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13 January 2000

PHYSICS LETTERS B

Physics Letters B 472 (2000) 101–104

Global path integral quantization of Yang–Mills theory

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Received 4 June 1999; received in revised form 6 October 1999; accepted 22 November 1999

Editor: L. Alvarez-Gaumé

Abstract

Based on a generalization of the stochastic quantization scheme recently a modified Faddeev–Popov path integral density for the quantization of Yang–Mills theory was derived, the modification consisting in the presence of specific *finite* contributions of the pure gauge degrees of freedom. Due to the Gribov problem the gauge fixing can be defined only locally and the whole space of gauge potentials has to be partitioned into patches. We propose a global path integral density for the Yang–Mills theory by summing over all patches, which can be proven to be manifestly independent of the specific local choices of patches and gauge fixing conditions, respectively. In addition to the formulation on the whole space of gauge potentials we discuss the corresponding global path integral on the gauge orbit space relating it to the original Parisi–Wu stochastic quantization scheme and to a proposal of Stora, respectively. © 2000 Elsevier Science B.V. All rights reserved.

The Faddeev–Popov [1] path integral procedure constitutes one of the most popular quantization methods for Yang–Mills theory and is widely used in elementary particle physics. It is, however, well known that at a non perturbative level due to the Gribov ambiguity [2] a unique gauge fixing in the full space of gauge fields is not possible so that the Faddeev–Popov path integral procedure is defined only locally in field space.

Several attempts were presented to generalize the above approach in order to establish global integral representations. We especially point out the construction of a regularized Feynman–Kac functional inte-

gral by Ref. [3], the use of equivariant cohomological techniques by Ref. [4] and the method of implementing BRST invariance globally by Ref. [5].

It is our aim to present a quite different argumentation based on a recently introduced generalization [6,7] of the stochastic quantization scheme [8–10].

Let $P(M,G)$ be a principal fiber bundle with compact structure group G over the compact Euclidean space time M . Let \mathcal{A} denote the space of all irreducible connections on P and let \mathcal{G} denote the gauge group, which is given by all vertical automorphisms on P reduced by the centre of G . Then \mathcal{G} acts freely on \mathcal{A} and defines a principal \mathcal{G} -fibration $\mathcal{A} \rightarrow \pi \mathcal{A}/\mathcal{G} =: \mathcal{M}$ over the paracompact [11] space \mathcal{M} of all inequivalent gauge potentials with projection π . Due to the Gribov ambiguity the principal \mathcal{G} -bundle $\mathcal{A} \rightarrow \mathcal{M}$ is not globally trivializable.

From Ref. [11] it follows that there exists a locally finite open cover $\mathcal{U} = \{U_\alpha\}$ of \mathcal{M} together

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² Supported by ‘‘Fonds zur Förderung der wissenschaftlichen Forschung in Österreich’’, project P10509-NAW.

with a set of background gauge fields $\{A_0^{(\alpha)} \in \mathcal{A}\}$ such that

$$\Gamma_\alpha = \left\{ B \in \pi^{-1}(U_\alpha) \mid D_{A_0^{(\alpha)}}^*(B - A_0^{(\alpha)}) = 0 \right\} \quad (1)$$

defines a family of local sections of $\mathcal{A} \rightarrow \mathcal{M}$. Here $D_{A_0^{(\alpha)}}^*$ is the adjoint operator of the covariant derivative $D_{A_0^{(\alpha)}}$ with respect to $A_0^{(\alpha)}$. Instead of analyzing Yang–Mills theory in the original field space \mathcal{A} we consider the family of trivial principal \mathcal{G} -bundles $\Gamma_\alpha \times \mathcal{G} \rightarrow \Gamma_\alpha$, which are locally isomorphic to the bundle $\mathcal{A} \rightarrow \mathcal{M}$, where the isomorphisms are provided by the maps

$$\chi_\alpha: \Gamma_\alpha \times \mathcal{G} \rightarrow \pi^{-1}(U_\alpha), \quad \chi_\alpha(B, g) := B^g, \quad (2)$$

with $B \in \Gamma_\alpha$, $g \in \mathcal{G}$ and B^g denoting the gauge transformation of B by g .

Using this mathematical setting we start with the Parisi–Wu approach for the stochastic quantization of the Yang–Mills theory in terms of the Langevin equation

$$dA = -\frac{\delta S}{\delta A} ds + dW. \quad (3)$$

Here S denotes the Yang–Mills action without gauge symmetry breaking terms and without accompanying ghost field terms, s denotes the extra time coordinate with respect to which the stochastic process is evolving, dW is the increment of a Wiener process.

Making use of the Ito stochastic calculus we locally transform the Langevin Eq. (3) into the adapted coordinates $\Psi = \begin{pmatrix} B \\ g \end{pmatrix}$, perform special geometrically distinguished modifications [7] of its drift and diffusion term – thereby leaving expectation values of gauge invariant observables unchanged – and finally arrive at

$$d\Psi = \left(-\tilde{G}_\alpha^{-1} \frac{\delta S_\alpha^{\text{tot}}}{\delta \Psi} + \frac{1}{\sqrt{\det G_\alpha}} \frac{\delta(\tilde{G}_\alpha^{-1} \sqrt{\det G_\alpha})}{\delta \Psi} \right) ds + \tilde{E}_\alpha dW. \quad (4)$$

Here \tilde{E}_α and $\tilde{G}_\alpha^{-1} = \tilde{E}_\alpha \tilde{E}_\alpha^*$ denote a specific vielbein and a (inverse) metric, respectively, which are associated to the above mentioned modifications. S_α^{tot} denotes a total Yang–Mills action

$$S_\alpha^{\text{tot}} = \chi_\alpha^* S + \text{pr}_\mathcal{G}^* S_\mathcal{G}, \quad (5)$$

defined by the original Yang–Mills action S and by $S_\mathcal{G} \in C^\infty(\mathcal{G})$ which is an arbitrary functional on \mathcal{G} such that $e^{-S_\mathcal{G}}$ is integrable with respect to the invariant volume density $\nu = \sqrt{\det(R_g^* R_g)}$ on \mathcal{G} . R_g is the differential of right multiplication transporting any tangent vector in $T_g \mathcal{G}$ back to the identity $\text{id}_\mathcal{G}$ on \mathcal{G} , $\text{pr}_\mathcal{G}$ is the projector $\Gamma_\alpha \times \mathcal{G} \rightarrow \mathcal{G}$. We recall furthermore that \mathcal{A} admits a natural metric which gives rise to an induced metric G_α on $\Gamma_\alpha \times \mathcal{G}$ where

$$\det G_\alpha = \nu^2 (\det \mathcal{F}_\alpha)^2 (\det \Delta_{A_0^{(\alpha)}})^{-1}. \quad (6)$$

$\mathcal{F}_\alpha = D_{A_0^{(\alpha)}}^* D_B$ denotes the Faddeev–Popov operator and $\Delta_{A_0^{(\alpha)}}^{-1}$ is the inverse of the covariant Laplacian $\Delta_{A_0^{(\alpha)}} = D_{A_0^{(\alpha)}}^* D_{A_0^{(\alpha)}}$.

The Fokker–Planck equation associated to Eq. (4) can easily be deduced and its (non-normalized) equilibrium distribution is obtained as [7]

$$\mu_\alpha e^{-S_\alpha^{\text{tot}}}, \quad \mu_\alpha = \sqrt{\det G_\alpha}. \quad (7)$$

It is the basic idea of the stochastic quantization scheme to interpret an equilibrium limit of a Fokker–Planck distribution as Euclidean path integral measure. Although our result implies unconventional *finite* contributions along the gauge group (arising from the $\text{pr}_\mathcal{G}^* S_\mathcal{G}$ term) it is equivalent to the usual Faddeev–Popov prescription for Yang–Mills theory. This follows from the fact that for expectation values of gauge invariant observables these contributions along the gauge group are exactly canceled out due to the normalization of the path integral, see below. We stress once more that due to the Gribov ambiguity the usual Faddeev–Popov approach as well as – presently – our modified version are valid only locally in field space.

In order to compare expectation values on different patches we consider the diffeomorphism in the overlap of two patches

$$\begin{aligned} \phi_{\alpha_1 \alpha_2}: (\Gamma_{\alpha_1} \cap \pi^{-1}(U_{\alpha_2})) \times \mathcal{G} \\ \rightarrow (\Gamma_{\alpha_2} \cap \pi^{-1}(U_{\alpha_1})) \times \mathcal{G}, \\ \phi_{\alpha_1 \alpha_2}(B, g) := (B^{\omega_{\alpha_2}(B)^{-1}}, g). \end{aligned} \quad (8)$$

Here $\omega_{\alpha_2}: \pi^{-1}(U_{\alpha_2}) \rightarrow \mathcal{G}$ is uniquely defined (see Ref. [7]) by $A^{\omega_{\alpha_2}(A)^{-1}} \in \Gamma_{\alpha_2}$. To the density μ_α there is associated a corresponding twisted top form on $\Gamma_\alpha \times \mathcal{G}$ (see e.g. Ref. [12]) which for simplicity we

denote by the same symbol. Using for convenience a matrix representation of G_α [7] we straightforwardly verify that

$$\phi_{\alpha_1\alpha_2}^* \mu_{\alpha_2} = \mu_{\alpha_1}. \quad (9)$$

This immediately implies that in overlap regions the expectation values of gauge invariant observables $f \in C^\infty(\mathcal{A})$ are equal when evaluated in different patches

$$\begin{aligned} & \int_{(\Gamma_{\alpha_2} \cap \pi^{-1}(U_{\alpha_1})) \times \mathcal{E}} \mu_{\alpha_2} e^{-S_{\alpha_2}^{\text{tot}}} \chi_{\alpha_2}^* f \\ &= \int_{(\Gamma_{\alpha_1} \cap \pi^{-1}(U_{\alpha_2})) \times \mathcal{E}} \mu_{\alpha_1} e^{-S_{\alpha_1}^{\text{tot}}} \chi_{\alpha_1}^* f. \end{aligned} \quad (10)$$

Suppose now that we consider a different locally finite cover $\{U'_\beta\}$ of \mathcal{M} together with a new set of background gauge fields $\{A_0^{(\beta)}\}$ so that a new family $\{\Gamma'_\beta\}$ of local sections, as well as new maps χ'_β , densities μ'_β , total actions S_{β}^{tot} and another partition of unity γ'_β are given. Applying the above integration formula in overlap regions we can prove furthermore that

$$\begin{aligned} & \int_{\Gamma_\alpha \times \mathcal{E}} \mu_\alpha e^{-S_\alpha^{\text{tot}}} \chi_\alpha^* (f \pi^* (\gamma_\alpha \gamma'_\beta)) \\ &= \int_{\Gamma'_\beta \times \mathcal{E}} \mu'_\beta e^{-S_{\beta}^{\text{tot}}} \chi_{\beta}^* (f \pi^* (\gamma_\alpha \gamma'_\beta)). \end{aligned} \quad (11)$$

Finally we propose the definition of the global expectation value of a gauge invariant observable $f \in C^\infty(\mathcal{A})$ by summing over all the partitions γ_α such that

$$\langle f \rangle = \frac{\sum_\alpha \int_{\Gamma_\alpha \times \mathcal{E}} \mu_\alpha e^{-S_\alpha^{\text{tot}}} \chi_\alpha^* (f \pi^* \gamma_\alpha)}{\sum_\alpha \int_{\Gamma_\alpha \times \mathcal{E}} \mu_\alpha e^{-S_\alpha^{\text{tot}}} \chi_\alpha^* \pi^* \gamma_\alpha} \quad (12)$$

(for a preliminary presentation of this result see Ref. [13]). Due to Eq. (9) it is trivial to prove that the global expectation value $\langle f \rangle$ is independent of the specific choice of the locally finite cover $\{U_\alpha\}$, of the choice of the background gauge fields $\{A_0^{(\alpha)}\}$ and of the choice of the partition of unity γ_α , respectively.

As already indicated in Ref. [7] these structures can equally be translated into the original field space

\mathcal{A} . With the help of the partition of unity the locally defined densities μ_α as well as $e^{-S_\alpha^{\text{tot}}}$ can be pieced together to give a globally well defined twisted top form Ω on \mathcal{A}

$$\Omega := \sum_\alpha \chi_\alpha^{-1*} (\mu_\alpha e^{-S_\alpha^{\text{tot}}}) \pi^* \gamma_\alpha. \quad (13)$$

The global expectation value (12) then reads

$$\langle f \rangle = \frac{\int_{\mathcal{A}} \Omega f}{\int_{\mathcal{A}} \Omega}, \quad (14)$$

which due to the discussion from above is independent of all the particular local choices.

In addition to the global expressions (12) and (14) for the path integral in the whole space of connections the stochastic quantization scheme also offers the possibility of deriving the corresponding formulation on the gauge orbit space \mathcal{M} : We consider the projections of either the original Parisi–Wu Langevin Eq. (3) or of the modified Eq. (4) onto the gauge invariant subspaces Γ_α described by the coordinate B ; in both cases we obtain (see Ref. [7])

$$\begin{aligned} dB &= \left(- (G_\alpha^{-1})^{\Gamma_\alpha \Gamma_\alpha} \frac{\delta S}{\delta B} \right. \\ &+ \left. \frac{1}{\sqrt{\det G_\alpha}} \frac{\delta \left((G_\alpha^{-1})^{\Gamma_\alpha \Gamma_\alpha} \sqrt{\det G_\alpha} \right)}{\delta B} \right) ds \\ &+ E_\alpha^{\Gamma_\alpha} dW. \end{aligned} \quad (15)$$

Notice that in local coordinates $(G_\alpha^{-1})^{\Gamma_\alpha \Gamma_\alpha}$ is the pullback of the restriction on U_α of the inverse of a globally defined metric on the gauge orbit space \mathcal{M} induced by the natural metric on \mathcal{A} . Since the locally defined Eqs. (15) are transforming covariantly under the local diffeomorphisms issued by the coordinate transformations, using Ref. [14] it is straightforward to check that their further projections onto \mathcal{M} are yielding a globally defined stochastic process.

In direct analogy to our derivation of the local Fokker–Planck densities (7) we obtain that the Fokker–Planck equation associated to the projected Langevin Eqs. (15) has an equilibrium distribution

given by just the gauge invariant part of the densities (7)

$$\det \mathcal{F}_\alpha (\det \Delta_{A_0^{(\alpha)}})^{-1/2} e^{-\chi_\alpha^* S}. \quad (16)$$

By using Eq. (9) we can prove explicitly that their projections onto \mathcal{M} on overlapping sets of \mathcal{U} agree giving rise to a globally well defined top form $\tilde{\Omega}$ on \mathcal{M} . Furthermore we can show that the above expectation values (12) and (14) of gauge invariant observables f can identically be rewritten as corresponding integrals over the gauge orbit space \mathcal{M} with respect to $\tilde{\Omega}$

$$\langle f \rangle = \frac{\int_{\mathcal{M}} \tilde{\Omega} f}{\int_{\mathcal{M}} \tilde{\Omega}}. \quad (17)$$

We note that this last expression shows agreement with the formulation proposed by Stora [4] upon identification of $\tilde{\Omega}$ with the Ruelle–Sullivan form [15]. Whereas in Ref. [4], however, this definition of expectation values on \mathcal{M} appeared as the starting point for a global formulation of Yang–Mills theory in the whole space of gauge potentials it appears now as our final result.

We directly aimed at the derivation of a global path integral formulation in the whole space of gauge potentials within the stochastic quantization approach; we recall that we first derived a path integral in terms of the local probability density $\mu_\alpha e^{-S_\alpha^{\text{tot}}}$ which assured gaussian decrease along the gauge fixing surface *as well as* along the gauge orbits. The inherent interrelation of the field variables on $\Gamma_\alpha \times \mathcal{G}$ subsequently led to simple relations of the local densities in the overlap regions and eventually to the global path integral formulations (12) and (14) on the whole field space, as well as to the expectation value (17) on the gauge orbit space, respectively.

It is remarkable that the projections onto the local gauge fixing surfaces Γ_α of in specific the original Parisi–Wu stochastic process induce a *globally defined stochastic process on the gauge orbit space* yielding the construction of the globally defined path integral density $\tilde{\Omega}$. In our opinion this relationship of the globally defined Parisi–Wu Langevin equation on the whole field space to the globally defined path

integral density on the gauge orbit space closes nicely one of the left open issues of the original paper [8].

We are aware that in a mathematically strict sense our results are rather formal due to the infrared and ultraviolet infinities inherent in the path integral; it seems challenging to investigate the applicability of a previously developed stochastic regularization scheme [16] within our generalized approach.

Acknowledgements

We thank C. Viallet for valuable discussions. G. Kelnhofer acknowledges support by ‘‘Fonds zur Förderung der wissenschaftlichen Forschung in Österreich’’, project P10509-NAW.

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