(Spinning) squashed extra dimensions, chiral fermions & hierarchy in $\mathcal{N} = 4$ SYM

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H.S., J. Zahn arXiv:1409.1440,
H.S. arXiv:1411.3139

Outline

- $SU(N)$ $\mathcal{N}=4$ SYM with cubic (flux) terms
- vacuum structure, Higgs effect & emergence of fuzzy sphere
- emergence of squashed (fuzzy) coadjoint orbits of $SU(3)$
- zero modes and chiral fermions
- spinning squashed branes
- would-be zero modes and heavy sector
- stacks of branes
  towards standard model ?

Motivation

- guide towards fundamental physics: simplicity

  established: relativity, electroweak unification, ...

  speculative: GUT models, string theory on $\mathbb{R}^4 \times \mathcal{K}$,...
- both ideas realized within 4D Yang-Mills gauge theory:

  dynamically develops extra dimensions $\mathbb{R}^4 \times \mathcal{K}_N$

  (Higgs mechanism!)

  Madore, Myers, Arkani-Hamed etal, Aschieri-HS-Zoupanos, ...
- aspects of string theory (Kaluza-Klein, intersecting branes ...) realized in controlled 4-D framework
- here: simplest of all 4D gauge theories: $\mathcal{N} = 4$ SYM

$\mathcal{N} = 4$ SYM and squashed fuzzy branes

$\mathcal{N} = 4$ SYM obtained by dimensional reduction from 10D SYM:

$$S_{10} = \int d^{10}x \; tr_N\Big(- \frac{1}{4g^2} F^{AB} F_{AB} + \bar{\Psi}\Gamma^A (i\partial_A + [A_A, .])\Psi \Big)$$

$F_{AB} = \partial_a A_B - \partial_B A_A + i[A_A, A_B]$, Majorana-Weyl fermion Ψ

reduce to $D = 4$, separate $A_A = (A_\mu, \phi_a)$

$$\begin{aligned} S_{\mathcal{N}=4} \quad &= \int d^4x \tfrac{1}{4g^2} tr\Big(- F^{\mu\nu} F_{\mu\nu} - 2D^\mu \Phi^a D_\mu \Phi_a + [\Phi^a, \Phi^b][\Phi_a, \Phi_b] \Big) \\ &\quad + tr\Big(\bar{\Psi}\gamma^\mu (i\partial_\mu + [A_\mu, .])\Psi + \bar{\Psi}\Gamma^a [\Phi_a, \Psi] \Big) \end{aligned}$$

- 6 scalar fields ϕ^a, $a \in \mathcal{I} = \{1, 2, 4, 5, 6, 7\}$
- $D_\mu \phi^a = (\partial_\mu + i[A_\mu, .])\phi^a$
- $(\gamma^\mu, \Gamma^a) = \Gamma^A$ ... 10D Clifford generators, $\Psi \to$ 4 Weyl fermions

symmetries: global $SO(6)$, $SU(N)$ gauge symmetry, $\mathcal{N} = 4$ SUSY

simplest, most symmetric 4D gauge theory, UV finite

$\mathcal{N} = 4$ SYM and squashed fuzzy branes

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$\mathcal{N} = 4$ SYM beautiful but too "round" for physics

possibilities to obtain structure:

- spontaneous symmetry breaking (SSB): still no "interesting" (chiral) low-energy physics
- consider rotating background H.S. arXiv:1411.3139
- add soft susy breaking terms to potential

H.S., J. Zahn arXiv:1409.1440

scalar potential:

$$V[\phi] = -\frac{1}{4g^2} tr_N \Big([\Phi^a, \Phi^b][\Phi_a, \Phi_b] + 2i f_{abc} \Phi^a \Phi^b \Phi^c \Big)$$

f_{abc} ... tot. antisymm., extra flux term added by hand (soft SUSY breaking)

$V[\phi]$ bounded from below

dimensionless fields $\Phi_a = mY_a$

saddle points $\frac{\delta V}{\delta \phi} = 0$:

$$\Box_Y Y^a = \frac{3}{4} i f_{abc}[Y^b, Y^c] \,, \qquad \Box_Y = [Y_a, [Y^a, .]]$$

fluctuations around background solution $\bar{Y}_a$

$$Y^a = \bar{Y}^a + \varphi^a$$

expand potential up to quadratic terms in fluctuations:

$$V[Y] = V[\bar{Y}] + \frac{1}{2}\mathrm{Tr}\Big(\varphi_a \Box \varphi^a + 2[\varphi_a, \varphi_b]\big([\bar{Y}^a, \bar{Y}^b] + \frac{3}{4} i f_{abc} \bar{Y}^c\big) + f^2 + O(Y^3)\Big)$$

(here $f = i[Y^a, \varphi_a]$ ... cf. gauge-fixing function)

eom for fluctuations φ_a:

$$\Big(\Box \delta^a_b + 2[([Y^a, Y^b] + \frac{3}{4} i f_{abc} Y^c), .] - [Y^a, [Y^b, .]]\Big)\varphi_b = 0 \,.$$

warm-up: the fuzzy sphere solution

Hoppe, Madore

$f_{abc} = \epsilon^{123}_{abc}$ ... $SU(2)$ structure constants

$\rightarrow$ fuzzy sphere solution S^2_N

$$Y_a = cJ_a^{(N)}, \quad a = 1,2,3 \qquad ...N - \text{dim. generators of} \quad SU(2)$$

(note: $\Box_Y = c^2[J_a,[J_a,.]]$ ... Casimir!)

flux term $tr(f\,YYY)$ preserves $SO(3)_{123} \times SO(3)_{456} \subset SO(6)_R$

S^2_N breaks gauge inv. and global $SO(3)_{123}$ spontaneously

$SO(3)_{123}$ acts on fluctuations φ_a

$$\begin{aligned} \mathfrak{su}(2) \times \varphi &\rightarrow \varphi \\ (J_a, \phi) &\mapsto [J_a^{(N)}, \phi] \end{aligned}$$

expand φ_a into harmonics

$$\begin{aligned} \varphi_a \in \mathrm{Mat}(N,\mathbb{C}) \cong (N) \otimes (\bar{N}) &= (1) \oplus (3) \oplus ... \oplus (2N-1) \\ &= \{\hat{Y}^0_0\} \oplus \{\hat{Y}^1_m\} \oplus ... \oplus \{\hat{Y}^{N-1}_m\}. \end{aligned}$$

... fuzzy spherical harmonics (=polynomials in Y_a);

observe:
decomposition of $Mat(N,\mathbb{C})$ into harmonics is precisely decomposition of $Pol_N(S^2)$ into spherical harmonics, up to cutoff

quantization map:

$$\begin{array}{rccl} \mathcal{Q}: & \mathcal{C}(S^2) & \to & \mathrm{Mat}(N,\mathbb{C}) \cong Fun_N(S^2) \\ & Y^l_m & \mapsto & \begin{cases} \hat{Y}^l_m, & l < N \\ 0, & l \geq N \end{cases} \end{array}$$

gives quantization of S^2 w.r.t. canonical symplectic form

in particular: Y_a is quantization of embedding map

$$\boxed{Y_a \sim y_a : \quad S^2 \hookrightarrow \mathbb{R}^3}$$

note

$$\boxed{\begin{array}{rcl} [y_a, y_b] & = & i\Lambda_N \, \varepsilon_{abc} \, y_c, \\ y_1^2 + y_2^2 + y_3^2 & = & 1. \end{array}} \quad \dots \quad S^2_N$$

$\Lambda_N = \frac{2}{\sqrt{N^2-1}} \sim \frac{2}{N}$ ... NC parameter, analog of $\hbar$

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back to $\mathcal{N} = 4$ SYM:

S^2_N - vacuum, SSB $\rightarrow$ Higgs effect $\rightarrow$ massive gauge bosons

$$D_\mu Y_a = (\partial_\mu + i[A_\mu, .])\, Y_a$$

decompose

$$A_\mu(x) = \sum_{l,m} A_\mu^{(lm)}(x)\, \hat{Y}^l_m \equiv A_\mu(x,y)$$

kinetic term $tr\left(D_\mu \phi_a D^\mu \phi^a\right)$ gives mass term,

$$\int tr(D_\mu \phi_a)^\dagger D_\mu \phi_a = \int tr(\partial_\mu \phi_a^\dagger \partial_\mu \phi_a + \sum_{l,m} m_l^2\, A^\dagger_{\mu,(lm)} A^\mu_{(lm)}) + S_{int}.$$

$\Rightarrow$ tower of massive KK modes $A_{\mu(lm)}(x)$, mass $m^2_{(l)} \sim l(l+1)$

$\mathfrak{su}(N)$ -valued fields $\rightarrow$ functions on $\mathbb{R}^4 \times S^2_N$
$\rightarrow$ Yang-Mills on $\mathbb{R}^4 \times S^2_N$!
mechanism at classical level (weak coupling), not holographic
Andrews, Dorey hep-th/0505107,
Aschieri, Grammatikopoulos,HS,Zoupanos hep-th/0606021

squashed fuzzy extra dimensions

problem: above backgrounds don't give chiral low-energy theory
(S^2 is not "space-filling" in transverse $\mathbb{R}^6$)
$\exists$ higher-dim fuzzy spaces, require higher-dim embedding

solution: squashed embedding !!
projected 4- and 6-dimensional coadjoint orbits of $SU(3)$

squashed fuzzy $\mathcal{C}[\mu]$ solutions H.S., J. Zahn arXiv:1409.1440

add flux term $\mathrm{tr}(f^{abc} Y_a Y_b Y_c)$ with

$f_{abc} = -\frac{8}{3} c_{abc}$ ... structure constants of $\mathfrak{su}(3)$
without Cartan generators

$\rightarrow$ solutions

$$Y_a = T_a^{(\Lambda)}, \qquad a = 1, 2, 4, 5, 6, 7$$

... generators of $\mathfrak{su}(3)$ (without Cartan generators T_3, T_8) on irrep $\mathcal{H}_\mu$

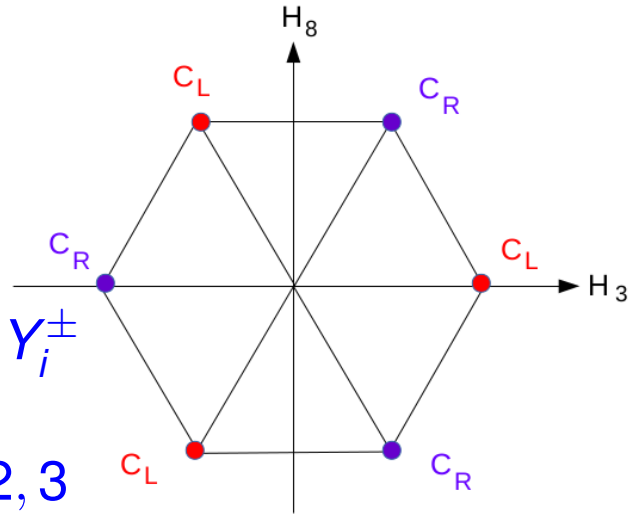
flux term $\mathrm{tr}(f^{abc} Y_a Y_b Y_c)$, breaks $SO(6) \rightarrow U(1)_3 \times U(1)_8 \subset SU(3)$

commutation relations of $\mathfrak{su}(3)$: $\boxed{[T_a, T_b] = if_{abc}T_c}$

Cartan generators $H_3 \equiv T_3,\ H_8 \equiv T_8$ commute $[H_3, H_8] = 0$
ladder operators

$$\begin{aligned} Y_1^{\pm} &= \tfrac{1}{2}(T_4 \pm iT_5), \\ Y_2^{\pm} &= \tfrac{1}{2}(T_6 \mp iT_7), \\ Y_3^{\pm} &= \tfrac{1}{2}(T_1 \pm iT_2) \end{aligned}$$

corresponding to 3 roots $\alpha_1^{\pm}, \alpha_2^{\pm}, \alpha_3^{\pm}$



satisfy

$$[H_\beta, Y_i^{\pm}] = \pm\alpha_i(H_\beta)Y_i^{\pm}$$

and

$$\begin{aligned} [Y_i^+, Y_i^-] &= H_{\alpha_i}, \quad i = 1,2,3 \\ [Y_1^+, Y_2^+] &= Y_3^+ \\ [Y_1^+, Y_3^-] &= -Y_2^- \\ [Y_2^+, Y_3^-] &= Y_1^- \\ [Y_1^+, Y_2^-] &= [Y_2^+, Y_3^+] = [Y_1^+, Y_3^+] = 0 \end{aligned}$$

eom: $\Box_Y Y_a = \frac{3}{4} i f_{abc}[Y_b, Y_c]$ follows from

$$\Box_Y Y_a = 8 Y_a, \qquad i c_{acb}[Y_a, Y_c] = -4 Y_b$$

include fluctuations:

$$\phi_a = Y_a + \varphi_a$$

fluctuation modes $\varphi_a \in End(\mathcal{H}_\mu)$ governed by

$$\Big(\Box \delta^a_b + 2[([Y_a, Y_b] + \frac{3}{4} i f_{abc} Y_c), .] - [Y_a, [Y_b, .]]\Big)\varphi_b = 0 \ .$$

background $Y_a = T_a^{(\Lambda)}$ breaks gauge invariance,

$\rightarrow$ $\dim \mathcal{H}^2 - 1$ trivial zero modes (gauge trafos),

eaten by massive gauge field modes $A_\mu(x)$: Higgs effect

massive gauge bosons & KK modes

kinetic term $tr\left(D_\mu Y_a D^\mu Y^a\right)$ gives mass term,

$$\int \mathrm{tr}(D_\mu Y_a)^\dagger D_\mu Y_a = \int \mathrm{tr}(\partial_\mu Y_a^\dagger \partial_\mu Y_a + \sum_{\Lambda,M} m^2_{\Lambda,M} A^\dagger_{\mu,(\Lambda M)} A^\mu_{(\Lambda,M)}) + S_{int}$$

using $D_\mu \phi_a = (\partial_\mu + i[A_\mu, .])\phi_a$ and decomposition

$$A_\mu(x) = \sum A_\mu^{(M,\Lambda)}(x)\, \hat{Y}^\Lambda_M$$

into eigenmodes of

$$\Box_Y \hat{Y}^\Lambda_M = m^2_{(\Lambda M)} \hat{Y}^\Lambda_M$$

$\Rightarrow$ tower of massive KK modes $A_{\mu,(\Lambda M)}(x)$, mass given by

$$m^2_{(\Lambda M)} \sim (\langle \Lambda + \rho, \Lambda + \rho \rangle - \langle \rho, \rho \rangle - \langle M, M \rangle) \ > 0$$

(no massless gauge modes)

internal geometry: squashed fuzzy coadjoint orbits

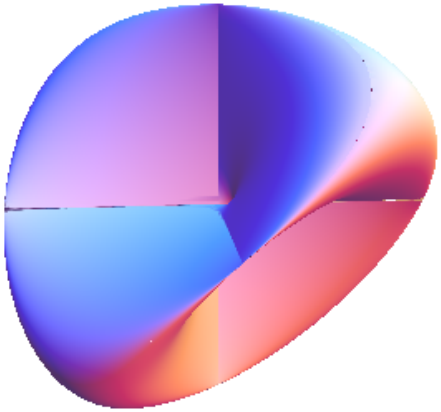
claim: $Y_a = T_a^{(\Lambda)}$ ... coadjoint $SU(3)$ orbits projected along Cartans

$$\begin{array}{ccc} \mathcal{C}[\mu] \hookrightarrow \mathbb{R}^8 & \overset{\Pi}{\to} & \mathbb{R}^6 \\ (y^a)_{a=1,\dots,8} & \mapsto & (y^a)_{a=1,2,4,5,6,7} \end{array}$$

4- or 6-dimensional variety, self-intersecting embedding in $\mathbb{R}^6$
multiple covering at origin, spanning all 6 directions

H.S., J. Zahn arxiv:1409.1440

e.g:
3-dimensional section of squashed $\mathbb{C}P^2$ through $y_2 = y_5 = y_7 = 0$:



triple self-intersection at origin

fuzzy $SU(3)$ branes

classical coadjoint orbits of $SU(3)$

$$\mathcal{C}[\mu] = \{p = g^{-1} H_\mu g;\ g \in SU(3)\} \ \cong\ SU(3)/_{\mathcal{K}} \ \subset \mathfrak{su}(3) \cong \mathbb{R}^8$$

$\mathcal{K}$ ... stabilizer group of $H_\mu \in \mathfrak{g}_0 \subset \mathfrak{g} = \mathfrak{su}(3)$ ($\mu \in \mathfrak{g}_0^*$ ... weight)

- H_μ 3 EV $\rightarrow$ $\mathcal{C}[\mu] \cong SU(3)/_{U(1)\times U(1)}$ ... 6-dim.
- H_μ 2 EV $\rightarrow$ $\mathcal{C}[\mu] \cong SU(3)/_{SU(2)\times U(1)} \ \cong\ \mathbb{C}P^2$ ... 4-dim.

parametrize $\mathcal{C}[\mu]$:

$$p = y^a \lambda_a \quad \in \mathcal{C}[\mu], \qquad a = 1, ..., 8$$

$$\boxed{y^a : \quad \mathcal{C}[\mu] \hookrightarrow \mathbb{R}^8 \ \cong\ \mathfrak{su}(3)}$$

quantized (fuzzy) coadjoint orbits:

$\mathcal{C}[\mu]$ is symplectic space (Kirillov-Kostant) $\rightarrow$ quantize it:

$\mu = n_1\Lambda_1 + n_2\Lambda_2$ ... dominant integral weight

$\mathcal{H}_\mu$ ... corresp. highest weight irrep

fuzzy $\mathcal{C}_N[\mu]$:

$$Y^a = \pi_\mu(T^a)$$

generate matrix algebra

$$\mathcal{A}_N = End(\mathcal{H}_\mu)$$

can show (in general):

decomposition into harmonics of $SU(3)$ agrees with classical harmonic analysis,

$$Pol(\mathcal{C}[\mu]) \cong \oplus_\Lambda m_\Lambda \mathcal{H}_\Lambda \cong \mathcal{H}_\mu \otimes \mathcal{H}_\mu^* = End(\mathcal{H}_\mu) = \mathcal{A}_N$$

up to cutoff Λ_{max}

quantization map

$$\begin{aligned} \mathcal{Q}: \quad Pol(\mathcal{C}[\mu]) &\to \mathcal{A}_N \\ Y^\Lambda_M &\mapsto \hat{Y}^\Lambda_M \end{aligned}$$

respects $SU(3)$, truncated at cutoff Λ_{max}.

$Y^a = \mathcal{Q}(y^a) = \pi_\mu(T^a)$ interpreted as quantized embedding functions

$$Y^a \sim y^a : \quad \mathcal{C}[\mu] \hookrightarrow \mathbb{R}^8.$$

commutation relations:

$$[Y^a, Y^b] = i\, c^{ab}_c\, Y^c$$

... quantizes $\mathfrak{su}(3)$ - invariant Poisson structure (Kirillov-Kostant)

$$\{y^a, y^b\} = c^{ab}_c\, y^c$$

Fluctuation modes in $\mathcal{N} = 4$ SYM

all fluctuations (scalar, gauge fields) take values in

$\mathfrak{su}(N) \subset \mathfrak{u}(N) = End(\mathcal{H}_\mu) \cong$ functions on $\mathbb{R}^4 \times \mathcal{C}_N[\mu]$

behaves like 8- or 10-dim gauge theory below Λ_{UV}

<u>scalar fluctuations</u>: $\phi_a = Y_a + \varphi_a$ governed by

$$\Big(\Box\delta^a_b + 2[([Y_a, Y_b] + \frac{3}{4} i f_{abc} Y_c), .] - [Y_a, [Y_b, .]]\Big)\varphi_b = 0 \ .$$

<u>can show</u>:

- respects decomposition into $SU(3)$ multiplets

$$End(\mathcal{H}_\mu) = \bigoplus_\Lambda m_\Lambda \, \mathcal{H}_\Lambda$$

- no negative modes (!!)
- $\exists$ zero modes (= flat deformations)

regular zero modes:

$\varphi^{(\Lambda+\rho)} = \lambda_\rho \otimes \varphi_\Lambda \quad \in (8) \otimes \mathcal{H}_\Lambda,$ φ_Λ...highest weight vector in $\mathcal{H}_\Lambda$

$\rho = \alpha_1 + \alpha_2 = \alpha_3$ ... Weyl vector

include images under Weyl group $\mathcal{W}$

6 zero modes for each $\mathcal{H}_\Lambda$

e.g. $\varphi_\Lambda = |\Omega\mu\rangle\langle\mu| \quad , \ |\mu\rangle$ ... coherent states,
string between coincident sheets of $\mathcal{C}[\mu]$ at/near origin

exceptional zero modes:

3+3 additional zero modes for $\varphi \in \mathcal{H}_\Lambda$ with $\Lambda = (n, 1)$ and $\Lambda = (1, n)$

fermions

Dirac operator on squashed $\mathcal{C}_{\mathcal{N}}[\mu]$:

$$\not{D}_{(6)}\Psi = \sum_{a\in\mathcal{I}} \Delta_a[Y_a,\Psi] \quad = 2\sum_{i=1}^{3}\left(\gamma_i[Y_i^+,.] + \gamma_i^\dagger[Y_1^-,.]\right)$$

spinorial ladder operators

$$\begin{aligned} 2\gamma_1 &= \Delta_4 - i\Delta_5, & 2\gamma_1^\dagger &= \Delta_4 + i\Delta_5, \\ 2\gamma_2 &= \Delta_6 + i\Delta_7, & 2\gamma_2^\dagger &= \Delta_6 - i\Delta_7, \\ 2\gamma_3 &= \Delta_1 - i\Delta_2, & 2\gamma_3^\dagger &= \Delta_1 + i\Delta_2, \end{aligned}$$

satisfy $\{\gamma_i,\gamma_j^\dagger\} = \delta_{ij}$ (Δ_a... internal 6D Clifford alg.)

zero modes:

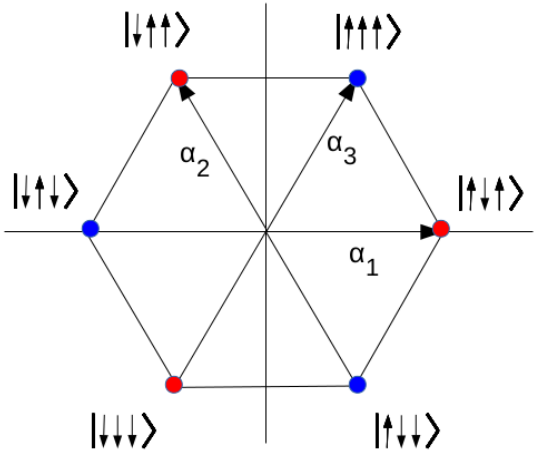
$$\not{D}_{(6)}\Psi_\Lambda = 0, \qquad \Psi_\Lambda = |\uparrow\uparrow\uparrow\rangle \otimes v_\Lambda$$

analogous zero modes $\Psi_{\Lambda'}$ by Weyl group:

$$\Psi_{w\Lambda} = |s_1,s_2,s_3\rangle w\cdot v_\Lambda, \qquad |s_1,s_2,s_3\rangle = \omega_1...\omega_k|\uparrow\uparrow\uparrow\rangle$$

for each extremal weight Λ' in $\mathcal{H}_\Lambda$ (strings connecting branes!)

chiral fermions



well-defined chirality

$$\Gamma^{(6)}\Psi_{w\Lambda} = (-1)^{|w|}\Psi_{w\Lambda} \ .$$

impose 10 Majorana-Weyl condition $\rightarrow$ four-dimensional spinors $\psi_\pm$

$$\Psi = \psi_+ \otimes \chi_+ + \psi_- \otimes \chi_-,$$

$$\not{D}_{(4)}\psi_\pm = 0, \qquad \gamma_5\psi_\pm = \pm\psi_\pm, \qquad \psi_\pm^C = \psi_\mp.$$

Weyl or Majorana spinor on $\mathbb{R}^4$, 3 generations (Weyl rotations $\frac{2\pi}{3}$)

spinning squashed branes

$\mathcal{C}_N[\mu]$ branes can be stabilized by rotation in $\mathcal{N} = 4$ SYM, no flux!

H.S, arXiv:1411.3139

complex scalar fields $\leftrightarrow$ roots $\alpha_i^\pm$ of $\mathfrak{su}(3)$

$$\begin{aligned} Y_1^\pm &= \tfrac{1}{2}(Y_4 \pm iY_5) \equiv Y_{\pm\alpha_1}, \\ Y_2^\pm &= \tfrac{1}{2}(Y_6 \mp iY_7) \equiv Y_{\pm\alpha_2}, \\ Y_3^\pm &= \tfrac{1}{2}(Y_1 \pm iY_2) \equiv Y_{\pm\alpha_3}. \end{aligned}$$

ansatz:

$$\boxed{Y_i^\pm = r_i e^{\pm i\omega_i x}\, \pi_\mu(T_i^\pm)}$$

$\omega_i = \omega_{i,\mu}$ ... 3 time-like vectors ("frequencies")

(cf. circularly polarized plane wave solutions)

have

$$\begin{aligned}\Box_Y Y_1^\pm &= 2(2r_1^2 + r_2^2 + r_3^2)\, Y_1^\pm, \\ \Box_Y Y_2^\pm &= 2(r_1^2 + 2r_2^2 + r_3^2)\, Y_2^\pm, \\ \Box_Y Y_3^\pm &= 2(r_1^2 + r_2^2 + 2r_3^2)\, Y_3^\pm\end{aligned}$$

... gives solution provided

$$\begin{aligned}\omega_1^2 &= -2m^2(2r_1^2 + r_2^2 + r_3^2) \\ \omega_2^2 &= -2m^2(r_1^2 + 2r_2^2 + r_3^2) \\ \omega_3^2 &= -2m^2(r_1^2 + r_2^2 + 2r_3^2)\end{aligned}$$

or

$$\begin{pmatrix} r_1^2 \\ r_2^2 \\ r_3^2 \end{pmatrix} = \frac{1}{8m^2} \begin{pmatrix} -3\omega_1^2 + \omega_2^2 + \omega_3^2 \\ \omega_1^2 - 3\omega_2^2 + \omega_3^2 \\ \omega_1^2 + \omega_2^2 - 3\omega_3^2 \end{pmatrix}$$

typically: spinning BG $\rightarrow$ Lorentz-breaking, bad dispersion relation

here: $\exists$ sector of zero modes with Lorentz-invariant kinematics

fermionic zero modes:

Dirac operator on spinning squashed $\mathcal{C}_{\mathcal{N}}[\mu]$:

$$\not{D}_{(6)}\Psi = \sum_{a\in\mathcal{I}} \Delta_a[Y_a,\Psi] \quad = 2\sum_{i=1}^{3} r_i\Big(\Delta_i^- e^{i\omega_i x}[T_i^+,.] + \Delta_i^+ e^{-i\omega_i x}[T_i^-,.]\Big)$$

zero modes as above:

$$\not{D}_{(6)}\Psi_\Lambda = 0, \qquad \Psi_\Lambda = |\uparrow\uparrow\uparrow\rangle \otimes v_\Lambda$$

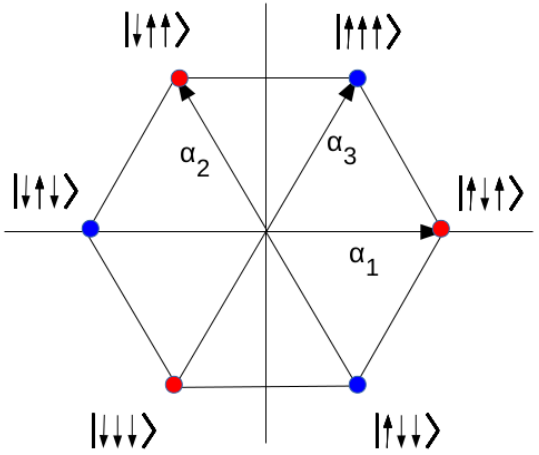
rotation $e^{i\omega_i x}$ drops out!
analogous zero modes $\Psi_{\Lambda'}$ via Weyl group:

$$\Psi_{w\Lambda} = |s_1,s_2,s_3\rangle w\cdot v_\Lambda, \qquad |s_1,s_2,s_3\rangle = \omega_1 ... \omega_k|\uparrow\uparrow\uparrow\rangle$$

for each extremal weight Λ' in $\mathcal{H}_\Lambda$ (strings connecting branes)

remaining "massive" modes feel rotation,

chiral fermions



well-defined chirality

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impose 10 Majorana-Weyl condition → four-dimensional spinors $\psi_\pm$

$$\Psi = \psi_+ \otimes \chi_+ + \psi_- \otimes \chi_-,$$

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Weyl or Majorana spinor on $\mathbb{R}^4$, 3 generations (Weyl rotations $\frac{2\pi}{3}$)

gauge bosons

KK modes determined by

$$\Box_Y = [Y_\alpha, [Y^\alpha, .]]$$

rotation drops out, same KK tower of massive gauge modes
Lorentz-invariant dispersion relation

particularly interesting "chiral" modes:

$$A_\mu^{L/R} \sim \chi^{L/R}$$

... act as ± 1 on weight states $C_{L/R}$ of fermionic zero modes,
according to their chirality

$\rightarrow$ signature of a "chiral gauge theory"

fermions with different chirality transform differently under
(spont. broken) gauge fields (cf. standard model in broken phase)

total index = 0

scalar modes

$Y_i^\pm = r_i e^{\pm i\omega_i x} T_i^\pm + \phi_i^\pm$ governed by

$$\Big(\frac{\Box_4}{m^2} + \Box_Y + 2(\not{D}_{\rm mix} + \not{D}_{\rm diag})\Big)\phi = 0,$$

with

$$\begin{aligned}(\not{D}_{\rm mix}\phi)_\alpha &= \pm 2 \sum_{\beta\neq\alpha} e^{i(\omega_a-\omega_\beta-\omega_{\alpha-\beta})x} r_\alpha r_\beta r_\gamma^{-1}[Y_{\alpha-\beta},\phi_\beta] \\ (\not{D}_{\rm diag}\phi)_\alpha &= 2r_\alpha[H_\alpha,\phi_\alpha] \qquad \text{(no sum)}.\end{aligned}$$

- <u>zero modes</u>:

$$\not{D}_{\rm mix}\phi^{(0)} = 0$$

... oblivious to rotation!! Lorentz-invariant

given by extremal weight states:

$$T_\alpha \triangleright \phi^{(0)}_\beta = [T_\alpha, \phi^{(0)}_\beta] = 0 \quad \text{for} \quad \alpha+\beta \in \mathcal{I}$$

6 zero modes in each $\mathcal{H}_\Lambda \subset End(\mathcal{H}_\mu)$,

$$\phi^{(0)}_{-\alpha} = |w_\alpha\Lambda,\Lambda\rangle, \qquad w_\alpha \in \mathcal{W}$$

masses of $\phi_\alpha^{(0)}$:

$$M_\alpha = 0 \qquad \text{if } \omega_i = \omega$$

in general:

$$\begin{pmatrix} M_{\alpha_3} \\ M_{-\alpha_2} \\ M_{-\alpha_1} \end{pmatrix} = 2 \begin{pmatrix} n_1 & n_2 & n_3 \\ n_2 & n_3 & n_1 \\ n_3 & n_1 & n_2 \end{pmatrix} \begin{pmatrix} r_1^2 \\ r_2^2 \\ r_3^2 \end{pmatrix}$$

sum rule

$$M^\phi_{-\alpha_1} + M^\phi_{-\alpha_2} + M^\phi_{\alpha_3} = 0$$

- "heavy" modes:

  ... generic, couple to time-dependent background, break Lorentz-invariance

  can be organized in terms of generalized translations

$$x^\mu \to x^\mu + c^\mu, \qquad Y_\alpha \to (\exp(i\omega_i c \tau_i) Y)_\alpha$$

  assume: no resonances (to be clarified)

second-stage SSB for $\omega_i \neq \omega_j$

$\exists$ negative modes for would-be zero modes $\phi_\alpha^{(0)}$
stabilized by self-interactions

$$V_{\text{int}}(\phi) = \frac{m^4}{g^4}\text{tr}\Big(Y_\alpha \Box_\phi \phi^\alpha + \frac{1}{4}[\phi_\alpha, \phi_\beta][\phi^\alpha, \phi^\beta]\Big)$$

eom

$$\big(\frac{\Box_4}{m^2} + \Box_Y + \Box_\phi + 2\not{D}^\phi_{ad}\big) Y_\alpha + \big(\frac{\Box_4}{m^2} + \Box_Y + \Box_\phi + 2\not{D}^Y_{ad}\big)\phi_\alpha = 0$$

ϕ_α decouple from rotating background Y_α if

$$\not{D}^\phi_{\text{mix}} Y_\alpha \equiv \sum_{\beta\neq\alpha}[[\phi_\alpha, \phi^\beta], Y_\beta] = 0$$

$\rightarrow$ 2 equations

$$\begin{aligned}\big(\tfrac{\Box_4}{m^2} + \Box_Y + \Box_\phi + 2\not{D}^\phi_{\text{diag}}\big) Y_\alpha &= 0\\ \big(\tfrac{\Box_4}{m^2} + \Box_\phi + \Box_Y + 2\not{D}^Y_{\text{diag}}\big)\phi_\alpha &= 0\end{aligned}$$

nontrivial solutions for ansatz $\phi_\alpha = \varphi_\alpha T_{-\alpha} \Rightarrow$ eom:

$$\begin{pmatrix} 2 & 1 & 1 & -2 & 1 & 1 \\ 1 & 2 & 1 & 1 & -2 & 1 \\ 1 & 1 & 2 & 1 & 1 & -2 \\ -2 & 1 & 1 & 2 & 1 & 1 \\ 1 & -2 & 1 & 1 & 2 & 1 \\ 1 & 1 & -2 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} r_1^2 \\ r_2^2 \\ r_3^2 \\ \varphi_1^2 \\ \varphi_2^2 \\ \varphi_3^2 \end{pmatrix} = -\frac{1}{2m^2} \begin{pmatrix} \omega_1^2 \\ \omega_1^2 \\ \omega_1^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\exists$ solution with $\varphi_3 \neq 0$ and $\varphi_1 = 0 = \varphi_2$:

$$\begin{pmatrix} r_1^2 \\ r_2^2 \\ r_3^2 \\ \varphi_3^2 \end{pmatrix} = \frac{1}{8m^2} \begin{pmatrix} -2\omega_1^2 + 2\omega_2^2 - \omega_3^2 \\ 2\omega_1^2 - 2\omega_2^2 - \omega_3^2 \\ -\omega_1^2 - \omega_2^2 + \omega_3^2 \\ -\omega_1^2 - \omega_2^2 + 2\omega_3^2 \end{pmatrix}$$

negative mass $M_3^\phi < 0$ leads to nontrivial VEV

$$m^2 \varphi_3^2 = \frac{1}{8}(2\omega_3^2 - \omega_1^2 - \omega_2^2) \approx \frac{\Delta\omega_i^2}{4} > 0$$

stable hierarchy!

towards interesting physics:

stacks of squashed $\mathcal{D}_i = \mathcal{C}[\mu_i]$ branes (=reducible rep. of $\mathfrak{su}(3)$)

$$Y^a = \begin{pmatrix} Y^a_{\mu_1} & \\ & Y^a_{\mu_2} \end{pmatrix}$$

off-diagonal fermions

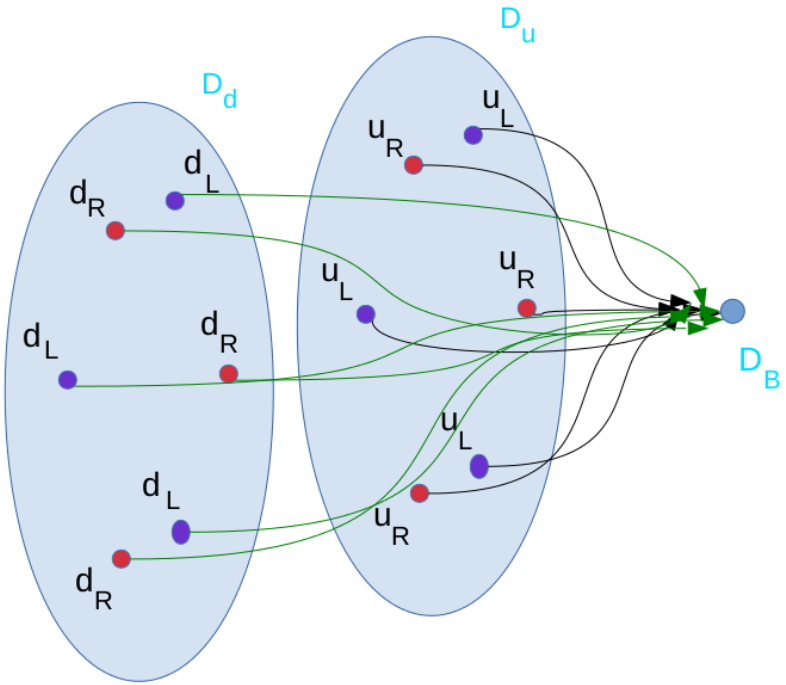
$$\Psi = \begin{pmatrix} 0 & \Psi_{12} \\ \Psi_{21} & 0 \end{pmatrix}$$

transform in bi-fundamental representation $H_1 \otimes \mathcal{H}_2^*$ of $U(N_1) \times U(N_2)$
$\rightarrow$ zero-modes as above

branes linked to point-branes: $\mathcal{H}_2 \cong \mathbb{C}$,

$$\Psi_{12} \in \mathcal{H}_\mu \otimes \mathbb{C} = \mathcal{H}_\mu$$

$\rightarrow$ 3+3 chiral zero-modes attached to extremal weights



- 3 generations, chiralities have different charges for lowest gauge modes
- can realize all standard model fermions
  e.g: baryons link $\mathcal{C}[\mu]$ with 3 coincident point branes ($\rightarrow$ color)
- same solutions in IKKT matrix model
  $\equiv$ noncommutative $\mathcal{N} = 4$ SYM, relation string theory, gravity

conclusion

- $\exists$ rich class of background solutions of $\mathcal{N} = 4$ SYM
  $\mathbb{R}^4 \times \mathcal{C}[\mu]$ ... self-intersecting fuzzy compact $SU(3)$ orbits
- interesting low-energy physics:
  space-filling $\rightarrow$ chiral fermions, scalars & gauge fields
- spinning branes $\rightarrow$ hierarchy,
  Lorentz-invariant low-energy sector
- not far from standard model (?!)
  cf. H.S., J. Zahn arXiv:1401.2020, H.S. arXiv:1411.3139
- open issues: stability (resonances?), full SSB
  gravity in matrix model version (?)
  realization in string theory?

coherent states on $\mathbb{C}[\mu]$: let $p \in \mathcal{C}[\mu]$ ... north pole

$$\begin{aligned} SU(3) &\to \mathcal{C}[\mu] \\ g &\mapsto g \triangleright p \end{aligned}$$

stabilizer $\mathcal{K} \subset SU(3) \Rightarrow \mathcal{C}[\mu] \cong SU(3)/\mathcal{K}$ ... coadjoint orbit

let $|\mu\rangle \in \mathcal{H}_\mu$ ... highest weight vector in $\mathcal{H}_\mu$

$$\langle\mu|\vec{y}|\mu\rangle = \vec{p} \qquad \text{...localized at north pole}$$

def. $|\psi_g\rangle := g \triangleright |\mu\rangle$

projector $\Pi_p = |\psi_g\rangle\langle\psi_g| \in End(\mathcal{H}_\mu)$ ... independent of $\mathcal{K} \subset SU(3)$

$$\begin{aligned} \mathcal{C}[\mu] \cong SU(3)/\mathcal{K} &\to End(\mathcal{H}_\mu) \\ p &\mapsto \Pi_p := |\psi_g\rangle\langle\psi_g| \quad =: \delta_N(y-p) \end{aligned}$$

$$\langle p|\vec{y}|p\rangle = tr(\Pi_p \vec{y}) = \vec{p} \qquad \text{... sweeps out } \mathcal{C}[\mu]$$

example: fuzzy $\mathbb{C}P^2_N$ (Grosse & Strohmaier, Balachandran etal, ...)

arises for $\mu = N\Lambda_1$ or $\mu = N\Lambda_2$, stabilizer $\mathcal{K} = SU(2) \times U(1)$

satisfy the relations

$$\begin{aligned} [Y^a, Y^b] &= \tfrac{i}{2}\Lambda_N f_{abc}\, Y^c \\ \delta_{ab} Y^a Y^b &= R^2, \\ d^c_{ab} Y^a Y^b &= R\, \tfrac{2N/3+1}{\sqrt{\frac{1}{3}N^2+N}}\, Y^c. \end{aligned}$$

harmonic decomposition

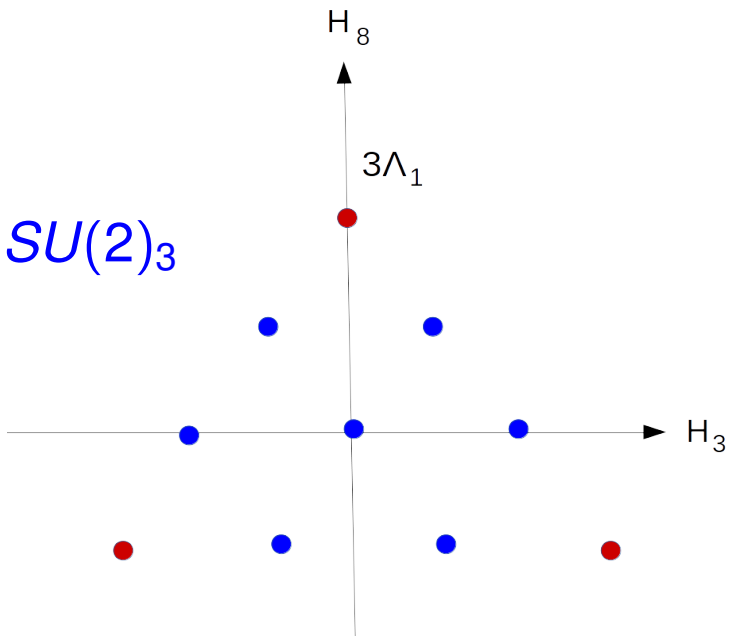
$$\mathcal{A} = End(\mathcal{H}_\mu) = \bigoplus_{n=0}^{N} \mathcal{H}_{(n,n)}.$$

squashed $\mathbb{C}P^2_N$:

projection $\Pi \equiv$ take away H_3, H_8

- $\langle\mu|\vec{y}|\mu\rangle = \vec{0}$, projected to 0 by Π
- acting with $SU(3)$ on $|\mu\rangle$: stabilizer $SU(2)_3$

  $\Rightarrow$ 4D orbit along 4567 plane
- analogous for 3 Weyl images of μ



- three 4D sheets intersecting at origin
- extremal weight states $|\mu\rangle$ = coherent states on different sheets at origin

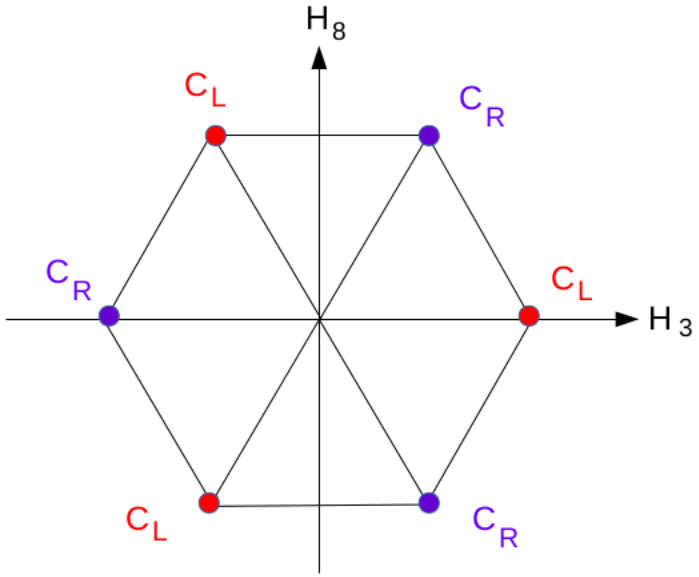
6-dimensional squashed $\mathcal{C}[\mu]$:

6 extremal weights $\mathcal{W}\mu$ located at origin, 6-dimensional orbits

$\Rightarrow$ 6-fold covering of $\mathbb{R}^6$ near origin,

$3+3$ oriented sheets C_L, C_R with opposite orientation

(defined by $Pfaff(\theta^{ab})$)



squashed $\mathcal{C}[(N,1)]$ ... squashed S^2 bundle over $\mathbb{C}P^2$