac{\lambda}{3!}\,\phi(x,t) + \eta(x,t) \tag{11.2}$$

with $\eta(x,t)$ being the usual noise field.

We can view eq. (11.2) as the equation of motion for the field $\phi(x,t)$. However, in contrast to the ordinary equations of motion for field theories, which are invalidated by quantum corrections, the Langevin equation (11.2) can be viewed as an *exact*, *classical* equation of motion which holds even at

the quantum level. This opens up new possibilities for approximation schemes, based directly on the Langevin equation itself. The approximation scheme we shall discuss briefly here is a simple extension [11.4] of a variational procedure introduced by Greensite [8.19] for the case of $N=\infty$ quenched master fields as discussed in section 8. However, as we shall see, the method is applicable to unquenched theories as well, and hence works also for N finite (and in particular for simple scalar theories).

Letting

$$G(m^2|x,t)=\theta(t)\int\frac{\mathrm{d}^Dp}{(2\pi)^D}\exp(\mathrm{i}px)\exp\{-(p^2+m^2)t\} \tag{11.3}$$

denote the Green function of eq. (11.2), we first introduce a variational 'potential' $V(\sigma)$ by the stochastic expectation value of a *squared* Langevin equation (in its integral form):

$$V(\sigma)=\left\langle\left[\phi_\sigma(x,t)-\int\mathrm{d}^Dx'\,\mathrm{d}t'\,G(m^2|x-x',t-t')\left\{\eta(x',t')-\frac{\lambda}{3!}\,\phi_\sigma^3(x',t')\right\}\right]^2\right\rangle \tag{11.4}$$

where $\sigma=\{\sigma_1,\sigma_2,\ldots\sigma_n\}$ is a set of variational parameters.

Clearly $V(\sigma)$ is positive definite, with a minimum at any solution of the Langevin equation (11.2). Then, if we minimize $V(\sigma)$ for all possible choices of variational fields $\phi_0(x,t)$ this will be equivalent to finding a solution of the Langevin equation. Taking the limit $t\to\infty$ we can subsequently calculate any vacuum expectation value or Green function from this solution.

In practice, one can choose a set of n parameters σ_i and a corresponding trial function $\phi_\sigma(x,t)$, and then solve (either analytically or numerically) the n equations $\partial V/\partial\sigma_i=0$ for $\sigma_1,\sigma_2,\ldots\sigma_n$. The absolute minimum of $V(\sigma)$ is to be chosen, and the corresponding set of σ_i substituted back into $\phi_\sigma(x,t)$.

One particularly simple one-parameter variational trial function is provided by [11.4]

$$\phi_\sigma(x,t)=\int\mathrm{d}^Dx'\,\mathrm{d}t'\,G(\sigma|x-x',t-t')\,\eta(x',t') \tag{11.5}$$

which is simply the free field solution with a shifted (renormalized) mass parameter. With this variational *ansatz* it is straightforward to evaluate the η-averages appearing in the definition of $V(\sigma)$ [eq. (11.4)] by use of Wick's theorem. For the two-point function one finds [11.4]

$$\langle\phi_\sigma(x)^2\rangle=\lim_{t\to\infty}\langle\phi_\sigma(x,t)^2\rangle=\int\frac{\mathrm{d}^Dp}{(2\pi)^D}\,\frac{1}{p^2+\sigma} \tag{11.6}$$

for the trial function (11.5). Not surprisingly, the variational parameter σ directly provides us with the *mass gap* of the theory [when the right-hand side of eq. (11.6) is well defined], as already anticipated from the form (11.5).

The potential $V(\sigma)$ is readily computed. For the general N-component version of the action (11.1) one finds [11.4]

$$\begin{aligned}V(\sigma)=N\Bigg[&\frac{(m^2-\sigma)^2}{\sigma m^2(m^2+\sigma)}+\frac{\lambda}{3N}\,\frac{(2+N)(m^2-\sigma)}{\sigma^2m^2(m^2+\sigma)}+\frac{\lambda^2}{(3!)^2N^2}\\&\times\frac{(8m^2+16\sigma)+N(6m^2+14\sigma)+N^2(m^2+3\sigma)}{\sigma^3m^2(m^2+\sigma)(m^2+3\sigma)}\Bigg]\end{aligned} \tag{11.7}$$

(where we have rescaled $\lambda \to \lambda/N$). This function is easily minimized for any value of λ, and the value of σ at that point can then be substituted back into eq. (11.5) to yield $\phi_\sigma(x, t)$.

Shown in fig. 11.1 is a plot of $\langle \boldsymbol{\phi} \cdot \boldsymbol{\phi} \rangle / N$ versus the coupling for $N = 5$. The dashed line shows the result of the variational procedure outlined above, and the fully drawn line represents the exact answer which, in this case, is easily found directly from the action (11.1). As can be seen, the variational solution agrees almost perfectly with the exact expression for all values of the couplings. For further examples of this kind, and in particular a variational scheme for quantum mechanical excitation energies (normally a very difficult problem), we refer the reader to ref. [11.4].

The simple example discussed here has been included mainly to illustrate the many new possibilities one has in working with a *classical* 'equation of motion' such as the Langevin equation. The particular choice of a variational calculation based on the square of this equation [ref. 8.19] is of course completely *ad hoc*, and one can imagine many other approaches. Also, one is not necessarily restricted to variational principles alone.

11.2. *The Langevin equation on a lattice*

For a scalar field theory, discretizing space-time is completely trivial: $\phi(x)$ is now just a function of a discrete number of points, and derivatives are replaced by differences. Discretizing the fictitious time of the Langevin equation is not difficult either. Instead of $\langle \eta(x, t)\, \eta(x', t') \rangle \propto \delta(t - t')$, we want to introduce the step-size δt and the Kronecker delta $\delta_{tt'}$. The most straightforward discretization of the Langevin equation for a scalar field theory then reads [11.5]

$$\phi(x, t + \delta t) - \phi(x, t) = -\frac{\delta S}{\delta \phi(x, t)}\, \delta t + \sqrt{\delta t}\, \eta(x, t) \tag{11.8}$$

where the factor of $\sqrt{\delta t}$ in front of $\eta(x, t)$ appears from requiring η-correlations of the simple form

$$\langle \eta(x, t)\, \eta(x', t') \rangle = 2\delta_{xx'}\delta_{tt'}\,. \tag{11.9}$$

Clearly, by continuity, in the limit $\delta t \to 0$ we recover the standard continuous-time formulation which

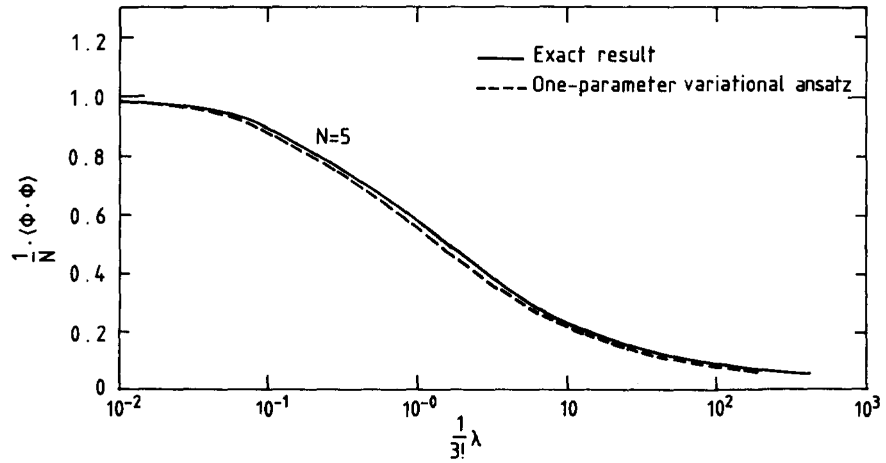


Fig. 11.1. The variational solution for $\langle \boldsymbol{\phi} \cdot \boldsymbol{\phi} \rangle / N$ (dashed line), compared with the exact result (fully drawn line).

in the infinite time limit has a probability distribution of the form $\exp(-S[\phi])$. For a numerical solution of (11.8) the question then is with what dimensionful quantity δt should be compared, and, once established, what is the 'error' introduced by keeping δt finite. We have put the term 'error' in quotes because, as has been emphasized in ref. [11.6], one can always view eq. (11.8) as defining a stochastic evolution equation for *any* step size. (Instabilities may of course occur for some ranges of δt, in which case this evolution equation will not have a well-defined equilibrium limit, see below.) From this point of view the discrete equation (11.8) is not fundamentally different from the continuous-time Langevin equation. The only difference, for sufficiently small step size δt, is that the equilibrium limit of the probability distribution will no longer be $\exp(-S[\phi])$ but rather $\exp(-\tilde{S}[\phi]) = \exp(-S[\phi])\, f[\phi]$ where $f[\phi]$ presumably will be δt-expandable,

$$f[\phi] = 1 + S_1[\phi]\,\delta t + \cdots \tag{11.10}$$

for sufficiently small δt. Thus, at least to this order in δt one can view (11.8) as defining a slightly modified theory with an effective action [11.6]

$$\tilde{S}[\phi] = S[\phi] - S_1[\phi]\,\delta t + \mathrm{O}(\delta t^2)\,. \tag{11.11}$$

Keeping the error small thus amounts to reducing the effects of $S_1[\phi]\,\delta t + \cdots$ on the computed correlation functions.

Pinning down what is meant by 'sufficiently small δt' requires a separate analysis. To illustrate, consider first a *free* scalar field theory with action $\frac{1}{2}\Sigma_{x,y}\, \phi(x)\, M(x, y)\, \phi(y)$. In this case the corresponding Langevin equation

$$\phi(x, t+\delta t) - \phi(x, t) = -\sum_y M(x, y)\, \phi(y)\, \delta t + \sqrt{\delta t}\, \eta(x, t) \tag{11.12}$$

is exactly solvable. For example, if we introduce a kernel $K(x, y) = M^{-1}(x, y)$,

$$\sum_z M^{-1}(x, z)\, M(z, y) = \delta_{x,y} \tag{11.13}$$

so that the time step δt becomes *dimensionless*, and the noise–noise correlation takes the form

$$\langle \tilde{\eta}(x, t)\, \tilde{\eta}(x', t') \rangle = 2\, M^{-1}(x, x')\, \delta_{tt'} \tag{11.14}$$

then we have immediately

$$\phi(x, t_n) = \sum_{j=0}^{n-1} (1-\delta t)^{n-j-1} \sqrt{\delta t}\, \tilde{\eta}(x, t_j) \tag{11.15}$$

and hence, for the two-point function:

$$\begin{aligned} \langle \phi(x, t_n)\, \phi(x', t_n) \rangle &= 2\,\delta t\, M^{-1}(x, x') \sum_{j=0}^{n-1} (1-\delta t)^{n-j-1} \sum_{i=0}^{n-1} (1-\delta t)^{n-i-1} \delta_{ij} \\ &= 2\,\delta t\, M^{-1}(x, x')\, (1-\delta t)^{-2(n-1)} \sum_{j=0}^{n-1} (1-\delta t)^{-2j} = M^{-1}(x, x') \left(\frac{1}{1-\frac{1}{2}\delta t}\right) [1-(1-\delta t)^{2n}]\,. \end{aligned} \tag{11.16}$$

We learn two things from this: First, as $n \to \infty$,

$$\langle \phi(x, t_n)\, \phi(x', t_n) \rangle \to M^{-1}(x, x') \left(\frac{1}{1 - \frac{1}{2}\delta t} \right) \tag{11.17}$$

so that even in the infinite time limit we do not recover the exact answer $M^{-1}(x, x')$. (For the error to be small, we require $\delta t \ll 1$.) Second, the approach to equilibrium follows the power law of eq. (11.16). As expected, taking δt smaller decreases the final error, but increases the number of steps n needed to reach equilibrium.

Having found the exact two-point function in the equilibrium limit, one has in this exactly solvable case also the explicit form of the effective action $\tilde{S}[\phi]$ of eq. (11.11). It follows from eq. (11.17) that all terms of order δt^2 or higher vanish, and one has simply:

$$\tilde{S}[\phi] = \frac{1}{2} \sum_{x,y} \phi(x)\, M(x, y)\, \phi(y) - \frac{1}{4}\, \delta t \sum_{x,y} \phi(x)\, M(x, y)\, \phi(y) \tag{11.18}$$

in the equilibrium limit. This action is unbounded from below if $\delta t > 2$.

The equilibrium action (11.18) is not unique. It depends not only on the particular discretization prescription for the Langevin equation, but also on our particular choice of kernel $K(x, y) = M^{-1}(x, y)$. (Note the contrast to the continuum case: there all kernels which can lead to an equilibrium action will have the *same* limit as $t \to \infty$.) If one does not use any kernel at all, but starts directly at eq. (11.12), one obtains instead [11.3, 11.6]:

$$\tilde{S}[\phi] = \tfrac{1}{2} \sum_{x,y} \phi(x)\, \tilde{M}(x, y)\, \phi(y) \tag{11.19}$$

with

$$\tilde{M}(x, y) = \sum_{z} M(x, z)\, [\delta_{z,y} - \tfrac{1}{2}\delta t\, M(z, y)]\,. \tag{11.20}$$

The error is in this case dependent on the operator M. This not only makes the analysis of stability and errors more complicated, but also has a serious negative effect on numerical simulations: as M generally will be x-dependent, long and short distance modes will equilibrate at different rates [11.6]. Roughly, one could expect that the short distance equilibration rate should be determined by the ultraviolet cut-off (i.e. the lattice spacing a), whereas the long distance equilibration rate should be given by the correlation length ξ. (This simple argument assumes that there is no connection between the ultraviolet and infrared scales, an assumption which for a general interacting theory is not valid when renormalization is taken into account. See ref. [11.6] for a short discussion of this problem.) This implies, from eq. (11.20), that in this case the relative error for the two-point function is roughly $\frac{1}{2}\delta\tau\, \lambda_{\max} \simeq \frac{1}{2}\delta\tau\, \xi^2$, where $\lambda_{\max} = \xi^2$, the largest eigenvalue of M, is the square of the correlation length of the system [11.3]. If one attempts to approach the continuum theory the correlation length diverges, $\xi \to \infty$, and the step size must then approach zero at least as fast as ξ^{-2} in order to keep the error bounded. The same conclusion is reached if one instead focuses on the number of iterations needed to obtain a new, statistically uncorrelated configuration [11.6] since, as we saw earlier, the number of steps needed to approach equilibrium diverges as $\delta\tau \to 0$.

The above problems were not present in our previous discussion, which made use of a kernel

$K(x, y) = M^{-1}(x, y)$. In fact, the use of such kernels is just one of the methods which have been advocated for the solution of the various problems associated with the simplest Langevin equations [11.6]. Conveniently expressed in momentum space rather than in real space, it makes use of field configuration updates on the lattice in momentum space by means of fast Fourier transforms. The idea of performing momentum space updates of this kind was first suggested by Parisi [11.7].

So far the discussion has been restricted to a trivial, solvable, free scalar field theory. In the general interacting case one can not expect to see clearly which kernel is the optimal one for removing (some of) the problems with the Langevin equation. However, it is reasonable to assume that the main features found for that case [i.e. $K(x, y) \sim M^{-1}(x, y)$] should persist to some degree in the full theory. This already has some support from numerical simulations [11.6, 11.8].

To determine the general form of the effective action $\tilde{S}[\phi]$ one must proceed via a Fokker–Planck analysis, as in the continuum case. With only a little more algebra one finds, using a discretized version of our derivation in section 3, the following Fokker–Planck equation [11.6]

$$\delta t^{-1}\{P[\phi, t+\delta t] - P[\phi, t]\} = \sum_x \frac{\delta}{\delta\phi(x)} \left\{ \frac{\delta P[\phi, t]}{\delta\phi(x)} + \frac{\delta\tilde{S}}{\delta\phi(x)} P[\phi, t] \right\} \tag{11.21}$$

up to order δt. This looks exactly as the continuum version. However, $\tilde{S}[\phi]$ is not the starting action; it contains a 'correction term':

$$\tilde{S}[\phi] = S[\phi] + \delta t \sum_{x,y} K(x, y) \left\{ 2 \frac{\delta^2 S}{\delta\phi(x)\,\delta\phi(y)} - \frac{\delta S}{\delta\phi(x)} \frac{\delta S}{\delta\phi(y)} \right\} \tag{11.22}$$

again keeping only terms of $O(\delta t)$. Clearly the $t \to \infty$ solution of (11.21) is

$$P[\phi, t \to \infty] = \exp(-\tilde{S}[\phi]) \tag{11.23}$$

up to order δt. The Fokker–Planck approach may seem to offer a systematic derivation of $O(\delta t^2)$ (and higher) terms in $\tilde{S}[\phi]$, but this could be deceptive: it is very difficult to see how the $O(\delta t^2)$ terms can be incorporated into an effective $\tilde{S}[\phi]$ without running into integrability problems. Nevertheless, we can now get a quick check of our effective action (11.18) which we derived there in a much more pedestrian approach. Indeed, inserting the action corresponding to eq. (11.12) into eq. (11.22) with $K(x, y) = M^{-1}(x, y)$ we reproduce eq. (11.18) up to an irrelevant constant. Similarly, if we choose $K(x, y) = \delta_{x,y}$ we obtain eq. (11.19).

The fact that $\tilde{S}[\phi] \neq S[\phi]$ means that even in the *equilibrium limit* we do not recover the original theory. This may sound like a serious shortcoming, but as has been emphasized in ref. [11.6] this is not necessarily so. For example, if one makes sure that the kernel $K(x, y)$ is chosen such that the Langevin equations respect all the symmetries of the original theory, then the extra terms in $\tilde{S}[\phi]$ can either just shift the original couplings, or introduce super-renormalizable interactions ('irrelevant operators'). In the scaling region neither of these effects should be observable; as long as only physical quantities are compared, the continuum limit (in the sense of the lattice spacing $a \to 0$) of the finite-δt theory is exactly the same as that of the original theory. Moreover, working always only to order δt, a simple shift of variables [11.6] can always leave an effective $\tilde{S}[\phi]$ which differs from $S[\phi]$ only by a calculable shift in the coupling constants.

11.3. Gauge theories

The Langevin equation for gauge fields on a lattice looks quite different from the standard gauge field Langevin equation in the continuum (see section 4). This is due to the fact that on a lattice it is much more natural (but of course not necessary) to work with elements of the Lie *group*, instead of elements of the Lie *algebra*. We shall not go into further details on this issue here, but simply assume that the reader has an at least rudimentary knowledge of the formalism of lattice gauge theories.

When integrating on group space we shall always be referring to the left and right invariant Haar measure for the group. (Only *compact* gauge groups will be considered, and we will of course mainly keep the SU(N) groups in mind.) This measure will be denoted by $\mathrm{d}U$. The fact that it is left and right invariant means that $\mathrm{d}(U \cdot V) = \mathrm{d}(V \cdot U) = \mathrm{d}U$, for any element V of the group. Differentiation (more precisely, right differentiation; a left differentiation can be defined similarly) of a function f can be defined by [11.9]

$$f(U \exp(\mathrm{i}\alpha_a T^a)) = f(U) + \alpha_a \nabla_a f(U) + \tfrac{1}{2}\alpha_a \alpha_b \nabla_a \nabla_b f(U) + \cdots \tag{11.24}$$

with α_a being an infinitesimal vector of, for SU(N), $N^2 - 1$ elements. The T_a's are the usual generators of the group, with $[T^a, T^b] = \mathrm{i}c^{abc}\, T^c$ and, by convention, $\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$.

Using the left or right invariance of the Haar measure we have, for example,

$$\int \mathrm{d}U\, f(U) = \int \mathrm{d}U\, f(U \exp(\mathrm{i}\alpha_a T^a))\,. \tag{11.25}$$

Choosing α_a infinitesimal, and inserting (11.24) this leads to

$$\alpha^a \int \mathrm{d}U\, \nabla_a f(U) = 0 \tag{11.26}$$

and hence a simple formula for integration by parts [11.9]:

$$\int \mathrm{d}U\, [\nabla_a f(U)]\, g(U) = -\int \mathrm{d}U\, f(U)\, \nabla_a\, g(U)\,. \tag{11.27}$$

From lattice gauge theory the reader may be more familiar with the derivative in the representation of a 'right derivative' on group elements:

$$\nabla^a = \mathrm{i}(UT^a)_{ij}\, \partial / \partial U_{ij} \tag{11.28}$$

where by definition

$$\frac{\partial}{\partial U_{kl}}\, U_{ij} = \delta_{ik}\delta_{jl}\,. \tag{11.29}$$

Derivatives of other quantities can be derived in various ways. For example, how can we compute $\partial U^{\dagger}_{ij} / \partial U_{kl}$? Using the chain rule on the unitarity constraint $U^{\dagger}U = 1$, we find

$$[\nabla^a U]\, U^{\dagger} = -U \nabla^a U^{\dagger} \tag{11.30}$$

i.e., by multiplying with $U^\dagger$ to the left:

$$(T^a U^\dagger)_{kl} = -\delta_{lm}(UT^a)_{ij} \frac{\partial}{\partial U_{ij}} U^\dagger_{mk} \tag{11.31}$$

and hence

$$\frac{\partial}{\partial U_{kl}} U^\dagger_{ij} = -U^\dagger_{ik} U^\dagger_{lj} \,. \tag{11.32}$$

Differentiation rules for more complicated objects can be worked out in similar ways. Care is always required when more than one derivative is involved, since the ∇^a's do not commute; rather, they satisfy the Lie algebra itself,

$$[\nabla^a, \nabla^b] = -f^{ab}_c \nabla^c \,. \tag{11.33}$$

In the continuum-time formalism a suitable Langevin equation reads [11.10] (see also ref. [11.11] for a more general discussion):

$$\frac{\partial}{\partial t} U(t) = [\nabla S[U] + \mathrm{i}\eta(t)]\, U(t) \tag{11.34}$$

for a group element $U(t)$ of an action $S[U]$. Here $\nabla \equiv T^a \nabla^a$ and $\eta(t) \equiv T^a \eta^a(t)$, with $\eta^a(t)$ having the usual Gaussian distribution

$$\langle \eta^a(t)\, \eta^b(t') \rangle = 2\, \delta^{ab} \delta(t - t') \,. \tag{11.35}$$

The equation (11.34) was constructed such that when $t \to \infty$ one obtains the standard $\exp\{-S[U]\}$ probability distribution. This can be seen by a Fokker–Planck argument [11.10], very similar to the scalar field case. It is also important to check that eq. (11.34) leads to a diffusion process which remains on the group manifold. Indeed, from eq. (11.34) we have, on account of the anti-Hermiticity of ∇^a (and the Hermiticity of the group generators),

$$\left[\frac{\partial}{\partial t} U(t)\right] U^\dagger(t) = \nabla\, S[U] + \mathrm{i}\, \eta(t) \tag{11.36a}$$

$$U(t) \frac{\partial}{\partial t} U^\dagger(t) = -\nabla\, S[U] - \mathrm{i}\, \eta(t) \tag{11.36b}$$

i.e.

$$\frac{\partial}{\partial t} [U(t)\, U^\dagger(t)] = 0 \tag{11.37}$$

so eq. (11.34) preserves the unitarity constraint.

The simplest choice of a discretized Langevin equation, which in the limit $\delta t \to 0$ reduces to eq. (11.34) and which preserves unitarity is [11.9, 11.10]

$$U(t + \delta t) = U(t) \exp\{\mathrm{i}W[U(t), \eta(t)]\} \tag{11.38}$$

where $W[U(t), \eta(t)]$ is Hermitian:

$$W[U(t), \eta(t)] = -\mathrm{i}\,\delta t\,\nabla\,S[U(t)] + \sqrt{\delta t}\,\eta(t) \tag{11.39}$$

and $\eta(t) = T^a\,\eta^a(t)$ has the usual discretized correlations:

$$\langle \eta^a(t)\,\eta^b(t')\rangle = 2\,\delta^{ab}\delta_{tt'}\,. \tag{11.40}$$

Clearly, eq. (11.38) also preserves the unitarity constraint. However, just as in the case of a scalar field theory, the equilibrium limit (i.e. the limit $t \to \infty$) will *not* be just $\exp\{-S[U]\}$, but will have $O(\delta t)$ corrections. Exponentiating these corrections, we have to $O(\delta t)$ an equilibrium probability distribution $P[U, t\to\infty] = \exp\{-\tilde{S}[U]\}$ with $\tilde{S}[U]$ being of the form $\tilde{S}[U] = S[U] - S_1[U]\,\delta t + O(\delta t^2)$. The general form of $S_1[U]$ for the Langevin equation (11.34) has been computed in refs. [11.6, 11.8, 11.9]:

$$S_1[U] = \sum_l \tfrac{1}{2}\nabla^2\,S[U] - \frac{C_A}{12}\,S[U] - \sum_l \tfrac{1}{4}\nabla^a\,S[U]\,\nabla^a\,S[U] \tag{11.41}$$

where $\nabla^2 \equiv \nabla^a\nabla^a$ and C_A is the quadratic Casimir in the adjoint representation, i.e. for SU(N) groups we have $C_A = N$.

For a standard lattice gauge theory action of the form

$$S[U] = \frac{\beta}{2N}\sum_p\,[\mathrm{Tr}\,U_p + \mathrm{Tr}\,U_p^\dagger] \tag{11.42}$$

with plaquette variables $U_p = U_\mu(x)\,U_\nu(x+\mu)\;U_\mu^\dagger(x+\nu)\,U_\nu^\dagger(x)$, the first two terms in eq. (11.41) simply shift the coupling β [11.6, 11.8]:

$$\beta \to \tilde{\beta} = \beta\left[1 - \left(C_F - \frac{C_A}{12}\right)\delta t\right] \tag{11.43}$$

with C_F being the quadratic Casimir invariant in the fundamental representation (in which the link variables $U_\mu(x)$ belong to the standard action), i.e. for SU(N), $C_F = (N^2-1)/2N$. The last term in eq. (11.41) can be removed by a change of variables [11.6]

$$U_\mu(x) \to \tilde{U}_\mu(x) = U_\mu(x)\exp\{-\mathrm{i}\tfrac{1}{4}\nabla\,S[U]\,\delta t\} \tag{11.44}$$

and all $O(\delta t)$ corrections are then absorbed. The equilibrium action up to order $O(\delta t^2)$ is indeed now of the form (11.42), but with shifted coupling and shifted link variables. Ample evidence has been given [11.6] that this procedure reproduces known results (from more standard Monte Carlo methods) of pure gauge theories. Other discretizations [11.6, 11.8] of the Langevin equation (e.g. of the Runge–Kutta type) also yield quite promising results with regards to reducing $O(\delta t)$ 'errors'. Such methods can in principle be used to eliminate corrections of arbitrary order in δt, but they rapidly become quite complicated.

The use of *kernels* turns out to be a particularly difficult issue when discussing numerical simulations of lattice gauge theories. We have already in section 4 discussed some of the problems with kernels in connection with gauge theories in the continuum. Although not understood in detail, the problems are

clearly connected with the fact that only very few kernels respect the gauge covariance of the original Langevin equation. On the lattice this problem remains. Furthermore, it is not obvious *which* kernel should be introduced in order to remedy, for example, the long equilibration rate for low-momentum modes (the simple scalar field example discussed earlier in this section clearly is no guideline due to the gauge invariance. Even in the continuum one cannot, of course, use a kernel such as the propagator itself without gauge fixing!). Some suggestions concerning the possibility of using particular gauge fixed Langevin equations to overcome these difficulties have been mentioned [11.6], but it is not yet clear to what extent they represent improvements.

11.4. Including fermions

Coupling scalar fields to the link variables discussed above is a rather trivial undertaking. The real challenge lies in finding an efficient way of incorporating *fermions*. Several algorithms which take care of this problem have been proposed [11.6, 11.8, 11.12, 11.13]. All these algorithms look quite different from the fermion Langevin equations we discussed in section 6. This is for good reasons: in numerical simulations it is quite difficult (but actually perhaps not impossible, at least on a finite number of lattice points) to operate with anticommuting numbers! The standard way out of this problem is to consider the generating functional *after* having formally integrated out all fermion fields, instead of the original generating functional itself. Of course, no approximation is involved in such a procedure [11.19], but the resulting effective piece of the action, $\mathrm{Tr}\ln[O]$ for a bilinear fermion term $\bar{\psi}O\psi$, is still difficult to evaluate. A tremendous amount of effort has gone into constructing efficient (fast) and accurate approximation schemes for this extra piece in the action. We shall not discuss these different techniques here at all, but instead refer to the recent comparative study of Weingarten [11.3].

Let us for the sake of convenience begin the discussion in continuum notation. The extra term we have to deal with in the case of a gauge theory reads

$$\int \mathrm{D}\bar{\psi}\,\mathrm{D}\psi \exp\left\{-\int \mathrm{d}^D x\, \bar{\psi}(x)\,[\mathrm{i}\not{D} - m]\,\psi(x)\right\} = \det[-\mathrm{i}\not{D} + m]\,. \tag{11.45}$$

Using $\det M = \exp(\mathrm{Tr}\ln M)$ this term can formally be introduced as an extra piece in the action, as mentioned above, but this does not really represent a simplification. Instead, one can try to 'bosonize' this term by means of the identity

$$\det[-\mathrm{i}\not{D} + m] = \det[\gamma_5(-\mathrm{i}\not{D} + m)\gamma_5] = \det[\mathrm{i}\not{D} + m] \tag{11.46}$$

which implies

$$\det[-\mathrm{i}\not{D} + m] = \{\det[-\mathrm{i}\not{D} + m]\det[\mathrm{i}\not{D} + m]\}^{1/2} = \{\det[\not{D}^2 + m^2]\}^{1/2} \tag{11.47}$$

and we are in fact now dealing with a positive definite operator. (Recall that in Euclidean space it is $-\partial^2 + m^2$ which leads to the positive definite inverse propagator $p^2 + m^2$.) This can conveniently be represented by a *bosonic* path integral [11.14]

$$\int \mathrm{D}\phi\,\mathrm{D}\phi^* \exp\left\{-\int \mathrm{d}^D x\, \phi^*(x)\,[-D^2 + m^2]^{-2}\,\phi(x)\right\} \tag{11.48}$$

and of course similar manipulations can be performed on the generating functional itself. This, then, leads to the first way of introducing fermions in numerical applications of the Langevin formalism [11.8]: for a Wilson fermion formulation the lattice equivalent of $(-\mathrm{i}\not{D} + m)$ is

$$D = 1 - k\,\tfrac{1}{2} \sum_{x,\mu} \{(1-\gamma_\mu)\, U_\mu(x) + (1+\gamma_\mu) U^\dagger_\mu(x)\} \tag{11.49}$$

which, for example, can be squared in order to make it positive definite: $D^2 = D^\dagger D$. This corresponds to doubling the number of fermions.

The Langevin equations for this case are eq. (11.34) with $S[U]$ replaced by $S[U, \phi^*, \phi]$ together with [11.8]

$$\phi(t+\delta t) = \phi(t) - \delta t\, D^{-2}[U(t)]\, \phi(t) + \eta(t)\, \delta t \tag{11.50}$$

and its complex conjugate equation. Here $\eta(t)$ is the usual Gaussian noise field.

Numerical applications of this Langevin algorithm, together with its possible Runge–Kutta improvements, have been presented by Ukawa, Fukugita and Oyanogi [11.8, 11.5]. A somewhat different fermion algorithm was suggested by the Cornell group [11.6], and developed into higher accuracy by Batrouni [11.12] and Kronfeld [11.13]. Here the fermion determinant is not exponentiated by means of auxiliary scalar fields, but is rather evaluated directly by means of just one (slightly modified) Langevin equation for the link variables, at the cost of an introduction of a complex scalar noise field for each of the components in the fermion field. In the most simple version, which is valid up to order δt only, this is done by adding an extra piece to $W[U(t), \eta(t)]$ of eq. (11.39):

$$W[U(t), \eta(t)] \to \tilde{W}[U(t), \eta(t), \theta(t)] = -\mathrm{i}\,\delta t\, \nabla S[U] + \mathrm{i}\, \frac{\delta t}{4}\, \theta^\dagger(t)\, D^{-1}\, (\nabla D^2)\, D^{-1}\, \theta(t) + \sqrt{\delta t}\, \eta(t) \tag{11.51}$$

where the fields $\theta^\dagger(t), \theta(t)$ are ordinary commuting noise fields with the usual Gaussian distribution. Indeed, since to lowest order in δt the leading contribution to the $t \to \infty$ equilibrium action comes from simply averaging (11.51) w.r.t. the fields $\theta^\dagger(t)$ and $\theta(t)$, one sees that with

$$\begin{aligned} \langle \tilde{W}[U, \eta, \theta] \rangle_{\theta^\dagger, \theta} &= -\mathrm{i}\,\delta t\, \nabla S[U] + \mathrm{i}\,\delta t\, (\nabla D)\, D^{-1} + \sqrt{\delta t}\, \eta(t) \\ &= -\mathrm{i}\,\delta t\, \nabla (S[U] - \mathrm{Tr}\ln D[U]) + \sqrt{\delta t}\, \eta(t) \end{aligned} \tag{11.52}$$

the main effect precisely has been to replace $S[U]$ by $S_{\mathrm{eff}}[U] = S[U] - \mathrm{Tr}\ln D[U]$. Schemes have been developed [11.12, 11.13] which eliminate corrections of higher order in δt; the advantages of such modifications not yet being known in terms of algorithm speed, storage problems, etc. We refer the reader to these two references for the technical details.

It should be clear at this point that feasible numerical Langevin algorithms for complete gauge theories, such as QCD with 'dynamical' quarks, exist. Full application to the variety of numerical problems in lattice gauge theories still remain to be carried out, but the framework clearly looks very promising.

11.5. Complex actions

We have always assumed that the Euclidean action, whose field derivative appears in the Langevin

equation, was *real*: $S[\phi] = S^*[\phi]$. The only places where we allowed for some departure from this restriction was in the case of Minkowski space stochastic quantization (see section 9) and stochastic quantization of gravity (or, more generally, strings) as in section 5. Complex terms can also appear in the Langevin equations for fermions (section 6), but can there usually be removed by suitable kernels. For a simple *scalar* field an interesting question is: What happens if the Langevin equation describes the diffusion around a *complex* action? Clearly, most of the results known from the *real* case (a positive definite Fokker–Planck Hamiltonian, exponential time damping, ergodicity, etc.) do not apply to this new situation, at least not without some modifications. That it nevertheless should be possible to get meaningful results out of Langevin diffusion processes around complex actions was first suggested by Klauder [11.16] and Parisi [11.17]. We have postponed our discussion of this interesting extension of stochastic quantization until now simply because most of the results are of numerical nature. Some analytic results have also been obtained [11.18, 11.19, 11.20], and we shall try to review them in some detail.

Why complex actions? It is a very general requirement that a well-defined Euclidean quantum field theory should be 'reflection positive', and this is normally equivalent to demanding that the action is real. However, there are exceptions. One is the case of a lattice gauge theory with the links in a complex representation of the gauge group [such as the fundamental representation of SU(3)] *and* at finite chemical potential for, e.g. the fermion density. Although the action itself is complex in this case, all physical observables can nevertheless be shown to be real: all imaginary quantities cancel pairwise between 'conjugate' gauge field configurations in the functional integral. Complex actions can also appear in a self-imposed way if, for example, one incorporates the effect of external charge probes (such as the Polyakov line at finite temperature) into the action itself. This can sometimes be advantageous in numerical simulations.

With this to serve as motivation, let us start by analysing a simple toy model which clearly illustrates the possibilities (and problems) connected with complex Langevin equations [11.18, 11.19]. Consider the 'action' of field theory in zero dimensions:

$$S(x) = \tfrac{1}{2}\sigma x^2 + \tfrac{1}{4}\lambda x^4 \,. \tag{11.53}$$

We shall assume that λ is real and positive, but that σ can take any value in the complex plane. Furthermore, although one can envisage a more general situation, we shall in this section always assume that the noise field of the Langevin equation is restricted to be real and with the usual noise–noise correlation of a Markov process. Even if one uses *real* initial conditions for $x(t = 0)$, and restricts oneself to a real noise, the diffusion process described by the Langevin equation corresponding to (11.53),

$$\frac{\partial}{\partial t} z(t) = -\sigma\, z(t) - \lambda\, z(t)^3 + \eta(t) \tag{11.54}$$

will of course not in general be restricted to the real axis if $\sigma \in \mathbb{C}$. It is therefore convenient to split the solution of (11.54) into its real and imaginary parts [11.19] $z(t) = x(t) + \mathrm{i}y(t)$, and write the complex differential equation (11.54) as

$$\frac{\partial}{\partial t} x(t) = -\mathrm{Re}[\sigma\, z(t) + \lambda\, z(t)^3] + \eta(t) \tag{11.55a}$$

$$\frac{\partial}{\partial t} y(t) = -\mathrm{Im}[\sigma\, z(t) + \lambda\, z(t)^3] \,. \tag{11.55b}$$

We now have two real, coupled, differential equations, and the situation is hence quite analogous to the case of two interacting real fields. In fact, it is *simpler*, since in this case there is no noise field associated with the imaginary part $y(t)$. The Fokker–Planck equation for this system then takes the simple form [see, for example, sections 3 and 9. Note the absence of a term $\partial^2 P/\partial y^2$ due to the noise field $\eta(t)$ being real]:

$$\frac{\partial}{\partial t} P(x, y, t) = \frac{\partial^2}{\partial x^2} P(x, y, t) + \frac{\partial}{\partial x} [\mathrm{Re}(\sigma\, z(t) + \lambda\, z(t)^3)\, P(x, y, t)] + \frac{\partial}{\partial y} [\mathrm{Im}(\sigma\, z(t) + \lambda\, z(t)^3)\, P(x, y, t)]\,. \tag{11.56}$$

For $\lambda = 0$ this becomes, with $\sigma = \sigma_{\mathrm{R}} + \mathrm{i}\sigma_{\mathrm{I}}$:

$$\frac{\partial}{\partial t} P(x, y, t) = \frac{\partial^2}{\partial x^2} P(x, y, t) + \frac{\partial}{\partial x} [(\sigma_{\mathrm{R}} x - \sigma_{\mathrm{I}} y)\, P(x, y, t)] + \frac{\partial}{\partial y} [(\sigma_{\mathrm{I}} x + \sigma_{\mathrm{R}} y)\, P(x, y, t)] \tag{11.57}$$

and the equilibrium distribution can in this case be found explicitly [11.20] (see also ref. [11.21]). To give a simple derivation, let us note that since in this case the starting action is a simple Gaussian, we do not expect the equilibrium distribution $P(x, y, t \to \infty)$ to be more than quadratic in x and y. We can then make the *ansatz*

$$P(x, y, t \to \infty) = \exp(-Ax^2 - By^2 - Cxy)\,. \tag{11.58}$$

Substituting (11.58) into (11.57), and using $\partial P(x, y, t \to \infty)/\partial t = 0$, we immediately find that demanding terms proportional to $P(x, y, t \to \infty)$ to cancel requires $A = \sigma_{\mathrm{R}}$. Using this, and demanding also that terms proportional to $x^2 P(x, y, t \to \infty)$ cancel gives $C = 2\sigma_{\mathrm{R}}^2/\sigma_{\mathrm{I}}$. Similarly, cancellation of terms proportional to $y^2 P(x, y, t \to \infty)$ leads to $B = \sigma_{\mathrm{R}}(1 + 2\sigma_{\mathrm{R}}^2/\sigma_{\mathrm{I}})$. Finally, as a check we see that also all terms proportional to $xy\, P(x, y, t \to \infty)$ cancel. An equilibrium distribution for the case of $\lambda = 0$ is hence [11.20]

$$P(x, y, t \to \infty) = \exp\left[-\sigma_{\mathrm{R}}\left\{x^2 + \left(1 + \frac{2\sigma_{\mathrm{R}}^2}{\sigma_{\mathrm{I}}}\right) y^2 + 2\,\frac{\sigma_{\mathrm{R}}}{\sigma_{\mathrm{I}}}\, xy\right\}\right] \Big/ N \tag{11.59}$$

with the normalization

$$N = \int \mathrm{d}x\, \mathrm{d}y\, P(x, y, t \to \infty)\,. \tag{11.60}$$

The result (11.59) may at first glance seem surprising, since the equilibrium distribution obviously is *not* of the form $\exp[-\frac{1}{2}\sigma z^2]$, with $z = x + \mathrm{i}y$. However, this latter quantity is of course simply a complex number which cannot be given a probabilistic interpretation, whereas $P(x, y, t \to \infty)$ of eq. (11.59) is a regular, normalized and real probability distribution. In fact, the distribution (11.59) is precisely of the 'right' form needed to obtain [11.20]

$$N_{\mathrm{compl.}}^{-1} \int \mathrm{d}x\, f(x) \exp(-\tfrac{1}{2}\sigma x^2) = \int \mathrm{d}x\, \mathrm{d}y\, f(x + \mathrm{i}y)\, P(x, y, t \to \infty) \tag{11.61}$$

with

$$N_{\text{compl.}} = \int \mathrm{d}x \exp(-\tfrac{1}{2}\sigma x^2) = (2\pi/\sigma)^{1/2} \tag{11.62}$$

(we assume that σ does not fall on the branch cut).

To see this, we note that since the left-hand side is a free (Gaussian) theory, we need only compute the 2-point function $\langle x^2 \rangle$ in order to obtain all other Green functions (i.e. $\langle x^{2n} \rangle$, with n positive and integer). Then, for the left-hand side:

$$N^{-1}_{\text{compl.}} \int \mathrm{d}x\, x^2 \exp(-\tfrac{1}{2}\sigma x^2) = 2\left(\frac{\sigma}{2\pi}\right)^{1/2} \frac{\partial}{\partial\sigma}\left(\frac{2\pi}{\sigma}\right)^{1/2} = \sigma^{-1}\,. \tag{11.63}$$

Similarly, introducing

$$M = \begin{pmatrix} A & C/2 \\ C/2 & B \end{pmatrix}, \qquad N = \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & -1 \end{pmatrix} \tag{11.64}$$

[in the notation of eq. (11.58)] we note that the right-hand side takes the form

$$\int \mathrm{d}\boldsymbol{x}\, \boldsymbol{x}^{\mathrm{T}} \cdot N \cdot \boldsymbol{x} \exp(-\boldsymbol{x} \cdot M \cdot \boldsymbol{x}) = \tfrac{1}{2}\, \mathrm{Tr}[M^{-1}N] \tag{11.65}$$

with $\boldsymbol{x}^{\mathrm{T}} = (x, y)$. Using eq. (11.64) this is trivial to evaluate, and one finds precisely $(\sigma_{\mathrm{R}} + \mathrm{i}\sigma_{\mathrm{I}})^{-1}$, in agreement with eq. (11.63).

Instead of working with the real probability distribution whose equilibrium distribution is given by eq. (11.59), we could attempt to set up a '*complex* Fokker–Planck formalism' by using

$$\langle f(z, t) \rangle = \int \mathrm{d}x\, f(x)\, P(x, t) \tag{11.66}$$

(with x a *real* parameter) as a definition of the *complex* Fokker–Planck distribution $P(x, t)$. For example, in the 'free' case $\lambda = 0$ we have already in eq. (11.61) seen how such a procedure may yield a well-defined equilibrium limit. In fact, using (11.66) as a definition, and going through all the standard steps [first taking the t-derivative on both sides of (11.66), then inserting the (complex) Langevin equation, etc.] one gets [11.17, 11.18] the usual Fokker–Planck equation

$$\frac{\partial}{\partial t} P(x, t) = \frac{\partial^2}{\partial x^2} P(x, t) + \frac{\partial}{\partial x}\left[\frac{\partial S(x)}{\partial x} P(x, t)\right]. \tag{11.67}$$

Now, however, convergence towards an equilibrium distribution as $t \to \infty$ is no longer obvious. If we make the usual change of variables to $\Psi(x, t) \equiv \exp[S(x)/2]\, P(x, t)$ (see section 3), we do obtain the Schrödinger-type equation

$$\frac{\partial}{\partial t} \psi(x, t) = -H\, \psi(x, t) \tag{11.68}$$

with

$$H = \left(\mathrm{i}\,\frac{\partial}{\partial x} - \frac{\mathrm{i}}{2}\,\frac{\partial S}{\partial x}\right)\left(\mathrm{i}\,\frac{\partial}{\partial x} + \frac{\mathrm{i}}{2}\,\frac{\partial S}{\partial x}\right) \tag{11.69}$$

but this Fokker–Planck Hamiltonian is clearly not positive definite, on account of $S(x) \neq S^*(x)$.

Intuitively, we might nevertheless expect that if $S(x)$ is 'almost real' (i.e. if H has eigenvalues whose real parts are much bigger than their imaginary parts), then (11.67) may still provide a well-defined convergence as $t \to \infty$. Can this be made more precise? Klauder and Petersen [11.18] find that if

$$\left[\int \mathrm{d}x\, |H_\mathrm{I}\, \psi(x)|^2\right]^{1/2} \le a\left[\int \mathrm{d}x\, |\psi(x)|^2\right]^{1/2} + b\left[\int \mathrm{d}x\, |H_\mathrm{R}\, \psi(x)|^2\right]^{1/2} \tag{11.70}$$

for some $a \ge 0, 0 \le b < 1$ (and $H = H_\mathrm{R} + \mathrm{i}H_\mathrm{I}$), then eigenfunctions of $H\,\Psi(x) = \lambda\,\Psi(x)$ do exist, and they form a complete set. Writing $\tilde{H} = H_\mathrm{R} + \mathrm{i}cH_\mathrm{I}$, eigenvalues $\tilde{\lambda}_n$ and eigenfunction $\tilde{\Psi}_n(x)$ of $\tilde{H}$ will in that case be analytic in the coefficient c about zero. In quantum mechanics the addition of an imaginary potential $\mathrm{i}H_\mathrm{I}$ to H corresponds to introducing sources or sinks of probability. Not much is known about such complex Hamiltonians, and it would clearly be of interest to establish more properties about the eigenfunctions of these types of Hamiltonians.

It follows from the discussion above that if (11.70) holds and $\mathrm{Re}\,\lambda_n > 0$ for all $n \ge 1$, then $P(x, t) \to \exp[-S(x)]$ as $t \to \infty$. This is so, because clearly $\exp[-S(x)/2]$ will be an eigenfunction of eigenvalue $\lambda_0 = 0$, i.e. it is the ground state.

Nevertheless, even if we operate only with complex actions such that (11.70) is obeyed, *and* if we assume that $\mathrm{Re}\,\lambda_n > 0$ for $n \ge 1$, then convergence towards equilibrium may still be very slow, and therefore impractical for numerical simulations. Worse, in the time-discretized case one might imagine instabilities to arise. In fact, a simple argument can be given as to why this might occur: consider again as an example $S(x) = \frac{1}{2}\sigma x^2 + \frac{1}{4}\lambda x^4$ with $\lambda \in \mathbb{R}_+$ and $\sigma \in \mathbb{C}$. Because of the complex σ, the solution $z(t)$ of the Langevin eq. (11.54) will also wander around in the complex plane. But this is potentially dangerous because if it hits the rays $z = \pm r \exp[\pm \mathrm{i}\pi/4]$, then the term $\frac{1}{4}\lambda x^4$ in the action has flipped its sign $\frac{1}{4}\lambda x^4 \to -\frac{1}{4}\lambda x^4$, and we are now in effect simulating a theory with *unbounded action*. This is an oversimplified argument, because we really should look at the Langevin equation itself to see if such potential instabilities can arise. For the case of the x^4-action mentioned above such a stability analysis has been performed by Klauder and Petersen [11.18], and by Ambjorn and Yang [11.20]. To search for the instabilities one can look for solutions to the Langevin equation

$$\frac{\partial}{\partial t}\, z(t) = -\sigma\, z(t) - \lambda\, z(t)^3 \tag{11.71}$$

without the noise term, since now we are interested in solutions with $|z| \to \infty$. The solution to (11.71) for the initial condition $z(0) = z_0$ has been found [11.20],

$$z(t) = z_0 \left[\mathrm{e}^{2\sigma t} + \frac{\lambda z_0^2}{\sigma}\,(\mathrm{e}^{2\sigma t} - 1)\right]^{-1/2} \tag{11.72}$$

and it shows clearly that with $\sigma_\mathrm{R} > 0$, $z(t) \to 0$ for all initial points except those of the form $(t' > 0)$:

$$z_0(t') = \pm \left[\frac{\sigma}{\lambda(1 - e^{-2\sigma t'})} \right]^{1/2} . \tag{11.73}$$

For z_0 of the form (11.73) the denominator in (11.72) vanishes for $t \to t'$, i.e. $|z(t \to t')| \to \infty$ in a *finite* time interval. The general flow pattern is then of convergence towards $z = 0$, *except* for starting points (11.73). The flow is of a four-leaf clover pattern centered around $z = 0$. Although the flow is pushed outwards along the rays $z = \pm r \exp[\pm i\pi/4]$, it nevertheless crosses these rays without diverging, and eventually converges towards $z = 0$. We refer the reader to ref. [11.20], where the flow has been mapped in detail.

Although seemingly singular at $\sigma = 0$, the limit $\sigma \to 0$ can be taken in eq. (11.72) and one of course recovers the solution [11.18]

$$z(t) = z_0 (1 + 2\lambda t z_0^2)^{-1/2} \tag{11.74}$$

of $\partial z(t)/\partial t = -\lambda\, z(t)^3$. For finite $t = t'$ this solution only diverges for purely imaginary starting points $z_0(t') = \pm i(2\lambda t)^{-1/2}$.

For $\sigma_R < 0$ the origin $z = 0$ now becomes *repulsive* [11.20], and the flow instead converges on what was previously the instability points (11.73). The flow is again pushed outwards when crossing the rays $z = \pm r \exp[\pm i\pi/4]$, hence again giving rise to a four-leaf clover pattern. This pattern is also reproduced in ref. [11.20].

In all cases the naive axes of instability $\pm r \exp[\pm i\pi/4]$ are crossed with encountering divergences. The reason is [11.20] that when $z(t)$ reaches these points the drift term $\partial S/\partial z$ is large and with the wrong sign, but the time [from eq. (11.71)]

$$t = -\int_{z_0}^{z} dz \left(\frac{\partial S}{\partial z} \right)^{-1} \tag{11.75}$$

spent there is then correspondingly small. In a sense, the instabilities caused by (11.71) always occur pairwise symmetrically around $z = 0$, and in a numerical simulation should always cancel out. However, such cancellations (which, for example, can be made finite by the introduction of a cut-off: $|z| < Z$) may be quite inconvenient since they introduce statistical errors in the cancellations between large numbers [11.18, 11.20].

Other zero-dimensional complex actions have been studied recently [11.22, 11.23] with even less positive conclusions. Thus, for example, if one attempts to simulate a theory with Boltzmann factor [11.22, 11.23]

$$\exp[\beta \cos\theta + \ln \cos\theta]$$

(integrated over $d\theta$ on the interval $0 \le \theta \le \pi$), *which can attain a zero* (at $\theta = \pi/2$), the Langevin method can fail completely for small β (<1). This is due to the fact that the available phase space has been split into two regions $0 \le \theta < \pi/2$ and $\pi/2 < \theta \le \pi$. The Langevin equation can keep moving around inside each of the domains, but if it does not cross the border $\theta = \pi/2$ in a finite amount of time, ergodicity is lost. By extending the region $0 \le \theta \le \pi$ to the full circle this problem can also be treated by complex Langevin equation techniques. Indeed, the method yields 'wrong' results for small β (due to the random paths being trapped inside a smaller domain) [11.22, 11.23], but the small-β Langevin results appear to fit nicely with expectations if one assumes that the complex Langevin

equation only probes phase space inside *one* of the smaller domains [11.23]. Although it is hence probably well understood, this clearly makes complex Langevin simulations problematic in these cases.

The complex Langevin equation has also been tested on more physical problems such as SU(3) lattice gauge theory in the presence of a chemical potential for quark (baryon) number [11.24], and U(1) lattice gauge theories in 2 and 3 dimensions with static sources [11.22, 11.23]. In the latter case good agreement has been found in the large-β region (strong coupling), but the method appears to be unreliable for smaller β-values, essentially for the reasons mentioned above. Some possible modifications for complex Langevin equations have also been considered in [11.25]. Further applications (as to the massive lattice Schwinger model) can be found in [11.26] and [11.27]. The tentative conclusion appears to be [11.22] that *when* the method works it works very well, but that it can fail completely in certain ranges of the coupling constant(s). If the imaginary part of the action is sufficiently large there is no analytic guarantee that Green functions will converge, and the convergence rate can in any case become very large. Clearly these problems should be brought under control before full use of this technique can be made.

12. Conclusions and future prospects

In all its many different aspects, stochastic quantization is a subject which is still being investigated actively in several areas of theoretical physics. This review has now come to its end, but it might be useful to recapitulate some of the main results which have been obtained. We would also like to give some indication of where future research may be particularly active in this field.

One of the principal motivations of Parisi and Wu was the application of stochastic quantization to gauge theories, and it is clearly here that some of the most intriguing aspects of this scheme come to light. With no gauge fixing needed at all, many 'disturbing' features of ordinary (be it canonical or path integral) quantization of gauge theories seem to vanish in the air. As in lattice gauge theories, only *gauge-invariant* quantities can be computed. In perturbation theory one encounters divergences as soon as one attempts to take the $t\rightarrow\infty$ limit of gauge *non*-invariant quantities in stochastic quantization. All gauge-invariant quantities which have been computed in perturbation theory (and there are actually not many; see section 4) have been found to agree with answers obtained from ordinary perturbation theory. However, gauge theories in stochastic quantization with no damping term for longitudinal modes, obviously represent a delicate issue, and it should perhaps be kept in mind that no general proof exists which guarantees the equivalence with other quantization methods – neither to all orders in perturbation theory nor at a non-perturbative level. It seems a real challenge to construct such an equivalence proof.

Stochastic quantization of gauge theories becomes a little more similar to standard methods when 'stochastic gauge fixing' by the so-called Zwanziger term is introduced. Since stochastic gauge fixing can eliminate $t\rightarrow\infty$ divergences for all Green functions, a Fokker–Planck analysis is now possible, and an equilibrium probability distribution can be found. Equivalence proofs are also more easily established in this case.

As we have shown in this report, quantization of many other types of field theories is possible within a stochastic scheme. This includes scalars, fermions, higher rank tensor fields and, perhaps, gravity or string field theories. Since in no cases gauge fixing terms have to be introduced, the stochastic quantization method can be set up to manifestly preserve *all* symmetries of the theories. In particular, Langevin equations can be set up which explicitly preserve simultaneously such symmetries as chiral invariance, gauge invariance, and supersymmetry.

This fact has led to the development of stochastic *regularization* schemes. Here, the theories are not only *quantized* stochastically, but their ultraviolet (and perhaps also infrared) divergences are regularized (i.e. kept finite) simultaneously. Most often this is accomplished by altering the noise–noise correlations in the fictitious time direction. This has the obvious advantage of manifestly preserving all symmetries of the underlying field theory. An alternative scheme consists in effectively changing the noise–noise correlations in the space-time directions instead. Also this scheme is set up in a way which manifestly preserves the symmetries. Stochastic regularization has also been applied to critical phenomena and has given an alternative method for the calculation of, for example, critical indices. Presumably many other applications in the border-line area of statistical mechanics/field theory are possible.

We remark that stochastic quantization is intrinsically related to supersymmetry. This issue is connected with the existence of a 'Nicolai map' for supersymmetric field theories; it allows for a deeper understanding of the dimensional reduction of the fictitious time coordinate in the equilibrium limit.

As already pointed out in the original paper of Parisi and Wu, the stochastic quantization method leads, in its discretized version, immediately to an alternative *numerical* scheme for quantum field theories. This very promising avenue of research, which now stands as strong competition to more standard Monte Carlo techniques in lattice gauge theories, is receiving considerable attention at the time of writing. In terms of new results or practical applicability this field is obviously the area where stochastic quantization has achieved most. It is very likely to remain an indispensable tool for numerical simulations, particularly on very large lattices, and for theories involving fermions.

It is often claimed that as far as more analytic approaches to quantum field theory are concerned, stochastic quantization has never really proved to be more efficient, convenient, or simply *better* than the old and more well-known techniques. This claim can be disputed. For example, although simple iterations of the Langevin equation lead to a perturbation expansion which in terms of 'stochastic diagrams' looks quite formidable in comparison with the set of standard Feynman diagrams, there exist now very fast and efficient techniques for the organization and evaluation of these stochastic diagrams. In the case of non-Abelian gauge theories the total elimination of ghosts makes up to a certain degree for the larger number of diagrams. The fact that gauge invariance can be made automatic at all steps of the calculations is another obvious advantage. Also, in cases where all known methods have failed as, for example, in the quantization of gravity, stochastic quantization may provide new hints of how to proceed.

In any case, we feel that stochastic quantization offers a beautiful and simple approach to quantum field theory. It provides an introduction to path integrals which is original, and which is easier to follow intuitively than the conventional approach. As such, stochastic quantization could serve as a pedagogical way of introducing students to the modern techniques of path integrals and quantum field theory. And as far as research in this field is concerned, we are sure that more is still to be seen.

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