Nonabelian Localization for Yang-Mills Theory on the Fuzzy Sphere

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DIAS Dublin, 2007

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Outline

- Background: Localization and Duistermaat – Heckmann formula
- A new model for Yang-Mills on the fuzzy 2-sphere
  symplectic structure, moment map
- Critical surfaces and local geometry
- Nonabelian localization for YM on S_N^2
  equivariant cohomology
- Evaluation of the instanton contributions

Background: Localization and Duistermaat – Heckmann

Consider X ... $2n$– dim. compact manifold with symplectic form ω.

Assume $U(1)$ acts on X, generated by a Hamiltonian vector field V

$$dH = -\iota_V \omega$$

for function H on X

<u>DH formula</u>: Partition function

$$Z = \int_X \frac{\omega^n}{n!} e^{-\beta H}$$

given exactly by semi-classical approximation, summing over all critical points P_i of H:

$$Z = \int_X \frac{\omega^n}{n!} e^{-\beta H} = \sum_i \frac{e^{-\beta H(P_i)}}{\alpha_i},$$

where α_i ... certain constant at P_i.

<u>Example:</u> $X = S^2$ given by $x^2 + y^2 + z^2 = 1$

polar coordinates $z = \cos\theta,\ x = \sin\theta\ \cos\varphi,\ y = \sin\theta\ \sin\varphi$

symplectic volume form $\omega = d\cos\theta\ d\varphi$

$U(1)$ action

$$\varphi \to \varphi + \text{constant}$$

generated by Hamiltonian (moment map)

$$H = \cos\theta + a, \qquad a\ldots\text{constant}$$

DH formula applies to

$$\begin{aligned} Z &= \int_X \omega\ e^{-\beta(\cos\theta + a)} \\ &= \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\varphi e^{-\beta(\cos\theta + a)} \\ &= \tfrac{2\pi}{\beta}\left(e^{\beta(1-a)} - e^{\beta(-1-a)}\right) \end{aligned}$$

two terms ... contributions from critical point of H at $\cos\theta = \pm 1$.

Witten (1992): nonabelian version of localization

(using equivariant cohomology) is applicable to 2D Yang-Mills:

- {gauge fields $A_i(x)$} ... symplectic space, $\omega = \delta A_i(x) \wedge \delta A_j(x)\, \varepsilon^{ij}$
- Hamiltonian $\to$ moment map $\to$ field strength F
- critical points $\to$ gauge orbits of $DF = 0$ ("2D instantons")
- difficult to evaluate ... only done for vacuum surface

recently: Beasley-Witten (hep-th/0503126) apply localization to Chern-Simons,

develop trick to evaluate higher critical surfaces

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here: develop new model for Yang-Mills on S^2_N:

localization can be applied in rigorous way

evaluate explicitly dominant contributions partition function

The fuzzy sphere S^2_N:

(J. Madore; ...)

quantization parameter: $\theta = \frac{1}{N}, \quad N \in \mathbb{N}$.

algebra of functions: $\mathcal{A} = Fun_N(S^2) = Mat(N, \mathbb{C})$

rotations: let X_i ... N-dim. rep. of $su(2)$: $[X_i, X_j] = i\epsilon_k^{ij} X_k, \quad X_i X^i = \frac{N^2-1}{4}$

$$f \in \mathcal{A} \to J_i \triangleright f := [X_i, f]$$

$\hookrightarrow$ space of functions decomposes into

$$\begin{aligned} f \in (N) \otimes (N) \quad &= (1) \ \oplus \ (3) \ \oplus ... \oplus \ (2N-1) \\ &= (\hat{Y}^0) + (\hat{Y}^1_m) + ... + (\hat{Y}^{N-1}_m) \end{aligned}$$

hence: $\exists$ map

$$\begin{aligned} S^2_N \quad &\hookrightarrow \quad S^2, \\ \hat{Y}^l_m \quad &\hookrightarrow \quad Y^l_m \end{aligned}$$

Integral: $\quad Tr(f) = \frac{N}{4\pi} \int_{S^2} d\Omega \ f$

A new model for YM on the fuzzy 2-sphere

Consider

$$\Xi = \tfrac{1}{2}\,\mathbf{1}_N \otimes \sigma^0 + X_i \otimes \sigma^i$$

satisfies

$$\Xi^2 = \tfrac{N^2}{4}\,\mathbf{1}_{\mathcal{N}} \qquad \text{and} \qquad Tr(\Xi) = N\ .$$

$\Rightarrow$ Ξ has eigenvalues $\pm\frac{N}{2}$ with multiplicities $N_\pm = N \pm 1$.

Introduce gauge fields through "covariant coordinates"

$$C_i = X_i + A_i \qquad \text{and} \qquad C_0 = \tfrac{1}{2}\,\mathbf{1}_N + A_0$$

gauge transformations $C_\mu \mapsto U^{-1}\,C_\mu\,U$ for $U \in G = U(N)$.

assemble them into a larger $\mathcal{N} \times \mathcal{N}$ matrix

$$C = C_\mu \otimes \sigma^\mu\ . \qquad\qquad (\mathcal{N} = 2N)$$

... four independent fields; want two tangential fields on S^2_N.

1st guess: constraints $A_0 = 0$ and $C_i\, C^i = \frac{N^2-1}{4}\, \mathbf{1}_{\mathcal{N}}$

better:

$$C^2 = \frac{N^2}{4}\, \mathbf{1}_{\mathcal{N}} \qquad \text{and} \qquad Tr(C) = N$$

... i.e. C has eigenvalues $\pm\frac{N}{2}$ with multiplicities $N_\pm = N \pm 1$.

claim: possible configuration space of $u(1)$ gauge fields given by

single coadjoint orbit

$$\boxed{\mathcal{O} := \mathcal{O}(\Xi) = \big\{ C = U^{-1}\, \Xi\, U \;\big|\; U \in U(\mathcal{N}) \big\} \quad \cong \quad U(2N)/U(N+1) \times U(N-1)}$$

dimension $\dim(\mathcal{O}) = 2(N^2-1) \approx 2N^2$ ... 2 (field) degrees of freedom

can show:

- $\mathcal{O}$ describes tangential $u(1)$ gauge fields on S^2_N
- $F := N A_0 = N(C_0 - \frac{1}{2})$ reproduces field strength in commutative limit

$$S_{YM} := \frac{N}{g}\, Tr\big(C_0 - \tfrac{1}{2}\, \mathbf{1}_{n\,N}\big)^2 \approx \frac{1}{4\pi\, g} \int_{S^2} d\Omega\; (F)^2 \;.$$

nonabelian case: take $\mathcal{N} = 2n\,N$, $C = C_\mu \sigma^\mu \in Mat(\mathcal{N}, \mathbb{C})$

configuration space of $u(n)$ gauge fields given by coadjoint orbit

$$\boxed{\mathcal{O} := \left\{ C = U^{-1}\,\Xi\,U \;\middle|\; U \in U(\mathcal{N}) \right\} \cong \quad U(2Nn)/U(n(N+1)) \times U(n(N-1))}$$

$\dim(\mathcal{O}) = 2n^2(N^2-1) \approx 2n^2N^2$ ... 2 $u(n)$–valued tangential fields

gauge group $G = U(nN) = U(n) \otimes U(N)$

can show:

- $\mathcal{O}$ describes tangential $u(n)$ gauge fields on S^2_N
- $F := NA_0 = N(C_0 - \frac{1}{2})$ reproduces $u(n)$ field strength,
  $S_{YM} = \frac{N}{g}\,Tr\big(C_0 - \frac{1}{2}\,\mathbf{1}_{n\,N}\big)^2$ ... YM action in commutative limit

Goal: compute partition function

$$\boxed{Z := \frac{1}{\mathrm{vol}(G)} \int_{\mathcal{O}} \mathrm{d}C\; e^{-S_{YM}(C)}}$$

justification: constraint $C^2 = \frac{N^2}{4}\, \mathbf{1}_{\mathcal{N}}$ equivalent to

$$C_i\, C^i + C_0^2 = \tfrac{N^2}{4}\, \mathbf{1}_{\mathcal{N}} \qquad \text{and} \qquad \mathrm{i}\, \epsilon_i{}^{jk}\, C_j\, C_k + \{C_0, C_i\} = 0\ .$$

Field strength

$$\begin{aligned} F_i \quad &:= \mathrm{i}\, \epsilon_i{}^{jk}\, C_j\, C_k + C_i \\ &= \mathrm{i}\, \epsilon_i{}^{jk}\, [X_j, A_k] + \mathrm{i}\, \epsilon_i{}^{jk} A_j\, A_k + A_i \end{aligned}$$

constraint implies $F_i = -\{A_0, C_i\}$, thus

$$\begin{aligned} F \quad &= F_r \approx -N\, A_0, \\ S_{YM} \quad &= \tfrac{N}{g}\, Tr\big(C_0 - \tfrac{1}{2}\, \mathbf{1}_{n\,N}\big)^2 \end{aligned}$$

Equivariant cohomology and nonabelian localization

(Witten)

based on

- $\mathcal{O}$ is coadjoint orbit $\Rightarrow$ (compact) symplectic space, Kirillov-Kostant symplectic form $\omega \in H^2(\mathcal{O}), \qquad d\omega = 0$.
- $G = U(N)$ action $C \to U^{-1}CU$ ... gauge transformation generated by vector fields

$$V_\phi = i[C, \phi], \qquad \phi = \phi_0 \sigma^0 \in \mathfrak{u}(N)$$

- V_ϕ are Hamiltonian vector fields, with generator (moment map) $H_\phi = Tr(\phi\, C) = Tr(\phi\, C_0)$:

$$dH_\phi = -\iota_{V_\phi}\omega$$

<u>however:</u> critical points of DH replaced by

G– invariant critical submanifolds (vacuum & instantons)

partition function

$$
\begin{aligned}
Z &:= \tfrac{1}{\mathrm{vol}(G)} \left(\tfrac{g}{4\pi\, N}\right)^{\dim(G)/2} \int\limits_{\mathcal{O}} \mathrm{d}C \;\; \exp\Big(- \tfrac{N}{g}\, Tr\big(C_0^2\big)\Big) \\
&= \tfrac{1}{\mathrm{vol}(G)} \int\limits_{\mathfrak{g}\times\mathcal{O}} \Big[\tfrac{\mathrm{d}\phi}{2\pi}\Big] \;\; \exp\Big(\omega - \mathrm{i}\, Tr(C_0\,\phi) - \tfrac{g'}{2}\, Tr\big(\phi^2\big)\Big) \;,
\end{aligned}
$$

Introduce the BRST operator

$$
\begin{aligned}
Q &= d - i\iota_{V_\phi} \\
Q^2 &= -\,\mathrm{i}\,\{\mathrm{d}, \iota_{V_\phi}\} = -\,\mathrm{i}\,\mathcal{L}_{V_\phi}
\end{aligned}
$$

Thus $Q^2 = 0$ on

$$
\Omega_G(\mathcal{O}) := \big(\mathbb{C}[[\mathfrak{g}]] \otimes \Omega(\mathcal{O})\big)^G
$$

... gauge invariant differential forms on $\mathcal{O}$ with values in $\mathbb{C}[[\mathfrak{g}]]$

$dTr(\phi\, C_0) = -\iota_{V_\phi}\omega$ implies

$$
Q\big(\omega - \mathrm{i}\, Tr(C_0\,\phi)\big) = 0
$$

$\Rightarrow$ $\boxed{\text{integrand defines } G\text{-equivariant cohomology class in } H_G(\mathcal{O})}$

Trick (Witten): add any Q-exact form to action

$$Z = \int_{\mathfrak{g}\times\mathcal{O}} \left[\frac{\mathrm{d}\phi}{2\pi}\right] \exp\left(\omega - \mathrm{i}\, Tr(C_0\,\phi) - \tfrac{g'}{2}\, Tr\big(\phi^2\big) + t\; Q\alpha\right)$$

... independent of $t \in \mathbb{R}$ for any G-invariant one-form α on $\mathcal{O}$,

$$Q\alpha = d\alpha - \mathrm{i}\,\langle \alpha, V_\phi\rangle\ .$$

take $t \to \infty$: integral localizes at stationary points of $\langle \alpha, V_\phi\rangle$ in $\mathfrak{g}\times\mathcal{O}$ determined by

$$\langle \alpha, V_a\rangle = 0, \qquad V_\phi = V_a\ \phi^a$$

(can assume $\phi = 0$). Consider

$$\alpha = -\,\mathrm{i}\, Tr\big(C_0\,[C, \mathrm{d}C]_0\big)$$

can show:

$$0 = \langle \alpha, V_a\rangle \qquad \Longleftrightarrow \qquad \boxed{[C, C_0] = 0}$$

$$\Longleftrightarrow \quad \text{critical surfaces of} \quad S_{YM} = \tfrac{N}{g}\, Tr\big(C_0 - \tfrac{1}{2}\ \mathbf{1}_{n\,N}\big)^2$$

explicit proof of localization:

$$
\begin{aligned}
Z &= \int_{\mathfrak{g}\times\mathcal{O}} \left[\tfrac{\mathrm{d}\phi}{2\pi}\right] \exp\Big(t\, d\alpha + \omega\Big) \\
&\times \exp\Big(- \mathrm{i}\, Tr(C_0\,\phi) - \tfrac{g'}{2} Tr\big(\phi^2\big) - \mathrm{i}\, t\, Tr\big([C\,,\,[C,C_0]\,]\,\phi\big)\Big) \\
&\sim \int_{\mathcal{O}} \exp\Big(t\, d\alpha + \omega\Big) \\
&\times \exp\Big(- \tfrac{1}{2g'}\, Tr\big(C_0^2\big) + \tfrac{t}{g'}\, Tr\big(C_0\,[C\,,\,[C,C_0]\,]\big) - \tfrac{t^2}{2g'}\, Tr\big([C\,,\,[C,C_0]\,]\big)^2\Big)
\end{aligned}
$$

$\Rightarrow$ for $t \to \infty$, only infintesimal neighborhood of configurations

$$[C\,,\,[C,C_0]\,] = 0 \;\Leftrightarrow\; [C,C_0] = 0 \;\Leftrightarrow\; C \in \mathcal{C}_{\underline{n}} \quad \text{...critical surfaces}$$

contribute,

$$Z = \sum_{\underline{n}} Z_{\underline{n}} = \sum_{\mathcal{C}_{\underline{n}}} w(\underline{n})\, e^{-S(\mathcal{C}_{\underline{n}})}$$

goal: compute $Z_{\underline{n}}$ explicitly for "dominant" $\mathcal{C}_{\underline{n}}$

need explicit local geometry of critical surfaces $\mathcal{C}_{\underline{n}}$

The critical surfaces

equation of motion $[C_i, C_0] = 0$

together with constraint implies

$$\begin{aligned} [C_i, C_j] &= \mathrm{i}\,\epsilon_{ij}{}^k\,(2C_0)\,C_k\ , \\ C_0^2 &= \tfrac{N^2}{4}\,\mathbf{1}_{n\,N} - C_i\,C^i\ . \end{aligned}$$

general solution = direct sum of irreps:

- $C_0 \neq 0$: $C_i = 2C_0 L_i$ with L_i ... n_i –dim. irrep of $\mathfrak{su}(2)$ with $C_0 = s_i \frac{N}{2n_i}$, $s_i = \pm 1$.
- Fluxons: one-dimensional blocks $C_0 = 0$, $C_i = c_i$ with $c_i\,c^i = \frac{N^2}{4}$, $c_i \in \mathbb{R}$ label position on S^2.

critical surfaces

$$\boxed{\mathcal{C}_{(n_1,s_1),\ldots,(n_k,s_k)}} \qquad \text{with} \qquad n_i \in \mathbb{N} \quad \text{and} \quad s_i \in \{\pm 1, 0\}$$

$$1 \leq n_1 \leq n_2 \leq \cdots \leq n_k\ , \quad \sum n_i = n\,N \quad \text{and} \quad \sum s_i = n\ ,$$

<u>action for critical points</u>:

dominant solution: $\mathcal{C}_{(n_1,1),\ldots,(n_n,1)}$ and $n_i = N - m_i \approx N$

$$S\big((n_1,1)\,,\,\ldots\,,\,(n_n,1)\big) \approx \frac{1}{4g}\sum_{i=1}^{n} m_i^2\ ,$$

... usual action of $U(n)$ YM on S^2 for instantons $(m_i) \in \mathbb{Z}^n$.

other non-classical solutions (fluxons, ...) suppressed by $e^{-\frac{N}{g}}$

$\Rightarrow$ (localization) only "classical" $\mathcal{C}_{(n_1,1),\ldots,(n_n,1)}$

contribute to $Z = \sum_{\underline{n}} w(\underline{n})\, e^{-S(\underline{n})}$ for $N \to \infty$, provided g finite.

to compute $Z_{\underline{n}}$: need local geometry near $\mathcal{C}_{(n_1,1),\ldots,(n_n,1)}$

The map $\mathcal{J}$

consider for $C \in \mathcal{O}$ the map

$$\begin{aligned} \mathcal{J} \,:\, \mathfrak{u}(\mathcal{N}) \quad &\longrightarrow \quad \mathfrak{su}(\mathcal{N}) \\ \phi \quad &\mapsto \quad \tfrac{1}{N}\, V_\phi = \tfrac{\mathrm{i}}{N}\, [C\,,\, \phi] \end{aligned}$$

satisfies

$$\mathcal{J}^3 = -\mathcal{J}$$

$\rightarrow$ Cartan decomposition of the symmetric space $\mathcal{O}$:

$$\mathfrak{u}(\mathcal{N}) = \ker(\mathcal{J}) \oplus \underbrace{\ker\left(\mathcal{J}^2 + \mathbf{1}_{\mathcal{N}}\right)}_{T_C\mathcal{O}}$$

$\mathcal{J}$ ... complex structure on $T_C\mathcal{O} = \mathrm{Im}(\mathcal{J})$

Consider

$$\mathfrak{g} \quad \to \quad \underset{\text{(gauge orbit)}}{\mathcal{J}(\mathfrak{g})} \quad \to \quad \mathcal{J}^2(\mathfrak{g})$$

for vacuum $C = \frac{1}{2} + X_i \otimes \mathbf{1}_n \sigma^i$:

$$\boxed{T_C\mathcal{O} = \mathcal{J}(\mathfrak{g}) \oplus \mathcal{J}^2(\mathfrak{g})}$$

in general:

$$\boxed{\mathcal{J}(\mathfrak{g} \ominus \mathfrak{h}) \oplus \mathcal{J}^2(\mathfrak{g} \ominus \mathfrak{h}) \oplus E_0 \oplus E_1 \;=\; T_C\mathcal{O}}$$

where

$$E_0 = \mathcal{J}(\mathfrak{g}) \cap \mathcal{J}^2(\mathfrak{g}) = \mathcal{J}(\mathfrak{h}) = \mathcal{J}^2(\mathfrak{h}) \ .$$

to determine E_0, E_1 explicitly, need decomposition under $SU(2)$

critical surface $\mathcal{C}_{(n_1,1),\ldots,(n_n,1)}$ defines $SU(2)$ generators

$$J_i = \frac{C_i}{2C_0} + \frac{1}{2}\sigma_i, \qquad [J_i, C] = 0$$

acting on

$$V \otimes \mathbb{C}^2 = \left(\bigoplus_{i=1}^{n} (n_i + 1) \right) \oplus \left(\bigoplus_{i=1}^{n} (n_i - 1) \right)$$

so that

$$\begin{aligned} C &= \tfrac{N}{2} \begin{pmatrix} \bigoplus_{i=1}^{n} \mathbf{1}_{(n_i+1)} & 0 \\ 0 & -\bigoplus_{i=1}^{n} \mathbf{1}_{(n_i-1)} \end{pmatrix} && \subset \mathfrak{u}(\mathcal{N}) \\ T_C\mathcal{O} &\cong \begin{pmatrix} 0 & X \\ X^\dagger & 0 \end{pmatrix} && \subset \mathfrak{u}(\mathcal{N}) \end{aligned}$$

thus

$$\begin{aligned} T_C\mathcal{O} &\cong \textstyle\bigoplus_{i,j=1}^{n} (n_i + 1) \otimes (n_j - 1) \ , \\ \mathfrak{g} &\cong \textstyle\bigoplus_{i,j=1}^{n} (n_i) \otimes (n_j) \end{aligned}$$

... allows to compute $\mathcal{J}, E_0, E_1$ explicity

1) vacuum surface $C = X_i \otimes \mathbf{1}_n \, \sigma^i, \quad n_i = N,$ stabilizer $[\mathfrak{u}(n), C] = 0$

$$\begin{aligned} \mathfrak{g} &\cong (N) \otimes (N) \otimes \mathfrak{u}(n) = \Big((1) \oplus (3) \oplus \cdots \oplus (2N-1) \Big) \otimes \mathfrak{u}(n) \\ &= \Big((1) \ \oplus \ (N+1) \otimes (N-1) \Big) \otimes \mathfrak{u}(n) \ , \end{aligned}$$

$$\Rightarrow \quad \boxed{T_C \mathcal{O} = \mathcal{J}(\mathfrak{g}) \oplus \mathcal{J}^2(\mathfrak{g})}$$

2) nondegenerate surface $\mathcal{C}_{(n_1,1),\ldots,(n_n,1)}$ with $n_1 > n_2 > \ldots > n_n$

then $T_C \mathcal{O}$ contains

$$(n_i+1) \otimes (n_j-1) \cong \Big(|n_i - n_j| + 3 \Big) \oplus \Big(|n_i - n_j| + 5 \Big) \oplus \cdots \oplus \Big(n_i + n_j - 1 \Big)$$

and

$$(n_j+1) \otimes (n_i-1) \cong \Big(|n_i - n_j| - 1 \Big) \oplus \Big(|n_i - n_j| + 1 \Big) \oplus \cdots \oplus \Big(n_i + n_j - 1 \Big)$$

while

$$\mathfrak{g} \cong \bigoplus_{i,j} (n_i) \otimes (n_j) = \bigoplus_{i,j} \big((|n_i - n_j| + 1) \oplus \cdots \oplus (n_i + n_j - 1) \big)$$

$$\Rightarrow \quad \boxed{E_1 = \bigoplus_{i,j} \big(|n_i - n_j| - 1 \big), \quad E_0 = \bigoplus_{i,j} \big(|n_i - n_j| + 1 \big)}$$

Localization at the vacuum surface

$$\boxed{\mathcal{O}_0 := \mathcal{C}_{(N,1),\ldots,(N,1)} = \left\{ g\, C\, g^{-1} \;\middle|\; g \in U(n\,N) \right\} \cong U(n\,N)/U(n)\ .}$$

gauge group $G = U(n\,N)$, stabilizer $\mathfrak{s} = \mathfrak{u}(n)$

tangent space $T_C\mathcal{O} = \underbrace{T_C\mathcal{O}_0}_{\mathcal{J}(\mathfrak{g}\ominus\mathfrak{s})} \quad \oplus \quad \mathcal{J}^2(\mathfrak{g}\ominus\mathfrak{s})$

let

$$\begin{array}{llll} J_i & = \mathcal{J}(g_i'), & \tilde{J}_j = \mathcal{J}^2(g_j') & \in T_C(\mathcal{O}) \qquad g_i' \ldots\text{ONB of } \mathfrak{g}\ominus\mathfrak{s} \\ \lambda^i, & & \tilde{\lambda}^j & \in \Omega^1(\mathcal{O}) \qquad \ldots\text{dual basis} \end{array}$$

Introduce functions $f_i = \langle \alpha, J_i \rangle$. Using $\langle \alpha, \mathcal{J}^2(\mathfrak{g}) \rangle \equiv 0$ it follows

$$\begin{array}{rcll} \alpha & = & f_i\ \lambda^i & \text{localization form} \\ \frac{1}{d!}\,(d\alpha)^d & = & \bigwedge_{i=1}^d \left(df_i \wedge \lambda^i \right) & \text{on-shell} \end{array}$$

local symplectic model $\mathcal{F}_0$ = equivariant V.B. over $\mathcal{O}_0$ with fibre $\mathcal{J}^2(\mathfrak{g} \ominus \mathfrak{s})$

$$
\begin{aligned}
Z_0 &= \tfrac{1}{\mathrm{vol}(G)} \int_{\mathfrak{g}\times\mathcal{F}_0} \left[\tfrac{\mathrm{d}\phi}{2\pi}\right] \tfrac{t^d}{d!} (\mathrm{d}\alpha)^d \; e^{-\mathrm{i}\, t\, \langle \alpha, V_\phi\rangle - \mathrm{i}\, Tr(C_0\, \phi) - \frac{g'}{2}\, Tr(\phi^2)} \\
&= \tfrac{1}{\mathrm{vol}(G)} \int_{\mathfrak{g}\times\mathcal{F}_0} \left[\tfrac{\mathrm{d}\phi}{2\pi}\right] t^d \; \textstyle\bigwedge_{i=1}^d \left(\mathrm{d}f_i \wedge \lambda^i\right) \; e^{-\mathrm{i}\, N\, t\, f_i\, \phi^i - \mathrm{i}\, Tr(C_0\, \phi) - \frac{g'}{2}\, Tr(\phi^2)} \\
&= \tfrac{1}{\mathrm{vol}(G)} \int_{\mathfrak{s}} \left[\tfrac{\mathrm{d}\phi}{2\pi}\right] e^{-\mathrm{i}\, Tr(C_0\, \phi) - \frac{g'}{2}\, Tr(\phi^2)} \; \tfrac{1}{N^d} \int_{\mathcal{O}_0} \wedge \lambda^i
\end{aligned}
$$

df_i integrals over $\mathcal{J}^2(\mathfrak{g} \ominus \mathfrak{s})$ produces $\frac{1}{N}\, \delta(\phi_i)$ (localization!), except for $\mathfrak{s}$.

can carry out integral over gauge orbit $\mathcal{O}_0$ observing that

$$
\frac{1}{N^d} \int_{\mathcal{O}_0} \wedge \lambda^i = \int_{G/S} \wedge \eta^i = \frac{\mathrm{vol}(G)}{\mathrm{vol}(S)} \; ,
$$

$$
\begin{aligned}
Z_0 &= \tfrac{1}{\mathrm{vol}(S)} \int_{\mathfrak{s}} \left[\tfrac{\mathrm{d}\phi}{2\pi}\right] e^{-\mathrm{i}\, Tr(C_0\, \phi) - \frac{g'}{2}\, Tr(\phi^2)} \\
&= \tfrac{1}{n!}\, \tfrac{1}{(2\pi)^{n^2+n}}\; e^{-\frac{n\, N^2}{4g}} \int_{\mathbb{R}^n} [\mathrm{d}s]\; \Delta(s)^2\; e^{-\frac{g}{4}\, \sum_i s_i^2} \\
&= w(g)\, e^{-S(\mathcal{O}_0)}
\end{aligned}
$$

... standard result (Minahan-Polychronakos)

Localization at the maximally irreducible surface

$$\mathcal{O}_{\max} := \mathcal{C}_{(n_1,1),\ldots,(n_n,1)} = \left\{ g\, C\, g^{-1} \;\middle|\; g \in U(n\,N) \right\} \cong U(n\,N)/U(1)^n$$

for $n_1 > n_2 > \cdots > n_n$

consider basis of $T_C\mathcal{O}_{\max}$

$$\begin{array}{llllllll} J_i & = \mathcal{J}(g_i') & \tilde{J}_j & = \mathcal{J}^2(g_j') & H_i & = \mathcal{J}(h_i) \in E_0, & K_i & \in E_1 \\ \lambda^i, & & \tilde{\lambda}^j, & & \beta^i, & & \gamma^i & \in \Omega^1(\mathcal{O}) \quad \text{...dual basis} \end{array}$$

can show

$$\langle \mathrm{d}\alpha, H_i \wedge H_j \rangle = 0 \qquad \text{and} \qquad \langle \mathrm{d}\alpha, K_i \wedge K_j \rangle = A_{ij} = 2\,\mathrm{i}\,Tr\big(K_i\ \mathrm{ad}_{C_0}(K_j)\big)$$

thus

$$\begin{aligned} \mathrm{d}\alpha &= \mathrm{d}f_i \wedge \lambda^i + \tfrac{1}{2}\, A_{ij}\ \gamma^i \wedge \gamma^j \\ \tfrac{(\mathrm{d}\alpha)^{d-d_0}}{(d-d_0)!} &= \mathrm{pfaff}(A) \left(\bigwedge \gamma^i \right) \wedge \left(\bigwedge \mathrm{d}f_j \wedge \lambda^j \right) \end{aligned}$$

need ω^{d_0} in order to define volume form on E_0

evaluate integral over $\mathcal{J}^2(\mathfrak{g}\ominus\mathfrak{s})$ and $\phi^i \in \mathfrak{g}\ominus\mathfrak{h}\ominus\mathfrak{s}$ as before,

$$Z_{\max} = \frac{1}{\mathrm{vol}(G)} \int_{\mathfrak{h}\oplus\mathfrak{s}} \left[\frac{\mathrm{d}\phi}{2\pi}\right] \frac{\mathrm{pfaff}(A)}{N^{d-d_0-d_1}} \int_{\mathcal{O}_{\max}\times E_1} t^{d_1} \left(\wedge\, \gamma^i\right) \wedge \left(\wedge\, \lambda^j\right) \wedge \frac{\omega^{d_0}}{d_0!} \times e^{-\mathrm{i}\, t\, \langle \alpha, V_\phi\rangle - \mathrm{i}\, Tr(C_0\, \phi) - \frac{g'}{2}\, Tr(\phi^2)} .$$

<u>difficulty</u>: integral over E_0, E_1 of $e^{-\mathrm{i}\, t\, \langle \alpha, V_\phi\rangle}$ is non-gaussian

<u>Trick</u> (Beasley-Witten): additional localization form α'

$$Z_{\max} = \frac{1}{\mathrm{vol}(G)} \int_{\mathfrak{g}\times\mathcal{N}_{\max}} \left[\frac{\mathrm{d}\phi}{2\pi}\right] \exp\left(\omega + t_1\, Q\alpha + t_2\, Q\alpha' - \mathrm{i}\, Tr(C_0\, \phi) - \tfrac{g'}{2}\, Tr\left(\phi^2\right)\right)$$

$$\alpha' := -\tfrac{2}{N}\, \mathcal{J}\, \mathrm{d}Tr(C\, \phi)\Big|_{E_0} .$$

can show

$$\begin{aligned} \mathrm{d}\alpha' &\sim \tilde{A}_{ij}\, \beta^i \wedge \beta^j, \qquad \tilde{A}_{ij} = Tr\big(H_i\, \mathrm{ad}_s(H_j)\big) \\ \frac{(\mathrm{d}\alpha')^{d_0}}{d_0!} &\sim \mathrm{pfaff}\big(\tilde{A}\big)\, \textstyle\bigwedge_{i=1}^{2d_0} \beta^i \\ \langle \alpha', V_{h_i}\rangle &= 2Tr\big(H_i\, H_j\big) = 2M_{ij} \qquad \text{for } \phi = h_j. \end{aligned}$$

thus

$$\int_{\mathfrak{h}} \left[\frac{\mathrm{d}\phi}{2\pi} \right] t_2^{d_0} \, \frac{(\mathrm{d}\alpha')^{d_0}}{d_0!} \, \epsilon^{-\,\mathrm{i}\, t_2 \, \langle \alpha', V_\phi \rangle} \sim \frac{\mathrm{pfaff}(\tilde{A})}{\sqrt{\det(M)}} \bigwedge \beta^i$$

Now ϕ-integration localized onto $\mathfrak{s}$, can now evaluate integral over E_1:

$$\langle \alpha, V_s \rangle = \Big(x^i \, , \, y^i \Big) \, \tilde{M}_{ij}(s) \, \begin{pmatrix} x^j \\ y^j \end{pmatrix} , \qquad x^i \, K_i + y^i \, \mathcal{J}(K_i) \in E_1$$

$$\tilde{M}_{ij}(s) = Tr\Big(K_i \, \mathrm{ad}_s \, \mathrm{ad}_{C_0}(K_j) \Big),$$

$$\int_{E_1} \prod_{i=1}^{d_1} \mathrm{d}x^i \, \mathrm{d}y^i \, t_1^{d_1} \, \epsilon^{-\,\mathrm{i}\, t_1 \, \langle \alpha, V_s \rangle} = \left(\tfrac{\pi}{\mathrm{i}} \right)^{d_1} \frac{1}{\sqrt{\det\Big(\tilde{M}(s) \Big)}} \, .$$

can show

$$\begin{aligned} \frac{\mathrm{pfaff}(A)}{\sqrt{\det\Big(\tilde{M}(s) \Big)}} &= 2^{d_1} \, \prod_{k>l} (s_k - s_l)^{1-|n_k - n_l|} \, , \\ \frac{\mathrm{pfaff}(\tilde{A})}{\sqrt{\det(M)}} &\sim \prod_{k>l} (s_k - s_l)^{|n_k - n_l|+1} \, . \end{aligned}$$

Now can evaluate everything using Gaussian integrals:

$$
\begin{aligned}
Z_{\max} &= \frac{1}{\mathrm{vol}(G)} \int_{\mathfrak{g}\times\mathcal{F}_{\max}} \left[\frac{\mathrm{d}\phi}{2\pi}\right] \exp\Big(\mathrm{d}(t_1\,\alpha + t_2\,\alpha'\,) - \mathrm{i}\,\langle t_1\,\alpha + t_2\,\alpha', V_\phi\rangle\Big) \\
&\qquad\qquad \times\, \epsilon^{-\,\mathrm{i}\,Tr(C_0\,\phi) - \frac{g'}{2}\,Tr(\phi^2)} \\
&\sim \frac{1}{\mathrm{vol}(G)} \prod_{k=1}^{n} \sqrt{n_k} \int_{\mathbb{R}^n} \left[\frac{\mathrm{d}s}{2\pi}\right] \Delta(s)^2\, \epsilon^{-\,\mathrm{i}\,Tr(C_0\,s) - \frac{g'}{2}\,Tr(s^2)} \\
&\qquad\qquad \times\, \frac{1}{N^{d+d_0-d_1}} \int_{\mathcal{O}_{\max}} \Big(\bigwedge \lambda^j\Big) \wedge \Big(\bigwedge \beta^i\Big)
\end{aligned}
$$

observing again

$$
\frac{1}{N^{d+d_0-d_1}} \int_{\mathcal{O}_{\max}} \Big(\wedge\, \lambda^j\Big) \wedge \Big(\wedge\, \beta^i\Big) = \frac{\mathrm{vol}(G)}{\mathrm{vol}(S)}\,,
$$

and

$$
\begin{aligned}
Z_{\max} &= \frac{\mathrm{i}^{\,n^2-n}}{(2\pi)^{n^2+n}} \frac{N^{n/2}}{\prod\limits_{k=1}^{n} \sqrt{n_k}} \int_{\mathbb{R}^n} [\mathrm{d}\tilde{s}\,] \prod_{k>l} \left(\sqrt{\frac{N}{n_k}}\,\tilde{s}_k - \sqrt{\frac{N}{n_l}}\,\tilde{s}_l\right)^2 e^{-\frac{\mathrm{i}}{2}\sum_i \sqrt{\frac{N^3}{n_i}}\,\tilde{s}_i - \frac{g}{4}\sum_i \tilde{s}_i^{\,2}} \\
&\approx \pm\frac{1}{(2\pi)^{n^2+n}}\, e^{-\frac{n\,N^2 - m\,N}{4g}} \int_{\mathbb{R}^n} [\mathrm{d}s]\, \Delta(s)^2\, e^{\frac{\mathrm{i}}{4}\,\sum_i\, m_i\, s_i - \frac{g}{4}\,\sum_i\, s_i^2} \\
&= \pm\, w(m_i, g)\, \epsilon^{-S(\mathcal{O}_{\max})}
\end{aligned}
$$

... again agrees with classical result for $N \to \infty$

Summary and outlook:

- new model for YM gauge theory on S^2_N,
  all solutions determined:
  $\exists$ classical solutions and non-classical solutions (fluxons, ...)
- path integral exactly solvably using localization techniques
- for g finite and $N \to \infty$:
  classical partition function of Migdal-Rusakov is recovered
  non-classical solutions are suppressed by $e^{-\frac{N}{g}}$
  may well contribute in scaling limit $S^2_N \to \mathbb{R}^2_\theta$! (matrix phase ...?)
- generalization to $\mathbb{C}P^2_N$:
  space of gauge fields is also compact Kähler space