Dynamical generation of fuzzy extra dimensions

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Outline

- background and motivation
- the 4-dimensional gauge theory
- vacuum structure, symmetry breaking
  and emergence of fuzzy sphere
- Kaluza-Klein modes and low-energy action
- fermions: index, (would-be) zero modes, chirality

Background: extra dimensions, Kaluza-Klein, etc.

- start with higher-dimensional theory on $M^4 \times K$, (K ... "small")
  internal excitations ("KK-modes") on K not visible for energies $E < \frac{1}{R}$
  $\Rightarrow$ effective 4-D theory on M^4 at low energies
- GUT models from compactified higher dim's *(Cremmer-Scherk '77)*
  Higgs from gauge field in extra dimensions

Problems:

- higher-dimensional (gauge) theories are typically *non-renormalizable*
- what determines or stabilizes extra dimensions?

recent developments

- Connes-Lott: “minimal” NC internal space $\mathbb{Z}_2$ (“2 points”)
- use of *matrices* to describe internal spaces *(Madore, Dubois-Violette, ...)*
- “deconstructing dimensions:” *(Arkani-Hamed, Cohen, Georgi)*
  4-dimensional gauge theory *dynamically develops* extra dimensions
- “coset space dimensional reduction” on fuzzy sphere
  (Aschieri, Madore, Manousselis, Zoupanos)

new: Quantization $\Rightarrow$ extra term in action,
mechanism for selecting & stabilizing unique vacuum (with flux)

The model

$SU(\mathcal{N})$ gauge theory on 4-dimensional Minkowski space M^4

Action:

$$\boxed{\mathcal{S}_{YM} = \int d^4y\, Tr\, \left(\frac{1}{4g^2}\, F_{\mu\nu}^\dagger F_{\mu\nu} + (D_\mu\phi_a)^\dagger D_\mu\phi_a\right) - V(\phi)}$$

where A_μ ... $su(\mathcal{N})$-valued gauge fields, $D_\mu = \partial_\mu + [A_\mu, .]$, and

3 antihermitian scalars in the adjoint of $SU(\mathcal{N})$ with global $SO(3)$ symmetry

$$\begin{aligned} \phi_a &= -\phi_a^\dagger\,, & a &= 1,2,3 \\ \phi_a &\to U^\dagger \phi_a U, & U &= U(y) \in SU(\mathcal{N}) \end{aligned}$$

$V(\phi)$ ... most general renormalizable potential respecting the symmetries

$$\begin{aligned} V(\phi) &= Tr\, (g_1\phi_a\phi_a\phi_b\phi_b + g_2\phi_a\phi_b\phi_a\phi_b - g_3\varepsilon_{abc}\phi_a\phi_b\phi_c + g_4\phi_a\phi_a) \\ &+ \frac{g_5}{\mathcal{N}}\, Tr(\phi_a\phi_a)Tr(\phi_b\phi_b) + \frac{g_6}{\mathcal{N}} Tr(\phi_a\phi_b)Tr(\phi_a\phi_b) + g_7. \end{aligned}$$

Main features:

- 4D renormalizable gauge theory on M^4
  dynamically develops extra-dimension: $M^4 \times S^2_N$
- dynamical choice of unique vacuum geometry, symmetry breaking,
  $SU(n_1) \times SU(n_2) \times U(1)$ arises naturally
- magnetic fluxes on internal space
  $\Rightarrow$ (would-be) zero modes, massless fermions

... within framework of renormalizable 4-D gauge theory potential can be rewritten as

$$\boxed{V(\phi) = Tr\left(a^2(\phi_a\phi_a + \tilde{b}\,\mathbf{1})^2 + c + \frac{1}{\tilde{g}^2}\,F_{ab}^\dagger F_{ab}\right) + \frac{h}{N}\,g_{ab}g_{ab}}$$

for suitable constants $a, b, c, d, \tilde{g}, h$, where

$$\begin{aligned} F_{ab} &= [\phi_a, \phi_b] - \varepsilon_{abc}\phi_c \\ \tilde{b} &= b + \tfrac{d}{N}\,Tr(\phi_a\phi_a), \\ g_{ab} &= Tr(\phi_a\phi_b) \\ R &= \tfrac{2g_2}{g_3} \end{aligned}$$

= action for Yang-Mills on a fuzzy sphere!

SSB, vacuum = minium of $V(\phi)$, Higgs effect

$\langle\phi_a\rangle$ depends on parameters of the potential

$\langle\phi_a\rangle$ forms fuzzy sphere, fluctuations = gauge fields on S^2_N

Type I vacuum and emergence of fuzzy sphere

assume for simplicity $d = h = 0$.

$$V(\phi) = Tr\left(a^2(\phi_a\phi_a + \tilde{b}\,\mathbf{1})^2 + \frac{1}{\tilde{g}^2}\,F_{ab}^\dagger F_{ab}\right) \quad \geq 0$$

$\Rightarrow$ global minimum $V(\phi) = 0$ achieved if

$$F_{ab} = [\phi_a, \phi_b] - \varepsilon_{abc}\phi_c = 0, \qquad -\phi_a\phi_a = \tilde{b},$$

$\Rightarrow$ ϕ_a representation of $SU(2)$ with Casimir $\tilde{b} = C_2(N)$ for some $N \in \mathbb{N}$

$$\boxed{\phi_a = X_a^{(N)} \otimes \mathbf{1}_n} \qquad \text{“type I vacuum”}$$

$X_a^{(N)}$ ... generator of the N-dimensional irrep of $SU(2)$,

defines fuzzy sphere S_N^2:

$$[\hat{x}_a, \hat{x}_b] = \varepsilon_{abc}\,\frac{2R}{N}\,\hat{x}_c \approx 0, \qquad \sum_a \hat{x}_a\hat{x}_a = R^2, \qquad \hat{x}_a = \frac{2R}{N}\,X_a^{(N)}$$

include fluctuations: $\phi_a = X_a^{(N)} \otimes \mathbf{1}_n + A_a$

$A_a = (A_a(y))_{\mathcal{N}\times\mathcal{N}} = \sum_\alpha A_{a,\alpha}(y,x)\,\lambda_\alpha$ ... $u(n)$-valued functions on $M^4 \times S_N^2$

$$\begin{aligned} F_{ab} &= [\phi_a,\phi_b] - \varepsilon_{abc}\phi_c \\ &= -iJ_aA_b + iJ_bA_a + [A_a,A_b] - \varepsilon_{abc}A_c \qquad \text{... Field strength on } S_N^2 \end{aligned}$$

fluctuations A_a provide components of higher-dimensional gauge field

$$A_M = (A_\mu, A_a) = A_M(y,x) \qquad \text{on} \quad M^4 \times S_N^2$$

gauge transformations

$$\begin{aligned} \phi_a &\to U^{-1}\phi_a U, \qquad\qquad U \in U(\mathcal{N}) \\ A_\mu &\to U^{-1}A_\mu U - U^{-1}\partial_\mu U, \\ A_a &\to U^{-1}A_a U - iU^{-1}J_a U, \end{aligned}$$

... local $U(n)$ gauge transformations on $M^4 \times S_N^2$.

$$\boxed{\mathcal{S}_{YM} = \int d^4y\, Tr\left(\tfrac{1}{4g^2}\, F_{\mu\nu}^\dagger F_{\mu\nu} + (D_\mu\phi_a)^\dagger D_\mu\phi_a - \tfrac{1}{\tilde{g}^2}\, F_{ab}^\dagger F_{ab} - a^2(\phi_a\phi_a + \tilde{b})^2\right)}$$

... action for $U(n)$ YM theory on $M^4 \times S_N^2$

Vacuum: generic case

in general, $\tilde{b} \neq C_2(N)$ or $\mathcal{N} \neq Nn \;\Rightarrow\;$ vacuum not obvious;

e.o.m.: $\frac{\delta V}{\delta \phi_a} = 0$

$$a^2\{\phi_a, \phi\cdot\phi + \tilde{b} + \frac{d}{\mathcal{N}}\, Tr(\phi\cdot\phi + \tilde{b})\} + \frac{1}{\tilde{g}^2}\,(2[F_{ab}, \phi_b] + F_{bc}\varepsilon_{abc}) = 0$$

solutions:

$$\phi_a = diag\Big(\alpha_1\, X_a^{(N_1)} \otimes \mathbf{1}_{n_1}, ..., \alpha_k\, X_a^{(N_k)} \otimes \mathbf{1}_{n_k}\Big)$$

for decomposition of $\mathcal{N} = n_1 N_1 + ... + n_k N_k$ into irreps of $SU(2)$, multiplicities n_i,

$$\alpha_i = 1 - \frac{m_i}{\tilde{N}} \; + O(\frac{1}{\tilde{N}^2}) \qquad \text{where } m = N_i - \tilde{N}, \quad \tilde{b} = \frac{1}{4}(\tilde{N}^2 - 1)$$

Assume that vacuum is of this form; determine the one with minimal energy for $m_i \ll \tilde{N}$, each block contributes

$$F_{ab} \sim \frac{1}{2} m_i \, \varepsilon_{abc} x_c, \qquad (\phi_a \phi_a + \tilde{b})^2 = O(\frac{1}{\tilde{N}^2})$$

therefore

$$V(\phi) = Tr\left(a^2 (\phi_a \phi_a + \tilde{b}\, \mathbf{1})^2 + \frac{1}{\tilde{g}^2} F_{ab}^\dagger F_{ab} \right) \approx \frac{1}{2\tilde{g}^2} \frac{\mathcal{N}}{k} \sum_i n_i \, m_i^2$$

with constraint $\sum n_i \, m_i = const$

discrete minimization problem; minimum of action achieved for

either

$\boxed{\text{one block of size } N \qquad \text{(type I vacuum),}}$

or

$\boxed{\text{2 different blocks with sizes } N_1 = N_2 \pm 1 \qquad \text{(type II vacuum, generic)}}$

type I vacuum:

$$\boxed{\phi_a = \alpha\, X_a^{(N)} \otimes \mathbf{1}_n} \qquad \text{... type I vacuum}$$

unbroken gauge group $U(n)$

interpretation: $U(n)$ Yang-Mills theory on $\mathcal{M}^4 \times S^2_N$

only possible for $\mathcal{N} = nN$

type II vacuum:

(generic case)

$$\boxed{\phi_a = \begin{pmatrix} \alpha_1\, X_a^{(N_1)} \otimes \mathbf{1}_{n_1} & 0 \\ 0 & \alpha_2\, X_a^{(N_2)} \otimes \mathbf{1}_{n_2} \end{pmatrix},}$$ ... type II vacuum

where $N_2 = N_1 + 1$ and $\mathcal{N} = N_1 n_1 + N_2 n_2$ (always possible) determined by $\tilde{b}$, etc.

interpretation: $SU(n_1) \times SU(n_2) \times U(1)$ Yang-Mills theory on $\mathcal{M}^4 \times S^2_N$

with monopole flux:

$$\phi_a = \alpha_1\, X_a^{(N_1)} \otimes \begin{pmatrix} \mathbf{1}_{n_1} & 0 \\ 0 & \mathbf{1}_{n_2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & A_a^{(k=1)} \otimes \mathbf{1}_{n_2} \end{pmatrix}$$

cf. GUT model $SU(5) \to SU(3) \times SU(2) \times U(1)$,

cf. Cremmer-Scherk 1977 : flux stabilizes vacuum

monopole on S^2_N is important for low-energy fermions (zero modes!)

Kaluza-Klein modes and symmetry breaking

1) type I vacuum $\phi_a = \alpha\, X_a^{(N)} \otimes \mathbf{1}_n$

unbroken low-energy gauge group (Higgs effect):

$$K := \text{ commutant of } \phi_a = SU(n)$$

KK expansion of gauge fields:

decompose 4-dimensional $su(\mathcal{N})$-valued gauge fields $A_\mu(y)$ into harmonics on S^2_N using

$$u(\mathcal{N}) = u(N) \otimes u(n) = (\oplus Y^{lm}(x)) \otimes u(n)$$

i.e.

$$\boxed{A_\mu(y) = \sum_{l,m} Y^{lm}(x) \otimes A_{\mu,lm}(y) = A_\mu(x,y) \;\; ... \;\; u(n)\text{-valued functions on } M^4 \times S^2_N}$$

$A_{\mu,lm}(y)$ ... KK modes , $u(n)$-valued gauge/vector fields on M^4

Masses of the KK modes: from Higgs effect!

$$\int Tr(D_\mu\phi_a)^\dagger D_\mu\phi_a = \int Tr(\partial_\mu\phi_a^\dagger\partial_\mu\phi_a + 2(\partial_\mu\phi_a^\dagger)[A_\mu,\phi_a] + [A_\mu,\phi_a]^\dagger[A_\mu,\phi_a])$$

(term $(\partial_\mu\phi_a^\dagger)[A_\mu,\phi_a]$ does not contribute)

$\Rightarrow$ set $\phi_a(y) = \langle\phi_a(y)\rangle = \alpha X_a^{(N)} \otimes \mathbf{1}_n$,

note: $i[X_a, A_\mu] = J_a A_\mu$ ... action of $SU(2)$

$\Rightarrow$ $Tr[X_a,A_\mu]^\dagger[X_a,A_\mu] = Tr A_\mu(J_aJ_a)A_\mu$ = Casimir on KK modes $A_{\mu,lm}$

$$\int Tr(D_\mu\phi_a)^\dagger D_\mu\phi_a = \int Tr(\partial_\mu\phi_a^\dagger\partial_\mu\phi_a + \sum_{l,m}\alpha^2\, l(l+1)\, A_{\mu,lm}(y)^\dagger A_{\mu,lm}(y)) + S_{int}.$$

$\Rightarrow$ expected tower of massive KK modes $A_{\mu,lm}(y)$, mass

$$m_l^2 = \frac{\alpha^2 g^2}{R^2}\, l(l+1)$$

Results for KK reduction of type I vacuum:

- single massless 4D $su(n)$ gauge field $A_{\mu,00}(y)$,
  tower of massive 4D $u(n)$ vector fields $A_{\mu,lm}(y)$, ($l > 0$)
- low-energy effective action

$$S_{LEA} = \int d^4y \, \frac{1}{4g^2} \, Tr_n \, F^{\dagger}_{\mu\nu} F_{\mu\nu},$$

  where $F_{\mu\nu}$ ... field strength of massless $su(n)$ gauge field
- radius of the internal S^2_N:

$$r_{S^2} \approx \frac{1}{g} R$$

- no massless 4D fields from KK modes of scalar fields

2) type II vacuum $\quad \phi_a = \begin{pmatrix} \alpha_1 \, X_a^{(N_1)} \otimes \mathbf{1}_{n_1} & 0 \\ 0 & \alpha_2 \, X_a^{(N_2)} \otimes \mathbf{1}_{n_2} \end{pmatrix}$

KK expansion of A_μ gauge fields:

$$A_\mu = \sum_{l,m} \begin{pmatrix} Y^{lm(N_1)} \, A^1_{\mu,lm}(y) & Y^{lm(+)} \, A^+_{\mu,lm}(y) \\ Y^{lm(-)} \, A^-_{\mu,lm}(y) & Y^{lm(N_2)} \, A^2_{\mu,lm}(y) \end{pmatrix} = A_\mu(x,y)$$

masses of diagonal KK modes $m^2_{l,i} = \frac{\alpha_i^2 g^2}{R^2} \, l(l+1)$, similar for off-diagonal modes

- massless 4D $su(n_1), su(n_2)$ gauge fields $A^{1,2}_{\mu,00}(y)$
  massless 4D $u(1)$ gauge field
  low-energy gauge group $SU(n_1) \times SU(n_2) \times U(1)$.
    cf. $SU(5) \to SU(3) \times SU(2) \times U(1)$ GUT model
    massive (“ultraheavy”) gauge fields in the bifundamental
- tower of massive 4D vector fields $A^{1,2,+,-}_{\mu,lm}(y)$

interpretation: gauge theory on $M^4 \times S^2$,

with magnetic flux $k = |N_1 - N_2|$ on S^2

Remarks on quantization and renormalization:

- model is renormalizable, even though behaves as gauge theory on $\mathcal{M}^4 \times S^2_N$
- running of parameters $a, b, \tilde{g}, ...$ under RG group
  $\Rightarrow$ *different vacua (geometry, gauge groups) at different energy scales* !
- radius of fuzzy sphere only logarithmically running
- large $\mathcal{N}$ gauge theory !

Adding Fermions

- Start from 4D: add fermions $\psi_{i,\alpha}$ to 4D gauge theory, (2) of $SU(2)$, consider most general renormalizable action respecting symmetries
- fermions in adjoint of $SU(\mathcal{N})$ (since: kinetic term on S^2_N can arise only through $[\phi_a, \psi_{i,\alpha}]$)

Minimal case: doublet of chiral 4-dimensional Weyl spinors

$$\Psi(y) = \begin{pmatrix} \psi_{1,\alpha}(y) \\ \psi_{2,\alpha}(y) \end{pmatrix}$$

kinetic term

$$S_K = \int d^4y\, Tr\, \Psi^\dagger i\gamma^\mu(\partial_\mu + g[A_\mu, .])\Psi = \int d^4y\, Tr\,(\psi_{i,\alpha})^\dagger i(\overline{\sigma}^\mu)^{\dot{\alpha}\beta}(\partial_\mu + g[A_\mu, .])\psi_{i,\beta}$$

mass term

$$S_m = \int d^4y\, Tr\, \psi_{i,\alpha}\varepsilon^{\alpha\beta}\varepsilon^{ij} m\psi_{j,\beta} + h.c. \quad \equiv 0$$

only possible (renormalizable) Yukawa interaction

$$\begin{aligned} S_Y &= \int d^4y\, Tr\, \psi_{i,\alpha}\varepsilon^{\alpha\beta}\varepsilon^{ij}(\sigma_a)^{jk}\phi_a\, \psi_{k,\beta} \,+ h.c. \\ &\sim \int d^4y\, Tr\, \psi_{i,\alpha}\varepsilon^{\alpha\beta}\varepsilon^{ij}\chi(\psi_\beta)_j \,+ h.c. \end{aligned}$$

where

$$\chi(\Psi) = \frac{2}{N}\,\sigma_a\{i\phi_a, \Psi\}$$

... fuzzy chirality operator

no 6D behavior obtained

emergence of 6D: doubling the fermions

to obtain 6D behavior: need 4 Weyl fermions

$$\Psi_{i,r;\alpha}(y) = \begin{pmatrix} \psi_{i,1;\alpha} \\ \psi_{i,2;\alpha} \end{pmatrix} \equiv \begin{pmatrix} \rho_{i,\alpha} \\ \eta_{i,\alpha} \end{pmatrix}$$

transforming under $SU(2) \times SU(2)_R$.

can be interpreted as 6D Dirac fermion on $M^4 \times S^2_N$.

6D Clifford algebra:

$$\Gamma^A = (\Gamma^\mu, \Gamma^a) = (\mathbf{1} \otimes \gamma^\mu, \sigma^a \otimes i\gamma_5) \qquad \text{acting on } \mathbb{C}^2 \otimes \mathbb{C}^4$$

Consider most general renormalizable action respecting $SU(2)_R$ (for simplicity)

Mass term:

$$S_m = \int d^4y\, Tr\, m\psi_{i,r,\alpha}\varepsilon^{\alpha\beta}\varepsilon^{ij}\varepsilon^{rs}\psi_{j,s,\beta} + h.c.$$

Kinetic term

$$S_K = \int d^4y\, Tr\, \Psi^\dagger i\gamma^\mu(\partial_\mu + g[A_\mu, .])\Psi$$

Yukawa coupling

$$\begin{aligned} S_Y &= \int d^4y\, Tr\, \psi_{i,r,\alpha}\varepsilon^{\alpha\beta}\varepsilon^{ij}\varepsilon^{rs} i\gamma_5(\not{D}_{(2)} - 1)\psi_{k,s,\beta} + h.c. \\ \not{D}_{(2)}\Psi &= i\sigma_a[\phi_a^{(N)}, \Psi] + \Psi \qquad \text{... Dirac operator on } S_N^2 \end{aligned}$$

combined action

$$D_K + D_Y = S_{6D} = \int d^4y\, Tr\, \overline{\Psi}_D \left(i\gamma^\mu(\partial_\mu + g[A_\mu, .]) + ai\gamma_5\, \not{D}_{(2)} + m + i\gamma_5\tilde{m}\right)\Psi_D$$

... interpreted as Dirac fermion on $M^4 \times S_N^2$

Effective 4D action and KK tower of fermions.

Consider eigenstates of Dirac operator on S^2_N

$$
\begin{aligned}
D\!\!\!/_{(2)}\psi_{\pm,(n)} &= E_{n,\pm}\psi_{\pm,(n)}, \\
\psi_{\pm,(n)} &\in (2)\otimes(N)\otimes(N) = (2)\otimes((1)\oplus(3)\oplus ... \oplus(2N-1)) \\
&= ((2)\oplus(4)\oplus ... \oplus(2N)) \oplus (\quad (2)\oplus ... \oplus(2N-2)) \\
&=: (\quad \psi_{+,(n)} \quad \oplus \quad \psi_{-,(n)}).
\end{aligned}
$$

Weyl fermions naturally pair up into 4D Dirac fermions,

$$
\Psi_{\pm,D,(n)} = \begin{pmatrix} \rho_{\pm,(n),\alpha} \\ i\,\overline{\eta}^{\dot\alpha}_{\pm,(n)} \end{pmatrix},
$$

provides KK tower of massive Dirac fermions with masses

$$
m_{\pm,D,n} = |E_{n,\pm} + m| \sim \frac{1}{2}n \neq 0
$$

since $E_{n,\pm} \neq 0$ on S^2 : no massless fermions in type I vacuum.

Fermions in type II vacuum:

decompose the spinors according to block structure

$$\Psi = \begin{pmatrix} \Psi^{11} & \Psi^{12} \\ \Psi^{21} & \Psi^{22} \end{pmatrix}$$

diagonal blocks Ψ^{11}, Ψ^{22}: same as before, massive

off-diagonal blocks Ψ^{12}, Ψ^{21} in $(n_1) \otimes (\overline{n}_2)$, feel magnetic monopole $k \neq 0$

$\Longrightarrow$ chiral (would-be) zero modes (index theorem):

$$\begin{aligned} \Psi_i^{12} &\in (2) \otimes (N_1) \otimes (N_2) = (2) \otimes ((1 + |N_2 - N_1|) \oplus (3 + |N_2 - N_1|) \oplus ...) \\ &= \Big((|N_2 - N_1| + 2) \oplus ... \oplus (N_1 + N_2) \Big) \oplus \Big((|N_2 - N_1|) \oplus ... \oplus (N_1 + N_2 - 2) \Big) \\ &=: (\Psi^{12}_{+,(n)} \oplus \Psi^{12}_{-,(n)}) \end{aligned}$$

zero modes: $\Psi^{12}_{(0)} \equiv (|N_2 - N_1|) = (k)$ are chiral on S^2_N:

$$\chi(\Psi^{12}_{(0)}) \approx -1, \quad \chi(\Psi^{21}_{(0)}) \approx 1, \quad \not{D}_{(2)} \psi_{(0)} = E_0 \psi_{(0)} \approx 0$$

Results for type II vacuum:

- 2 sets of 4D "special" fermions

$$\begin{cases} \Psi_{(0)}^{12} \in (n_1) \times (\overline{n}_2), \\ \Psi_{(0)}^{21} \in (n_2) \times (\overline{n}_1) \end{cases} \quad \text{of } SU(n_1) \times SU(n_2), \text{ opposite 4D chirality}$$

  "would-be zero modes"; "mirror fermions"

- However, can acquire a mass term $\int Tr \overline{\Psi}_{(0)}^{12} \Psi_{(0)}^{21}$, induced by renormalization (because 6D theory is non-chiral)

- KK tower of massive states as expected.

6D chiral case

Assume chiral 6D model: impose something like

$$\begin{aligned}\Gamma\Psi &\equiv \gamma_5\chi\Psi = \Psi \\ \gamma_5\chi\Psi &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \chi\rho^{12} \\ -\overline{\chi\eta^{12}} \end{pmatrix} = \begin{pmatrix} \rho^{12} \\ \overline{\eta}^{12} \end{pmatrix}\end{aligned}$$

$\Rightarrow$ only $\rho^{12}, \eta^{12} \in (n_1) \times (\overline{n}_2)$ survive

no way to write down any mass term

$\Rightarrow$ exactly massless chiral 4D fermions, "zero modes" (cp. left-handed quarks)

not clear how to impose 6D chirality constraint on the quantum level

Chirality on the quantum level

Problem: cannot impose $\Gamma = \gamma_5 \chi = 1$ since $\chi^2 \neq 1$, field-dependent

better alternative: note $$\Phi^2 = (\phi_a \sigma^a + \frac{1}{2})^2 \sim \frac{N^2}{4}\,\mathbf{1}$$

$\Rightarrow$ can use modified chirality operator

$$\tilde{\chi}\Psi \equiv \frac{\Phi}{|\Phi|}\Psi \sim \frac{2}{N}\,\Phi\Psi \approx \chi\Psi$$

most natural for constraint

$$\Phi^2 = (\phi_a \sigma^a + \phi_0)^2 = \frac{N^2}{4}\,\mathbf{1}$$

- gives good definition for YM on S^2_N! *(Steinacker, Szabo hep-th/0701041)*
- suggests SUSY, $SU(2N)$ structure: $\Psi = (\psi_{i,r}) = \psi_\alpha \sigma^a$ *(Andrews, Dorey hep-th/0601098)*
- how to impose it on QFT? might add $MTr(\Phi^2 - \frac{N^2}{4})^2, \qquad M \to \infty$
- nevertheless: 6D chirality constraint not clear

Fluxons

solution $X_a = c_a$, $\quad \sum_a c_a^2 = R^2$ ... localized flux tube on S^2_N

Consider type II vacuum with one fluxon:

$$\phi_a = \begin{pmatrix} \alpha_1\, X_a^{(N_1)} \otimes \mathbf{1}_3 & 0 & 0 \\ 0 & \alpha_2\, X_a^{(N_2)} \otimes \mathbf{1}_2 & D_a \\ 0 & -D_a^\dagger & c_a \end{pmatrix}$$

assume furthermore a nontrivial D_a (""electroweak" Higgs")

leads to further breaking $SU(3) \times SU(2) \times U(1) \times U(1) \to SU(3) \times U(1) \times U(1)$

expect zero modes

$$\psi_{eff} = \begin{pmatrix} 0 & \psi_{32} & \psi_{31} \\ \psi_{23} & 0 & \psi_{21} \\ \psi_{13} & \psi_{12} & 0 \end{pmatrix}$$

$\psi_{31} \approx$ right-handed quarks, $\psi_{21} \approx$ left-handed leptons

$\psi_{32} \approx$ left-handed quarks (not quite)

Summary and outlook:

- simple 4D gauge theory dynamically develops fuzzy extra dimensions, behaves like 6D gauge theory on $\mathcal{M}^4 \times S^2_N$
  confirmed by full KK analysis
- is renormalizable large $\mathcal{N}$ gauge theory, no particular fine-tuning ...
  naturally leads to $SU(n_1) \times SU(n_2) \times U(1)$
- can study ideas of compactification within renormalizable framework
  magnetic fluxes arise automatically, leads to ("would-be" ?) zero modes
- chirality on quantum level unclear
- running of parameters $a, \tilde{b}, \tilde{g}, ...$ under RG group
  $\Rightarrow$ *different vacua (geometry, gauge group) at different energy scales*
- many possible generalizations (other fuzzy spaces, etc.)

Monopoles from matrices

consider $U(1)$ YM on $S^2_{\tilde{N}}$ with vacuum $\phi_a = X_a^{(\tilde{N})}$

different solution of e.o.m:

$$\phi_a^{(m)} = \alpha X_a^{(N)} = X_a^{(\tilde{N})} + A_a^{(m)} \qquad m = N - \tilde{N}$$

corresponds to classical gauge field $A^{(cl)} = \frac{m}{2} \frac{1}{1+x_3} (ydx - xdy)$

(except at the south pole, Dirac string)

...magnetic monopole, monopole charge m.