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Generalized Stochastic Quantization of Yang–Mills Theory

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We perform the stochastic quantization of Yang–Mills theory in configuration space and derive the Faddeev–Popov path integral density. Based on a generalization of the stochastic gauge fixing scheme and its geometrical interpretation this result is obtained as the exact equilibrium solution of the associated Fokker–Planck equation. Included in our discussion is the precise range of validity of our approach. © 1998 Academic Press

1. INTRODUCTION

The stochastic quantization method of Parisi and Wu [1] was introduced 1981 as a new method for quantizing field theories. It is based on concepts of non-equilibrium statistical mechanics and provides novel and alternative insights into quantum field theory, see Refs. [2, 3] for comprehensive reviews and referencing. One of the most interesting aspects of this new quantization scheme lies in its rather unconventional treatment of gauge field theories, in specific of Yang–Mills theories. We do not intend to review here the basic facts, benefits,or problems of the stochastic quantization scheme of gauge field theories (see, however, [4]) but just recall that originally it was formulated by Parisi and Wu without the introduction of gauge fixing terms and the usual Faddeev–Popov ghost fields; later on a modified approach named stochastic gauge fixing was given by Zwanziger [5] where again no Faddeev–Popov ghost fields where introduced. Our focus is based on extending a previously introduced generalization [6, 7, 4] of Zwanziger's stochastic gauge fixing scheme. We so far studied the helix model [8–11] which is an abelian gauge theory coupled to bosonic matter fields in $0+1$ dimensions which does not suffer from a Gribov ambiguity [12]. By this generalized stochastic gauge fixing scheme it was possible to derive a nonperturbative proof of the equivalence between the conventional path integral formulation of this model and the equilibrium limit of the corresponding stochastic correlation functions. The method mainly is based on the possibility of introducing adapted coordinates which means separating the original gauge fields into gauge independent and gauge dependent degrees of freedom.

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In the present article we straightforwardly generalize our formalism to the nonabelian Yang–Mills theory. In comparison with the helix model, however, the geometrical structure of Yang–Mills theory obstructs a global separation of the field variables as mentioned above due to the well known Gribov ambiguity. Therefore we have to restrict our investigation to sufficiently small regions in the space of Yang–Mills fields.

The main difficulty in the previous investigations of the stochastic quantization of Yang–Mills theory for deriving a conventional field theory path integral density was to solve the Fokker–Planck equation in the equilibrium limit. In the original Parisi–Wu approach this equilibrium limit could not even be attained due to unbounded diffusions of the gauge modes. Zwanziger [5, 13, 14] suggested introducing a specific additional nonholonomic stochastic force term to suppress these gauge modes yet keeping the expectation values of gauge invariant observables unchanged. The approach to equilibrium and the discussion of the conditions of applicability to the nonperturbative regime, however, do not seem to have been fully completed.

Our analysis is distinguished by the above approaches by exploiting a more general freedom to modify both the drift term and the diffusion term of the stochastic process again leaving all expectation values of gauge invariant variables unchanged. Due to this additional structure of modification the equilibrium limit could be obtained immediately using the fluctuation dissipation theorem proving equivalence with the well known Faddeev–Popov path integral density. In deriving this result the gauge degrees of freedom were fully under control, no infinite gauge group volumes arose. However, this equivalence proof has been performed only for those gauge field configurations satisfying a unique gauge fixing condition leaving the option for further investigations concerning the Gribov issue.

In Section 2 the geometrical setting for Yang–Mills theory is introduced. The relevant objects are identified and the nontrivial bundle structure of the space of gauge potentials is outlined.

Section 3 offers a brief review on the stochastic gauge fixing scheme issued by Zwanziger.

A new generalized stochastic gauge fixing method for Yang–Mills theory is presented in Section 4. We exploit the most general form of the Langevin equation such that the expectation values of gauge invariant observables remain unchanged. The adapted coordinates, the corresponding vielbeins, and metrics are defined.

The geometrical structure of the generalized stochastic gauge fixing is revealed in depth in Section 5. Due to the extension of the stochastic process a new metric is induced with respect to which the space spanned by the gauge invariant fields becomes orthogonal to the gauge orbits. The relation between this metric and several horizontal bundles in the space of gauge fields is elucidated.

Section 6 is devoted to the derivation of the path integral density as an equilibrium solution of the Fokker–Plank equation. Thereby the equivalence with the Faddeev–Popov approach is proved.

Finally an outlook is presented in Section 7.

2. THE GEOMETRICAL SETTING OF YANG–MILLS THEORY

In this section we present the major geometrical structures of pure Yang–Mills theory. We collect in a somewhat formal style all the necessary ingredients which are needed later on for a compact and transparent formulation of the stochastic quantization scheme of Yang–Mills theory.

Let $P(M, G)$ be a principal fiber bundle with structure group G over the compact Euclidean space time M. Let $\mathscr{A}$ denote the space of all irreducible connections on P and let $\mathscr{G}$ denote the gauge group, which is given by all vertical automorphisms on P reduced by the centre of G. Then $\mathscr{G}$ acts freely on $\mathscr{A}$ and defines a principal $\mathscr{G}$-fibration $\mathscr{A} \xrightarrow{\pi} \mathscr{A}/\mathscr{G} =: \mathscr{M}$ over the space $\mathscr{M}$ of all inequivalent gauge potentials with projection π [15–17]. A Riemannian structure on $\mathscr{A}$ can be introduced as follows: Let g denote the Lie algebra of G and consider the adjoint bundle $\operatorname{ad} P = P \times_{\mathrm{ad}} g$ which is associated to the principal bundle P via the adjoint action of G on g. Choosing the natural Killing form on g an inner product can be defined on the space $\Omega^q(M, \operatorname{ad} P)$ of $\operatorname{ad} P$-valued q-forms on M by

$$\langle \varphi, \varphi' \rangle_{(q)} = \int_M \operatorname{tr}(\varphi \wedge *\varphi'), \tag{2.1}$$

where $*$ is the Hodge operation with respect to the given metric on M and tr denotes the trace on g. Locally a form in $\Omega^q(M, \operatorname{ad} P)$ is just a g-valued q-form on M.

Since $\mathscr{A}$ is an affine space modelled on $\Omega^1(M, \operatorname{ad} P)$ the tangent bundle of $\mathscr{A}$ is given by $T\mathscr{A} = \mathscr{A} \times \Omega^1(M, \operatorname{ad} P)$. Hence one can define a Riemannian structure h on $\mathscr{A}$ by

$$h_A \colon T_A\mathscr{A} \times T_A\mathscr{A} \to R, \qquad h_A(\tau^1, \tau^2) := \langle \tau^1, \tau^2 \rangle_{(1)}, \qquad \tau^1, \tau^2 \in \Omega^1(M, \operatorname{ad} P). \tag{2.2}$$

The space $\Omega^0(M, \operatorname{ad} P)$ can be identified with the Lie algebra $\operatorname{Lie}\mathscr{G}$ of the gauge group $\mathscr{G}$ and a natural inner product on $\operatorname{Lie}\mathscr{G}$ is given by $\langle\ ,\ \rangle_{(0)}$.

Due to the Gribov ambiguity [12] the principal $\mathscr{G}$-bundle $\mathscr{A} \to \mathscr{M}$ is not globally trivializable. In order to define a local section, we choose a fixed background connection $A_0 \in \mathscr{A}$ and consider a sufficient small neighbourhood $U(A_0)$ of $\pi(A_0)$ in $\mathscr{M}$. Then the subspace

$$\Sigma = \{B \in \pi^{-1}(U(A_0)) / D^*_{A_0}(B - A_0) = 0\} \tag{2.3}$$

defines a local section of $\mathscr{A} \to \mathscr{M}$ [18, 17]. Here $D^*_{A_0}$ is the adjoint operator of the covariant derivative D_{A_0} with respect to A_0. A tangent vector $\zeta_B \in T_B\Sigma$ is uniquely characterized by the property $D^*_{A_0}\zeta_B = 0$.

Notice that in the zero instanton sector (for $M = S^4$) this background field A_0 can be set to zero, yielding the familiar covariant gauge condition

$$\partial_\mu B^\mu = 0. \tag{2.4}$$

3. STOCHASTIC GAUGE FIXING

We start with the Parisi–Wu approach for the stochastic quantization of the Yang–Mills theory in terms of the Langevin equation

$$dA = -\frac{\delta S}{\delta A}\, ds + dW. \tag{3.1}$$

Here S denotes the Yang–Mills action without gauge symmetry breaking terms and without accompanying ghost field terms

$$S = \tfrac{1}{2}\langle F, F\rangle_{(2)}, \tag{3.2}$$

where F denotes the curvature of A, s denotes the extra time coordinate ("stochastic time" coordinate) with respect to which the stochastic process is evolving, and dW is the increment of a Wiener process (for a detailed presentation see, e.g., [19]).

We now discuss Zwanziger's modified formulation [5] of the Parisi–Wu scheme: The stochastic gauge fixing procedure consists in adding an additional drift force to the Langevin equation (3.1) which acts tangentially to the gauge orbits. This additional term generally can be expressed by the gauge generator D_A and an arbitrary function α so that the modified Langevin equation reads as

$$dA = \left[-\frac{\delta S}{\delta A} + D_A \alpha \right] ds + dW. \tag{3.3}$$

The expectation values of gauge invariant observables remain unchanged for any choice of the function α (see below for the explicit demonstration contained in the discussion of our generalized stochastic gauge fixing procedure). For specific choices of the—in principle—arbitrary function α the gauge modes' diffusion is damped along the gauge orbits. As a consequence the Fokker–Planck density can be normalized; we recall that this situation is in contrast to the Parisi–Wu approach, where for expectation values of gauge variant observables no equilibrium values could be attained.

In contrast to the approach of [5] where no equilibrium distribution of the Fokker–Planck equation could be derived as well as in contrast to [13] where the full Fokker–Planck operator

$$L = \frac{\delta}{\delta A}\left[\frac{\delta S}{\delta A} - D_A \alpha + \frac{\delta}{\delta A} \right] \tag{3.4}$$

was needed to obtain an equilibrium distribution we present a quite different strategy.

As the Fokker–Planck operator factorizes into first order differential operators the question arises whether it is possible to derive the equilibrium distribution directly by solving a simpler first order problem. However, for this to be possible a necessary integrability condition imposed on the drift term $\delta S/\delta A - D_A \alpha$ has to be fulfilled. It is well known that for the Yang–Mills case this is violated.

In the following we want to clarify the relationship of this integrability condition and the underlying geometrical structure of the space of gauge potentials.

We recall that any bundle metric on a principal fiber bundle which is invariant under the corresponding group action gives rise to a natural connection whose horizontal subbundle is orthogonal to the corresponding group. The natural connection induced by $\langle\ ,\ \rangle_{(1)}$ in Yang–Mills theory is given by the following Lie $\mathscr{G}$ valued one form

$$\gamma = \Delta_A^{-1} D_A^*. \tag{3.5}$$

The projection $\mathbf{P}$ onto the corresponding horizontal subbundle is given by

$$\mathbf{P} = \mathbf{1} - D_A \gamma. \tag{3.6}$$

The curvature Ω of γ, $\Omega = \delta_{\mathscr{A}}\gamma + \frac{1}{2}[\gamma, \gamma]$, where $\delta_{\mathscr{A}}$ denotes the exterior derivative on $\mathscr{A}$, however, does not vanish [15] so that there does not exist (even locally) a manifold whose tangent bundle is isomorphic to this horizontal subbundle. Moreover this also implies that any vector field along the gauge group cannot be written as a gradient of a function with respect to the metric $\langle\ ,\ \rangle_{(1)}$.

To verify this explicitly let us assume that the one form $\langle D_A\alpha(A), \cdot\rangle$, where $\alpha(A)$ is any Lie $\mathscr{G}$-valued function on $\mathscr{A}$ is the differential of a function f on $\mathscr{A}$, i.e.,

$$\langle D_A\alpha(A), \cdot\rangle_{(1)} = \delta_{\mathscr{A}} f. \tag{3.7}$$

For two vector fields τ^1, τ^2 in $T\mathscr{A}$ being horizontal with respect to γ (i.e., $\gamma(\tau^1) = \gamma(\tau^2) = 0$) we have

$$\Omega(\tau^1, \tau^2) = -\gamma([\tau^1, \tau^2]), \tag{3.8}$$

so that the one form on the left hand side of (3.7) gives on the vector field commutator $[\tau^1, \tau^2]$

$$\begin{aligned} \langle D_A\alpha(A), [\tau^1, \tau^2]\rangle_{(1)} &= \langle D_A\alpha(A), (\mathbf{1} - \mathbf{P}_A)[\tau^1, \tau^2]\rangle_{(1)} \\ &= -\langle \alpha(A), \Delta_A \Omega(\tau^1, \tau^2)\rangle \neq 0. \end{aligned} \tag{3.9}$$

But $\delta_{\mathscr{A}} f[\tau^1, \tau^2] = 0$ since $\delta_{\mathscr{A}} f(\tau^1) = \delta_{\mathscr{A}} f(\tau^2) = 0$ hence giving a contradiction. However, it should be remarked that the vanishing of the curvature is only a necessary condition.

It is our intention to modify the stochastic process (3.3) for the Yang–Mills theory in such a way that the factorization of the modified Fokker–Planck operator indeed allows the determination of the equilibrium distribution as a solution of a first order differential equation in a consistent manner.

4. GENERALIZED STOCHASTIC GAUGE FIXING

In this section we apply our recently introduced [6, 7, 4] generalization of the stochastic gauge fixing procedure of Zwanziger to the Yang–Mills theory. It is advantageous to avoid the complicated nonabelian dynamics of the Yang–Mills fields by transforming them into a set of adapted coordinates [20, 14]. This means separating the original gauge fields into gauge independent and gauge dependent degrees of freedom. However, this is only locally possible due to the nontriviality of the bundle $\mathscr{A} \to \mathscr{M}$ so that we are forced to consider the trivializable bundle $\pi^{-1}(U(A_0)) \to U(A_0)$. In concrete, our analysis will be performed on the isomorphic trivial principal $\mathscr{G}$-bundle, $\Sigma \times \mathscr{G} \to \Sigma$, where the isomorphism is given by the map

$$\chi\colon \Sigma \times \mathscr{G} \to \pi^{-1}(U(A_0)), \qquad \chi(B, g) := B^g \tag{4.1}$$

with $B \in \Sigma$, $g \in \mathscr{G}$, and B^g denoting the nonabelian gauge transformation of B by g

$$B^g = g^{-1} B g + g^{-1}\, dg. \tag{4.2}$$

Evidently the inverse map χ^{-1} is given by the expression

$$\chi^{-1}\colon \pi^{-1}(U(A_0)) \to \Sigma \times \mathscr{G}, \qquad \chi^{-1}(A) := (A^{\omega(A)^{-1}}, \omega(A)), \tag{4.3}$$

where $\omega\colon \pi^{-1}(U(A_0)) \to \mathscr{G}$ is uniquely defined by the requirement that $A^{\omega(A)^{-1}} \in \Sigma$, i.e.,

$$D^*_{A_0}(A^{\omega(A)^{-1}} - A_0) = 0. \tag{4.4}$$

Although an explicit expression for ω can be given only in terms of a perturbative expansion [20], it is nevertheless easy to derive in closed form its differential, which is necessary to calculate the corresponding vielbeins. To do this, we begin by calculating the differential $T\chi$ of χ. $T\chi$ provides an isomorphism $T\chi\colon T(\Sigma \times \mathscr{G}) \to T\pi^{-1}(U(A_0))$ given by

$$T\chi(\zeta_B, Y_g) = \mathrm{ad}(g^{-1})(\zeta_B + D_B R_g(Y_g)), \tag{4.5}$$

where $\zeta_B \in T_B\Sigma \subset \Omega^1(M, \mathrm{ad}\, P)$, Y_g is a tangent vector on the gauge group $\mathscr{G}$ in point g, ad denotes the adjoint action of $\mathscr{G}$ on Lie $\mathscr{G}$, and R_g denotes the invertible operator on $T\mathscr{G}$ which transports a tangent vector in $T_g\mathscr{G}$ back to the identity by the differential of right multiplication.

From (4.1) the vielbeins e corresponding to the change of coordinates $(B, g) \to A$ are given by

$$e = (e_\Sigma, e_{\mathscr{G}}) = \mathrm{ad}(g^{-1})(\mathbf{P}^\Sigma, D_B R_g). \tag{4.6}$$

Here $\mathbf{P}^{\Sigma} = \mathbf{1} - D_{A_0} \Delta_{A_0}^{-1} D_{A_0}^*$ denotes the projector onto the subspace $T_B\Sigma$ and $\Delta_{A_0}^{-1}$ is the inverse of the covariant Laplacian $\Delta_{A_0} = D_{A_0}^* D_{A_0}$.

Now it is an easy task to verify that the following map is the inverse of $T\chi$, hence giving the tangent map of χ^{-1}, namely

$$\begin{aligned} &T\chi^{-1}\colon T(\pi^{-1}(U(A_0))) \to T(\Sigma \times \mathcal{G}), \\ &T\chi^{-1}(\tau_A) = (\mathbf{P}^{\Sigma}(\mathbf{1} - D_B \mathcal{F}_B^{-1} D_{A_0}^*) \operatorname{ad}(g)\, \tau_A,\ R_g^{-1} \mathcal{F}_B^{-1} D_{A_0}^* \operatorname{ad}(g)\, \tau_A), \end{aligned} \tag{4.7}$$

where $\tau_A \in T_A(\pi^{-1}(U(A_0)))$. Here $B = A^{\omega(A)^{-1}}$, $g = \omega(A)$, and

$$\mathcal{F}_B\colon \Omega^0(M, \operatorname{ad} P) \to \Omega^0(M, \operatorname{ad} P), \qquad \mathcal{F}_B = D_{A_0}^* D_B \tag{4.8}$$

denotes the Faddeev–Popov operator. Since for sufficiently small $U(A_0)$ the coordinate transformation onto the adapted coordinates is regular the Faddeev–Popov operator $\mathcal{F}_B$ is invertible, and Σ thus is completely contained within one Gribov horizon. Notice that the Faddeev–Popov operator is self-adjoint for all $B \in \Sigma$.

From (4.3) the vielbeins E corresponding to the change of coordinates $A \to (B, g)$ are given by

$$E = \begin{pmatrix} E^{\Sigma} \\ E^{\mathcal{G}} \end{pmatrix} = \begin{pmatrix} \mathbf{P}^{\Sigma}(\mathbf{1} - D_B \mathcal{F}_B^{-1} D_{A_0}^*) \operatorname{ad}(g) \\ R_g^{-1} \mathcal{F}_B^{-1} D_{A_0}^* \operatorname{ad}(g) \end{pmatrix}. \tag{4.9}$$

Before we defined a Riemannian structure on $\mathcal{A}$ by the inner product $\langle\ ,\ \rangle_{(1)}$. However, in the adapted coordinates (B, g) this metric G is given as follows (pullback of h by χ),

$$\begin{aligned} G_{(B,g)}((\zeta_B^1, Y_g^1), (\zeta_B^2, Y_g^2)) &= \langle \zeta_B^1, \zeta_B^2 \rangle_{(1)} + \langle \zeta_B^1, D_B R_g(Y_g^2) \rangle_{(1)} + \langle D_B R_g(Y_g^1), \zeta_B^2 \rangle_{(1)} \\ &\quad + \langle R_g(Y_g^1), \Delta_B R_g(Y_g^2) \rangle_{(1)}, \end{aligned} \tag{4.10}$$

where $\zeta_B^1, \zeta_B^2 \in T_B\Sigma$ and $Y_g^1, Y_g^2 \in T_g\mathcal{G}$.

Formally, the metric G can be written in matrix form

$$G = e^* e = \begin{pmatrix} \mathbf{P}^{\Sigma} & \mathbf{P}^{\Sigma} \cdot D_B \cdot R_g \\ R_g^* \cdot D_B^* \cdot \mathbf{P}^{\Sigma} & R_g^* \cdot \Delta_B \cdot R_g \end{pmatrix}, \tag{4.11}$$

where R_g^* is the adjoint operation of R_g with respect to the inner product $\langle\ ,\ \rangle_{(0)}$ on Lie $\mathcal{G}$. We also mention the inverse metric G^{-1}

$$\begin{aligned} G^{-1} &= \begin{pmatrix} (G^{-1})^{\Sigma\Sigma} & (G^{-1})^{\Sigma\mathcal{G}} \\ (G^{-1})^{\mathcal{G}\Sigma} & (G^{-1})^{\mathcal{G}\mathcal{G}} \end{pmatrix} = EE^* \\ &= \begin{pmatrix} \mathbf{P}^{\Sigma} - \mathbf{P}^{\Sigma} D_B \mathcal{F}_B^{-1} \Delta_{A_0} \mathcal{F}_B^{-1} D_B^* \mathbf{P}^{\Sigma} & \mathbf{P}^{\Sigma} D_B \mathcal{F}_B^{-1} \Delta_{A_0} \mathcal{F}_B^{-1} R_g^{*\,-1} \\ R_g^{-1} \mathcal{F}_B^{-1} \Delta_{A_0} \mathcal{F}_B^{-1} D_B^* & -R_g^{-1} \mathcal{F}_B^{-1} \Delta_{A_0} \mathcal{F}_B^{-1} R_g^{*\,-1} \end{pmatrix}. \end{aligned} \tag{4.12}$$

The determinant of G is then given by

$$\det G = \det(R_g^* R_g)(\det \mathscr{F}_B)^2 (\det \Delta_{A_0})^{-1}, \tag{4.13}$$

where $\sqrt{\det(R_g^* R_g)}$ can be identified with the volume density on $\mathscr{G}$, associated to the (right) invariant metric R^*R on $\mathscr{G}$.

In the following we transform the Parisi–Wu Langevin equation (3.1) into the adapted coordinates $\Psi = \binom{B}{g}$. As this transformation is not globally possible the region of definition of (3.1) has to be restricted to $\pi^{-1}(U(A_0))$. Making use of the Ito stochastic calculus [19, 4] the above Langevin equation now reads

$$d\Psi = \left(-G^{-1} \frac{\delta S}{\delta \Psi} + \frac{1}{\sqrt{\det G}} \frac{\delta (G^{-1} \sqrt{\det G})}{\delta \Psi} \right) ds + E\, dW, \tag{4.14}$$

where the vielbein E, the metric G, its inverse, and its determinant were introduced in the previous section.

The generalized stochastic quantization procedure amounts—as a direct consequence of our previous investigations [4] on the abelian helix model—to considering the modified Langevin equation

$$d\Psi = \left(-G^{-1} \frac{\delta S}{\delta \Psi} + \frac{1}{\sqrt{\det G}} \frac{\delta (G^{-1} \sqrt{\det G})}{\delta \Psi} + E\, D_A \alpha \right) ds + E(\mathbf{1} + D_A \beta)\, dW, \tag{4.15}$$

where $A = B^g$. Here $\alpha\colon \pi^{-1}(U(A_0)) \to \operatorname{Lie} \mathscr{G}$ and the $\operatorname{Lie} \mathscr{G}$ valued one form $\beta \in \Omega^1(\pi^{-1}(U(A_0)), \operatorname{Lie} \mathscr{G})$ are a priori arbitrary and will be fixed later on.

The above Langevin equation is the most general Langevin equation for Yang–Mills theory which leads to the same expectation values of gauge invariant variables as the original Parisi–Wu equation (3.1) written in adapted coordinates.

Let us recall that the stochastic time evolution of expectation values of observables is described by the adjoint Fokker–Planck operator $L^\dagger$ which corresponding to (4.15) is given by

$$L^\dagger = \left[-\frac{\delta S}{\delta \Psi} + \frac{1}{\sqrt{\det G}} \frac{\delta \sqrt{\det G}}{\delta \Psi} + \frac{\delta}{\delta \Psi} \right] G^{-1} \frac{\delta}{\delta \Psi} + L^\dagger_{\text{extra}}. \tag{4.16}$$

We introduce

$$\tilde{E} = E(\mathbf{1} + D_A \beta), \qquad \tilde{G}^{-1} = \tilde{E} \tilde{E}^*, \tag{4.17}$$

with $A = B^g$ and have

$$L^\dagger_{\text{extra}} = (E\, D_A \alpha)^* \frac{\delta}{\delta \Psi} + (\tilde{G}^{-1} - G^{-1}) \frac{\delta^2}{\delta \Psi\, \delta \Psi}, \tag{4.18}$$

where again $A = B^g$ and where the second term in (4.18) reads explicitly

$$(\tilde{G}^{-1} - G^{-1}) \frac{\delta^2}{\delta\Psi\,\delta\Psi} = (\tilde{G}^{-1} - G^{-1})^{\Sigma\Sigma} \frac{\delta^2}{\delta B\,\delta B} + (\tilde{G}^{-1} - G^{-1})^{\Sigma\mathcal{G}} \frac{\delta^2}{\delta B\,\delta g} + (\tilde{G}^{-1} - G^{-1})^{\mathcal{G}\Sigma} \frac{\delta^2}{\delta g\,\delta B} + (\tilde{G}^{-1} - G^{-1})^{\mathcal{G}\mathcal{G}} \frac{\delta^2}{\delta g\,\delta g}. \tag{4.19}$$

Our proof consists in showing that the α, β dependent extra term $L^{\dagger}_{\text{extra}}$ of $L^{\dagger}$ annihilates on gauge invariant observables. Indeed we obtain from (4.9)

$$(E\, D_A \alpha)^{*\Sigma} = 0, \tag{4.20}$$

where $A = B^g$. Furthermore we have

$$(\tilde{G}^{-1} - G^{-1})^{\Sigma\Sigma} = 0 \tag{4.21}$$

so that the action of $L^{\dagger}_{\text{extra}}$ on gauge invariant observables, which are purely functions $f(B)$ when written in terms of adapted coordinates, identically vanishes

$$L^{\dagger}_{\text{extra}} f(B) = 0. \tag{4.22}$$

Alternatively we observe that due to (4.20) the α and β dependent terms in the modified Langevin equation (4.15) drop out after projecting on the gauge invariant subspace Σ described by the coordinate B

$$dB = \left[-(G^{-1})^{\Sigma\Sigma} \frac{\delta S}{\delta B} + \frac{1}{\sqrt{\det G}} \frac{\delta((G^{-1})^{\Sigma\Sigma} \sqrt{\det G})}{\delta B} \right] ds + E^{\Sigma}\, dW. \tag{4.23}$$

In deriving the above Langevin equation we have used the fact that the divergence of the generator of right group transformations corresponding to the invariant group measure induced by the metric $R^*_g R_g$ vanishes, i.e.,

$$\frac{1}{\sqrt{\det(R^*_g R_g)}} \frac{\delta(\sqrt{\det(R^*_g R_g)}\, R^{*\,-1}_g)}{\delta g} = 0. \tag{4.24}$$

We close by transforming back the Langevin equation (4.15) into the original coordinates A. Invoking the Ito stochastic calculus once more again we have

$$dA = \left[-\frac{\delta S}{\delta A} + D_A \alpha + \frac{\delta^2 A}{\delta\Psi\,\delta\Psi} (\tilde{G}^{-1} - G^{-1}) \right] ds + (\mathbf{1} + D_A \beta)\, dW. \tag{4.25}$$

In the above equation it is understood to take $B = A^{\omega(A)^{-1}}$ and $g = \omega(A)$. Let us recall that the above Langevin equation is valid only in the restricted domain $\pi^{-1}(U(A_0))$.

5. ON THE GEOMETRICAL INTERPRETATION OF GENERALIZED STOCHASTIC GAUGE FIXING

As a consequence of our generalized stochastic gauge fixing procedure not only Zwanziger's original term $D_A\alpha$ is appearing in the Langevin equation (4.25) for the Yang–Mills field A, but also an additional β-dependent drift term as well as a specific modification of the Wiener increment, described by the operator

$$\hat{e} = \mathbf{1} + D_A\beta. \tag{5.1}$$

We regard $\hat{e}$ as a $T\pi^{-1}(U(A_0))$-valued one form on $\pi^{-1}(U(A_0))$ by setting $\hat{e}(\tau) = \tau + D_A\beta(\tau)$ for all tangent vectors $\tau \in T_A\pi^{-1}(U(A_0))$. The idea is to view $\hat{e}$ as a vielbein giving rise to the inverse of a yet not specified metric $\hat{g}$ on the space $\pi^{-1}(U(A_0))$. We note that the inverse vielbein $\hat{e}^{-1}$ is given by the $T\pi^{-1}(U(A_0))$-valued one form on $\pi^{-1}(U(A_0))$

$$\hat{e}^{-1} = \mathbf{1} - D_A(\mathbf{1} + \beta\, D_A)^{-1}\,\beta, \tag{5.2}$$

provided the operator $\mathbf{1} + \beta\, D_A$: Lie $\mathcal{G} \to$ Lie $\mathcal{G}$ is invertible for all $A \in \pi^{-1}(U(A_0))$. Hence the metric $\hat{g} = \hat{e}^{-1*}\hat{e}^{-1}$ is given by

$$\hat{g}(\tau^1, \tau^2) = \langle \hat{e}^{-1}(\tau^1), \hat{e}^{-1}(\tau^2)\rangle \qquad \forall \tau^1, \tau^2 \in T_A\pi^{-1}(U(A_0)). \tag{5.3}$$

Corresponding to the Langevin equation (4.25) this metric appears when considering the associated Fokker–Planck operator L. We rewrite it by a simple manipulation so that it becomes similar to a Fokker–Planck operator for a stochastic process on a manifold described by the metric $\hat{g}$,

$$\begin{aligned} L &= \frac{\delta}{\delta A}\left[\frac{\delta S}{\delta A} - D_A\alpha - \frac{\delta^2 A}{\delta\Psi\,\delta\Psi}(\tilde{G}^{-1} - G^{-1}) + \frac{\delta}{\delta A}\,\hat{g}^{-1}\right] \\ &= \frac{\delta}{\delta A}\left\{\hat{g}^{-1}\left[\frac{\delta S}{\delta A} - (\mathbf{1} - \hat{g})\frac{\delta S}{\delta A} - \hat{g}\,D_A\alpha + \frac{\delta}{\delta A}\right] - \frac{\delta^2 A}{\delta\Psi\,\delta\Psi}(\tilde{G}^{-1} - G^{-1}) + \frac{\delta\hat{g}^{-1}}{\delta A}\right\}. \end{aligned} \tag{5.4}$$

Using the gauge invariance of the Yang–Mills action $D_A^*(\delta S/\delta A) = 0$ and (5.2)–(5.3) we find

$$(\mathbf{1} - \hat{g})\frac{\delta S}{\delta A} - \hat{g}\,D_A\alpha = \hat{g}\,D_A\left(\beta\frac{\delta S}{\delta A} - \alpha\right) \tag{5.5}$$

so that instead of the one form $\langle D_A\alpha(A), \cdot\rangle_{(1)}$ corresponding to the original Zwanziger term the modified one form given in (5.5) appears in the Fokker–Planck operator. The last two terms in (5.4) arise due to the rules of Ito-stochastic calculus; they will turn out later on to give a contribution of the form $D_A\xi$.

At this point we want to draw attention to the appearance of the metric $\hat{g}$. Since any of the $\hat{g}$ (parametrized by the yet not specified β) gives rise to a specific connection one has an analogous obstruction as in (3.9) when trying to have (5.5) as a closed one form. A necessary requirement to overcome this obstruction is therefore that the corresponding curvature has to vanish. The question of how to find such a metric $\hat{g}$ is reduced to the question of how to find a flat connection.

Indeed, we can show now that there exists a flat connection in our bundle. The gauge fixing surface Σ gives rise to a natural notion of horizontal vector spaces in the bundle $\pi^{-1}(U(A_0)) \to U(A_0)$, by declaring all those vectors $\tau \in T_A\pi^{-1}(U(A_0))$ in the tangent space in $A \in \pi^{-1}(U(A_0))$ to be horizontal, which can be written in the form $\tau = \mathrm{ad}(g^{-1})\,\zeta_B$, where $A = B^g$ and $\zeta_B \in T_B\Sigma$ is a tangent vector of Σ in point B. Let us denote the corresponding subbundle by $\mathscr{H}$. It is evident by inspection that the corresponding connection one form $\tilde{\gamma}$ is given by the expression

$$\tilde{\gamma} = \mathrm{ad}(g^{-1})\,\mathscr{F}_B^{-1}\,D^*_{A_0}\,\mathrm{ad}(g), \qquad A = B^g. \tag{5.6}$$

This connection is the pull-back of the Maurer–Cartan form $\theta = \mathrm{ad}(g^{-1})\,R_g$ on the gauge group via the local trivialization χ of the bundle $\pi^{-1}(U(A_0)) \to U(A_0)$. The corresponding curvature vanishes, in other terms expressing the fact that the horizontal subbundle $\mathscr{H}$ is isomorphic to the tangent bundle $T\Sigma$ and hence integrable. The projector onto the horizontal subbundle is given by

$$\tilde{\mathbf{P}} = \mathbf{1} - D_A\tilde{\gamma}. \tag{5.7}$$

It has to be mentioned that the connection $\tilde{\gamma}$ cannot be extended to a globally defined flat connection on the whole bundle $\mathscr{A} \to \mathscr{M}$ due to its nontriviality.

Now we shall fix the new metric $\hat{g}$ in such a way that the already introduced connection $\tilde{\gamma}$ is exactly the induced connection imposed by itself. In other words this means that the horizontal subbundle $\mathscr{H}$ should be orthogonal to the gauge orbits with respect to $\hat{g}$. In particular the gauge fixing surface is then orthogonal to the gauge orbits. Hence $\hat{g}$ has to be chosen such that

$$\hat{g}(\tilde{\mathbf{P}}(\tau), D_A\xi) = \langle \hat{e}^{-1}(\tilde{\mathbf{P}}(\tau)), \hat{e}^{-1}(D_A\xi)\rangle_{(1)} = 0 \tag{5.8}$$

$\forall \xi \in \mathrm{Lie}\,\mathscr{G}$ and $\forall \tau \in T_A\pi^{-1}(U(A_0))$. Using that

$$\hat{e}^{-1}\,D_A = D_A(\mathbf{1} + \beta\,D_A)^{-1} \tag{5.9}$$

one has to conclude that $\hat{e}^{-1}\tilde{\mathbf{P}}$ must be horizontal with respect to the connection γ. Hence β has to satisfy

$$(\mathbf{1} - \mathbf{P})\,\hat{e}^{-1}\tilde{\mathbf{P}} = 0. \tag{5.10}$$

Using that $\mathbf{P} \cdot \tilde{\mathbf{P}} = \mathbf{P}$ we finally obtain

$$\beta\mathbf{P} = \gamma - \tilde{\gamma}. \tag{5.11}$$

Notice that β is only fixed on the horizontal bundle with respect to γ. In the vertical direction, however, β has only to satisfy that $\mathbf{1}+\beta\, D_A \neq 0$ in order to guarantee the existence of $\hat{e}^{-1}$. The solution for β thus obtains as

$$\beta = c\gamma - \tilde{\gamma}, \qquad c \neq 0, \tag{5.12}$$

where c is a nonsingular map from Lie $\mathcal{G}$ to Lie $\mathcal{G}$. Obviously there is left a freedom for the choice of β along the gauge group. This, however, can be proven to be irrelevant in the derivation of the equilibrium distribution of the Fokker–Planck equation, out of which the path integral density is constructed. Choosing for c the identity operator the following appealing expressions for the vielbein $\hat{e}^{-1}$

$$\hat{e}^{-1} = \mathbf{1} - D_A(\gamma - \tilde{\gamma}) = \mathbf{P} + (\mathbf{1} - \tilde{\mathbf{P}}) \tag{5.13}$$

as well as for the metric $\hat{g}$

$$\hat{g} = \langle \mathbf{P}(\tau_1), \mathbf{P}(\tau_2) \rangle_{(1)} + \langle (\mathbf{1} - \tilde{\mathbf{P}})(\tau_1), (\mathbf{1} - \tilde{\mathbf{P}})(\tau_2) \rangle_{(1)} \tag{5.14}$$

are easily derived. Notice that $\langle \mathbf{P}(\cdot), \mathbf{P}(\cdot) \rangle_{(1)}$ induces a metric on the space $\mathcal{M}$.

Similarly as in the case of the helix model there does not exist a coordinate transformation ϕ such that the Jacobian gives rise to the vielbein $\hat{e}^{-1}$. In order to prove this fact let us assume the contrary, i.e.,

$$\hat{e}^{-1}(\tau) = \phi_* \tau = T\phi(\tau_{\phi^{-1}}) \tag{5.15}$$

for $\tau \in T_A \pi^{-1}(U(A_0))$. But then we get for all vector fields τ^1, τ^2 on $\pi^{-1}(U(A_0))$

$$[\hat{e}^{-1}(\tau^1), \hat{e}^{-1}(\tau^2)] - \hat{e}^{-1}([\tau^1, \tau^2]) = [\phi_* \tau^1, \phi_* \tau^2] - \phi_*[\tau^1, \tau^2] = 0 \tag{5.16}$$

using the properties of the push-forward ϕ_*. On the other hand, using (5.13) we find for τ^1, τ^2 being horizontal with respect to $\tilde{\gamma}$ that the above difference of commutators gives

$$[\hat{e}^{-1}(\tau^1), \hat{e}^{-1}(\tau^2)] - \hat{e}^{-1}([\tau^1, \tau^2]) = [\mathbf{P}(\tau^1), \mathbf{P}(\tau^2)] - \mathbf{P}[\tau^1, \tau^2]. \tag{5.17}$$

That this expression is not vanishing can explicitly be shown by applying the connection γ on the left hand side of (5.17), yielding

$$\gamma([\hat{e}^{-1}(\tau^1), \hat{e}^{-1}(\tau^2)] - \hat{e}^{-1}([\tau^1, \tau^2])) = -\Omega(\tau^1, \tau^2) \tag{5.18}$$

hence proving that (5.15) cannot be true.

In the adapted coordinates the orthogonality condition of the gauge fixing surface and the gauge orbit with respect to the metric $\hat{g}$ is transformed into simply

$$(\tilde{G}^{-1})^{\Sigma\mathcal{G}} = (\tilde{G}^{-1})^{\mathcal{G}\Sigma} = 0. \tag{5.19}$$

This condition is fulfilled provided β is chosen as above in (5.12). Note that for the choice $c=1$ we obtain

$$\tilde{E}^{\mathscr{G}} = D_A^*, \qquad (\tilde{G}^{-1})^{\mathscr{G}\mathscr{G}} = \Delta_A. \tag{5.20}$$

6. THE PATH INTEGRAL DENSITY AS EQUILIBRIUM DISTRIBUTION

This section is devoted to the derivation of the Fokker–Planck equilibrium distribution which—according to the general principles of the stochastic quantization scheme—will be identified with the path integral density for the Yang–Mills field.

We previously have already worked out in (4.23) the Langevin equation for the B-field. Now we derive from the general Langevin equation (4.15), inserting the special value (5.20), the corresponding g-field equation

$$\begin{aligned} dg = \Bigg[&-(G^{-1})^{\mathscr{G}\Sigma} \frac{\delta S}{\delta B} + \frac{1}{\sqrt{\det G}} \frac{\delta((G^{-1})^{\mathscr{G}\mathscr{G}} \sqrt{\det G})}{\delta g} \\ &+ \frac{1}{\sqrt{\det G}} \frac{\delta((G^{-1})^{\mathscr{G}\Sigma} \sqrt{\det G})}{\delta B} + R_g^{-1}\, \mathrm{ad}(g)\, \alpha \Bigg] ds + \tilde{E}^{\mathscr{G}}\, dW. \end{aligned} \tag{6.1}$$

We choose α as

$$\begin{aligned} \alpha = \mathrm{ad}(g^{-1})\, R_g \Bigg[&-(\tilde{G}^{-1})^{\mathscr{G}\mathscr{G}} \frac{\delta S_{\mathscr{G}}[g]}{\delta g} + (G^{-1})^{\mathscr{G}\Sigma} \frac{\delta S}{\delta B} - \frac{1}{\sqrt{\det G}} \frac{\delta((G^{-1})^{\mathscr{G}\mathscr{G}} \sqrt{\det G})}{\delta g} \\ &- \frac{1}{\sqrt{\det G}} \frac{\delta((G^{-1})^{\mathscr{G}\Sigma} \sqrt{\det G})}{\delta B} + \frac{1}{\sqrt{\det G}} \frac{\delta((\tilde{G}^{-1})^{\mathscr{G}\mathscr{G}} \sqrt{\det G})}{\delta g} \Bigg], \end{aligned} \tag{6.2}$$

where $S_{\mathscr{G}}[g]$ is an arbitrary damping function with the property that

$$\int_{\mathscr{G}} \mathscr{D}g \sqrt{\det(R_g^* R_g)}\, e^{-S_{\mathscr{G}}[g]} < \infty. \tag{6.3}$$

The choice of α is in fact suggestive: the drift term of the g-field Langevin equation (6.1) is totally exchanged by the damping term $-(\tilde{G}^{-1})^{\mathscr{G}\mathscr{G}} (\delta S_{\mathscr{G}}[g]/\delta g)$; in addition a judiciously chosen Ito-term $(1/\sqrt{\det G})(\delta((\tilde{G}^{-1})^{\mathscr{G}\mathscr{G}} \sqrt{\det G})/\delta g)$ is added. Due to the choice (6.2) and (6.3), α serves as the integrating factor to obtain the well damped Langevin equation

$$dg = \left[-(\tilde{G}^{-1})^{\mathscr{G}\mathscr{G}} \frac{\delta S_{\mathscr{G}}[g]}{\delta g} + \frac{1}{\sqrt{\det G}} \frac{\delta((\tilde{G}^{-1})^{\mathscr{G}\mathscr{G}} \sqrt{\det G})}{\delta g} \right] ds + \tilde{E}^{\mathscr{G}}\, dW. \tag{6.4}$$

For fixed B, the above equation describes a stochastic process on the gauge group.

The Langevin equations (4.23) and (6.4) for B and g, respectively, can be recast into

$$d\Psi = \left[-\tilde{G}^{-1} \frac{\delta S_{\text{tot}}[\Psi]}{\delta\Psi} + \frac{1}{\sqrt{\det G}} \frac{\delta(\tilde{G}^{-1}\sqrt{\det G})}{\delta\Psi} \right] ds + \tilde{E}\, dW, \tag{6.5}$$

where

$$S_{\text{tot}}[\Psi] = S[B] + S_{\mathscr{G}}[g]. \tag{6.6}$$

The associated Fokker–Planck equation is derived in a straightforward manner

$$\frac{\partial\rho[\Psi, s]}{\partial s} = L[\Psi]\, \rho[\Psi, s], \tag{6.7}$$

where now the Fokker–Planck operator $L[\Psi]$ is appearing in just factorized form

$$L[\Psi] = \frac{\delta}{\delta\Psi} \tilde{G}^{-1} \left[\frac{\delta S_{\text{tot}}[\Psi]}{\delta\Psi} - \frac{1}{\sqrt{\det G}} \frac{\delta(\sqrt{\det G})}{\delta\Psi} + \frac{\delta}{\delta\Psi} \right]. \tag{6.8}$$

Due to the positivity of $\tilde{G}$ the fluctuation dissipation theorem applies and the equilibrium Fokker–Planck distribution $\rho^{\text{eq}}[\Psi]$ obtains by direct inspection as

$$\begin{aligned} \rho^{\text{eq}}[\Psi] &= \frac{\sqrt{\det G}\, e^{-S_{\text{tot}}[\Psi]}}{\int_{\Sigma\times\mathscr{G}} \mathscr{D}B\, \mathscr{D}g\, \sqrt{\det G}\, e^{-S_{\text{tot}}}} \\ &= \frac{\det \mathscr{F}_B e^{-S[B]} \sqrt{\det R_g^* R_g}\; e^{-S_{\mathscr{G}}[g]}}{\int_\Sigma \mathscr{D}B \det \mathscr{F}_B e^{-S[B]} \int_{\mathscr{G}} \mathscr{D}g \sqrt{\det R_g^* R_g}\; e^{-S_{\mathscr{G}}[g]}}. \end{aligned} \tag{6.9}$$

This result is completely equivalent to the Faddeev–Popov prescription [21] for Yang–Mills theory. The additional *finite* contributions of the gauge degrees of freedom always cancel out when evaluated on gauge invariant observables.

On the bundle $\pi^{-1}(U(A_0)) \to U(A_0)$ in the original coordinates the Langevin equation takes the simple form

$$dA = \left[-\hat{g}^{-1} \frac{\delta S_{\text{tot}}}{\delta A} + \frac{\delta \hat{g}^{-1}}{\delta A} \right] ds + \hat{e}\, dW, \tag{6.10}$$

where the total action reads $S_{\text{tot}}[A] = S[A] + S_{\mathscr{G}}[\omega(A)]$. The dependence of the gauge fixing surface occurs through the form of $\omega(A)$ as defined in (4.4). The Fokker–Planck operator in the original coordinates is given by

$$L[A] = \frac{\delta}{\delta A} \hat{g}^{-1} \left[\frac{\delta S_{\text{tot}}[A]}{\delta A} + \frac{\delta}{\delta A} \right]. \tag{6.11}$$

With the same argument as above we obtain as a new result that the equilibrium distribution for the original variables A is given by

$$\rho^{\mathrm{eq}}[A]=\frac{e^{-S_{\mathrm{tot}}[A]}}{\int_{\pi^{-1}(\mathbf{U}(\mathbf{A}_0))}\mathscr{D}A\,e^{-S_{\mathrm{tot}}}}. \tag{6.12}$$

7. OUTLOOK

In this paper we proposed a new stochastic gauge fixing procedure for Yang–Mills theory. We were led by the paradigm that instead of the stochastic process itself the expectation values of gauge invariant variables should be the right focus. We succeeded in modifying the original Parisi–Wu as well as Zwanziger's approach such that the Faddeev–Popov path integral density could be obtained as the Fokker–Planck equilibrium distribution in a geometrically transparent way. Distinguished by its concept it is the forthcoming task to extend the procedure which so far has been performed only locally to cover the whole space of gauge potentials.

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