Quantization of the NC ϕ^3 model in 2 and 4

and the Kontsevich model

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Outline:

- <u>Noncommutative Quantum Field Theory</u>

  UV/ IR mixing and renormalization

  the Grosse-Wulkenhaar term

  Matrix model formulation of the ϕ^3 model

- <u>The Kontsevich model</u>

  review of relevant facts

- <u>Quantization, renormalization and "solution" of the NC</u>

  2 dimensions

  4 dimensions

- <u>Outlook</u>

Noncommutative Quantum Field The

Motivation and relevance

1. Standard model of high-energy physics, gravity based o
   spacetime-continuum

   idealization, seems unplausible;

   $\Rightarrow$ quantized space? Heisenberg 1938

2. Gravity & Quantum Mechanics $\Rightarrow$ space should have

3. String theory: strings ending on D-branes

D-branes in B–field background $\Rightarrow$ strings induce
NC field theory (NCFT) on D-brane

$\Rightarrow$ D-branes = NC space (independent of l_p!)

Non-commutative geometry, field theory

- Manifold $\mathcal{M} \to$ NC algebra $\mathcal{A}$ of functions on $\mathcal{M}$ (with

  "pointless geometry" (von Neumann 1955)

  NC (differential) geometry (Connes)

  simplest example $\mathbb{R}^n_\theta$:
  $$[\hat{x}_i, \hat{x}_j] = i\theta_{ij}$$

  (cp. Quantum Mechanics, phase space)

  usually $\exists$ derivatives ∂_i, integral = trace, some symmet

- Field theory on NC space:
  $$\begin{array}{rll} \mathcal{C}(\mathcal{M}) & \to \mathcal{A} & \to L(\mathcal{H}) \\ \phi(x) & \to \hat{\phi}(\hat{x}) & \to \phi \end{array}$$

  e.g. plane waves $e^{ikx} \to e^{ik\hat{x}}$, spherical harmonics (fuzzy

  Formulation of field theory is possible, many examples

Example: the quantum plane $\mathbb{R}^2_\theta$

"coordinate-functions" $\hat{x}_i,\ i = 1, 2$ satisfy CCR

$$[\hat{x}_i, \hat{x}_j] = i\theta_{ij},$$

θ_{ij} ... a.s. tensor, "background-field"

generate algebra $\mathcal{A}_\theta \cong$ Heisenberg-algebra ($(\hat{x}_1, \hat{x}_2) \leftrightarrow (x$

representation on Hilbert space $\mathcal{H} \cong L^2(\mathbb{R})$ as in Quantum

Scalar field: $\phi = \phi(\hat{x}) \in \mathcal{A}_\theta$ resp. $\phi \in L(\mathcal{H})$ ... lin. operato

e.g. localized wave-packets: coherent states $\phi_{\vec{a}} =$

differential calculus

$$\hat{\partial}_i \phi = -i[\tilde{x}_i, \phi] \qquad \text{for} \quad \tilde{x}_i := \theta^{-1}_{ij}\hat{x}_j$$

note: a priori, NC does not imply existence of UV - cutoff,

UV/IR relation (cp. Quantum mechanics: squeezed st

NC scalar field theory

consider some NC space, algebra $\mathcal{A}$ (e.g. $\mathbb{R}^d_\theta, T^2_\theta, S^2_N, \mathbb{C}P^2_N$

use representation of algebra $\mathcal{A}$ on Hilbert space $\mathcal{H}$

Field $\phi(x) \rightsquigarrow \phi \in L(\mathcal{H})$ ... Hermitian operator on $\mathcal{H}$

trace replaces integral

<u>Example:</u>

$$S = Tr\left(\frac{1}{2}\partial_i\phi\partial_i\phi + \frac{1}{2}m^2\phi^2 + \frac{g}{4}\phi^4\right)$$

can write e.g. $\phi(x) = \int dk\ \phi_k : e^{ikx} :$ etc.,

Quantization

formally defined by (Euclidean) path

$$\langle\phi_{k_1}\cdots\phi_{k_l}\rangle = \frac{\int[\mathcal{D}\Phi]e^{-S}\ \phi_{k_1}\cdots\phi_{k_l}}{\int[\mathcal{D}\Phi]e^{-S}}, \qquad [\mathcal{D}\Phi] =$$

$\Rightarrow$Wick's theorem, however distinction planar $\leftrightarrow$ nonplanar

<u>propagator</u>: as usual, $\langle\phi_k\phi_{k'}\rangle = \delta_{kk'}\frac{1}{k^2+m^2}$

one-loop planar and non-planar self-energy diagrams:

$$\begin{aligned} \Gamma_P^{(2)} &:= g\int\frac{d^dk}{(2\pi)^d}\ \frac{1}{k^2+m^2}\ \sim g\Lambda^{d-2}, \\ \Gamma_{NP}^{(2)}(p) &:= g\int\frac{d^dk}{(2\pi)^d}\ \frac{e^{ik\theta p}}{k^2+m^2}\ \sim g\left(\frac{1}{1/\Lambda^2+p^2\theta^2}\right) \end{aligned}$$

$\Gamma_{NP}^{(2)}(p)$ is finite as long as $p\neq 0$, but IR singularity as $p\rightarrow$

... UV/IR mixing *(Minwalla, Van Raam*

central feature of NC field theories,

serious obstacle to perturbative renormalization!

nontrivial relation UV $\leftrightarrow$ IR momentum dependence of effektive action

$\Gamma^{(2)}(p)$

$\Rightarrow$ modes $p \to 0$ are suppressed (UV/IR)

spontaneous symmetry breaking, phase transition

$\langle p \rangle \neq 0$ ... "striped" phase *(Gubser, Sondhi)*

verified numerically *(Ambjorn, Catterall; Martin; Bietenho Nishimura)* one way to overcome this problem:

$\boxed{\text{The Grosse-Wulkenhaar term}}$

add "confining" potential to action, consider

$$S = \int \Big(\frac{1}{2}\partial_i\phi\partial_i\phi + \Omega^2\,\tilde{x}_i\phi\tilde{x}_i\phi + \frac{1}{2}m^2\phi^2 + \frac{g}{4}\phi^4$$

suppresses IR !

Observation: there is a duality $x \leftrightarrow p$ at $\Omega = 1$ *(Langm*

Result (Grosse - Wulkenhaar): perturbatively renormalizab

in 2 and 4 dimensions (RG techniques)

technically difficult, uses matrix formulation:

scalar field $\equiv$ hermitian matrix $\phi_{ij} = \langle i|\phi|j\rangle$

Matrix model formulation:

scalar field $\phi \in L(\mathcal{H})$

recall: $\partial_i \phi = -i[\tilde{x}_i, \phi] \quad \Rightarrow$

$$S = \int -(\tilde{x}_i\phi\tilde{x}_i\phi - \tilde{x}_i\tilde{x}_i\phi\phi) + \Omega^2\tilde{x}_i\phi\tilde{x}_i\phi + \frac{\mu^2}{2}\phi^2 +$$

simplifies for $\Omega = 1$ to

$$\boxed{S = \int (\tilde{x}_i\tilde{x}_i + \tfrac{\mu^2}{2})\phi^2 + \tfrac{i\tilde{\lambda}}{3!}\ \phi^3 \ = \ Tr\Big(\tfrac{1}{2}J\phi^2 + \tfrac{i}{3}}$$

where

$$J = 2(2\pi\theta)^2(\sum_i \tilde{x}_i\tilde{x}_i + \frac{\mu^2}{2}) \quad \ldots \quad \text{harmonic oscilla}$$

choose basis of eigenstates:

in $\underline{d=2}$: $\quad J|n\rangle = 4\pi\,(n + \frac{1}{2} + \frac{\mu^2\theta}{2})|n\rangle, \qquad n \in \{0, 1, 2, ...\}$

$\underline{d=4}$: $\quad J|n_1, n_2\rangle = 8\pi^2\theta\,(n_1 + n_2 + 1 + \frac{\mu^2\theta}{2})|n_1, n_2\rangle,\ n_i \in$

The regularized (Euclidean) NC ϕ^3 model for $\Omega =$

regularization (cutoff): $\mathcal{H} = \mathbb{C}^N$ such that

$\underline{d=2}$: $\quad J|n\rangle = 4\pi\,(n + \frac{1}{2} + \frac{\mu^2\theta}{2})|n\rangle, \qquad n \in \{0, 1, 2, ...N\}$

$\underline{d=4}$: $\quad J|n_1, n_2\rangle = 8\pi^2\theta\,(n_1 + n_2 + 1 + \frac{\mu^2\theta}{2})|n_1, n_2\rangle,\ n_i \in$

introduce counterterms $\int A\phi + \frac{1}{2}\delta\mu^2\phi^2$ $\quad$ (+ one more in d

can eliminate either linear or quadratic term:

$$\boxed{S = Tr\Big(-\tfrac{1}{2i\lambda}M^2\tilde{\phi} + \tfrac{i\lambda}{3!}\,\tilde{\phi}^3\Big) = \ Tr\Big(\tfrac{1}{2}MX^2 + \tfrac{i\lambda}{3!}\,X^3 -}$$

using shift

$$\tilde{\phi} = \phi + \frac{1}{i\lambda}J \ = \ X + \frac{1}{i\lambda}M$$

where

$$M = \sqrt{J^2 + 2(i\lambda)A}$$

= Kontsevich model !

Quantization

$$Z(M) = \int D\tilde{\phi} \, \exp(-Tr\Big(- \frac{1}{2i\lambda} M^2 \tilde{\phi} + \frac{i\lambda}{3!} \, \tilde{\phi}^3 \Big)) =$$

Kontsevich model, for fixed given (diagonal) matrix M as a

Correlators or "n-point functions"

$$\langle \phi_{i_1 j_1} ... \phi_{i_n j_n} \rangle = \frac{1}{Z} \int D\phi \, \exp(-S) \, \phi_{i_1 j_1} .... \phi_{i_n j}$$

(recall: $\phi_{ij} \sim \langle i|\phi|j\rangle$ ... evaluation of field, cp. $\sim \langle x|\phi|y\rangle$)

Renormalization condition (as for free case $\lambda = 0$):

$$\langle \phi_{00} \phi_{00} \rangle = \frac{1}{2\pi} \frac{1}{\mu_R^2 \theta + 1}, \qquad \langle \phi_{00} \rangle = 0$$

Nontrivial task: show that all n-point functions have a

well-defined limit $N \to \infty$ (with nontrivial dependence on i

Computation of correlators

obtained simply by taking derivatives of $F(M)$:

$$\begin{aligned}
\langle\phi_{ik}\rangle &= \langle\tilde{\phi}_{ik}\rangle - \tfrac{J_{ik}}{i\lambda} \\
&= -\tfrac{J_{ik}}{i\lambda} + \tfrac{1}{Z}\, 2i\lambda \tfrac{\partial}{\partial(M^2)_{ik}} \int D\tilde{\phi}\, \exp(-Tr\Big(- \tfrac{1}{2i\lambda} M^2 \\
&= -\tfrac{J_{ik}}{i\lambda} + 2i\lambda \tfrac{\partial}{\partial(M^2)_{ik}}\, F(M)
\end{aligned}$$

etc. (this is particular for the ϕ^3 model!)

$\Rightarrow$ only need to show: $Z(M) = e^{F(M)}$ depends smoothly o

well-defined limit $N \to \infty$.

however: nontrivial, requires renormalization.

first: perform perturbative computations to get better fe

Perturbative computations:

rewrite action

$$\begin{aligned} S &= Tr\Big(\tfrac{1}{4}(J\phi^2 + \phi^2 J) + \tfrac{i\lambda}{3!}\,\phi^3 - A\phi\Big) \\ &= Tr(\tfrac{1}{2}\phi^i_j\,(G_R)^{j;l}_{i;k}\,\phi^k_l + \tfrac{i\lambda}{3!}\,\phi^3 - A\phi + \tfrac{1}{4}(\delta J\phi^2 + \end{aligned}$$

finite (renormalized) kinetic term $(G_R)^{j;l}_{i;k} = \frac{1}{2}\delta^i_l\delta^k_j(J^R_i + J^R_j$

propagator:

$$\Delta^{i;k}_{j;l} = \langle\phi^i_j\phi^k_l\rangle = \delta^i_l\delta^k_j\frac{2}{J^R_i + J^R_j} = \delta^i_l\delta^k_j\frac{1/(4\pi^2\theta}{\underline{i} + \underline{j} + (\mu_R^2\theta}$$

where $\underline{n} = n$ in 2D, $\underline{n} = n_1 + n_2$ in 4D.

$$\begin{aligned} J^R|n_1, n_2\rangle &= 8\pi^2\theta(\underline{n} + 1 + \tfrac{\mu_R^2\theta}{2})|n_1, n_2\rangle \qquad (4 \\ J^R|n\rangle &= 4\pi(n + \tfrac{1+\mu_R^2\theta}{2})|n\rangle, \qquad (2 \end{aligned}$$

$\delta J \sim \delta\mu^2 = (\mu^2 - \mu_R^2)$ ... part of the counter-term

1-point function

one-loop contribution to the 1-point function:

$$\langle \phi_{ii} \rangle = \frac{1}{J_i^R} A_i - \frac{i\lambda}{2} \frac{1}{J_i^R} \sum_k \frac{2}{J_i^R + J_k^R} \quad + O(\lambda$$

... divergent, unless canceled by counterterm A.

<u>2D</u>: $A \sim (i\lambda) \log(N)$

<u>4D</u>: $A_i \sim (i\lambda) N \log(N) \mathbf{1} + (i\lambda) \log(N) J_i$.

$\Rightarrow$ need further counterterm: either $A = a\mathbf{1} + cJ$ or e

$\phi \to \phi + c, \quad c \sim i\lambda \log(N)$.

can be absorbed by redefinition of Kontsevich-mode

$$\begin{aligned} M &= \sqrt{\tilde{J}^2 + 2(i\lambda)a - (i\lambda)^2 c^2} \\ \tilde{J} &= J + (i\lambda)c. \end{aligned}$$

2-point function

$\langle \phi_{kl} \phi_{lk} \rangle$ for $l \neq k$: has one-loop planar contribution

gives

$$\begin{aligned} \langle \phi_{kl} \phi_{lk} \rangle \quad &= \tfrac{2}{J_k^R + J_l^R} - 2(i\lambda) \tfrac{\langle \phi_{kk} + \phi_{ll} \rangle}{(J_k^R + J_l^R)^2} \\ &+ \tfrac{4}{(J_k^R + J_l^R)^2} \Big(\textstyle\sum_j \tfrac{(i\lambda)^2}{J_k^R + J_j^R} \tfrac{1}{J_l^R + J_j^R} - \tfrac{\delta J_l + \delta J_k}{2} \Big) \end{aligned}$$

implies mass renormalization in 4D

$$\delta \mu^2 \sim \frac{(i\lambda)^2}{256\, \pi^6 \theta^4} \log N$$

(no mass renormalization in 2D)

$\underline{\langle \phi_{ll}\phi_{kk}\rangle \text{ for } l \neq k}$: vanishes at tree level,

one-loop nonplanar contribution

which gives

$$\langle \phi_{ll}\phi_{kk}\rangle = \langle \phi_{kk}\rangle\langle \phi_{ll}\rangle + \frac{1}{4}\frac{(i\lambda)^2}{J_k^R\, J_l^R}\left(\frac{2}{J_k^R + J_l^R}\right)$$

is finite.

Goal: $\underline{\text{nonperturbative proof of renormalizability}}$:

need renormalized Kontsevich model $Z(M)$,

with eigenvalues m_i appropriately rescaled as $N \to \infty$

The Kontsevich model

defined by

$$\boxed{\begin{aligned} Z^{Kont}(M) &= e^{F^{Kont}(M)} = \tfrac{1}{\mathcal{N}} \int dX \exp\left\{Tr\left(-\frac{MX^2}{2}\right.\right. \\ &= \tfrac{1}{\mathcal{N}} \int d\tilde{\phi} \exp\left\{Tr\left(\frac{1}{2i\lambda} M^2\tilde{\phi} - \frac{i}{3!}\,\tilde{\phi}^3 - \frac{1}{3}\right.\right. \end{aligned}}$$

where $\mathcal{N} = \int dX \exp\left\{-Tr\left(\frac{MX^2}{2}\right)\right\}$.

depends only on eigenvalues m_i of M ... hermitian $N \times N$

introduced by Kontsevich 1991

suitable variables:

$$t_r = -(2r-1)!! \sum_i m_i^{-(2r+1)},$$

Remarkable fact (Kontsevich): $F^{Kont}(M) = F^{Kont}(t_r)$ is ge
function of intersection numbers (topological characteristics
spaces of punctured Riemann surfaces (⇒rational coefficien

more precisely: (Kontsevich, Itzykson-Zuber)

consider perturbative expansion

$$Z^{Kont\,(N)} = \int d\mu_M(X) \exp(\frac{i}{6} Tr X^3) = \sum_{k\geq 0} Z_k^{Kont\,(}$$

$$Z_k^{Kont\,(N)}(M) = \frac{(-1)^k}{(2k)!} \int d\mu_M(X) \left(\frac{Tr X^3}{6}\right)^{2}$$

then

- $Z_k^{Kont\,(N)}(M)$ is polynomial in the t_r of degree $3k$, for
- $Z_k^{Kont\,(N)}(t_r)$ is independent of N for $N \geq 3k$,

  depends only on t_r, $2r+1 \leq 3k$.

$\Rightarrow Z^{Kont}(t_r) = \sum_{k\geq 0} Z_k^{Kont}(t_r)$ well-defined for $N \to \infty$

as asymptotic expansion.

Further facts: (Kontsevich)

- $Z^{Kont}(t_r)$ is a τ-function for the Korteweg-de Vries equ

  i.e. $u = \frac{\partial^2}{\partial t_0^2} \ln Z^{Kont}$ satisfies

$$\frac{\partial u}{\partial t_1} = \frac{\partial}{\partial t_0}\left(\frac{1}{12}\frac{\partial^2 u}{\partial t_0^2} + \frac{1}{2}u^2\right), \qquad \frac{\partial}{\partial t_n} u = \frac{\partial}{\partial t_0}$$

- Virasoro constraints: $\qquad L_m Z = 0, \qquad m \geq -1$

  for suitable operators L_m (differential operators in the

- genus expansion: (generally for matrix models...)

$$\ln Z^{Kont} = F^{Kont} = \sum_{g \geq 0} F_g^{Kont}$$

  by drawing the Feynman diagrams on a Riemann surfa

... allows to obtain explicit solution:

genus 0: (Makeenko and Semenoff, ...)

$$
\begin{aligned}
F_0^{Kont} &= \tfrac{1}{3}\sum_i m_i^3 - \tfrac{1}{3}\sum_i (m_i^2 - 2u_0)^{3/2} - u_0 \sum_i (m_i^2 \\
&+ \tfrac{u_0^3}{6} - \tfrac{1}{2}\sum_{i,k} \ln\left\{ \tfrac{(m_i^2-2u_0)^{1/2} + (m_k^2-2u_0)^{1/2}}{m_i+m_k} \right\}
\end{aligned}
$$

higher genus: (Itzykson-Zuber)

$$
\begin{aligned}
F_1^{Kont} &= \tfrac{1}{24} \ln \tfrac{1}{1-I_1}, \\
F_2^{Kont} &= \tfrac{1}{5760} \left[5 \tfrac{I_4}{(1-I_1)^3} + 29 \tfrac{I_3 I_2}{(1-I_1)^4} + 28 \tfrac{I_2^3}{(1-I_1)^5}
\end{aligned}
$$

etc. where

$$
\begin{aligned}
I_k &= -(2k-1)!! \sum_i \tfrac{1}{(m_i^2-2u_0)^{k+\frac{1}{2}}}, \\
u_0 &= -\sum_i \tfrac{1}{\sqrt{m_i^2-2u_0}} = I_0.
\end{aligned}
$$

generally: all F_g^{Kont} with $g \geq 2$ are given by *finite* sums of

$$I_k/(1-I_1)^{\frac{2k+1}{3}}.$$

explicitly (just for illustration ...) (Itzykson-Zu

$$
\begin{aligned}
F_0^{Kont} &= \frac{t_0^3}{3!} + t_1\frac{t_0^3}{3!} + \left(t_2\frac{t_0^4}{4!} + 2\frac{t_1^2}{2!}\frac{t_0^3}{3!}\right) + \left(t_3\frac{t_0^5}{5!} + 3t_1t_2\frac{t_0^4}{4!}\right.\\
&+ \left[t_4\frac{t_0^6}{6!} + \left(6\frac{t_2^2}{2!} + 4t_1t_3\right)\frac{t_0^5}{5!} + 24\frac{t_0^3}{3!}\frac{t_1^4}{4!} + 12t_2\frac{t_1^2}{2!}\frac{t_0^4}{4!}\right]\\
&+ \ldots\\
24F_1^{Kont} &= t_1 + \left(\frac{t_1^2}{2!} + t_0t_2\right) + \left(2\frac{t_1^3}{3!} + t_3\frac{t_0^2}{2!} + 2t_0t_1t_2\right)\\
&+ \left(6\frac{t_1^4}{4!} + t_4\frac{t_0^3}{3!} + 4\frac{t_0^2}{2!}\frac{t_2^2}{2!} + 6t_0t_2\frac{t_1^2}{2!} + 3t_1t_3\frac{t_0^2}{2!}\right)\\
&+ \left(24\frac{t_1^5}{5!} + t_5\frac{t_0^4}{4!} + 24t_0t_2\frac{t_1^3}{3!} + (4t_1t_4 + 7t_2t_3)\frac{t_0^3}{3!} + \right.\\
&+ \ldots
\end{aligned}
$$

Application to the NC ϕ^3 model

set $m_i = \sqrt{\tilde{J}_i^2 + const} \approx \begin{cases} i, & d=2 \\ (i_1+i_2), & d=4 \qquad \text{(degenera} \end{cases}$

as given by the ϕ^3 model with harmonic oscillator poten

<u>note:</u>

- $t_0, (t_1)$ and $u_0 = -\sum_i \frac{1}{\sqrt{m_i^2 - 2u_0}}$ and correlation func divergent a priori
- only the combination

$$\sqrt{m_i^2 - 2u_0} = \lambda^{-2/3}\sqrt{\tilde{J}_k^2 + 2b}$$

enters, where $b = (i\lambda)a - \lambda^{4/3}\, u_0 + \lambda^2 c^2/2 \qquad (a, c \ \ldots$ ba

<u>will show</u>: $\tilde{J}$ and b are finite after renormalization,

for suitable $\boxed{a = a_N,\, c = c_N,\, \mu^2 = \mu_N^2}$ (dive

rendering correlation functions well-defined in the limit

Renormalization and finiteness

1-point function at genus 0:

$$\begin{aligned}\langle\phi_{kk}\rangle_{g=0} &= 2i\lambda\frac{\partial}{\partial\tilde{J}_k^2}\,F_0(\tilde{J}^2) - \frac{J_k}{i\lambda}\\ &= \frac{1}{i\lambda}(\sqrt{\tilde{J}_k^2+2b} - \tilde{J}_k) + c + \sum_j \frac{(i\lambda)}{\sqrt{\tilde{J}_k^2+2b}\sqrt{\tilde{J}_j^2+2b+}}\end{aligned}$$

$$\Rightarrow \quad c = \begin{cases} 0 \quad , & d=2 \\ -\frac{i\lambda}{(8\pi^2\theta)^2}\,\ln(N) + c', & d=4 \end{cases}, \qquad \text{and } J$$

b determined by renormalization condition $\langle\phi_{00}\rangle = 0$, solu

in 4D: $\tilde{J} = J + (i\lambda)c$ finite $\Rightarrow$ mass renormalization

$$\mu^2 = \frac{(i\lambda)^2}{256\pi^6\theta^4}\,\ln(N)\, + \mu_R^2$$

complete agreement with 1-loop result

counterterm a: (recall $\int a\phi$) determined through implic

$$\frac{b}{i\lambda} - a + (i\lambda)^2 c^2/2 = -\sum_i \frac{(i\lambda)}{\sqrt{\tilde{J}_i^2 + 2b}}$$

$$\Rightarrow \quad a = \begin{cases} \frac{(i\lambda)}{4\pi} \ln N + (finite), \\ \frac{(i\lambda)}{8\pi^2\theta} N \ln N - \frac{\lambda^2}{(8\pi^2\theta)^4} \ln(N)^2 + (finite), \end{cases}$$

renormalized 1-point function in 4D:

$$\boxed{\langle \phi_{kk} \rangle_{g=0} = \frac{1}{i\lambda}(\sqrt{\tilde{J}_k^2 + 2b} - \tilde{J}_k) + c' + (i\lambda) f}$$

finite and well-defined as $N \to \infty$

For certain point in moduli space ($b = c' = 0$):

$$\langle \phi_{kk} \rangle_{g=0} = (i\lambda) \sum_j \frac{1}{\tilde{J}_j} \Big(\frac{1}{\tilde{J}_k + \tilde{J}_j} - \frac{1}{\tilde{J}_0 + \tilde{J}_j} \Big)$$

coincides with one-loop result

Finiteness of general n-point function for an

1) diagonal case:

$$\langle \phi_{i_1 i_1} ... \phi_{i_n i_n} \rangle_c \sim \frac{\partial}{\partial m^2_{i_1 i_1}} ... \frac{\partial}{\partial m^2_{i_n i_n}} \; F(M)$$

easy to show using explicit form of F_g (polynomial in $\frac{I_k}{(1-I_1)}$

above renormalization $\quad a = a_N,\, c = c_N,\, \mu^2 = \mu_N^2$

guarantees that all derivatives of $F_g(M)$ w.r.t. m_i^2 are

well-defined as $N \to \infty$, as long as $I_1 = -\lambda^2 \sum_i \frac{1}{(\tilde{J}_i^2 + 2b}$

2) can be extended to general n-point functions

$$\langle \phi_{i_1 j_1} .... \phi_{i_n j_n} \rangle_c \sim \frac{\partial}{\partial m^2_{i_1 j_1}} ... \frac{\partial}{\partial m^2_{i_n j_n}} \; F(M)$$

also well-defined for each genus g, using $F(M^2) = F(U^{-1} di$

(slight complication due to degeneracy in 4D, but no essent

Theorem: *The (connected) genus g contribution to an function* $\langle\phi_{i_1 j_1}....\phi_{i_n j_n}\rangle_c$ *is finite and has a well-defined for all g, provided the couplings are renormalized as abo*

all correlation functions can be computed in principle for ar

<u>2D</u>:

$$a = \frac{(i\lambda)}{4\pi}\ \ln N \qquad (+finite)$$

<u>4D</u>:

$$\begin{aligned} c &= -\tfrac{i\lambda}{(8\pi^2\theta)^2}\ \ln(N) \qquad (+finite) \\ \mu^2 &= \tfrac{(i\lambda)^2}{256\pi^6\theta^4}\ \ln(N) \qquad (+finite) \\ a &= \tfrac{(i\lambda)}{8\pi^2\theta} N\ln N + \tfrac{(i\lambda)}{(8\pi^2\theta)^4}\ \ln(N)^2 \qquad (+fini \end{aligned}$$

<u>note:</u> model originally defined for imaginary coupling $\int i\lambda\phi$

can analytically continue to (sufficiently small) real co

all correlation functions real

Singularity (phase transition, instability) for real coupling λ simultaneously for all genera $g \geq 1$:

$$I_1 = -(i\lambda)^2 \sum_i \frac{1}{(\tilde{J}_i^2 + 2b)^{3/2}} = 1$$

<u>2D</u>:

$$\frac{1 + \mu^2\theta}{\lambda} \approx \pm 0.0646989$$

<u>4D</u>:

$$\mu_R^2\theta + 2 = \frac{\lambda^2}{2(8\pi^2\theta)^3}.$$

Example: 2-point function at genus 0 (in 4D)

using identity $\left\langle \tilde{\phi}_{kl}\tilde{\phi}_{lk} \right\rangle = \frac{2i\lambda}{m_k^2 - m_l^2} \left\langle \tilde{\phi}_{kk} - \tilde{\phi}_{ll} \right\rangle$

find

$$\boxed{\left\langle \phi_{kl}\phi_{lk} \right\rangle_{g=0} = 2\, \frac{\sqrt{\tilde{J}_k^2+2b} - \sqrt{\tilde{J}_l^2+2b} + (i\lambda)^2\, (f_R(\underline{k}) - j}{\tilde{J}_k^2 - \tilde{J}_l^2}}$$

coincides (!) with 1-loop for special point ($b = c' = 0$) in mo

$$\overset{(b=c'=0)}{=} \frac{2}{\tilde{J}_k + \tilde{J}_l} - (i\lambda)^2 \frac{2}{\tilde{J}_k + \tilde{J}_l} \sum_j \frac{1}{\tilde{J}_j(\tilde{J}_k + \tilde{J}_j)}$$

similarly

$$\boxed{\langle \phi_{ll}\phi_{kk} \rangle - \langle \phi_{kk} \rangle \langle \phi_{ll} \rangle = (i\lambda)^2 \frac{1}{\sqrt{\tilde{J}_k^2+2b}} \frac{1}{\sqrt{\tilde{J}_l^2+2b}} \left(\frac{1}{\sqrt{\tilde{J}_k^2+2b+}} \right.}$$

in agreement with perturbative computations (recall $b = b($

Remarks and outlook:

- nontrivial interacting "solvable" NC 4D field theory
- renormalization determined by genus 0 contribution alo
  (cp. general NCFT; IR divergences are suppressed here
  expect to hold generally for (scalar) NCFT with oscilla
  genus 0 is accessible more generally (matrix model tech
- genus 0 contribution coincides exactly with 1-loop for s
  moduli space (not for higher genera) (??)
- is $\sum_g$ convergent ?
- generalizations:
  - D=6 ? (no longer super-renormalizable)
  - extend to $\Omega \neq 1$ (i.e. remove the oscillator potential
    certainly doable "perturbatively" ... ?

NCFT are accessible through new analytical (matrix) meth