String states, loops and effective actions in NC field theory and matrix models

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Action MP 1405
Quantum Structure of Spacetime

H.S., arXiv:1606.00646

Motivation

- practical way of performing loop integrals (quantum corrections) in NC gauge theory

  for generic NC geometries, no symmetries
- understand better nature of UV/IR mixing in NC field theory
- derive supergravity from matrix models ($\leftrightarrow$ string theory)
- quantum corrections on fuzzy S^4_N

  ($\rightarrow$ emergent 4D gravity in IKKT model)

Outline

- coherent states on S^2_N and $\mathbb{R}^2_\theta$
- string states
- some trace computations
- 1-loop computations on S^2_N, UV/IR mixing
- IIB supergravity in the IKKT model

Fuzzy S^2_N

$$[X^a, X^b] = i\varepsilon^{abc} X^c, \qquad X^a X_a = \frac{1}{4}(N^2-1) \ =: R_N^2 \ .$$

$X^a = J^a_{(N)}$ ... irrep of $SU(2)$ on $\mathcal{H} = \mathbb{C}^N$

functions on S^2_N ... $\mathcal{A} = End(\mathcal{H}) = \bigoplus_{l=0}^{N-1} (2l+1)$

matrix Laplacian

$$\Box\phi = [X^a, [X_a, \phi]], \qquad \phi \in End(\mathcal{H})$$

quantized space, $SO(3)$-invariant symplectic form ω

$$\int_{S^2} \omega = 2\pi \dim(\mathcal{H}) = 2\pi N$$

fuzzy sphere contains N quantum cells of area $A_N = \frac{4\pi R_N^2}{N} \sim L^2_{NC}$

S^2 as group orbit: let $p \in S^2$ ... north pole

$$\begin{aligned} SU(2) &\to S^2 \\ g &\mapsto g \cdot p =: x \end{aligned}$$

stabilizer $\mathcal{K} \subset SU(2) \Rightarrow S^2 \cong SU(2)/U(1)$

coherent states on S^2_N:

$|p\rangle \in \mathcal{H}_N$ ... highest weight vector in $\mathcal{H}_N$

def.

$$\begin{aligned} |x\rangle &= g_x \cdot |p\rangle, \qquad g_x \in SU(2) \text{ ... coherent states} \\ x^a &= \langle x|X^a|x\rangle \equiv \langle X^a\rangle \in S^2, \qquad x^a x_a = \tfrac{1}{4}(N-1)^2 =: r_N^2 \end{aligned}$$

$|x\rangle$ ... one-to-one correspondence to points x on S^2 (up to phase)

relation with $\mathbb{R}^2_\theta$, Q.M: focus at north pole $p \in S^2$

rescale $R \to \infty$ s.t. $\theta = \frac{2R^2}{N} = const$

$$[X^i, X^j] = i\theta\epsilon^{ij} \quad + \mathcal{O}(\frac{1}{N})$$

coherent state at "origin" = north pole:

$$|0\rangle \equiv |p\rangle, \qquad a|0\rangle \sim (X_1 + iX_2)|0\rangle = 0 \qquad \text{highest weight state}$$

shifted (rotated) coherent states:

$$|x\rangle = U_x|0\rangle \quad \text{where} \qquad U_x = \exp(i\phi_i J^i), \qquad x^i = R\,\epsilon^{ij}\phi_j$$

localization:

$$\langle x'|x\rangle = e^{-\frac{i}{2\theta}x^i\varepsilon_{ij}x'^j} e^{-\frac{|x-x'|^2}{4\theta}}$$

covers area $A_N = \theta$

back to S^2_N:

coherent states are optimally localized:

$$\begin{aligned} \Delta^2 &= \sum_a \langle (X^a)^2 \rangle - \langle X^a \rangle^2 = R_N^2 - r_N^2 = \tfrac{N-1}{2} \\ &=: L_{NC}^2 \approx \tfrac{2}{N} R_N^2 \end{aligned}$$

$$\begin{aligned} |\langle x|y\rangle|^2 = (\tfrac{1+x\cdot y}{2})^{N-1} &\approx \exp(-\tfrac{1}{4}\phi^2(N-1)) = \tfrac{1}{c_N}\,\delta_N(x,y) \\ \phi &= \measuredangle(x,y) \end{aligned}$$

char. angle $\phi_N = \frac{\pi}{\sqrt{N}}$
char. angular momentum $l \sim \sqrt{N}$

overcompleteness:

$$\mathbf{1}_{\mathcal{H}} = c_N \int_{S^2} dx |x\rangle\langle x|, \qquad c_N = \frac{\dim \mathcal{H}}{\mathrm{Vol}\mathrm{S}^2}$$

(by $SU(2)$ invariance)

trace of any operator $\mathcal{O} \in End(\mathcal{H})$

$$\mathrm{tr}\mathcal{O} = \frac{\dim \mathcal{H}}{\mathrm{Vol}\mathrm{S}^2} \int_{S^2} dx \langle x|\mathcal{O}|x\rangle$$

generalizes to any quantized coadjoint orbit

operators and symbols

For an operator $\mathcal{O} \in End(\mathcal{H})$, define *symbol* of $\mathcal{O}$ as

$$\mathcal{O}(x) = \langle x|\mathcal{O}|x\rangle$$

... de-quantization of $\mathcal{O}$, "semi-classical limit"

conversely:

$$\mathcal{O} = c_N \int_{S^2} dx \tilde{O}(x)|x\rangle\langle x|$$

in particular

$$\hat{Y}^I_m = c_N \int_{S^2} dx Y^I_m(x)|x\rangle\langle x|$$

however very delicate for large momenta, misleading in UV

wavefunctions and UV / IR sectors

- IR ("semi-classical") sector:

  non-local matrix elements decay at distances $|x-y| \sim L_{NC}$

$$\langle x|\mathcal{O}|y\rangle \approx \langle x|\mathcal{O}|x\rangle \, \langle x|y\rangle \;\approx\; \frac{1}{c_N} \langle x|\mathcal{O}|x\rangle \, \tilde{\delta}_N(x,y)$$

  i.e.

$$\langle x|f(X)|y\rangle \approx f(x)\langle x|y\rangle \approx f(y)\langle x|y\rangle$$

max. angular momentum $l \leq \sqrt{N}$, uncertainty $\Delta^2 = L_{NC}^2$

optimally localized function $\quad |p\rangle\langle p| =: \frac{1}{c_N}\, \delta_N(X;p)$

- <u>UV sector:</u>

  most $\mathcal{O} \in End(\mathcal{H})$ have $l > \sqrt{N}$, not in semi-classical sector.

  best described by *non-local* string states

$$\psi_{x,y} := |x\rangle\langle y| \qquad \in End(\mathcal{H})$$

most extreme "function" on S^2_N:

$$Y^{N-1}_{N-1} = |p\rangle\langle -p|,$$

has $l_{UV} = N$, maximally de-localized

most NC "functions" are non-local
$\Rightarrow$ non-local contributions in loops!

string states

Iso Kawai Kitazawa hep-th/0001027; H.S., arXiv:1606.00646

$$\begin{aligned} \left|{x \atop y}\right) &:= \psi_{x,y} := |x\rangle\langle y| \qquad \in End(\mathcal{H}) \\ \left({x \atop y}\right| &:= \psi^\dagger_{x,y} := |y\rangle\langle x| \end{aligned}$$

momentum operators

$$\begin{aligned} \mathcal{P}^a \mathcal{O} &:= [X^a, \mathcal{O}], \\ \Box \mathcal{O} &:= \mathcal{P}^a \mathcal{P}_a \mathcal{O} \end{aligned}$$

expectation values

$$\begin{aligned} \left({x \atop y}\right| \mathcal{P}^a \left|{x \atop y}\right) &= \operatorname{tr} \psi_{y,x}[X^a, \psi_{x,y}] = \vec{\mathbf{x}}(x) - \vec{\mathbf{x}}(y) \\ \left({x \atop y}\right| \mathcal{P}^a \mathcal{P}_a \left|{x \atop y}\right) &= \operatorname{tr} \psi_{y,x}[X^a, [X_a, .]]\psi_{x,y} \quad = E_{xy} \end{aligned}$$

$$\boxed{E_{xy} = (\vec{\mathbf{x}}(x) - \vec{\mathbf{x}}(y))^2 + 2\Delta^2}$$

energy of string state = length ^{2}+ zero point energy

general matrix elements

$$\begin{aligned}
\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right| \mathcal{P}^a \mathcal{P}_a \left|\begin{smallmatrix} x' \\ y' \end{smallmatrix}\right) &= \langle x|X^a X^a|x'\rangle\langle y'|y\rangle + \langle x|x'\rangle\langle y'|X^a X^a|y\rangle - 2\langle x|X^a|x'\rangle\langle y'|X_a|y\rangle \\
&\approx (2\Delta^2 + \vec{x}^2 + \vec{y}^2 - 2\vec{x}\vec{y}))\,\langle x|x'\rangle\langle y'|y\rangle \\
&\approx E_{xy}\,\langle x|x'\rangle\langle y'|y\rangle
\end{aligned}$$

nearly diagonal

good localization properties in both position and momentum !!

propagator

claim:

$$\boxed{(\bar{\Box}+\mu^2)^{-1} := c_N^2 \int\limits_{\mathcal{M}} dxdy \left|{x \atop y}\right) \frac{1}{E_{xy}+\mu^2} \left({x \atop y}\right| \approx (\Box+\mu^2)^{-1}}$$

is excellent approximation to the propagator

because:

$$\begin{aligned}
(\tilde{\Box}+\mu^2)^{-1}(\Box+\mu^2)\left|{x \atop y}\right) &= c_N^2 \int dx'dy' \left|{x' \atop y'}\right) \tfrac{1}{E_{x'y'}+\mu^2} \left({x' \atop y'}\right| (\Box+\mu^2)\left|{x \atop y}\right) \\
&\approx c_N^2 \int dx'dy' \left|{x' \atop y'}\right) \tfrac{1}{E_{x'y'}+\mu^2} (E_{xy}+\mu^2)\langle x'|x\rangle\langle y|y'\rangle \\
&\approx \left|{x \atop y}\right)
\end{aligned}$$

completely regular since $E_{xy} \geq \Delta^2$

trace formula for $End(\mathcal{H})$

$$\boxed{\mathrm{Tr}_{End(\mathcal{H})}\mathcal{O} = \frac{(\dim\mathcal{H})^2}{(\mathrm{Vol}\mathcal{M})^2}\int\limits_{\mathcal{M}\times\mathcal{M}} dxdy\, \left({x \atop y}\middle|\, \mathcal{O}\, \middle|{x \atop y}\right)}$$

<u>proof</u>:

rhs = unique functional on $End(End(\mathcal{H}))$ invariant under $G_L \times G_R$

= Tr

example:

$$\begin{aligned}
\mathrm{Tr}_{End(\mathcal{H})}[X^a,[X_a,.]] &= \frac{N^2}{(\mathrm{VolS}^2)^2}\int\limits_{S^2\times S^2} dxdx'\mathrm{tr}(|x\rangle\langle x'|)\big(|x'^a-x^a|^2+2\Delta^2\big)(|x'\rangle\langle x|)\\
&= \frac{N^2}{(\mathrm{VolS}^2)^2}\int\limits_{S^2\times S^2} dxdx'(|x'^a-x^a|^2+2\Delta^2)\\
&= \frac{N^2}{\mathrm{VolS}^2}\int\limits_{S^2} dx(|x^a(e)-x^a(x)|^2+2\Delta^2)\\
&\approx \frac{1}{4}\frac{N^2(N^2-1)}{4\pi^2}\int\limits_{S^2} dx(|e_3-x|^2+O(\tfrac{1}{N}))\\
&= \frac{1}{2}N^2(N-1)^2(1+O(\tfrac{1}{N}))
\end{aligned}$$

using $r_N^2 = x^a x_a = \frac{1}{4}(N-1)^2$ and $\Delta^2 \approx \frac{N}{2}$

good agreement with exact result:

$$\mathrm{Tr}_{End(\mathcal{H})}[X^a,[X_a,.]] = \sum_{j=0}^{N-1} j(j+1)(2j+1) = \frac{1}{2}N^2(N^2-1).$$

more generally:

for any smooth function f

$$
\begin{aligned}
\mathrm{Tr}_{End(\mathcal{H})} f(\Box) &= \frac{N^2}{(\mathrm{Vol}S^2)^2} \int_{S^2} dx \int_{S^2} dy\, f(R_N^2 |x-y|^2 + 2\Delta^2) \\
&= \frac{N^2}{\mathrm{Vol}S^2} \int_{S^2} dx f(r_N^2 |e_3 - x|^2 + 2\Delta^2) \\
&= 2\pi \frac{N^2}{\mathrm{Vol}S^2} \int_0^\pi d\vartheta \sin\vartheta f(r_N^2 (1-\cos\theta)^2 + \sin^2\theta) + 2\Delta^2) \\
&= \frac{N^2}{2} \int_{-1}^1 du f(2r_N^2(1-u) + 2\Delta^2) \\
&\approx \int_0^N dj\, 2j f(j^2 + 2\Delta^2) \approx \sum_{j=0}^{N-1} (2j+1) f\big(j(j+1) + 2\Delta^2\big) \\
&= \mathrm{Tr}_{j_{\max}} f(\Box_g + 2\Delta^2) \\
&= \mathrm{Tr}_{j_{\max}} f(\Box_g)
\end{aligned}
$$

shift by $2\Delta^2 = N - 1$ negligible for $N \gg 1$

UV dominates!

works because $\mathrm{spec}(\Box) = \mathrm{spec}(\Box_g)$ also in UV

one-loop propagator for ϕ^4 on S^2_N

$$S[\phi] = \frac{1}{N}\mathrm{tr}\Big(\frac{1}{2}\phi(\Box + \mu^2)\phi + \frac{g}{4!}\phi^4\Big) \;= S_0[\phi] + S_{\mathrm{int}}[\phi]\;.$$

1-loop effective action (= Gaussian approx.)

$$\begin{aligned}\Gamma_{\mathrm{eff}}[\phi] &= S[\phi] + \tfrac{1}{2}\mathrm{Tr}_{End(\mathcal{H})}\log\Big(S''[\phi]\Big)\\ (\psi, S''[\phi]\psi) &= \tfrac{1}{N}\mathrm{tr}\Big(\psi(\Box + \mu^2)\psi + \tfrac{g}{3}\phi^2\psi^2 + \tfrac{g}{6}\psi\phi\psi\phi\Big)\end{aligned}$$

expanded:

$$\begin{aligned}\Gamma_{1-loop}[\phi] &= \mathrm{Tr}\log(.(\Box + \mu^2). + \tfrac{g}{3}.\phi^2. + \tfrac{g}{6}.\phi.\phi)\\ &= \mathrm{Tr}\log(\Box + \mu^2) + \mathrm{Tr}\Big(.\tfrac{1}{\Box+\mu^2}(\tfrac{g}{3}\phi^2. + \tfrac{g}{6}\phi.\phi)\Big) + O(\phi^4)\end{aligned}$$

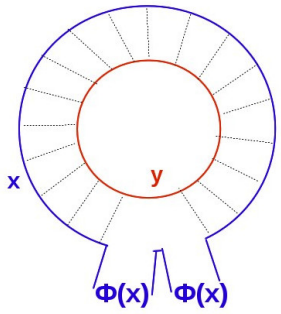
assume $\phi = \phi(X)$ slowly varying, IR regime

$\Rightarrow \phi\psi_{yx} \approx \phi(y)\psi_{yx}$ in string basis

$$\begin{aligned} \mathrm{Tr}(.\phi^2.) &= \frac{N^2}{Vol(\mathcal{M})^2} \int_{\mathcal{M}\times\mathcal{M}} dxdy\, \mathrm{tr}(\psi_{y,x}\phi^2\psi_{x,y}) \\ &= \frac{N^2}{Vol(\mathcal{M})} \int_{\mathcal{M}} dx\, \langle x|\phi^2|x\rangle \ . \end{aligned}$$

Similarly, "planar" contribution

$$\begin{aligned} Tr(.\Box^{-1}\phi^2.) &= \frac{N^2}{Vol(\mathcal{M})^2} \int_{\mathcal{M}\times\mathcal{M}} dxdy\, \mathrm{tr}(\psi_{y,x}(\Box+\mu^2)^{-1}(\phi^2\psi_{x,y})) \\ &\approx \frac{N^2}{Vol(\mathcal{M})^2} \int_{\mathcal{M}\times\mathcal{M}} dxdy\, \frac{1}{r_N^2|x-y|^2+2\Delta^2+\mu^2}\, \mathrm{tr}(\psi_{y,x}\phi^2\psi_{x,y}) \\ &= \frac{N^2}{Vol(\mathcal{M}^2)} \int_{\mathcal{M}\times\mathcal{M}} dxdy\, \frac{1}{r_N^2|x-y|^2+\tilde{\mu}^2}\, \langle x|\phi^2|x\rangle \\ &= \frac{\mu_N^2}{Vol(\mathcal{M})} \int_{\mathcal{M}} dx\, \phi^2(x) \end{aligned}$$



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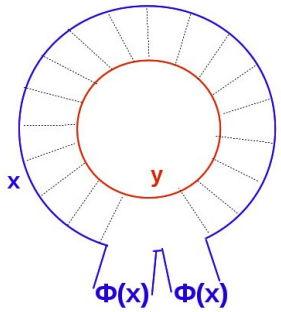
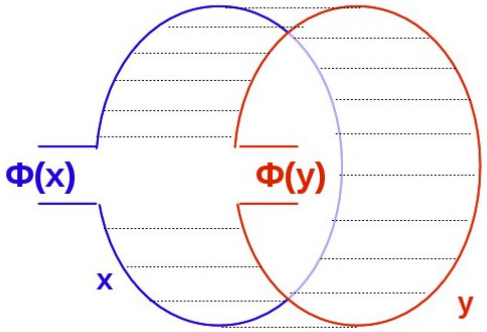


1-loop "planar" mass renormalization

$$
\begin{aligned}
\mu_N^2 &= \frac{N^2}{Vol(S^2)} \int_{S^2} dy \, \frac{1}{r_N^2 |e-y|^2 + \tilde{\mu}^2} \\
&= \frac{N^2}{2 r_N^2} \int_0^\pi d\vartheta \sin\vartheta \frac{1}{(1-\cos\vartheta)^2 + \sin\vartheta^2 + \frac{\tilde{\mu}^2}{r_N^2}} \\
&= 2 \int_{-1}^{1} du \frac{1}{2 - 2u + \frac{\tilde{\mu}^2}{r_N^2}} \\
&\approx \sum_{j=0}^{N} \frac{2j+1}{j(j+1)+\mu^2}
\end{aligned}
$$

"nonplanar" contribution

$$
\begin{aligned}
\mathrm{Tr}(.(\Box+\mu^2)^{-1}\phi.\phi) &= \frac{N^2}{Vol(\mathcal{M})^2}\int\limits_{\mathcal{M}\times\mathcal{M}} dxdy\,\mathrm{tr}(\psi_{y,x}(\Box+\mu^2)^{-1}(\phi\psi_{x,y}\phi)) \\
&= \frac{N^2}{Vol(\mathcal{M})^2}\int\limits_{\mathcal{M}\times\mathcal{M}} dxdy\,\langle x|(\Box+\mu^2)^{-1}\phi|x\rangle\langle y|\phi|y\rangle \\
&= \frac{N^2}{Vol(\mathcal{M})^2}\int\limits_{\mathcal{M}\times\mathcal{M}} dxdy\,\frac{1}{r_N^2|x-y|^2+\tilde{\mu}^2}\phi(x)\phi(y)
\end{aligned}
$$



one-loop quantum effective action:

$$\boxed{S_{1-loop} \sim S_0 + \frac{g}{3}\frac{1}{\mathrm{Vol}(\mathcal{M})}\int_{\mathcal{M}} dx \mu_N^2 \phi(x)^2 + \frac{g}{6}\frac{N^2}{\mathrm{Vol}(\mathcal{M})^2 \mathrm{r_N^2}}\int_{\mathcal{M}\times\mathcal{M}} dxdy \frac{\phi(x)\phi(y)}{|x-y|^2 + \frac{\tilde{\mu}^2}{r_N^2}} + O(\phi^4)}$$

long-range non-locality from UV sector (UV/IR mixing)

applies to any compact fuzzy space

check for S_N^2: agrees with traditional mode expansion

$$S_{1-loop} = S_0 + \int \frac{1}{2}\,\Phi(\mu_N^2 - \frac{g}{12\pi}h(\tilde{\Box}))\Phi + o(1/N)$$

Chu Madore HS hep-th/0106205

where

$$h(L) = -\frac{1}{2}\int_{-1}^{1} dt \frac{1}{1-t}(P_L(t) - 1) = \sum_{k=1}^{L}\frac{1}{k}$$

less transparent, requires asymptotics of 6J symbols etc.

Moyal-Weyl plane limit $\mathbb{R}^2_\theta$ $\qquad R^2 = r^2 R_N^2 = \frac{N\theta}{4}$

$$\begin{aligned}\Gamma_{NP} &\approx \frac{gN^2}{6\mathrm{Vol}(\mathcal{M})^2} \int\limits_{\mathcal{M}\times\mathcal{M}} dxdy \frac{\phi(x)\phi(y)}{|x-y|^2+\mu^2} \\ &= \frac{g}{6\pi^2\theta^2} \int\limits_{\mathcal{M}\times\mathcal{M}} dxdy \frac{\phi(x)\phi(y)}{|x-y|^2+\mu^2}\end{aligned}$$

where $\mathrm{Vol}\mathcal{M} = 4\pi R^2 = \pi N\theta$

plane wave basis $\phi(x) = \int \frac{d^2k}{2\pi}\phi_k(e^{ixk} + e^{-ikx})$.

$$\begin{aligned}\Gamma_{NP} &\approx \frac{g}{6\pi^2\theta^2} \int d^2k\phi(k)^2 \int d^2z \frac{1}{|z|_g^2+\mu^2} e^{ik_i z^i} \\ &= \frac{g}{6\pi^2\theta^2} \int d^2k\phi(k)^2 \int d^2p \frac{1}{p_i p_j G^{ij}+\mu^2} e^{ik_i\theta^{ij}p_j}.\end{aligned}$$

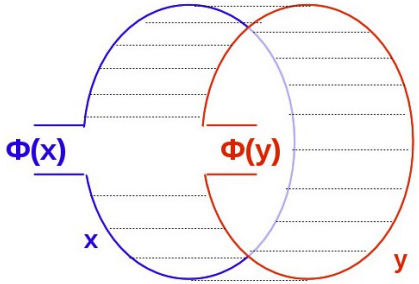
replacing $z^i = \theta^{ij}p_j$, and $G^{ij} = \theta^{ii'}\theta^{jj'}\delta_{i'j'}$

... familiar form in NCFT, IR divergence for $k \to 0$ from UV loop,

UV/IR mixing

significance of UV/IR mixing:

UV sector in loops = virtual long strings $|x\rangle\langle y|$
lead to long-range non-locality in $\int\limits_{\mathcal{M}\times\mathcal{M}} dxdy \frac{\phi(x)\phi(y)}{|x-y|^2+\mu^2}$



interpret NCFT as (non-critical) string theory!
open strings beginning and ending on D-branes

universal, same on any fuzzy space $\mathcal{M}$, any dimension
accumulates at higher loops, inacceptable as fundamental theory
except in SUSY case: cancellations!

higher loops

t'Hooft double line formalism, ribbon graphs

lines labeled by positions x, preserved by propagators (!!)

much simpler than in ordinary QFT, directly in position space !

H.S., arXiv:1606.00646

(to be developed)

claim:

analogous (quasi-) coherent states exist on generic fuzzy spaces

(=quantized symplectic manifolds $\mathcal{M} \subset \mathbb{R}^n$)

L. Schneiderbauer HS arXiv:1601.08007; cf. Ishiki arXiv:1503.01230

defined as ground states of point branes on $\mathcal{M}$

$$\begin{aligned} \Box_{\mathfrak{X}}\Phi &= \textstyle\sum_a [\mathfrak{X}^a, [\mathfrak{X}^a, \Phi]] = \sum_a (\mathfrak{X}^a\mathfrak{X}^a\Phi + \Phi\mathfrak{X}^a\mathfrak{X}^a - 2\mathfrak{X}^a\Phi\mathfrak{X}^a) \\ &= \begin{pmatrix} \begin{matrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{matrix} & \sum\limits_a (X^a - x^a)^2 |\phi\rangle \\ \langle\phi| \sum\limits_a (X^a - x^a)^2 & 0 \end{pmatrix} \end{aligned}$$

... shifted harmonic oscillator for $|\phi\rangle$ on $\mathcal{M}$

$$\mathfrak{X}^a = \begin{pmatrix} X^a & 0 \\ 0 & x^a \end{pmatrix}, \qquad a = 1, \ldots, d$$

conjecture: formula / approx. for $Tr_{End\mathcal{H}}$ holds quite generally

maximally SUSY case: IKKT model

Ishibashi, Kawai, Kitazawa, Tsuchiya 1996

$$S[X,\Psi] = -Tr\left([X^a,X^b][X^{a'},X^{b'}]\eta_{aa'}\eta_{bb'} + \bar{\Psi}\gamma_a[X^a,\Psi]\right)$$

$$X^a = X^{a\dagger} \in Mat(N,\mathbb{C}), \qquad a = 0,...,9 \qquad (N\to\infty)$$

gauge symmetry $X^a \to UX^aU^{-1}$, $SO(9,1)$, SUSY

$$\begin{cases} 1)\ \text{nonpert. def. of IIB string theory (on } \mathbb{R}^{10}) \qquad (IKKT) \\ 2)\ \mathcal{N}=4 \ \text{SUSY Yang-Mills gauge thy. on } \text{“}noncommutative\text{“}\ \mathbb{R}^4_\theta \end{cases}$$

- aim: evaluate & interpret 1-loop effective action on background: $\mathbb{R}^4_\theta$ and generic (curved) branes $\mathcal{M}\subset\mathbb{R}^{10}$
- will see: non-local UV/IR mixing $\equiv$ IIB supergravity (as predicted in string theory)

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background: generic fuzzy space

$$X^a \sim x^a : \ \mathcal{M} \ \hookrightarrow \mathbb{R}^{10}$$

fluctuation around background $X^a \to X^a + \mathcal{A}^a(X^a)$

one-loop effective action = Gaussian approx. around background

$$Z[X] = \int\limits_{\text{Gauss}} d\mathcal{A} d\Psi e^{-S[X+\mathcal{A},\Psi]} = e^{-\Gamma_{\text{eff}}[X]}$$

where

$$\Gamma_{\text{eff}}[X] = S[X] + \Gamma_{1\text{loop}}[X]$$

expand bare action to $\mathcal{O}(\mathcal{A}^2)$:

$$S[X+\mathcal{A}] \ = S[X] + \tfrac{2}{g^2} Tr\Big(2\mathcal{A}^a(\Box + \mu^2)X_a + \mathcal{A}_a\big((\Box + \mu^2)\delta^a_b + 2i[\Theta^{ab}, .] - [X^a, [X^b, .]]\big)\mathcal{A}_b\Big)$$

with $i\Theta^{ab} = [X^a, X^b]$

one-loop effective action in IKKT model

$$
\begin{aligned}
\Gamma_{1\text{loop}}[X] &= \tfrac{1}{2}\mathrm{Tr}\Big(\log(\Box+\mu^2-M^{(\mathcal{A})}_{ab}[\Theta^{ab},.])-\tfrac{1}{2}\log(\Box-M^{(\psi)}_{ab}[\Theta^{ab},.])-2\log(\Box)\Big)\\
&= \tfrac{1}{2}\mathrm{Tr}\Big(\textstyle\sum_{n>0}\tfrac{1}{n}\Big((\Box^{-1}(-M^{(\mathcal{A})}_{ab}[\Theta^{ab},.]+\mu^2))^n-\tfrac{1}{2}(-\Box^{-1}M^{(\psi)}_{ab}[\Theta^{ab},.])^n\Big)\Big)\\
&\overset{SUSY}{=} \tfrac{1}{2}\mathrm{Tr}\Big(\tfrac{1}{4}(\Box^{-1}(M^{(\mathcal{A})}_{ab}[\Theta^{ab},.])^4-\tfrac{1}{8}(\Box^{-1}M^{(\psi)}_{ab}[\Theta^{ab},.])^4+\mathcal{O}(\Box^{-1}[\Theta^{ab},.])^5\Big)\\
&\quad+\tfrac{1}{2}\mu^2\mathrm{Tr}\Box^{-1}+O(\mu^4)\\
&= \tfrac{1}{4}\mathrm{Tr}\Big(\Box^{-1}[\Theta^{a_1b_1},\ldots\Box^{-1}[\Theta^{a_4b_4},.]]]]\Big)\\
&\quad(-4g_{b_1a_2}g_{b_2a_3}g_{b_3a_4}g_{b_4a_1}-4g_{b_1a_2}g_{b_2a_4}g_{b_4a_3}g_{b_3a_1}-4g_{b_1a_3}g_{b_3a_2}g_{b_2a_4}g_{b}\\
&\quad+g_{b_1a_2}g_{b_2a_1}g_{b_3a_4}g_{b_4a_3}+g_{b_1a_3}g_{b_3a_1}g_{b_2a_4}g_{b_4a_2}+g_{b_1a_4}g_{b_4a_1}g_{b_2a_3}g_{b_3a_2})\\
&\quad-\tfrac{1}{4}\mu^2\mathrm{Tr}(\Box^{-1})
\end{aligned}
$$

where

$$
\begin{aligned}
(M^{(\psi)}_{ab})^{\alpha}_{\beta} &= \tfrac{1}{4i}[\gamma_a,\gamma_b]^{\alpha}_{\beta}\\
(M^{(\mathcal{A})}_{ab})^{c}_{d} &= i(\delta^c_b\delta_{ad}-\delta^c_a\delta_{bd})
\end{aligned}
$$

use string states to evaluate this !!

$$\begin{aligned} \Box^{-1}(|x\rangle\langle y|) &\sim \frac{1}{|x-y|^2+2\Delta^2}\,|x\rangle\langle y| \\ \Box^{-1}[\Theta^{ab},.](|x\rangle\langle y|) &\sim \frac{1}{|x-y|^2+2\Delta^2}\,\delta\Theta^{ab}(x,y)|x\rangle\langle y| \end{aligned}$$

on slowly-varying background, where

$$\delta\Theta^{ab}(x,y) := \Theta^{ab}(x) - \Theta^{ab}(y)$$

evaluate trace as

$$\begin{aligned} \Gamma_{1\mathrm{loop};4}[X] &\sim \frac{1}{4}\frac{(\dim\mathcal{H})^2}{(\mathrm{Vol}\mathcal{M})^2}\int\limits_{\mathcal{M}\times\mathcal{M}}\Omega_x\Omega_y\frac{\delta\Theta^{a_1b_1}(x,y)\delta\Theta^{a_2b_2}(x,y)\delta\Theta^{a_3b_3}(x,y)\delta\Theta^{a_4b_4}(x,y)}{(|x-y|^2+2\Delta^2)^4} \\ &\qquad 3\Big(-4g_{b_1a_2}g_{b_2a_3}g_{b_3a_4}g_{b_4a_1}+g_{b_1a_2}g_{b_2a_1}g_{b_3a_4}g_{b_4a_3}\Big) \\ &= \frac{3}{4}\int\limits_{\mathcal{M}\times\mathcal{M}}dxdy\,\rho(x)\rho(y)\frac{S_4[\delta\Theta(x,y)]}{(|x-y|^2+2\Delta^2)^4} \\ \Gamma_{1\mathrm{loop};\mu^2}[X] &\sim \frac{5}{2}\int\limits_{\mathcal{M}\times\mathcal{M}}dxdy\,\rho(x)\rho(y)\frac{\mu^2}{|x-y|^2+2\Delta^2} \end{aligned}$$

where

$$S_4[\delta\Theta] = -4tr\delta\Theta^4 + (tr\delta\Theta^2)^2$$

induced IIB supergravity

physics of effective action:

- conjecture (IKKT): gives IIB supergravity on $\mathbb{R}^{10}$

  (relation with string theory !)

- in particular:

  "*D*-branes" = (soliton) solutions of IIB sugra

  (open) strings end on *D*-branes, lead to NC gauge theory

- direct verification (starting from M.M.) so far only for special geometries:
  - interaction between flat ∞ branes
  - long-distance interaction between "D-blobs"
  - ...

IKKT, Kabat-Taylor, van Raamsdonk, Chepelev-Tseytlin, ...

rewrite as

$$\begin{aligned}\Gamma_{1\text{loop};4}[X] \quad \sim \quad & \frac{\pi^5}{2} \int\limits_{\mathcal{M}\times\mathcal{M}} dxdy\,\rho(x)\rho(y)\Big(2S_4(\Theta(x))\; D(x-y) \\ & +16\big(\Theta^{ae}(x)\Theta_{ef}(x)\Theta^{fb}(x) + \tfrac{1}{4}\Theta^{ab}(x)\Theta^{ef}(x)\Theta_{ef}(x)\big)\; D^{(AS)}_{ab;cd}(x,y)\,\Theta^{dc}(y) \\ & -8T^{ab}(x)D^{(S)}_{ab;cd}(x,y)\; T^{cd}(y) \\ & +4\Theta^{aab}(x)\Theta^{ef}(x)D^{(AS)}_{abef;cdgh}(x,y)\,\Theta^{cd}(y)\Theta^{gh}(y)\Big).\end{aligned}$$

Here

$$D^{(S)}_{ab;cd}(x,y) \quad = \quad \big(g_{ac}g_{bd} + g_{ad}g_{bc} - \tfrac{1}{4}g_{ab}g_{cd}\big)D(x-y)$$

... graviton propagator in 10D

$$D^{(AS)}_{abef;cdgh}(x,y) \quad = \quad \big(g_{ac}g_{bd}g_{eg}g_{fh} + g_{ac}g_{bh}g_{ed}g_{fg} - g_{ac}g_{bg}g_{ed}g_{fh}\big)D(x-y)$$

... prop. for rank 4 AS tensor

$$D^{(AS)}_{ab;cd}(x,y) \quad = \quad \big(g_{ac}g_{bd} - g_{ad}g_{bc}\big)D(x-y)$$

... prop. for rank 2 AS tensor

$$D(x-y) \quad = \quad \frac{3}{2\pi^5}\frac{1}{(|x-y|^2+\Delta^2)^4} \sim \frac{3}{2\pi^5}\frac{1}{|x-y|^8}$$

... 10D propagator

$$T^{ab} = \Theta^{ac}\Theta^{cb} \quad \ldots \quad \text{"closed string" 10D e-m tensor}$$

... consistent with interpretation in terms of IIB sugra

similarly for fields on fuzzy branes:

$$\Theta^{ij} = \bar{\Theta}^{ij} + \mathcal{F}^{ij}$$

expand $\Gamma_{1\text{loop};4}[X]$ in $\mathcal{F}$

leading interaction between separated blobs $\mathcal{F}(x) = \mathcal{F}_A(x) + \mathcal{F}_B(x)$:

$$\Gamma_{1\text{loop};4}[\mathcal{F}_A, \mathcal{F}_B] \sim \pi^5 \int\limits_{\mathcal{M}\times\mathcal{M}} dxdy\,\rho(x)\rho(y)\Big(-8\,T_A^{ab}(x)\,D^{(S)}_{ab;cd}(x,y)\;T_B^{cd}(y) + \ldots\Big)$$

gravitational interaction of gauge fields etc.

however, is not physical gravity, decays like r^{-8}, short range!

way out:

- compactify extra dimensions $\rightarrow$ "orthodox string theory"
- conjecture: 4D gravity emerges on suitable 4-dim. NC branes

  works indeed e.g. on fuzzy S^4_N !

H.S., arXiv:1606.00769

summary & outlook

- coherent states $|x\rangle$, "string states" $|x\rangle\langle y|$ useful on fuzzy spaces
- greatly simplify loop computations on NC spaces, position space works for generic (curved) geometries
- explains UV/IR mixing $\leftrightarrow$ long-range non-locality
- supersymm. IKKT model $\rightarrow$ mild non-locality, IIB supergravity candidate for theory of fundamental interactions including gravity