From Matrices to Quantum Geometry

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Geometry and physics without space-time continuum

aim: (toy-?) model for

- quantum theory of all fund. interactions (gravity!)
- pre-geometric
  $\rightarrow$ geometry, gravity "emerge" at low energies
- quantum structure of space-time at L_P

candidates:

- string theory:
  - vast landscape of possible vacua... (??)
- here: Matrix Models
  - related to string theory, more predictive
  - dynamical NC space-time, matrix geometry
  - accessible, novel tools!
- ...

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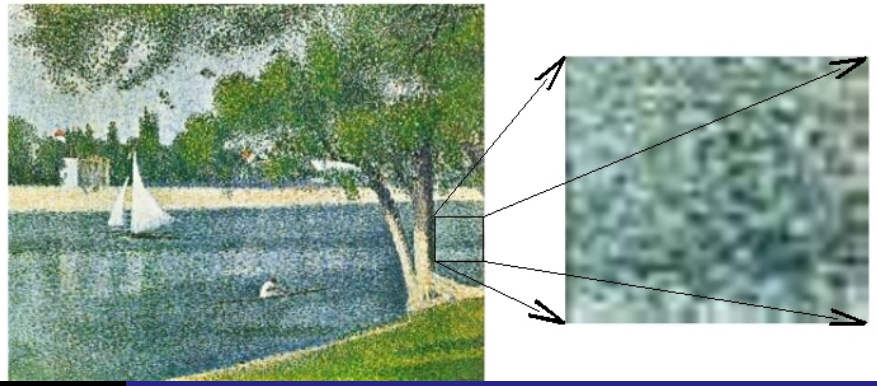
Quantized phase space in quantum mechanics

classical mechanics	quantum mechanics
phase space $\mathbb{R}^2$	"quantized phase space" $\mathbb{R}^2_\hbar$
functions $f(q,p) \in \mathcal{A}$, $\mathcal{A} = \mathcal{C}(\mathbb{R}^2)$...commutative algebra	observables $f(Q,P) \in \mathcal{A}_\hbar$ $\mathcal{A}_\hbar = L(\mathcal{H})$...noncommutative algebra (Heisenberg algebra)
Poisson bracket $\{q,p\} = 1$	$[Q,P] = i\hbar\,\mathbf{1}$ minimal area $= 2\pi\hbar$

quantization map: $\mathcal{Q}: \quad \mathcal{A} \quad \to \quad \mathcal{A}_\hbar$

$$\begin{aligned} \mathcal{Q}(f)\,\mathcal{Q}(g) &= \mathcal{Q}(fg) + O(\hbar) \\ [\mathcal{Q}(f), \mathcal{Q}(g)] &= \mathcal{Q}(i\{f,g\}) + O(\hbar^2) \end{aligned}$$



Quantization of Poisson (symplectic) manifolds

$(\mathcal{M}, \{.,.\})$ ... $2n$-dimensional manifold with Poisson structure

$$\{f,g\} \quad = \quad i\theta^{\mu\nu}(x)\partial_\mu f\, \partial_\nu g$$

quantization map:

$$\mathcal{Q}: \ \mathcal{C}(\mathcal{M}) \quad \to \quad L(\mathcal{H})$$

such that

$$\begin{aligned} \mathcal{Q}(f)\,\mathcal{Q}(g) &= \mathcal{Q}(fg) \ + O(\theta) \\ [\mathcal{Q}(f), \mathcal{Q}(g)] &= \mathcal{Q}(i\{f,g\}) \ + O(\theta^2) \end{aligned}$$

("nice") $\Phi \in L(\mathcal{H}) \cong Mat(\infty, \mathbb{C}) \quad \leftrightarrow \quad$ quantized function on $\mathcal{M}$

here: assume $\mathcal{M}$ compact (torus, sphere, ...)

$\to$ Hilbert space $\mathcal{H} = \mathbb{C}^N$ finite-dimensional, $\mathcal{A}_\theta = Mat(N, \mathbb{C})$

$$(2\pi\hbar)\mathrm{Tr}\,\mathcal{Q}(\phi) \quad \sim \quad \int_{\mathcal{M}} d^2x \ \phi(x) \qquad \text{(Bohr-Sommerfeld)}$$

$$2\pi N = 2\pi \mathrm{Tr}\,\mathbf{1} \quad \sim \quad \int_{\mathcal{M}} \omega \ = \mathrm{Vol}(\mathcal{M})$$

ω ... sympletic (volume) form on $\mathcal{M}$

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NC (matrix) geometry:

same math as Q.M.,

Poisson (sympletic) manifold $\mathcal{M}$ interpreted as physical space(-time)

quantized space(-time) $\mathcal{M}_\theta \leftrightarrow$ NC algebra of functions $\mathcal{A}_\theta$

---

beyond Q.M:

need metric structure, dynamical (from M.M.!)

math background, NC geometry:

Gelfand-Naimark theorem:

commutative C^* - algebra $\mathcal{A}$ with 1 is isomorphic to C^* - algebra of (continuous) functions on compact Haussdorf-space $\mathcal{M}$.

NC geometry: $\mathcal{A} =$ NC (operator-) algebra

$\hat{=}$ "functions on quantum space"

$+$ additional structure (metric, ...)

many possibilities (A. Connes; matrix models; ...)

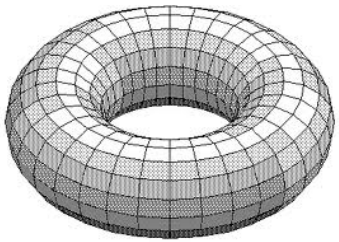
guideline:

physically relevant models of QFT, gauge theory

$\overset{\text{here}}{\Longrightarrow}$ matrix-models

Example: the fuzzy torus T^2_N

The embedded 2D torus



$$x^a : T^2 \hookrightarrow \mathbb{R}^4, \qquad a = 1,2,3,4$$

via

$$\begin{aligned} x^1 + ix^2 &= e^{i\varphi}, \\ x^3 + ix^4 &= e^{i\psi} \end{aligned}$$

so that $x^a = x^a(\varphi, \psi)$ ... functions on T^2,

$$(x^1)^2 + (x^2)^2 = 1, \qquad (x^3)^2 + (x^4)^2 = 1$$

$U(1)_L \times U(1)_R$ symmetry $\varphi \to \varphi + \alpha, \quad \psi \to \psi + \beta$

invariant Poisson structure

$$\{\varphi, \psi\} = \theta, \quad \Leftrightarrow \quad \{e^{i\varphi}, e^{i\psi}\} = -\theta\, e^{i\varphi} e^{i\psi}$$

symplectic form $\omega = \theta^{-1} d\varphi \wedge d\psi$

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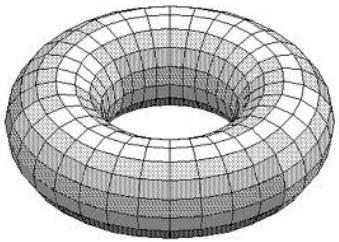

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The fuzzy torus T^2_N: unitary matrices

$$U = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \ddots & & \\ 0 & & \dots & 0 & 1 \\ 1 & 0 & \dots & & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & & & \\ & e^{2\pi i \frac{1}{N}} & & \\ & & e^{2\pi i \frac{2}{N}} & \\ & & & \ddots \\ & & & & e^{2\pi i \frac{N-1}{N}} \end{pmatrix} \text{ satisfy}$$

$$UV = qVU, \qquad U^N = V^N = 1, \qquad q = e^{\frac{2\pi i}{N}}$$

generate $\mathcal{A} = \mathrm{Mat}(\mathrm{N}, \mathbb{C})$ ... quantiz. algebra of functions on T^2_N

$\mathbb{Z}_N \times \mathbb{Z}_N$ action:

$$\begin{array}{rcl} \mathbb{Z}_N \times \mathcal{A} & \to & \mathcal{A} \\ (\omega^k, \phi) & \mapsto & U^k \phi U^{-k} \end{array}$$

similar other $\mathbb{Z}_N$

$\mathcal{A} = \oplus_{n,m=0}^{N-1} U^n V^m$ ... harmonics

quantization map:

$$\begin{aligned} \mathcal{Q}: \quad \mathcal{C}(T^2) &\to \mathcal{A} = \mathrm{Mat}(\mathrm{N}, \mathbb{C}) \\ e^{in\varphi} e^{im\psi} &\mapsto \begin{cases} U^n V^m, & |n|, |m| < N/2 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

satisfies

$$\begin{aligned} \mathcal{Q}(fg) &= \mathcal{Q}(f)\mathcal{Q}(g) + O(\tfrac{1}{N}), \\ \mathcal{Q}(i\{f,g\}) &= [\mathcal{Q}(f), \mathcal{Q}(g)] + O(\tfrac{1}{N^2}) \end{aligned}$$

in particular

$$[U, V] = (q-1)UV \sim \frac{2\pi i}{N} UV$$

$\to$ Poisson structure $\{e^{i\varphi}, e^{i\psi}\} = \frac{2\pi}{N} e^{i\varphi} e^{i\psi}$ on T^2

$$\{\varphi, \psi\} = -\frac{2\pi}{N} \equiv \theta$$

T^2_N ... quantization of $(T^2, N\omega)$

need something like a metric on T^2_N

several possibilities:

- metric encoded in Laplacian $\Delta_g \phi = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} g^{\mu\nu} \partial_\nu \phi)$

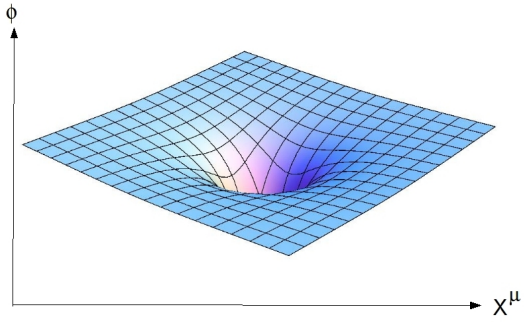
  $\rightarrow$ define Laplace operator (or Dirac operator ... (Connes)) on $\mathcal{A}_\theta$

  can recover (almost ...) metric $g_{\mu\nu}$ from spectrum of Δ_g

  ("can you hear the shape of a drum"?)

- induced by embedding ($\rightarrow$ matrix models!)

$$X^a \sim x^a : \quad \mathcal{M} \hookrightarrow \mathbb{R}^{10}$$



more transparent, work with Poisson manifolds $(\mathcal{M}, \theta^{\mu\nu}, g_{\mu\nu})$

- differential calculus (Madore), ...

metric on T^2_N: encoded in embedding $X^a = \mathcal{Q}(x^a) : \; T^2 \hookrightarrow \mathbb{R}^4$

$$\begin{aligned} U &= X^1 + iX^2 = \mathcal{Q}(x^1 + ix^2) = \mathcal{Q}(e^{i\varphi}), \\ V &= X^3 + iX^4 = \mathcal{Q}(x^3 + ix^4) = \mathcal{Q}(e^{i\psi}) \end{aligned}$$

... quantization of embedding maps

Laplace operator:

$$\begin{aligned} \Delta\phi &= [X^a, [X^b, \phi]]\delta_{ab} \\ &= [U, [U^\dagger, \phi]] + [V, [V^\dagger, \phi]] = 2\phi - U\phi U^\dagger - U^\dagger \phi U - (\% V) \\ \Delta(U^n V^m) &\sim ([n]_q^2 + [m]_q^2)\, U^n V^m \sim (n^2 + m^2)\, U^n V^m \end{aligned}$$

where

$$[n]_q = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} = \frac{\sin(n\pi/N)}{\sin(\pi/N)} \sim n \qquad \text{(“q-number”)}$$

$\mathrm{spec}\Delta \approx \mathrm{spec}\Delta_{T^2}$ below cutoff $\Rightarrow$ geometry = flat torus
UV cutoff $|n| \leq N/2$

The fuzzy sphere

classical S^2 :

$$\left.\begin{array}{rcl} x^a : S^2 & \hookrightarrow & \mathbb{R}^3 \\ x^a x^a & = & 1 \end{array}\right\} \Rightarrow \mathcal{A} = \mathcal{C}^\infty(S^2)$$

fuzzy sphere S^2_N : (Hoppe, Madore)

let $X^a \in \mathrm{Mat}(\mathrm{N}, \mathbb{C})$ ... 3 hermitian matrices

$$\boxed{\begin{array}{rl} [X^a, X^b] & = \frac{i}{\sqrt{C_N}} \varepsilon^{abc} X^c \,, \qquad C_N = \frac{1}{4}(N^2 - 1) \\ X^a X^a & = \mathbf{1}, \end{array}}$$

realized as $X^a = \frac{1}{\sqrt{C_N}} J^a$ ... $N-$ dim irrep of $\mathfrak{su}(2)$ on $\mathbb{C}^N$,

generate $\mathcal{A} \cong \mathrm{Mat}(\mathrm{N}, \mathbb{C})$ ... alg. of functions on S^2_N

$SO(3)$ action:

$$\begin{array}{rcl} \mathfrak{su}(2) \times \mathcal{A} & \to & \mathcal{A} \\ (J^a, \phi) & \mapsto & [X^a, \phi] \end{array}$$

decompose $\mathcal{A} = \mathrm{Mat}(\mathrm{N}, \mathbb{C})$ into irreps of $SO(3)$:

$$\begin{aligned} \mathcal{A} = \mathrm{Mat}(\mathrm{N}, \mathbb{C}) \cong (\mathrm{N}) \otimes (\bar{\mathrm{N}}) &= (1) \oplus (3) \oplus \ldots \oplus (2N-1) \\ &= \{\hat{Y}^0_0\} \oplus \{\hat{Y}^1_m\} \oplus \ldots \oplus \{\hat{Y}^{N-1}_m\}. \end{aligned}$$

... fuzzy spherical harmonics (polynomials in X^a); UV cutoff !

<u>quantization map:</u>

$$\begin{aligned} \mathcal{Q}: \quad \mathcal{C}(S^2) &\to \mathcal{A} = \mathrm{Mat}(\mathrm{N}, \mathbb{C}) \\ Y^l_m &\mapsto \begin{cases} \hat{Y}^l_m, & l < N \\ 0, & l \geq N \end{cases} \end{aligned}$$

satisfies

$$\begin{aligned} \mathcal{Q}(fg) &= \mathcal{Q}(f)\mathcal{Q}(g) + O(\tfrac{1}{N}), \\ \mathcal{Q}(i\{f,g\}) &= [\mathcal{Q}(f), \mathcal{Q}(g)] + O(\tfrac{1}{N^2}) \end{aligned}$$

$SO(3)$-inv. Poisson structure $\{x^a, x^b\} = \frac{2}{N}\varepsilon^{abc} x^c$ on S^2

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S^2_N ... quantization of $(S^2, N\omega)$

metric structure of fuzzy sphere

metric encoded in NC Laplace operator

$$\Delta : \ \mathcal{A} \to \mathcal{A}, \qquad \Delta\phi = [X^a, [X^b, \phi]]\delta_{ab}$$

$$\Rightarrow \qquad \boxed{\Delta \hat{Y}^l_m = \frac{1}{C_N} l(l+1) \hat{Y}^l_m}$$

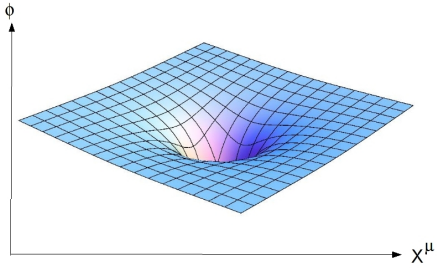
spectrum identical with classical case $\Delta_g \phi = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} g^{\mu\nu} \partial_\nu \phi)$

$\Rightarrow$ effective metric of Δ = round metric on S^2

geometry from matrices:

given a suitable set of matrices $X^a \in Mat(\infty, \mathbb{C}) \cong L(\mathcal{H})$

defines
- algebra $\mathcal{A} \cong \langle f(X^a) \rangle \subset \mathrm{Mat}(\infty, \mathbb{C})$
- quantized embedding $X^a \sim x^a : \mathcal{M} \hookrightarrow \mathbb{R}^{10}$



<u>Lemma:</u>

$$\Delta f(X) := [X_a, [X^a, f(X)]] \sim -e^{\sigma} \Delta_G f(x)$$

... Matrix Laplace- operator, effective metric (H.S. Nucl.Phys. B810 (2009))

$$G^{\mu\nu}(x) = e^{-\sigma} \theta^{\mu\mu'}(x) \theta^{\nu\nu'}(x)\, g_{\mu'\nu'}(x)$$ effective metric (cf. open string m.)

$$g_{\mu\nu}(x) = \partial_\mu x^a \partial_\nu x^b \eta_{ab}$$ induced metric on $\mathcal{M}^4_\theta$ (cf. closed string m.)

$$e^{-2\sigma} = \frac{|\theta^{-1}_{\mu\nu}|}{|g_{\mu\nu}|}$$

classical geometry $\leftrightarrow$ NC geometry

Poisson–manifolds $(\mathcal{M}, \{,\})$ $\mathcal{C}^\infty(\mathcal{M}) = \{f : \mathcal{M} \to \mathbb{C}\}$ $\{x^\mu, x^\nu\} = \theta^{\mu\nu}$ additional geom. structures diff. calculus, metric ... embedded manifolds $x^a : \mathcal{M} \hookrightarrow \mathbb{R}^D$	NC space $(\mathcal{A}, \mathcal{H})$ NC algebra $\mathcal{A}$, rep. on $\mathcal{H}$ z.B. $[X^\mu, X^\nu] = i\theta^{\mu\nu}\mathbf{1}$ NC diff. calculus (A. Connes) Dirac operator $\not{D}$, Laplacian Δ_g matrices X^a, $a = 1, ..., D$
field theory: e.g. $\Delta_g \phi = \lambda\phi$ $\phi \in \mathcal{C}^\infty(\mathcal{M})$	NC field theory: $\Delta\phi = \lambda\phi$, $\phi \in \mathcal{A}$
QFT $\int\limits_{\mathcal{C}(\mathcal{M})} d\phi\, e^{-S(\phi)}$	NC QFT $\int\limits_{\mathcal{A}} d\phi\, e^{-S(\phi)}$
quantum gravity: e.g. (?) $\int\limits_{geometries} dg_{\mu\nu}\, e^{-S_{EH}[g]}$	matrix models NC embedded manif. $\mathcal{M} \subset \mathbb{R}^D$, $\int\limits_{matrices} dX\, e^{-S_{YM}[X]}$

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IKKT (IIB) matrix model

Ishibashi, Kawai, Kitazawa, Tsuchiya 1996

$$S[X] = -Tr\left([X^a, X^b][X^{a'}, X^{b'}]\eta_{aa'}\eta_{bb'} + \bar{\Psi}\gamma_a[X^a, \Psi]\right)$$

$$X^a = X^{a\dagger} \in Mat(N, \mathbb{C}), \qquad a = 0, ..., 9$$
$$N \to \infty$$

gauge symmetry $X^a \to UX^aU^{-1}$, $SO(9,1)$, SUSY

$$\begin{cases} 1)\ \text{nonpert. def. of IIB string theory (on } \mathbb{R}^{10}) \qquad (IKKT) \\ 2)\ \mathcal{N} = 4 \ \text{ SUSY Yang-Mills gauge thy. on } \text{“}noncommutative\text{“}\ \mathbb{R}^4_\theta \end{cases}$$

dynamical NC branes $\mathcal{M} \subset \mathbb{R}^{10}$ ($\to$ 4D gravity ? H.S. 2007 ff)

$\to$ brane-world scenarios

Space-time from matrix models:

e.o.m.: $\delta S = 0 \;\Rightarrow\; [X^a, [X^{a'}, X^{b'}]]\eta_{aa'} = 0$

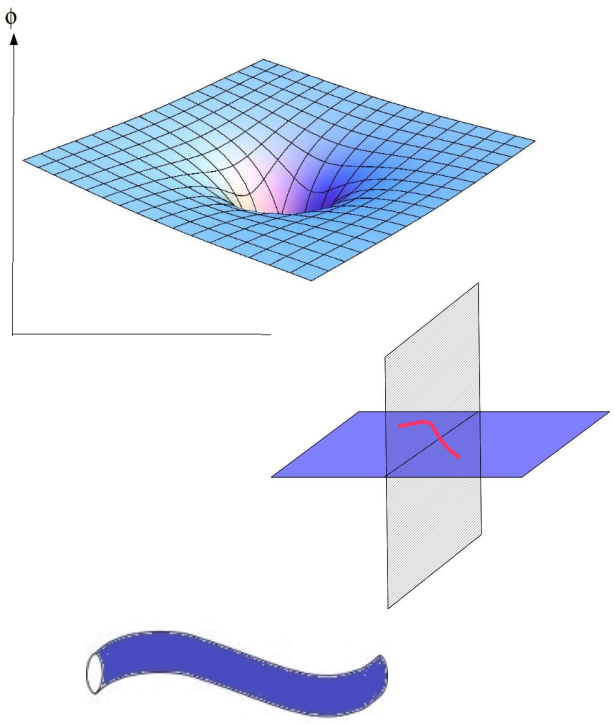
solutions:

- $[X^a, X^b] = i\theta^{ab}\,\mathbf{1}$, "quantum plane" $\mathbb{R}^4_\theta$
- $[X^a, X^b] \sim i\{x^a, x^b\} = i\theta^{ab}(x)$, generic quantum space

  $\rightarrow$ space-time as 3+1-dim. brane solution

  $X^a \sim x^a : \; \mathcal{M}^4 \hookrightarrow \mathbb{R}^{10}$

- intersecting branes, stacks (as in string theory)
- compact extra dim $\mathcal{M}^4 \times T^2$, etc.



basic solution of $[X_a,[X^a,X^b]]=0$: $\quad X^a = \begin{pmatrix} \bar{X}^\mu \\ \bar{X}^i \equiv 0 \end{pmatrix}$

$$[\bar{X}^\mu, \bar{X}^\nu] \;=\; i\bar{\theta}^{\mu\nu}\,\mathbf{1}, \qquad \mu,\nu = 0,\dots,3$$

... Heisenberg algebra $\quad \mathcal{A} = Mat(\infty,\mathbb{C}) \;=$ functions on $(\mathbb{R}^4_\theta, \theta^{\mu\nu})$

$\bar{X}^\mu \in Mat(\infty,\mathbb{C})$ ... coordinate functions on quantum plane $\mathbb{R}^4_\theta$

$$\Delta\bar{X}^\mu \Delta\bar{X}^\nu \geq |\bar{\theta}^{\mu\nu}|$$

quantization map (Weyl):

$$\mathcal{Q}: \;\; \mathcal{C}(\mathbb{R}^4) \;\to\; \mathcal{A}$$

$$f(x) = \int d^4k\, \tilde{f}(k) e^{ik_\mu x^\mu} \;\mapsto\; \int d^4k\, \tilde{f}(k) e^{ik_\mu \bar{X}^\mu} =: F(\bar{X})$$

derivatives:

$$[\bar{X}^\mu, F(\bar{X})] =: \theta^{\mu\nu}\partial_\nu F$$

tangential deformations: gauge fields

$$X^a = \bar{X}^a + A^a = \begin{pmatrix} \bar{X}^\mu \\ 0 \end{pmatrix} + \begin{pmatrix} A^\mu(\bar{X}^\mu) \\ 0 \end{pmatrix}$$

$S = Tr([X^a, X^b][X_a, X_b])$ is gauge-invariant: $X^a \to U^{-1} X^a U$

$\to$ fluctuations $X^\mu = \bar{X}^\mu + \theta^{\mu\nu} A_\nu$ transform as
$A_\mu \to U^{-1} A_\mu U + i U^{-1} \partial_\mu U$ gauge fields!

$$\begin{aligned} [X^\mu, X^\nu] &= i\theta^{\mu\nu} + i\theta^{\mu\mu'}\theta^{\nu\nu'} \left(\partial_{\mu'} A_{\nu'} - \partial_{\nu'} A_{\mu'} + [A_{\mu'}, A_{\nu'}]\right) \\ &= i\theta^{\mu\nu} + i\theta^{\mu\mu'}\theta^{\nu\nu'} F_{\mu'\nu'} \qquad \text{field strength} \end{aligned}$$

$\Rightarrow$ eff. action $\boxed{S = \text{const} + \int d^4x \sqrt{G}\, e^\sigma\, G^{\mu\mu'} G^{\nu\nu'} F_{\mu\nu} F_{\mu'\nu'}}$

tangential perturbations $\to$ gauge fields on $\mathbb{R}^4_\theta$, eff. metric $G^{\mu\nu}$

similarly: transversal deformations $\rightarrow$ scalar fields

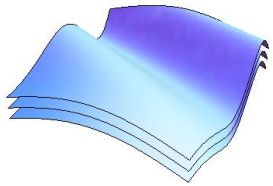
$$X^a = \bar{X}^a + A^a = \begin{pmatrix} \bar{X}^\mu \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \phi^i(\bar{X}^\mu) \end{pmatrix}$$

transversal fluctuations $\rightarrow$ scalar fields on $\mathbb{R}^4_\theta$, eff. metric $G^{\mu\nu}$

nonabelian gauge theory: stack of coincident branes

tangential fluctuation $\rightarrow$ $su(n)$ gauge fields background

$$Y^a = \begin{pmatrix} Y^\mu \\ Y^i \end{pmatrix} = \begin{pmatrix} X^\mu \otimes \mathbf{1}_n \\ \phi^i \otimes \mathbf{1}_n \end{pmatrix}$$



include fluctuations:

$$Y^a = (1 + \mathcal{A}^\rho \partial_\rho) \begin{pmatrix} X^\mu \otimes \mathbf{1}_n \\ \phi^i \otimes \mathbf{1}_n + \Phi^i \end{pmatrix}$$

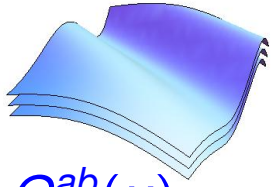
$\Rightarrow$ effective action:

$$\boxed{S_{YM} = \int d^4x \sqrt{G}\, e^\sigma \, tr \langle F, F \rangle_G + 2 \int \eta(x)\, tr\, F \wedge F}$$

(H.S., JHEP 0712:049 (2007), JHEP 0902:044,(2009))

IKKT model on stack of branes $\rightarrow$ $SU(n)$ $\mathcal{N} = 4$ SYM coupled to metric $G^{\mu\nu}(x)$

main results:

- universal effective metric $G^{ab}(x)$ on such branes, dynamical
- *fluctuations* of matrices X^A around stack of branes 

  $\rightarrow$ $SU(n)$ NC Yang-Mills gauge theory coupled to $G^{ab}(x)$
- Poisson structure $\theta^{\mu\nu}$ invisible ($U(1)$ is sterile)

prospects:

- intersecting branes $\rightarrow$ chiral fermions

  A. Chatzistavrakidis, H.S., G. Zoupanos (2011)

  all ingredients for physics ($\rightarrow$ brane-world picture)
- well suited for quantization, predictive

Quantization

$$Z = \int dX^a d\Psi \, e^{-S[X]-S[\Psi]}$$

2 interpretations:

1. on $\mathbb{R}^4_\theta$: NC gauge theory on $\mathbb{R}^4_\theta$, UV/IR mixing in $U(1)$ sector

   almost all models are sick (loops probe UV, too "wild")

   except IKKT model: $\mathcal{N} = 4$ SYM, perturb. finite !(?)

2. on $\mathcal{M}^4 \subset \mathbb{R}^{10}$: $U(1)$ absorbed in $\theta^{\mu\nu}(x)$, $G^{\mu\nu}(x)$
   $\rightarrow$ induced E-H. action

$$S_{eff} \sim \int d^4x \sqrt{|G|} \, (\Lambda^4 + c\Lambda_4^2 R[G] + ...)$$

- IKKT $\rightarrow$ good quantization for theory with gravity! (SUSY)
- 4 noncompact dimensions preferred (higher dim unstable)

can be put on computer (Monte Carlo; Lorentzian) !

can measure effective dimensions Kim, Nishimura, Tsuchiya PRL 108 (2012)

result:

3 out of 9 spatial directions expand, 3+1 dims at late times

towards (emergent) gravity

brane gravity (not bulk gravity); propagation on 4D brane
complicated dynamics, not well understood
Einstein equations not established

several mechanisms:

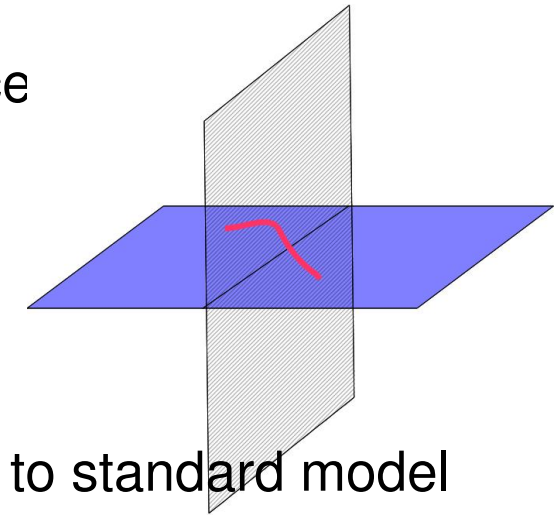
- tang. modes $\rightarrow$ NC $U(1)$ gauge fields $\partial^\mu F_{\mu\nu} = 0 \;\Rightarrow\; \delta R_{\mu\nu} = 0$
  Ricci-flat vacuum perturbations (around $\mathbb{R}^4$) V.Rivelles (2003)
- similar for fluctuations of $M^4 \times \mathcal{K} \subset \mathbb{R}^{10}$
  A.Polychronakos, H.S., J.Zahn (2013)
- $T_{\mu\nu}$ induces perturbation of $R_{\mu\nu}$ in presence of extrinsic curvature
  $\rightarrow$ (Newtonian) gravity (without E-H action !) H.S. (2009,2012)
- quantum effects $\rightarrow$ induced gravity (?)

towards particle physics

intersecting brane solutions

chiral fermions at intersection = 4D space

$$\begin{pmatrix} X^a_{(11)} & \psi_{(12)} \\ \psi_{(21)} & X^a_{(22)} \end{pmatrix}$$

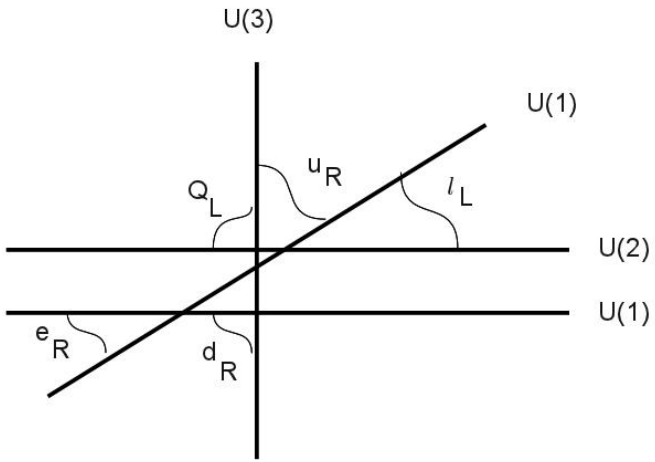


stacks of intersecting branes $\rightarrow$ close to standard model

A. Chatzistavrakidis, H.S., G. Zoupanos JHEP 1109 (2011)

(cf. string theory)
clear-cut, predictive framework

1-loop $\rightarrow$ intersecting branes can form bound system!



Summary, conclusion

- matrix geometry:
  can describe space-time & geometry with (finite-dim.!) matrices

  $\rightarrow$ quantum structure of space(time)
- matrix-models $Tr[X^a, X^b][X^{a'}, X^{b'}]\,\eta_{aa'}\eta_{bb'}$ + fermions

  dynamical NC branes, emergent gauge theory (& gravity ?!)

  background independent, all ingredients for physics
- *not* same as general relativity, but might be close enough (?)
- suitable for quantizing gauge theory & geometry (gravity?)
  (IKKT model, $\mathcal{N} = 4$ SUSY in $D = 4$)

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- D. N. Blaschke and H. Steinacker, “Curvature and Gravity Actions for Matrix Models,” arXiv:1003.4132 [hep-th].*Class. Quant. Grav.* **27**:165010,2010.

higher-order terms, curvature

$$
\begin{aligned}
H^{ab} &:= \tfrac{1}{2}[[X^a, X^c], [X^b, X_c]]_+ \\
T^{ab} &:= H^{ab} - \tfrac{1}{4}\eta^{ab} H, \quad H := H^{ab}\eta_{ab} = [X^c, X^d][X_c, X_d]\,, \\
\Delta X &:= [X^b, [X_b, X]]
\end{aligned}
$$

result:

for 4-dim. $\mathcal{M} \subset \mathbb{R}^D$ with $g_{\mu\nu} = G_{\mu\nu}(Euclidean!)$:

$$
\begin{aligned}
&Tr\left(2T^{ab}\Delta X_a \Delta X_b - T^{ab}\Delta H_{ab}\right) \sim \tfrac{2}{(2\pi)^2}\int d^4x\sqrt{g}\, e^{2\sigma} R \\
&Tr([[X^a, X^c], [X_c, X^b]][X_a, X_b] - 2\Delta X^a \Delta X^a) \\
&\qquad \sim \tfrac{1}{(2\pi)^2}\int d^4x\sqrt{g}\, e^{\sigma}\left(\tfrac{1}{2}e^{-\sigma}\theta^{\mu\eta}\theta^{\rho\alpha}R_{\mu\eta\rho\alpha} - 2R + \partial^\mu\sigma\partial_\mu\sigma\right)
\end{aligned}
$$

(Blaschke, H.S. arXiv:1003.4132)

(cf. Arnlind, Hoppe, Huisken arXiv:1001.2223)

$\Rightarrow$ Einstein-Hilbert- type action for gravity as matrix model
pre-geometric version of (quantum?) gravity, background indep.!