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Quantized Gauge Theory on Fuzzy S^2 and $\mathbb{C}P^2$

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hep-th/0307075, Nucl.Phys. B679 (2004) (H. S.)
hep-th/0407089, (H. Grosse, H. S.)

Outline:

- fuzzy spaces, fuzzy S^2
- Yang-Mills on fuzzy sphere as matrix model,

  monopole sectors,

  "path integral" and partition function
- 4 dimensions: gauge theory on fuzzy $\mathbb{C}P^2$

Fuzzy spaces:

(Madore...)

- $\mathcal{M}_N$ ... NC space with finitely many degrees of freedom (N)
- $\mathcal{M}_N \to \mathcal{M}$ (manifold) as $N \to \infty$.
- (NC) geometrical structures, symmetries, ...
- typically matrix algebras

Examples: fuzzy S^2_N, $\mathbb{C}P^2_N$, $S^2_N \times S^2_N$, ...

any (co)adjoint orbit of a Lie group G can be quantized in terms of a finite matrix algebra.

Goal: Study (quantum) field theory on $\mathcal{M}_N$

is regularized (cp. lattice field theory)

existence of limit $N \to \infty$ is nontrivial (UV/IR mixing, ...)

- coadjoint orbits $\Rightarrow$ can use group theory!!
- can be used to "regularize" other noncommutative spaces,
  $S^2_N \to \mathbb{R}^2_\theta$,
  $S^2_N \times S^2_N \to \mathbb{R}^4_\theta$,
  $\mathbb{C}P^2_N \to \mathbb{R}^4_\theta$, etc
- occur in string theory:
  - D-branes of group manifolds
  - solutions of IKKT matrix model, ...
- Gauge theory on fuzzy spaces can be formulated as (multi)-matrix model
  - well-defined
  - simple: connections, bundles, topology arises automatically
  - useful? computable?

Noncommutative gauge theory in a nutshell

assume: there exist elements $\lambda_i \in \mathcal{A}$ such that

$$\partial_i f(x) = [\lambda_i, f(x)], \qquad f(x) \in \mathcal{A}$$

(derivatives are INNER)!

assume $\qquad [\lambda_i, \lambda_j] = \theta_{ij}(\lambda)$

consider "covariant coordinates" (one-forms, ...)

$$\boxed{B_i = \lambda_i + A_i}, \qquad A = A_i \ldots \text{ gauge field}$$

consider

$$\begin{aligned} [B_i, B_j] - \theta_{ij}(B_i) &= [\lambda_i, A_j] - [\lambda_j, A_i] + [A_i, A_j] \\ &= \quad F_{ij} \quad \ldots \text{ field strength} \end{aligned}$$

gauge transformations: $\qquad B_i \to U^{-1} B_i U, \quad A_i \to U^{-1}\partial_i U + U^{-1} A U$

Yang-Mills action on $\mathcal{A}$: $\quad$ (1st guess)

$$\boxed{S \quad = Tr(F_{ij}F^{ij}) = Tr(([B_i, B_j] - \theta_{ij}(B))([B^i, B^j] - \theta^{ij}(B)))}$$

obviously gauge invariant under

$$B_i \to U^{-1} B_i U$$

$U \in U(\mathcal{A}) \qquad \Rightarrow U(1)$ gauge theory

$\mathcal{A} \to \mathcal{A} \otimes Mat(n) \Rightarrow U(n)$ gauge theory

simpler than on commutative spaces: don't need connection

... essentially Matrix Models: $\quad B_i \in \mathcal{A}$ resp. $Gl(\mathcal{H})$.

<u>trouble</u>: usually $\dim(\mathcal{A}) = \infty \qquad$ (e.g. for $\mathbb{R}^4_\theta$)

<u>however</u>: on "fuzzy spaces", $\dim(\mathcal{A}) = finite$.

The fuzzy sphere S^2_N:

(Madore; Grosse, Klimcik, Presnajder ...)

quantization parameter: $\theta = \frac{1}{N}, \quad N \in \mathbb{N}$.

algebra of functions: $\mathcal{A} = Fun_N(S^2) = Mat(N, \mathbb{C})$

rotations: let λ_i ... N-dim. rep. of $su(2)$.

$$f \in \mathcal{A} \to J_i \triangleright f := [\lambda_i, f]$$

$\hookrightarrow$ space of functions decomposes into

$$\begin{aligned} f \in (N) \otimes (N) &= (1) \ \oplus \ (3) \ \oplus ... \oplus \ (2N-1) \\ &= (\hat{Y}^0) + (\hat{Y}^1_m) + ... + (\hat{Y}^{N-1}_m) \end{aligned}$$

hence: $\exists$ map

$$\begin{aligned} S^2_N &\to S^2, \\ \hat{Y}^l_m &\to Y^l_m \end{aligned}$$

dimensionless coordinates $\lambda_i = x_i/\Lambda_N$ satisfy

$$\varepsilon_k^{ij}\lambda_i\lambda_j = i\lambda_k, \quad \lambda_i\lambda^i = \frac{N^2-1}{4}$$

$$\boxed{\begin{aligned} [x_i, x_j] &= i\Lambda_N\, \varepsilon_{ijk}\, x_k, \\ x_1^2 + x_2^2 + x_3^2 &= R^2. \end{aligned}} \quad \ldots\ S_N^2$$

$$\Lambda_N \approx \frac{R}{N} \quad \ldots \quad \text{NC parameter [length]}$$

integral:

$$\int f(x) := \frac{4\pi}{N}\mathrm{Tr}[f(x)]$$

action for scalar fields:

$$S_0 = \int \frac{1}{2}\,\Phi(\Delta + m^2)\Phi + \frac{g}{4!}\Phi^4$$

$\Phi \in \mathcal{A}$ ... hermitian matrix, $\Delta = \sum J_i^2 = [\lambda_i, [\lambda_i, .]]$

Gauge theory on S^2_N as Matrix Model

(Grosse, Klimcik, Watamuras, H.S. ...)

consider "covariant coordinates"

$$B_i = \lambda_i + A_i \qquad \in Fun(S^2_N) = Mat(N, \mathbb{C})$$

note $[\lambda_j, \lambda_k] = i\varepsilon_{ijk}\lambda_i$, define

$$\boxed{\begin{aligned} F_{jk} &= [B_j, B_k] - i\varepsilon_{ijk}B_i \\ &= [\lambda_j, A_k] - [\lambda_k, A_j] - i\varepsilon_{ijk}A_i \\ &\cong dA + AA \end{aligned}} \qquad \ldots. \qquad \text{field strength}$$

gauge transformations:

$$\boxed{\begin{aligned} B_i &\to U^{-1}B_iU, \\ A_i &\to U^{-1}A_iU + U^{-1}[\lambda_i, U] \end{aligned}} \qquad U \in SU(N)$$

try <u>action</u>:

$$\boxed{\begin{aligned} S_{YM} &= \tfrac{1}{g^2}\int([B_j,B_k]-i\varepsilon_{ijk}B_i)([B^j,B^k]-i\varepsilon^{ijk}B^i) \\ &= \tfrac{1}{g^2}\int F_{jk}F^{jk} \end{aligned}}$$

invariant under rotations $SO(3)$ and gauge trafos

<u>Solutions</u>:

$$F_{jk}=0 \quad \Leftrightarrow \quad [B_j,B_k]=i\varepsilon_{ijk}B_i$$

$$B_i=\lambda_i$$

hence: <u>fuzzy sphere is solution of S_{YM}.</u>

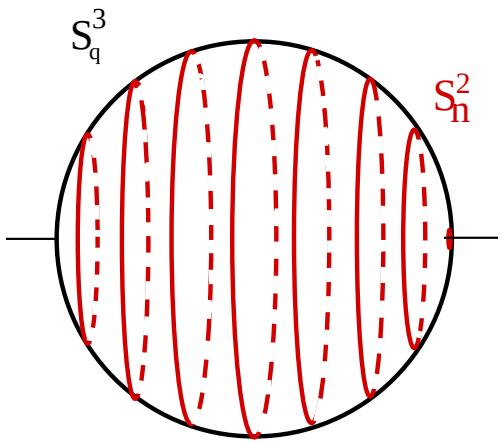
other solutions: any other rep. of $su(2)$!

general solution of $F = 0$:

direct sum

$$B_i = \begin{pmatrix} \lambda_i^{(M_1)} & 0 & \dots & 0 \\ 0 & \lambda_i^{(M_2)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_i^{(M_k)} \end{pmatrix},$$

"multi-brane configurations" in string theory



geometry is dynamical !

additional terms possible:

- Chern-Simons term (string theory)

$$S_{CS} = \int AdA + A^3 = \int \frac{1}{3} B_i B_j B_k \varepsilon^{ijk} + \frac{1}{2} B_i B^i$$

... same saddle-points, but different action

- $$S_4 = \frac{1}{N} Tr\Big((B_i B^i - \lambda_i \lambda^i)^2 \Big)$$

prefers solution $B_i = \lambda_i$, suppresses other (reducible) solutions

$\Rightarrow$ $S_{YM} + S_4$ ... YM on single fuzzy sphere

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note: NC gauge theory: action determines geometry,

gauge fields = fluctuations of NC geometry

Observation:

$$C = \frac{1}{2} + B_i \sigma^i \quad \in Mat(2N, \mathbb{C})$$

for $B_i = \lambda_i$, satisfies

$$C^2 = (\frac{N}{2})^2, \qquad Tr(C) = N$$

include gauge fields $B_i = \lambda_i + A_i$ $\hookrightarrow$ consider

$$\boxed{\begin{aligned} S &= TrV(C) = \tfrac{1}{Ng^2} Tr\left((C^2 - \tfrac{N^2}{4})^2\right) \\ &= \tfrac{1}{Ng^2} Tr\Big((B_i B^i - \tfrac{N^2-1}{4})^2 \\ &\qquad + (B_i + i\varepsilon_{ijk} B^j B^k)(B^i + i\varepsilon^{irs} B_r B_s)\Big) \end{aligned}}$$

need constraint $C_0 = \frac{1}{2}$ ... gauge theory on fuzzy sphere

Note: $Tr(V(C))$ invariant under $U(2N)$

constraint $C_0 = \frac{1}{2}$ breaks $U(2N) \quad \rightarrow \quad U(N)$

physical meaning of action:

decompose

$$A_i \approx \frac{2}{N} x_i \, \varphi + A_i^{(t)}, \quad x_i A_i^{(t)} = 0$$

φ ... auxiliary scalar field

$$B_i B^i - (\frac{N}{2})^2 = 2\varphi - [\lambda^i, A_i^{(t)}] + (A_i^{(t)})^2 \; + o(\frac{1}{N})$$

$$\begin{aligned} F_{kl} &= [B_k, B_l] - i\varepsilon_{klm} B^m \\ &\approx [\lambda_k, A_l^{(t)}] - [\lambda_l, A_k^{(t)}] + [A_k^{(t)}, A_l^{(t)}] - i\varepsilon_{klm} A_m^{(t)} \; + O(\varphi/N) \end{aligned}$$

$$\boxed{\begin{aligned} S &= \tfrac{1}{g^2} \int F_{kl} F^{kl} + (2\varphi - [\lambda^i, A_i] + A_i^2)^2 \\ &\cong \tfrac{1}{g^2} \int F_{kl} F^{kl} \end{aligned}}$$

... YM action on sphere.

φ decouples, F_{kl} ... usual (tangential) field strength

Monopole sectors:

new saddle-points for $S = \mathrm{Tr} V(C)$:

$$C = \frac{1}{2} + \begin{pmatrix} \alpha_m \lambda_i^{(M)} \sigma^i & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2} + \lambda_i^{(N)} \sigma^i + A_i \sigma^i,$$

where

$$\begin{aligned} M &= N - m \\ \lambda_i^{(M)} &\ \ldots\ M - \text{dim. rep. of } su(2) \\ \alpha_m &\sim 1 + \tfrac{m}{N} \end{aligned}$$

written as fluctuation over previous "vacuum" $\lambda_i^{(N)}$:

Result (for $N \to \infty$, fixed m):

$$\vec{A} = \frac{m}{2} \frac{1}{1+x_3} \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix}$$

... usual monopole field with charge $m \in \mathbb{Z}$

action:

$$S(C^{(M)}) = \frac{m^2}{2g^2}$$

<u>note</u>: no need to introduce connections on fiber bundles etc., just matrices of different size

simpler than classical!

Nonabelian $U(n)$ Yang-Mills

same action $S = \mathrm{Tr}V(C)$, but with matrices of size

$$M = nN - m \quad m \in \mathbb{Z}.$$

write again

$$C = \frac{1}{2} + B_i\sigma^i, \qquad B_i = \lambda_i^{(N)} + A_{i,0} + A_{i,a}t^a$$

t^a ... $su(n)$ Gell-Mann matrices

Action:

$$\boxed{S = \mathrm{Tr}V(C) = \tfrac{1}{g^2}\int(F_{kl,0}F^{kl,0} + F_{kl,a}F^{kl,a})} \quad ...U(n) \text{ YM theory}$$

saddle points: Block-matrices parametrized by integers $m_1, ..., m_n$.

Action:

$$S(C^{(m_1,...,m_n)}) = \frac{1}{2g^2}\sum_i m_i^2 \qquad \text{(for large } N\text{)}$$

Quantization

$$\boxed{Z[J] = \int dB_i \; e^{-S(B_i) \; + B_i J^i},} \qquad B_i \in Mat(N, \mathbb{C})$$

nice properties:

- well-defined (finite!)
- invariant under gauge-trafos and $SO(3)$ rotation
- no gauge fixing (Faddeev-Popov) necessary

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goal: evaluate path integral using matrix model techniques

known: Z can be calculated for gauge theory on S^2

(Migdal, Rusakov)

Recall <u>single-matrix models</u>:

$$Z = \int dC e^{-TrV(C)} = \int \prod_{i=1}^{N} dc_i \, \Delta^2(c) \, e^{-\sum_i V(c_i)}$$

c_i ... eigenvalues of C, $\Delta(c) = \prod_{i<j}(c_i - c_j)$ ... Vandermonde det.

<u>here</u>: more complicated: 3 matrices B_i

<u>Trick</u>: one $2M \times 2M$ matrix $C = \frac{1}{2} + B_i \sigma^i = C_\alpha \sigma^\alpha$, constraint $C_0 = \frac{1}{2}$.

$$\begin{aligned} Z &= \int dB_i \, \exp(-S(B))) \\ &= \int dC \, \delta(C_0 - \tfrac{1}{2}) \, \exp(-\mathrm{Tr}V(C)) \\ &= \int dK \, Z_0[J] \, e^{-\frac{i}{2} TrJ} \end{aligned}$$

where

$$Z_0[J] = \int dC \, \exp(-\mathrm{Tr}V(C) + i\mathrm{Tr}(CJ)), \quad J = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} = K \, \sigma^0$$

use Itzykson-Zuber-Harish-Chandra formula:

$$\int dU \exp(i\mathrm{Tr}(U^{-1}CUJ)) = const\ \frac{\det(e^{i\Lambda_i J_j})}{\Delta(\Lambda_i)\Delta(J_j)}$$

$$\boxed{Z = \int dk_i \Delta^2(k) \int d\Lambda_i \Delta^2(\Lambda_i) \exp(-\mathrm{Tr}V(\Lambda)) \frac{\det(e^{i(\Lambda_i J_j)})}{\Delta(\Lambda_i)\Delta(J_j)}}$$

in large N limit (n fixed): .... (considerable effort) ...

$$Z = \sum_{p_1,\ldots,p_n \in \mathbb{Z}} \Delta^2(p)\ \exp(-2\pi^2 g^2 \sum_i p_i^2)$$

equivalent to result of Migdal, Rusakov on S^2:

$$\boxed{Z = \sum_R (d_R)^2\ \exp(-4\pi^2 g^2 C_{2R}),}$$

R ... irrep of $su(n)$, C_{2R} ... quadratic casimir

Hence:

YM on fuzzy sphere $\rightarrow$ YM on classical sphere, is calculable

Gauge theory on fuzzy $\mathbb{C}P^2$

1.) Classical $\mathbb{C}P^2$

$$\mathbb{C}P^2 = SU(3)/_{SU(2)\times U(1)} \quad \subset Lie(su(3)) \cong \mathbb{R}^8$$

4-dim. (co)adjoint orbit, compactification of $\mathbb{R}^4$ with S^2

8-dimensional isometry group $SU(3)$ (cp. Poincare/Euclidean group)

classical coordinate form: x_a ... coords of $\mathbb{R}^8 \cong su(3)$

$$\begin{aligned} g^{ab} x_a x_b &= 1, \\ d_c^{ab} x_a x_b &= \tfrac{2}{\sqrt{3}}\, x_c. \end{aligned}$$

d_{abc} ... d-tensor of $su(3)$

2.) Fuzzy $\mathbb{C}P^2_N$ (Grosse & Strohmaier, Balachandran etal, ...)

$$\begin{aligned} \mathbb{C}P^2_N &= Mat(D_N, \mathbb{C}) = Hom(V_N), \\ D_N : &= dim(V_N) = (N+1)(N+2)/2 \end{aligned}$$

V_N ... irrep of $su(3)$ with highest weight $N\Lambda_2$.

$$\xi_a = T_a|_{V_N}, \quad x_a = \Lambda_N \, \xi_a, \qquad \Lambda_N \approx \frac{R}{N}$$

and satisfy the relations

$$\begin{aligned} [x_a, x_b] &= \tfrac{i}{2}\Lambda_N f_{abc} \, x_c \\ g^{ab} x_a x_b &= R^2, \\ d^{ab}_c x_a x_b &= R \, \tfrac{2N/3+1}{\sqrt{\frac{1}{3}N^2+N}} \, x_c. \end{aligned}$$

Gauge theory on fuzzy $\mathbb{C}P^2_N$

(H. Grosse & H.S.)

8 coordinates $\rightarrow$ (4 gauge fields, 4 auxiliary fields)

$$B_a = \xi_a + A_a$$

Yang-Mills action: multi-matrix model

$$\boxed{S = \frac{1}{g} \int F_{ab}F_{ab} + \frac{1}{N} D_a D_a}$$

where

$$\begin{aligned} F_{ab} &= i[B_a, B_b] + \tfrac{1}{2} f_{abc} B_c \\ D_a &= d_{abc} B_b B_c - (\tfrac{2N}{3} + 1) B_a \end{aligned}$$

note: additional term

$$\frac{1}{N} D_a D_a = 4N(A_1^2 + A_2^2 + A_3^2 + \frac{1}{9} A_8^2)$$

gives large mass to the transversal fields, 4 gauge fields $A_{4,5,6,7}$ survive

Monopole solutions:

Ansatz

$$C_a = \alpha \xi_a^{(N-m)} = \alpha \pi_{(0,N-m)}(T_a), \qquad \alpha = 1 + \frac{m}{N} + o(1/N^2)$$

is exact solution of equation of motion.

Gauge field:

$$C_a = \begin{pmatrix} \alpha \xi_a^{(N-m)} & 0 \\ 0 & 0 \end{pmatrix} = \xi_a + A_i \sigma^i$$

A_a finite, explicit except at "sphere at infinity"

Field strength:

$$F = -m\,\eta$$

η ... symplectic form

$$c_1 = m \qquad \text{... } 1^{st} \text{ Chern number}$$

"Instanton" solutions:

Ansatz

$$C''_a = \zeta_a^{(m)} = \pi_{(N-m)\Lambda_2+\Lambda_1}(T_a)$$

must be projected on tangential space

$\hookrightarrow$ exact solution of equation of motion,

gauge field finite, explicit except at "sphere at infinity"

Field strength: $|F_{ab}| = const$

first and second Chern number $c_1, c_2 \neq 0$

nontrivial rank 2 bundle: "instanton + monopole"

$$L^m \otimes F$$

L ... basic line bundle (monopole)

F .. nontrivial rank 2 bundle, $F \oplus L = I^3$.

Quantization

$$\boxed{Z[J] = \int dB_a \; e^{-S(B_a) \; + B_a J^a},} \quad B_a \in Mat(D_N, \mathbb{C})$$

nice properties:

- well-defined (finite!)
- invariant under gauge-trafos and $SU(3)$ rotation

Conclusions:

- gauge theory on fuzzy spaces = (multi-) Matrix Model
- provides nonperturbative definition for finite N
- "smart" lattice gauge theory, preserving symmetries
- $\exists$ higher-dim. analogs: fuzzy $\mathbb{C}P^2$ (4-dim), ...
- $\exists$ scaling limit $S^2_N \to \mathbb{R}^2_\theta$, $\mathbb{C}P^2_N \to \mathbb{R}^4_\theta$, ...

Questions/Problems:

- does $N \to \infty$ exist on other spaces? properties?
- calculate observables, develop tools